# Two results on Goldbach–Linnik problems for cubes of primes *

Yuhui Liu
School of Science, Jiangnan University
Wuxi 214122, Jiangsu, China
Email: tjliuyuhui@outlook.com

**Abstract** In this paper, it is proved that every pair of sufficiently large even integers can be represented in the form of a pair of equations of eight prime cubes and 609 powers of 2, and each sufficiently large even integer is the sum of eight cubes of primes and 157 powers of 2. These results constitute refinements upon those of Z. X. Liu [6] and of X. D. Zhao and W. X. Ge [10].

**2010 Mathematics Subject Classification**: 11P32, 11P55.

**Key Words**: Waring–Goldbach problem, Hardy–Littlewood method, additive number theory.

## 1 Introduction

In 1938, Hua [2] proved that each large odd integer is the sum of nine cubes of primes. Later, Roth [9] showed that the number of variables in Hua's theorem can be reduced to eight, if one of them is allowed to be a positive interger $m$, i.e. every large integer $N$ can be written as

$$N = m^3 + p_2^3 + \cdots + p_8^3.$$

Let $P_r$ denote an almost-prime with at most $r$ prime factors, counted according to multiplicity. The previous best result of $m$ is due to Kawada [3] who obtained $m$ be an almost-prime $P_3$.

In view of this result, it is reasonable to propose the conjecture that every sufficiently large even integer satisfying some necessary congruence conditions can be expressed as the sum of eight prime cubes. However, this conjecture is perhaps out of reach at present times.

In 2001, Liu and Liu [4] proved that every large even integer $N$ can be written as a sum of eight cubes of primes and a bounded number $K$ of powers of 2,

$$N = p_1^3 + p_2^3 + \cdots + p_8^3 + 2^{v_1} + 2^{v_2} + \cdots + 2^{v_K}. \tag{1.1}$$

*Project supported by the National Natural Science Foundation of China (grant no.11771333) and the CSC Scholarship (grant no.201906260145).

An explicit value of $K = 358$ was firstly obtained by Liu, Liu [4]. Later $K$ was improved by Platt and Trudgian [7], where $K = 341$. While in 2020, Zhao and Ge [10] sharpened the value of $K$ to 204.

As a comparison, Liu [6] considered the equations

$$\begin{aligned} N_1 &= p_1^3 + \cdots + p_8^3 + 2^{v_1} + 2^{v_2} + \cdots + 2^{v_k}, \\ N_2 &= p_9^3 + \cdots + p_{16}^3 + 2^{v_1} + 2^{v_2} + \cdots + 2^{v_k}, \end{aligned} \tag{1.2}$$

and proved that for $k = 1432$, the equations (1.2) are solvable. In 2015, Platt and Trudgian [7] improved the value of $k$ to 1364. In this paper, we obtain a further improvement of the value of $k$ and $K$ by giving the following Theorems 1 and 2.

**Theorem 1** *For $k = 609$, the equations* (1.2) *are solvable for every pair of sufficiently large positive even integers $N_1$ and $N_2$.*

**Theorem 2** *For $K = 157$, the equation* (1.1) *is solvable for every sufficiently large positive even integer $N$.*

**Remark**: In order to prove Theorems 1 and 2, we follow the line in Liu [5] and use the method of Zhao and Ge [10] to handle the minor arcs. The difference is the estimates of major arcs. We obtain a better lower bound on $J(n)$ and $\mathfrak{S}(n)$ (see Lemmas 2.3 and 2.4). And also we get a better upper bound on the integral in Lemma 2.6, which further results in our improvements.

## 2 Notation and some preliminary lemmas

For the proof of the theorems, in this section we introduce the necessary notation and lemmas.

Throughout this paper, by $N_1$ and $N_2$ we denote sufficiently large even integers. In addition, let $\delta < 10^{-4}$ be a fixed positive constant, and let $\varepsilon < 10^{-10}$ be an arbitrarily small positive constant not necessarily the same in different formulae. The letter $p$, with or without subscripts, is reserved for a prime number. We use $e(\alpha)$ to denote $e^{2\pi i\alpha}$ and $e_q(\alpha) = e(\alpha/q)$. By $A \sim B$ we mean that $B < A \leqslant 2B$. We denote by $(m, n)$ the greatest common divisor of $m$ and $n$. As usual, $\varphi(n)$ stands for Euler's function. For $i = 1, 2$, as the values in Ren [8] (which we will use in Lemma 2.6), let

$$U_i = \left(\frac{N_i}{16(1+\delta)}\right)^{\frac{1}{3}}, \quad V_i = U_i^{\frac{5}{6}}, \quad L = \frac{\log(\frac{N_1}{\log N_1})}{\log 2},$$

$$S_i = S_i(\alpha_i, U_i) = \sum_{p\sim U_i} (\log p) e(\alpha_i p^3), \quad T_i = T_i(\alpha_i, V_i) = \sum_{p\sim V_i} (\log p) e(\alpha_i p^3),$$

$$H(\alpha_i) = \sum_{v\le L} e\left(\alpha_i 2^v\right), \quad \mathcal{E}(\lambda) = \{\alpha_i \in (0,1] : |H(\alpha_i)| \ge \lambda L\},$$

$$A(n,q) = \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \left( \sum_{\substack{r=1 \\ (r,q)=1}}^{q} e_q(ar^3)\right)^8 e_q(-an).$$

For the application of the Hardy–Littlewood method, we need to define the Farey dissection. For this purpose, as the values in Liu and Lü [5] (which we will use in Lemma 2.1), we set

$$P_i = N_i^{\frac{1}{9}-2\varepsilon}, \;\; Q_i = N_i^{\frac{8}{9}+\varepsilon},$$

and for $(a_i, q_i) = 1$, $1 \le a_i \le q_i$, put

$$\mathfrak{M}_i(q_i, a_i) = \left(\frac{a_i}{q_i} - \frac{1}{q_i Q_i}, \frac{a_i}{q_i} + \frac{1}{q_i Q_i}\right], \;\; \mathfrak{M}_i = \bigcup_{1\le q_i \le P_i} \bigcup_{\substack{a_i=1 \\ (a_i,q_i)=1}}^{q_i} \mathfrak{M}_i(q_i, a_i),$$

$$\mathfrak{I}_0 = \left(\frac{1}{Q_i}, 1 + \frac{1}{Q_i}\right], \;\; \mathfrak{m}_i = \mathfrak{I}_0 \setminus \mathfrak{M}_i.$$

Then it follows from orthogonality that

$$\begin{aligned} R(N_1, N_2) &= \sum_{\substack{N_1 = p_1^3+\cdots+p_8^3+2^{v_1}+2^{v_2}+\cdots+2^{v_k} \\ N_2 = p_9^2+\cdots+p_{16}^3+2^{v_1}+2^{v_2}+\cdots+2^{v_k} \\ p_1,\cdots,p_4\sim U_1,\; p_5,\cdots,p_8\sim V_1,\; p_9,\cdots,p_{12}\sim U_2, \\ p_{13},\cdots,p_{16}\sim V_2,\; v_j\le L\,(j=1,2,\cdots,k)}} (\log p_1)(\log p_2)\cdots(\log p_{16}) \\ &= \int_0^1\int_0^1 S_1^4 T_1^4 S_2^4 T_2^4 H^k(\alpha_1+\alpha_2) e(-\alpha_1 N_1 - \alpha_2 N_2) d\alpha_1 d\alpha_2. \end{aligned} \tag{2.1}$$

Now we state the lemmas required in this paper.

**Lemma 2.1**. *For* $\frac{N_i}{2} < n \leqslant N_i\,(i=1,2)$, *we have*

$$\int_{\mathfrak{M}_i} S_i^4 T_i^4 e(-\alpha n) d\alpha_i = \frac{1}{3^8}\mathfrak{S}(n) J(n) + O\left(U_i V_i^4 L^{-1}\right).$$

*Here $\mathfrak{S}(n)$ is defined as $\mathfrak{S}(n) = \sum_{q=1}^{\infty} A(n,q)$, and satisfies $\mathfrak{S}(n) \gg 1$ for $n \equiv 0 \,(\bmod\ 2)$. $J(n)$ is defined as*

$$J(n) = \sum_{\substack{n=m_1+\cdots+m_8 \\ {U_i}^3 < m_1,\cdots,m_4 \le (2U_i)^3,\ {V_i}^3 < m_5,\cdots,m_8 \le (2V_i)^3}} (m_1 \cdots m_8)^{-\frac{2}{3}},$$

*and satisfies $U_i V_i^4 \ll J(n) \ll U_i V_i^4$.*

**Proof**. This can be found in Lemma 2.1 in Liu and Lü [5].

**Lemma 2.2**. i) *Let $\alpha = \frac{a}{q} + \lambda$ subject to $1 \le a \le q \le N_i^{\frac{1}{2}}$, $(a,q) = 1$, and $|\lambda| \le 1/qN_i^{\frac{1}{2}}$, we have*

$$S_i \ll U_i^{1-\frac{1}{12}+\varepsilon} + \frac{U_i^{1+\varepsilon}}{\sqrt{q\left(1 + N\left|\alpha - \frac{a}{q}\right|\right)}}.$$

ii) *For $\alpha \in \mathfrak{m}_i$ $(i = 1, 2)$, we have $\max\limits_{\alpha \in \mathfrak{m}_i} |S_i| \ll N_i^{\frac{11}{36}+\varepsilon}$.*

**Proof**. i) is Lemma 2.4 in Zhao [11].

For ii), by Dirichlet's lemma on rational approximations, each real number $\alpha \in \mathfrak{m}_i$ can be written as $\alpha = \frac{a}{q} + \lambda$ with $1 \le q \le N_i^{\frac{1}{2}}$, $(a,q) = 1$, and $|\lambda| \le 1/qN_i^{\frac{1}{2}}$. If $q \le P_i = N_i^{\frac{1}{9}-2\varepsilon}$, since $\alpha \in \mathfrak{m}_i$, we have $|\lambda| > 1/qQ_i$; Otherwise, $q > P_i$. In either case, we have

$$\sqrt{q\left(1 + N_i\left|\alpha - \frac{a}{q}\right|\right)} > \min\left(P_i^{\frac{1}{2}}, \left(\frac{N_i}{Q_i}\right)^{\frac{1}{2}}\right) = N_i^{\frac{1}{18}-\varepsilon}.$$

Then it follows from Lemma 2.2 i) that

$$\begin{aligned} S_i &\ll U_i^{1-\frac{1}{12}+\varepsilon} + \frac{U_i^{1+\varepsilon}}{N_i^{\frac{1}{18}-\varepsilon}} \\ &\ll (N_i^{\frac{1}{3}})^{\frac{11}{12}+\varepsilon} + N_i^{\frac{5}{18}+\varepsilon} \\ &\ll N_i^{\frac{11}{36}+\varepsilon}, \end{aligned}$$

which completes the proof of Lemma 2.2.

**Lemma 2.3**. *We have $J(n) \ge 90.9653 U_i V_i^4$.*

**Proof**. The domain of $J(n)$ can be written as

$$\mathfrak{D} = \{(m_1, \cdots, m_8) : U_i^3 < m_1, \cdots, m_4 \le (2U_i)^3,\ V_i^3 < m_5, \cdots, m_8 \le (2V_i)^3\}$$

with $m_1 = n - m_2 - \cdots - m_8$. Define

$$\mathfrak{D}^* = \{(m_2, \cdots, m_8) : \frac{8}{3}U_i^3 < m_2, \cdots, m_4 \le 5U_i{}^3,\ V_i^3 < m_5, \cdots, m_8 \le 8V_i{}^3\}.$$

We choose the numbers 8/3 and 5 in the above definition of $\mathfrak{D}^*$ to obtain the following inequality (2.2). Then for $(m_2, \cdots, m_8) \in \mathfrak{D}^*$, we can deduce from $(1-\eta)N_i \leqslant n \leqslant N_i$ that

$$U_i^3 < m_1 = n - m_2 - \cdots - m_8 \le 8U_i^3. \tag{2.2}$$

Thus $\mathfrak{D}^*$ is a subset of $\mathfrak{D}$. Without loss of generality, we set $m_2 = \min\{m_2, m_3, m_4\}$. Then we have

$$\begin{aligned} m_2^{-\frac{2}{3}} &\ge \left\{\frac{1}{3}(n - m_1 - m_5 - m_6 - m_7 - m_8)\right\}^{-\frac{2}{3}} \\ &\ge \left(\frac{1}{3}\right)^{-\frac{2}{3}} (16U_i^3 - m_1)^{-\frac{2}{3}}. \end{aligned}$$

It follows from Euler–Maclaurin summation that

$$\begin{aligned} J(n) &\ge \sum_{\substack{U_i^3 < m_1 \le 8U_i{}^3,\ \frac{8}{3}U_i^3 < m_3, m_4 \le 5U_i{}^3, \\ V_i^3 < m_5, \cdots, m_8 \le 8V_i{}^3}} (m_1 \cdots m_8)^{-\frac{2}{3}} \\ &\ge \left(\frac{1}{3}\right)^{-\frac{2}{3}} \sum_{U_i^3 < m_1 \le 8U_i{}^3} (m_1)^{-\frac{2}{3}} (16U_i^3 - m_1)^{-\frac{2}{3}} \sum_{\frac{8}{3}U_i^3 < m_3, m_4 \le 5U_i^3} (m_3 m_4)^{-\frac{2}{3}} \\ &\quad \times \sum_{V_i^3 < m_5, \cdots, m_8 \le 8V_i^3} (m_5 \cdots m_8)^{-\frac{2}{3}} \\ &\ge \left(\frac{1}{3}\right)^{-\frac{2}{3}} \left(\int_1^8 t^{-\frac{2}{3}} (16-t)^{-\frac{2}{3}}\right) \cdot 3^6 \cdot \left(5^{\frac{1}{3}} - \left(\frac{8}{3}\right)^{\frac{1}{3}}\right)^2 U_i V_i{}^4 \\ &\ge 90.9653 U_i V_i{}^4. \end{aligned}$$

The proof of Lemma 2.3 is completed.

**Lemma 2.4**. *Let* $\Xi(N_i, k) = \{(1-\eta)N_i \le n_i \le N_i : n_i = N_i - 2^{v_1} - 2^{v_2} - \cdots - 2^{v_k}\}$ $(i = 1, 2)$. *For* $k \ge 2$ *and* $N_1 \equiv N_2 \equiv 0 \pmod 2$, *we have*

$$\sum_{\substack{n_1 \in \Xi(N_1, k) \\ n_2 \in \Xi(N_2, k) \\ n_1 \equiv n_2 \equiv 0 \pmod 2}} \mathfrak{S}(n_1)\mathfrak{S}(n_2) \ge 0.8909 L^k.$$

**Proof**. We follow the notations and argument as Lemma 2.3 in Zhao and Ge [10]. Let $C_0 = 0.8206744593$. And for convenience, we set $q = \prod_{3<p<14} p = 5005$. Now we have

$$\sum_{\substack{n_1\in\Xi(N_1,k)\\ n_2\in\Xi(N_2,k)\\ n_1\equiv n_2\equiv 0 \pmod 2}} \mathfrak{S}(n_1)\mathfrak{S}(n_2)$$

$$\geq \left(2(1-\frac{1}{2^8})C_0\right)^2 \sum_{1\leq j_1\leq q}\prod_{3<p_1<14}\big(1+A(j_1,p_1)\big)\sum_{1\leq j_2\leq q}\prod_{3<p_2<14}\big(1+A(j_2,p_2)\big)\sum_{\substack{n_1\in\Xi(N_1,k)\\ n_2\in\Xi(N_2,k)\\ n_1\equiv n_2\equiv 0 \pmod 2\\ n_1\equiv j_1 \pmod q\\ n_2\equiv j_2 \pmod q}} 1. \tag{2.3}$$

Let $S$ denote the last inner sum of (2.3), we have

$$S = \sum_{\substack{1\leq v_1,\cdots,v_k\leq L,\ i=1,2\\ 2^{v_1}+\cdots+2^{v_k}\equiv N_i \pmod 2\\ 2^{v_1}+\cdots+2^{v_k}\equiv N_i-j_i \pmod q}} 1.$$

Noting the fact that for $N_1 \equiv N_2 \equiv 0 \pmod 2$, $N_2 \equiv N_1 + t \pmod q$, $j_2 \equiv j_1 + t \pmod q$ $(1 \leq t \leq q)$, we have

$$\begin{aligned} S &= \sum_{\substack{1\leq v_1,\cdots,v_k\leq L\\ 2^{v_1}+\cdots+2^{v_k}\equiv N_1-j_1 \pmod q}} 1 \\ &= \left(\frac{L}{\rho(3q)}+O(1)\right)^k \sum_{\substack{1\leq v_1,\cdots,v_k\leq \rho(3q)\\ 2^{v_1}+\cdots+2^{v_k}\equiv a_j \pmod{3q}}} 1, \end{aligned}$$

where $a_j$ is the natural number in $[1, 3q]$ satisfying $a_j \equiv 0 \pmod 3$ and $a_j \equiv j \pmod q$; for odd $q$, let $\rho(q)$ denote the smallest positive integer $\rho$ such that $2^\rho \equiv 1 \pmod q$.
By Lemma 2.3 in Zhao and Ge [10], we obtain

$$S \geq \frac{(1-0.00006439)L^k}{3q} + O(L^{k-1}). \tag{2.4}$$

Moreover, we have

$$\begin{aligned} &\sum_{1\leq j_1\leq q}\prod_{3<p_1<14}\big(1+A(j_1,p_1)\big)\prod_{3<p_2<14}\big(1+A(j_2,p_2)\big) \\ &= \prod_{3<p<14}\sum_{1\leq j\leq q}\big(1+A(j,p)\big)^2 \\ &= \prod_{3<p<14}\left(p+\sum_{1\leq j\leq p}A(j,p)^2\right) \geq \prod_{3<p<14} p = q. \end{aligned} \tag{2.5}$$

On combining (2.3)-(2.5), we have

$$\begin{aligned}&\sum_{\substack{n_1\in\Xi(N_1,k)\\ n_2\in\Xi(N_2,k)\\ n_1\equiv n_2\equiv 0 \pmod 2}}\mathfrak{S}(n_1)\mathfrak{S}(n_2)\\ &\geq\left(2\left(1-\frac{1}{2^8}\right)C_0\right)^2\times q\times\frac{(1-0.0000643971)L^k}{3q}\\ &\geq 0.8909L^k. \qquad (2.6)\end{aligned}$$

Now we complete the proof of Lemma 2.4.

**Lemma 2.5**. *We have*

$$\text{meas}\left(\mathcal{E}(\lambda)\right)\ll N_i^{-E(\lambda)}, \text{with } E(0.9570253)>\frac{8}{9}+10^{-10}.$$

**Proof**. Thanks to the help of Professor Platt, Lemma 2.5 follows from running the Pari program from Alessandro Languasco in Platt and Trudgian [7].

**Lemma 2.6**. *We have*

$$\int_0^1|S_i^4T_i^4|d\alpha\leq 7.3909U_i{V_i}^4.$$

**Proof**. We follow the notations in Ren [8]. By (2.6) and (2.7) in Ren [8] and Proposition 2 in Elsholtz and Schlage-Puchta [1], we obtain

$$\sum_{\frac{N_i}{8}<l\leq N_i}r^2(l)\leq\rho(0)\leq\big(C+o(1)\big)U_i{V_i}^4L^{-8},$$

where $C=100552$ is defined in Elsholtz and Schlage-Puchta [1, Proposition 2]. Then we have

$$\begin{aligned}\int_0^1|S_i^4T_i^4|d\alpha&\leq(\log 2U_i)^4(\log 2V_i)^4\rho(0)\\ &\leq\left(\frac{1}{3}\right)^4\left(\frac{5}{18}\right)^4\big(C+o(1)\big)U_i{V_i}^4\\ &\leq 7.3909U_i{V_i}^4.\end{aligned}$$

## 3 Auxiliary estimates

We initiate our proof by recalling the Farey dissections (2.1) that

$$\begin{aligned} R(N_1,N_2) &= \int_0^1\int_0^1 S_1^4T_1^4S_2^4T_2^4H^k(\alpha_1+\alpha_2)e(-\alpha_1N_1-\alpha_2N_2)d\alpha_1d\alpha_2 \\ &= \left(\int_{\mathfrak{M}_1}+\int_{\mathfrak{m}_1\bigcap\mathcal{E}(\lambda)}+\int_{\mathfrak{m}_1\backslash\mathcal{E}(\lambda)}\right)\left(\int_{\mathfrak{M}_2}+\int_{\mathfrak{m}_2\bigcap\mathcal{E}(\lambda)}+\int_{\mathfrak{m}_2\backslash\mathcal{E}(\lambda)}\right) \\ &\quad S_1^4T_1^4S_2^4T_2^4H^k(\alpha_1+\alpha_2)e(-\alpha_1N_1-\alpha_2N_2)d\alpha_1d\alpha_2 \\ &= \sum_{s=1}^{3}\sum_{t=1}^{3}R_{st}, \end{aligned}$$

where $R_{st}$ denotes the combination of $s$-th term in the first bracket and the $t$-th term in the second bracket.

**Proposition 1**. *We have* $R_{11}\geq 0.0001712U_1U_2V_1^4V_2^4L^k$.

**Proof**. By Lemmas 2.3, 2.4 and 2.1, we get

$$\begin{aligned} R_{11} &= \int_{\mathfrak{M}_1} S_1^4T_1^4H^k(\alpha_1)e(-\alpha_1N_1)d\alpha_1\int_{\mathfrak{M}_2} S_2^4T_2^4H^k(\alpha_2)e(-\alpha_2N_2)d\alpha_2 \\ &= \sum_{\substack{n_1\in\Xi(N_1,k)\\ n_2\in\Xi(N_2,k)}}\int_{\mathfrak{M}_1} S_1^4T_1^4e(-\alpha_1n_1)d\alpha_1\int_{\mathfrak{M}_2} S_2^4T_2^4e(-\alpha_2n_2)d\alpha_2 \\ &\geq \left(\frac{1}{3^8}\right)^2 J(n_1)J(n_2)\sum_{\substack{n_1\in\Xi(N_1,k)\\ n_2\in\Xi(N_2,k)}}\mathfrak{S}(n_1)\mathfrak{S}(n_2)+O\left(U_1V_1^4U_2V_2^4L^{k-1}\right) \\ &\geq \left(\frac{1}{3^8}\right)^2\times(90.9653)^2\times 0.8909U_1V_1^4U_2V_2^4L^k \\ &\geq 0.0001712U_1U_2V_1^4V_2^4L^k. \end{aligned}$$

This completes the proof of Proposition 1.

We now introduce three lemmas essential in our proof of the following propositions and the theorems.

**Lemma 3.1** *We have*

$$I_{1i}=\int_{\mathfrak{M}_i}\left|S_i^4T_i^4H^{\frac{k}{2}}(2\alpha_i)\right|d\alpha_i\leq 7.3909U_iV_i^4L^{\frac{k}{2}}.$$

**Proof**. It follows from Lemma 2.6 and the definition of $\mathcal{E}(\lambda)$ that

$$\begin{aligned}
I_{1i} &\le \left(\max_{\alpha\in\mathfrak{M}_i}|H(2\alpha_i)|\right)^{\frac{k}{2}}\int_{\mathfrak{M}_i}|S_i^4T_i^4|d\alpha_i\\
&\le L^{\frac{k}{2}}\int_0^1|S_iT_i|^4d\alpha_i\\
&\le 7.3909U_iV_i^4L^{\frac{k}{2}}.
\end{aligned}$$

**Lemma 3.2** *We have*

$$I_{2i}=\int_{\mathfrak{m}_1\bigcap\mathcal{E}(\lambda)}\left|S_i^4T_i^4H^{\frac{k}{2}}(2\alpha_i)\right|d\alpha_i\ll U_iV_i^4L^{\frac{k}{2}-1}.$$

**Proof**. Using the definition of $\mathcal{E}(\lambda)$, the trivial bound of $H(2\alpha_i)$ and Lemma 2.2 ii), we get

$$\begin{aligned}
I_{2i} &\ll \left(\max_{\alpha\in\mathfrak{m}_i}|S_iT_i|\right)^4L^{\frac{k}{2}}\int_{\mathcal{E}(\lambda)}1d\alpha_i\\
&\ll U_iV_i^4L^{\frac{k}{2}-1}.
\end{aligned}$$

**Lemma 3.3** *We have*

$$I_{3i}=\int_{\mathfrak{m}_i\setminus\mathcal{E}(\lambda)}\left|S_i^4T_i^4H^{\frac{k}{2}}(2\alpha_i)\right|d\alpha_i\ \le 7.3909U_iV_i^4\lambda^{\frac{k}{2}}L^{\frac{k}{2}}.$$

**Proof**. Similar to Lemma 3.1, we deduce from the trivial bound $\left|H(2\alpha_i)\right|\le\left|H(\alpha_i)\right|+2\le\big(1+o(1)\big)\lambda L$ and Lemma 2.6 that

$$I_{3i}\le(\lambda L)^{\frac{k}{2}}\int_0^1|S_iT_i|^4d\alpha_i\ \le 7.3909U_iV_i^4\lambda^{\frac{k}{2}}L^{\frac{k}{2}}.$$

According to Cauchy's inequality, we have

$$|H(\alpha_1+\alpha_2)|\le\sqrt{|H(2\alpha_1)H(2\alpha_2)|}.$$

Now we turn to give an upper bound of $R_{12},R_{21},R_{22},R_{23}$ and $R_{32}$.

**Proposition 2**. *We have* $R_{12}$, $R_{21}$, $R_{22}$, $R_{23}$, $R_{32}\ll U_1U_2V_1^4V_2^4L^{k-1}$.
**Proof**.

$$\begin{aligned}
R_{12} &= \int_{\mathfrak{M}_1}\int_{\mathfrak{m}_2\bigcap\mathcal{E}(\lambda)}S_1^4T_1^4S_2^4T_2^4H^k(\alpha_1+\alpha_2)e(-\alpha_1N_1-\alpha_2N_2)d\alpha_1d\alpha_2\\
&\ll I_{11}I_{22}\ \ll U_1U_2V_1^4V_2^4L^{k-1}. \qquad (3.1)
\end{aligned}$$

Similiarily to (3.1), we obtain $R_{21} \ll I_{12}I_{21} \ll U_1U_2V_1^4V_2^4L^{k-1}$.
Moreover,

$$R_{22} = \int_{\mathfrak{m}_1 \bigcap \mathcal{E}(\lambda)} \int_{\mathfrak{m}_2 \bigcap \mathcal{E}(\lambda)} S_1^4T_1^4S_2^4T_2^4H^k(\alpha_1+\alpha_2)e(-\alpha_1N_1-\alpha_2N_2)d\alpha_1d\alpha_2$$
$$\ll I_{21}I_{22} \ll U_1U_2V_1^4V_2^4L^{k-1}. \tag{3.2}$$

Besides, we have

$$R_{23} = \int_{\mathfrak{m}_1 \bigcap \mathcal{E}(\lambda)} \int_{\mathfrak{m}_2 \backslash \mathcal{E}(\lambda)} S_1^4T_1^4S_2^4T_2^4H^k(\alpha_1+\alpha_2)e(-\alpha_1N_1-\alpha_2N_2)d\alpha_1d\alpha_2$$
$$\ll I_{21}I_{32} \ll U_1U_2V_1^4V_2^4L^{k-1}. \tag{3.3}$$

Analogously to (3.3), we get $R_{32} \ll I_{22}I_{31} \ll U_1U_2V_1^4V_2^4L^{k-1}$.

Next we give the estimation for $R_{13}$ and $R_{31}$.

**Proposition 3**. *We have* $R_{13}$, $R_{31} \le (7.3909)^2U_1U_2V_1^4V_2^4L^k\lambda^{\frac{k}{2}}$.
**Proof**.

$$R_{13} = \int_{\mathfrak{M}_1} \int_{\mathfrak{m}_2 \backslash \mathcal{E}(\lambda)} S_1^4T_1^4S_2^4T_2^4H^k(\alpha_1+\alpha_2)e(-\alpha_1N_1-\alpha_2N_2)d\alpha_1d\alpha_2$$
$$\le I_{11}I_{32} \le (7.3909)^2U_1U_2V_1^4V_2^4L^k\lambda^{\frac{k}{2}}.$$

In the similar manner, we get $R_{31} \le I_{12}I_{31} \le (7.3909)^2U_1U_2V_1^4V_2^4L^k\lambda^{\frac{k}{2}}$.
It remains to estimate $R_{33}$.

**Proposition 4**. *We have* $R_{33} \le (7.3909)^2U_1U_2V_1^4V_2^4\lambda^kL^k$.
**Proof**.

$$R_{33} = \int_{\mathfrak{m}_1 \backslash \mathcal{E}(\lambda)} \int_{\mathfrak{m}_2 \backslash \mathcal{E}(\lambda)} S_1^4T_1^4S_2^4T_2^4H^k(\alpha_1+\alpha_2)e(-\alpha_1N_1-\alpha_2N_2)d\alpha_1d\alpha_2$$
$$\le I_{31}I_{32} \le (7.3909)^2U_1U_2V_1^4V_2^4\lambda^kL^k.$$

# 4 Proof of Theorems 1 and 2

On combining Propositions 1-4, we arrive at the conclusion that

$$R(N_1,N_2) \ge \Big(0.0001712 - 2(7.3909)^2\lambda^{\frac{k}{2}} - (7.3909)^2\lambda^k\Big)U_1U_2V_1^4V_2^4L^k.$$

When $k \ge 609$ and $\lambda = 0.9570253$, we get $R(N_1,N_2) > 0$ for all sufficiently large even integers $N_1, N_2$, which proves Theorem 1.

For Theorem 2, we use the same notations as in Zhao and Ge [10]. On replacing Lemma 2.1, Lemma 2.5 and Lemma 2.6 in Zhao and Ge [10] with Lemma 2.5, Lemma 2.6 and Lemma 2.3 in this paper, respectively. We have

$$R(N) > \left(\frac{1}{3^8} \times 0.5449 \times 90.9653 - 7.3909\lambda^k\right) UV^4L^k.$$

When $k \geq 157$ and $\lambda = 0.9570253$, we get $R(N) > 0$ for all sufficiently large even integer $N$, which proves Theorem 2.

**Acknowledgement**. The author would like to thank the anonymous referee for his/her patience and time in refereeing this manuscript.

# References

[1] C. Elsholtz, J.-C. Schlage-Puchta, The density of integers representable as the sum of four prime cubes, Acta Arith., (4) 192 (2020), 363-369.

[2] L. K. Hua, Some results in the additive prime number theory, Quart. J. Math. Oxford 9 (1938), 68-80.

[3] K. Kawada, Note on the sum of cubes of primes and an almost prime, Arch. Math., 69 (1997), 13-19.

[4] J. Y. Liu, M. C. Liu, Representation of even integers by cubes of primes and powers of 2, Acta Math. Hungar., (3) 91 (2001), 217-243.

[5] Z. X. Liu, G. S. Lü, Eight cubes of primes and powers of 2, Acta Arith., 145 (2010), 171-192.

[6] Z. X. Liu, On pairs of quadratic equations in primes and powers of 2, J. Number Theory 133 (2013), 3339-3347.

[7] D. Platt and T. Trudgian, Linnik's approximation to Goldbach's conjecture, and other problems, J. Number Theory 153 (2015), 54-62.

[8] X. M. Ren, Sums of four cubes of primes, J. Number theory 98 (2003), 156-171.

[9] K. F. Roth, On Waring's problem for cubes, Proc. Lond. Math. Soc., 53 (1951), 268-279.

[10] X. D. Zhao, W. X. Ge, Eight cubes of primes and 204 powers of 2, Int. J. Number Theory (7) 16 (2020), 1547-1555.

[11] X. D. Zhao, Goldbach–Linnik type problems on cubes of primes, Ramanujan J., (2020), https://doi.org/10.1007/s11139-020-00303-9.