NEW CHARACTERIZATIONS OF COMPLETENESS AND HAUSDORFF UNIFORMITY

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Abstract. Let $(X,\mathcal{U})$ be a uniform space and $(K(X),\mathcal{U}_H)$ be a hyperspace of non-empty compact subsets of X equipped with the Hausdorff uniformity. We show that $(X,\mathcal{U})$ is complete iff every $\mathcal{U}_H$-Cauchy net converges to its Kuratowski-Painlevé limit in $(K(X),\mathcal{U}_H)$. We also show that a metrizable locally convex space X is complete, iff for every compact subset of X, its closed convex hull is also compact. This is not necessarily true for non-metrizable spaces.

1. Introduction

In this paper, we present a novel proof demonstrating the completeness of a uniform space $(X,\mathcal{U})$ if and only if $(K(X),\mathcal{U}_H)$ is complete, where $(K(X),\mathcal{U}_H)$ is a hyperspace of non-empty compact subsets of X equipped with the Hausdorff uniformity.

The definition of Hausdorff uniformity is based on the idea of Hausdorff distance. Let (X,d) be a metric space and $A,B \subseteq X$. The excess of a set A over B is $e(A,B) = \sup\{d(a,B) : a \in A\}$. Pompeiou [16] has considered the distance of two sets defined as $e(A,B)+e(B,A)$. Later development led to a slightly different *Hausdorff distance* defined as

$$\rho_H(A,B) = \max\{e(A,B), e(B,A)\} = \inf\{\varepsilon > 0 : A \subseteq S_\varepsilon(B) \text{ and } B \subseteq S_\varepsilon(A)\},$$

where $S_\varepsilon(A)$ is an ε-enlargement of the set A. Note that Hausdorff distance can be infinite for unbounded sets. The result by Hahn [6] asserts that if X is a complete metric space, then the

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space of its closed and bounded subsets, equipped with the Hausdorff distance, is complete, too. This also holds for the space $CL(X)$, of all non-empty closed subsets of X. The classical proof of the completeness of the Hausdorf distance in the metrizable setting shows that the Kuratowksi-Painleve limit superior of a Cauchy sequence of non-empty closed sets is non-empty and then that the sequence actually converges to this limit superior. A good source for this proof is [13], and another earlier source is [2].

Another approach can be found in [1]. The assignment $A \mapsto d(\cdot, A)$ is an isometry between $(CL(X), \rho_H)$ and the space of distance functions equipped with a natural distance generated by the topology of uniform convergence. The function space approach to establish the completeness of ρ_H on $CL(X)$ is based on the fact that the uniform limit of a sequence of distance functions $(d(\cdot, A_n))$ on a complete metric space must be a distance function [1, Lemma 3.1.1].

Given a uniform space $(X,\mathcal{U})$, we can define a Hausdorff uniformity on the space $CL(X)$; the details are in the section below. However, completeness of X does not ensure the completeness of $CL(X)$; in fact, completeness of $CL(X)$ amounts to *supercompleteness* of X, see [5, 11]. The authors have found that space is supercomplete, iff it is paracompact, and the coarsest *locally fine* uniformity finer than the original uniformity is *fine*. One can easily find a complete uniform space that is not paracompact and hence not supercomplete. This leaves the question whether $K(X)$ is always complete, provided X is complete. While Morita previously established this result in [15], our proof offers a more constructive approach, showing that in a complete uniform space $(X,\mathcal{U})$ every $\mathcal{U}_H$-Cauchy net converges to its Kuratowski-Painlevé limit in $(K(X),\mathcal{U}_H)$.

In the second part of this paper, we investigate the interrelation between the completeness of a topological vector space and the property that the closed convex hull of a compact space is again compact. We can mention here that Hörmander's theorem [1, Theorem 3.2.9] ties Hausdorff distance to convexity. On the other hand, the above property is important in the study of spaces of minimal usco and minimal cusco maps, particularly in understanding the connections between these two spaces. Also, the fact that the Hausdorff uniformity is complete on the compact sets is significant in the examination of these spaces, [8, 9, 10] (the result itself is used in [10]). Thus, these applications can provide another connection of this part to the first one. It is worth noting that minimal usco and minimal cusco maps play a crucial role in various fields, such as functional analysis, optimization, and the study of the differentiability of Lipschitz functions, among others. For further applications and insights regarding these maps, we refer readers to the book [7].

2. Preliminaries

All topological spaces are assumed to be at least Hausdorff. By $\mathbb{Z},\mathbb{N},\mathbb{R},\mathbb{R}^+,\mathbb{C}$, we will denote the sets of integers, natural numbers (positive integers), real numbers, positive real numbers, and complex numbers, respectively. They will be considered to be equipped with their usual structures when needed. We will denote uniform space as $(X,\mathcal{U})$, where $\mathcal{U}$ is a uniformity (a system of entourages). We will also always assume that it is Hausdorff.

Whenever we consider topological vector space X, it will be over the field $\mathbb{K}$, that is, either $\mathbb{R}$ or $\mathbb{C}$. The field will be mentioned only if necessary. Its zero element will be denoted as 0

21 Mar 2024 04:54:53 PDT
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and the system of neighborhoods of 0 as $\mathcal{N}$. Recall that every topological vector space has a compatible uniformity generated by the entourages of the form

$$V_O = \{(x,y) \in X \times X : x - y \in O\},$$

where $O \in \mathcal{N}$.

The symbols $\overline{A}$ and $\operatorname{co} A$ stand for the closure and the convex hull of the set A, respectively. Denote $\overline{\operatorname{co}}\, A = \overline{\operatorname{co} A}$, a closed convex hull of A. A subset A of a topological vector space X is *bounded* (*totally bounded*) if for every $O \in \mathcal{N}$, there is $\lambda \in \mathbb{R}^+$ (finite $F \subseteq X$) such that $A \subseteq \lambda O$ ($A \subseteq F + O$). A set $A \subseteq Y$ is *absolutely convex* if it is convex and for every $\lambda \in \mathbb{K}$, such that $|\lambda| \leq 1$ it holds $\lambda A \subseteq A$.

The topological vector space is called *locally convex*, iff $\mathcal{N}$ has a base consisting of convex sets. In this case, it has a base consisting of absolutely convex sets. A completely metrizable locally convex topological vector space is called *Fréchet* space. For other basic notions, we refer to [3, 4, 12].

We will denote by

$\mathcal{P}(X)$	the system of all subsets of X;
$CL(X)$	the system of non-empty closed subsets of X;
$K(X)$	the system of non-empty compact subsets of X;
$CK(X)$	the system of non-empty compact and convex subsets of X.

Let $(X,\mathcal{U})$ be a uniform space. We will use the *Hausdorff uniformity* $\mathcal{U}_H$ on $CL(X)$, see [14], which is generated by entourages of the form

$$W_U = \{(A,B) \in CL(X) \times CL(X) : A \subseteq U[B] \text{ and } B \subseteq U[A]\},$$

where $U \in \mathcal{U}$. Note the similarity to the alternative definition of the Hausdorff distance mentioned in the introduction. Also note that we could define this uniformity on $\mathcal{P}(X)$, but it would not necessarily have the Hausdorff property, while on $CL(X)$, it does. It can be inherited to $K(X)$ and $CK(X)$. We will denote the topology generated by $\mathcal{U}_H$ as τ_H.

If X is a topological vector space, then we can generate Hausdorff uniformity from the standard uniformity on X, which is generated from the system of neighborhoods of 0, $\mathcal{N}$. In particular, $\mathcal{U}_H$ has a base consisting of the elements of the form

$$W_O = \{(A,B) \in CL(X) \times CL(X) : A \subseteq O + B \text{ and } B \subseteq O + A\},$$

where $O \in \mathcal{N}$. Note that it is sufficient when O runs through some basis on $\mathcal{N}$.

Another well known topology on $CL(X)$ is the *Vietoris topology*, denoted by τ_V, which is generated by the following subbase:

$$\{V^+ : V \text{ is an open subset of } X\} \cup \{V^- : V \text{ is an open subset of } X\},$$

where

$$V^+ = \{A \in CL(X) : A \subseteq V\} \text{ and } V^- = \{A \in CL(X) : A \cap V \neq \emptyset\}.$$

Proposition 2.1 ([14]). *Let $(X,\mathcal{U})$ be a uniform space, then τ_H and τ_V coincide on $K(X)$.*

3. Completeness of Hausdorff uniformity

Let X be a topological space and $(F_\sigma)_{\sigma\in\Sigma}$ be a net of non-empty closed subsets of X. Then $\operatorname{Li} F_\sigma$ is the set of all points $x \in X$ such that for every neighborhood U of x, there is $\sigma_0 \in \Sigma$ such that $F_\sigma \cap U \neq \emptyset$ for every $\sigma \geq \sigma_0$. $\operatorname{Ls} K_\sigma$ is the set of all points $x \in X$ such that for every neighbourhood U of x and every $\sigma \in \Sigma$ there is $\eta \geq \sigma$ such that $F_\eta \cap U \neq \emptyset$. One can easily see that $\operatorname{Li} F_\sigma \subseteq \operatorname{Ls} F_\sigma$. If $\operatorname{Li} F_\sigma = \operatorname{Ls} F_\sigma$, then $F = \operatorname{Li} F_\sigma = \operatorname{Ls} F_\sigma$ is the *Kuratowski-Painlevé limit* of $(F_\sigma)_{\sigma\in\Sigma}$, see [1].

Lemma 3.1. *Let X be a regular space. Let $(F_\sigma)_{\sigma\in\Sigma}$ be a net of non-empty closed sets which converges to a non-empty closed set F in $(CL(X), \tau_V)$. Then F is the Kuratowski-Painlevé limit of $(F_\sigma)_{\sigma\in\Sigma}$.*

Proof. It is easy to verify that $F \subseteq \operatorname{Li} F_\sigma$. Suppose that $x \in F$. We will show that $\operatorname{Ls} F_\sigma \subseteq F$. Suppose there is $x \in \operatorname{Ls} F_\sigma \setminus F$. The regularity of X implies that there are two disjoint open sets U, V in X such that $x \in U$ and $F \subseteq V$. There is $\sigma_0 \in \Sigma$ such that $F_\sigma \subseteq V$ for every $\sigma \geq \sigma_0$, a contradiction since $x \in \operatorname{Ls} F_\sigma$. □

Lemma 3.2. *Let $(X, \mathcal{U})$ be a uniform space and $(F_\sigma)_{\sigma\in\Sigma}$ be a $\mathcal{U}_H$-Cauchy net of subsets of X. Then $\operatorname{Li} F_\sigma = \operatorname{Ls} F_\sigma$.*

Proof. We want to prove that $\operatorname{Ls} F_\sigma \subseteq \operatorname{Li} F_\sigma$. Let $x \in \operatorname{Ls} F_\sigma$ and V be an open neighborhood of x. Let U be an open symmetric element fom $\mathcal{U}$ such that $(U \circ U \circ U)[x] \subseteq V$. Since $(F_\sigma)_{\sigma\in\Sigma}$ is a $\mathcal{U}_H$-Cauchy net, there is $\sigma_0 \in \Sigma$ such that $F_\sigma \subseteq U[F_{\sigma_0}]$ and $F_{\sigma_0} \subseteq U[F_\sigma]$ for every $\sigma \geq \sigma_0$. Since $x \in \operatorname{Ls} F_\sigma$, there is $\eta \geq \sigma_0$ such that $F_\eta \cap U[x] \neq \emptyset$, so $x \in U[F_\eta] \subseteq (U \circ U)[F_{\sigma_0}] \subseteq (U \circ U \circ U)[F_\sigma]$ and thus $\emptyset \neq F_\sigma \cap (U \circ U \circ U)[x] \subseteq F_\sigma \cap V$ for every $\sigma \geq \sigma_0$. □

A family $\mathcal{A}$ of subsets of a uniform space $(X, \mathcal{U})$ *contains small sets* (resp. *almost small sets*) iff for each $U \in \mathcal{U}$ there are $A \in \mathcal{A}$ and $x \in X$ (resp. finite $D \subseteq X$) such that $A \subseteq U[x]$ (resp. $A \subseteq U[D]$).

Theorem 3.3 ([12])**.** *A uniform space is complete if and only if each family of closed sets, which has the finite intersection property and contains small sets, has a non-empty intersection.*

Theorem 3.4. *A uniform space is complete if and only if each family of closed sets, which has the finite intersection property and contains almost small sets, has a non-empty intersection.*

Proof. Let $(X, \mathcal{U})$ be a complete uniform space. Let $\mathcal{F}$ be a family of closed sets in X, which has the finite intersection property and contains almost small sets. Let $\mathcal{F}'$ be the family of all finite intersections of elements of $\mathcal{F}$. Then $\mathcal{F}'$ is a filter base and let $\mathcal{A}$ be an ultrafilter which contains $\mathcal{F}'$. We will show that $\mathcal{A}$ contains small sets. Let $U \in \mathcal{U}$ be arbitrary and $V \in \mathcal{U}$ be a closed set in $X \times X$ such that $V \subseteq U$. Since $\mathcal{F}$ contains almost small sets, there is $F \in \mathcal{F}$ such that $F \subset V[D]$ for a finite set $D = \{x_1, \ldots, x_n\}$. Then $F = (F \cap V[x_1]) \cup \ldots \cup (F \cap V[x_n]) \in \mathcal{A}$. Since $\mathcal{A}$ is an ultrafilter, there is $i \in \{1, 2, \ldots, n\}$ such that $F \cap V[x_i] \in \mathcal{A}$. Since $F \cap V[x_i] \subseteq U[x_i]$, $\mathcal{A}$ contains small sets. Put $\mathcal{A}' = \{\overline{A} : A \in \mathcal{A}\}$ and observe that $\mathcal{F} \subseteq \mathcal{F}' \subseteq \mathcal{A}' \subseteq \mathcal{A}$. Then $\mathcal{A}'$ is the family of closed sets, which has the finite intersection property and contains small sets. Thus by Theorem 3.3 we have $\emptyset \neq \bigcap \mathcal{A}' \subseteq \bigcap \mathcal{F}$. The reverse implication follows from Theorem 3.3. □

A net $(A_\sigma)_{\sigma\in\Sigma}$ of subsets of a uniform space $(X,\mathcal{U})$ is *nested* iff for every $\eta,\sigma\in\Sigma$ such that $\eta\geq\sigma$ holds $A_\eta\subseteq A_\sigma$. It is easy to see that a for a nested net, the corresponding system $\{A_\sigma:\sigma\in\Sigma\}$ has the finite intersection property iff it does not contain an empty set.

Lemma 3.5. *Let $(X,\mathcal{U})$ be a complete uniform space. Let $(F_\sigma)_{\sigma\in\Sigma}$ be a nested net of non-empty closed sets such that $\mathcal{F}=\{F_\sigma:\sigma\in\Sigma\}$ contains almost small sets. Then $(F_\sigma)_{\sigma\in\Sigma}$ converges to a non-empty compact set $F=\bigcap\mathcal{F}$ in $(CL(X),\tau_V)$.*

Proof. By Theorem 3.4, the set F is non-empty, and it is obviously closed. Since $\mathcal{F}$ contains almost small sets, it is easy to verify that F is totally bounded and thus compact. We will now prove that $(F_\sigma)_{\sigma\in\Sigma}$ converges to F in $(CL(X),\tau_V)$.

Let $U\subseteq X$ be open and $F\subseteq U$. We claim that there is $\sigma_0\in\Sigma$ such that for every $\sigma\geq\sigma_0$ holds $F_\sigma\subseteq U$. If this is not true, then the net $(F_\sigma\setminus U)_{\sigma\in\Sigma}$ is nested, does not contain an empty set, and contains almost small sets. Thus by Theorem 3.4 we have $\emptyset\neq\bigcap\{F_\sigma\setminus U:\sigma\in\Sigma\}\subseteq F\setminus U$, a contradiction.

Now let $U\subseteq X$ be open and $F\cap U\neq\emptyset$. Since for every $\sigma\in\Sigma$ it holds $F\subseteq F_\sigma$, then $F_\sigma\cap U\neq\emptyset$. □

The following theorem is a generalization of the result of Morita [15]. He did not specify that if $(X,\mathcal{U})$ is complete a $\mathcal{U}_H$-Cauchy net converges to its Kuratowski-Painlevé limit.

Theorem 3.6. *Let $(X,\mathcal{U})$ be a uniform space. Then $(X,\mathcal{U})$ is complete if and only if every $\mathcal{U}_H$-Cauchy net $(K_\sigma)_{\sigma\in\Sigma}$ in $(K(X),\mathcal{U}_H)$ has a non-empty Kuratowski-Painlevé limit and converges to it in $(K(X),\mathcal{U}_H)$.*

Proof. If every $\mathcal{U}_H$-Cauchy net converges in $(K(X),\mathcal{U}_H)$, then $(K(X),\mathcal{U}_H)$ is complete. Then also $(X,\mathcal{U})$ is complete, since X is a closed subspace of $(K(X),\mathcal{U}_H)$.

Suppose now that $(X,\mathcal{U})$ is complete. Let $(K_\sigma)_{\sigma\in\Sigma}$ be a $\mathcal{U}_H$-Cauchy net of non-empty compact subsets of X. For every $\sigma\in\Sigma$ put $F_\sigma=\overline{\bigcup\{K_\eta:\eta\geq\sigma\}}$. Notice that $\bigcap\{F_\sigma:\sigma\in\Sigma\}=\operatorname{Ls}K_\sigma$, see [1].

By Lemma 3.2 we have $K:=\operatorname{Li}K_\sigma=\operatorname{Ls}K_\sigma$. The net $(F_\sigma)_{\sigma\in\Sigma}$ is nested with non-empty closed members. To prove that it contains almost small sets take any $U\in\mathcal{U}$. Let $V\in\mathcal{U}$ be closed such that $V\circ V\subseteq U$. Since $(K_\sigma)_{\sigma\in\Sigma}$ is a $\mathcal{U}_H$-Cauchy net, there is $\sigma_0\in\Sigma$ such that $K_\sigma\subseteq V[K_{\sigma_0}]$ for every $\sigma\geq\sigma_0$. Compactness of K_{σ_0} implies that there is a finite $D\subseteq K_{\sigma_0}$ such that $K_{\sigma_0}\subseteq V[D]$. Thus $F_{\sigma_0}\subseteq U[D]$. By Lemma 3.5 the net $(F_\sigma)_{\sigma\in\Sigma}$ converges to a non-empty compact set K in $(CL(X),\tau_V)$.

We will now prove that K is a cluster point of the net $(K_\sigma)_{\sigma\in\Sigma}$ in $(K(X),\tau_V)$. Let $W\subseteq X$ be open and $K\subseteq W$. Then there is $\sigma_0\in\Sigma$ such that for every $\sigma\geq\sigma_0$ we have $W\supseteq F_\sigma\supseteq K_\sigma$. Now let $W\subseteq X$ be open and $K\cap W\neq\emptyset$. Then for every $\sigma\in\Sigma$ there is $\sigma'\geq\sigma$ such that $F_{\sigma'}\cap W\neq\emptyset$ and thus there is $\sigma''\geq\sigma'$ such that $K_{\sigma''}\cap W\neq\emptyset$.

From Proposition 2.1, we have that K is a cluster point of the net $(K_\sigma)_{\sigma\in\Sigma}$ in $(K(X),\mathcal{U}_H)$, which is by the assumption Cauchy and thus K is its limit. □

Proposition 3.7 ([10, Lemma 6.2])**.** *Let X be a locally convex topological vector space, then $CK(X)$ is a closed subset of $(K(X),\mathcal{U}_H)$.*

Corollary 3.8. *Let $(X,\mathcal{U})$ be a uniform space. The following are equivalent:*

(1) $(X,\mathcal{U})$ is complete;

(2) $(K(X),\mathcal{U}_H)$ is complete;

moreover if X is a locally convex topological vector space and $\mathcal{U}$ is its standard uniformity, the above are equivalent to

(3) $(CK(X),\mathcal{U}_H)$ is complete.

4. Closed convex hull of a compact set

We will work with a locally convex topological vector space X, which will be equipped with its standard uniformity, thus uniform notions like *completeness* are with respect to it.

The operator of the closed convex hull $\overline{\mathrm{co}}: \mathcal{P}(X) \to CL(X)$; $\overline{\mathrm{co}}\, A = \overline{\mathrm{co}\, A}$ is an important tool. We will be interested in the following property

$$(*) \qquad \text{for every compact set } K \subseteq X, \text{ the set } \overline{\mathrm{co}}\, K \text{ is compact.}$$

It is a well-known fact that in a locally convex topological vector space X, the convex hull of any totally bounded set is totally bounded. Moreover, if X is complete, then for every compact set $K \subseteq X$, the set $\overline{\mathrm{co}}\, K$ is compact. In other words, complete locally convex spaces fulfill the property $(*)$.

We will say that a topological vector space X is *quasi-complete*, see [17] if every closed and bounded subset of X is complete. The following lemma is well known.

Lemma 4.1 ([17, 4.3 Corollary])**.** *Let X be a quasi-complete, locally convex topological vector space. Then for every compact set $K \subseteq X$, the set $\overline{\mathrm{co}}\, K$ is compact.*

Note that $\overline{\mathrm{co}}\, K$ in the above lemma is closed, totally bounded, by the assumption also complete, and hence compact.

Lemma 4.2 ([9, Lemma 6.4])**.** *Let X be a topological vector space, $O \in \mathcal{N}$ be closed and convex, and let $A, B \subseteq X$. If $\overline{\mathrm{co}}\, B$ is compact and $A \subseteq O + B$, then $\overline{\mathrm{co}}\, A \subseteq O + \overline{\mathrm{co}}\, B$.*

The importance of the property $(*)$ is shown in the following statement.

Theorem 4.3. *Let X be a locally convex topological vector space fulfilling property $(*)$. Then*

$$\overline{\mathrm{co}}: (K(X),\mathcal{U}_H) \to (CK(X),\mathcal{U}_H)$$

is a (uniformly continuous) retraction.

Proof. From the property $(*)$, we have that the closed convex hull of a compact set is again compact and obviously also convex, thus the codomain of the function $\overline{\mathrm{co}}$ is indeed $CK(X) \subseteq K(X)$. Note that if $K \in CK(X)$, then $\overline{\mathrm{co}}\, K = K$, so it suffices to prove that $\overline{\mathrm{co}}$ is uniformly continuous.

Take an arbitrary $O \in \mathcal{N}$ that is closed and convex. Suppose that $A, B \in K(X)$ and $(A,B) \in W_O$, then $A \subseteq O + B$. Since $\overline{\mathrm{co}}\, B$ is compact, then from the Lemma 4.2 we have that $\overline{\mathrm{co}}\, A \subseteq O + \overline{\mathrm{co}}\, B$. Similarly, for the order B, A and thus $(\overline{\mathrm{co}}\, A, \overline{\mathrm{co}}\, B) \in W_O$, i.e., $\overline{\mathrm{co}}$ is uniformly continuous. □

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Recall that the Krein–Milman theorem says that if X is a locally convex topological vector space and $K \in CK(X)$, then the set $\mathcal{E}(K)$ of extreme points of K is non-empty and $\overline{\mathrm{co}}\,\mathcal{E}(K) = K$. This means that the preimage $\overline{\mathrm{co}}^{-1}(K)$ contains all sets $A \in K(X)$ such that $\mathcal{E}(K) \subseteq A \subseteq K$.

It is now interesting to investigate the property $(*)$. If X is not complete, then it does not have to have it, as the following example shows.

Example 4.4. Let $X \subseteq \ell^2$ be such that $(x_n)_{n\in\mathbb{N}} \in X$, iff finitely many $x_n \neq 0$. Let $\{e_n : n \in \mathbb{N}\}$ be the standard orthonormal basis of ℓ^2 and

$$K = \{0\} \cup \left\{ v_n = \frac{e_n}{n} : n \in \mathbb{N} \right\} \subseteq X.$$

Observe that K is compact and

$$\mathrm{co}\,K = \left\{ \sum_{k=1}^{n} \alpha_k v_k : n \in \mathbb{N}, \alpha_k \geq 0, \sum_{k=1}^{n} \alpha_k \leq 1 \right\}.$$

Since

$$\sum_{k=1}^{\infty} 2^{-k} = 1, \quad \text{then} \quad x_n = \sum_{k=1}^{n} 2^{-k} v_k \in \mathrm{co}\,K \subseteq \overline{\mathrm{co}}\,K,$$

where the closure is in the space X. Observe that in the space ℓ^2 the sequence $(x_n)_{n\in\mathbb{N}}$ converges to $\sum_{k=1}^{\infty} 2^{-k} v_k \notin X$. Thus $(x_n)_{n\in\mathbb{N}}$ has no cluster point in X, i.e. $\overline{\mathrm{co}}\,K$ is not compact.

This example can be generalized into the following theorem.

Theorem 4.5. *Let X be a metrizable locally convex space. If for every compact $K \subseteq X$, the set $\overline{\mathrm{co}}\,K$ is compact, then X is complete.*

Proof. Let $\{U_n : n \in \mathbb{N}\}$ be a countable base of neighborhoods of 0, consisting of absolutely convex sets. For every $n \in \mathbb{N}$ let p_n be the Minkowski functional of U_n; i.e., it is a seminorm. Following [4, Proposition 4.10], for every $x, y \in X$, we can define the following:

$$d(x,y) = \sum_{n=1}^{\infty} 2^{-n} \frac{p_n(x-y)}{1 + p_n(x-y)}.$$

One can easily check that

- d is a metric, that generates the standard uniformity on X;
- d is translation invariant; i.e. for every $x, y, z \in X$: $d(x-z, y-z) = d(x,y)$;
- for every $x, y \in X$ and $\alpha \in K$, such that $|\alpha| \geq 1$: $d(\alpha x, \alpha y) \leq |\alpha| d(x,y)$.

Note that the first two facts are proven in [4, Proposition 4.10].

Suppose $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in X. We want to prove that it is convergent. For every $k \in \mathbb{N}$ there is n_k such that for every $p, q \geq n_k$: $d(x_p, x_q) \leq 3^{-k}$. Without loss of generality the sequence $(n_k)_{k\in\mathbb{N}}$ is non-decreasing. Put $z_1 = 2x_{n_1}$ and $z_k = 2^k(x_{n_k} - x_{n_{k-1}})$ for $k > 1$, then

$$x_{n_t} = \sum_{k=1}^{t} 2^{-k} z_k,$$

21 Mar 2024 04:54:53 PDT
231122-Novotny Version 2 - Submitted to Rocky Mountain J. Math.

for every $t \in \mathbb{N}$. For $k > 1$, we have that

$$d(z_k, 0) = d(2^k(x_{n_k} - x_{n_{k-1}}), 0) \le 2^k d(x_{n_k}, x_{n_{k-1}}) \le 3(2/3)^k \to 0,$$

thus the sequence $(z_k)_{k\in\mathbb{N}}$ converges to 0 and so $K = \{z_k : k \in \mathbb{N}\} \cup \{0\}$ is compact. Observe that

$$\operatorname{co} K = \left\{ \sum_{k=1}^{t} \alpha_k z_k : t \in \mathbb{N}, \alpha_k \ge 0, \sum_{k=1}^{t} \alpha_k \le 1 \right\}.$$

Since $\sum_{k=1}^{\infty} 2^{-k} = 1$ then $x_{n_t} \in \operatorname{co} K$ for every $t \in \mathbb{N}$. By the assumption $\overline{\operatorname{co}}\, K$ is compact and thus $(x_{n_t})_{t\in\mathbb{N}}$ has a cluster point, i.e., $(x_n)_{n\in\mathbb{N}}$ has to be convergent. □

This means that for metrizable spaces, the property (∗) is equivalent to completeness. The natural question is if this is also true in the non-metrizable case. The answer is no, as the following example shows.

Example 4.6 ([17, p. 148])**.** Let X be an infinite-dimensional Fréchet space. Let X' be its weak dual, i.e., continuous dual with the weak* topology. Then X' is non-metrizable, non-complete, but it is quasi-complete, and hence it fulfills the property (∗).

5. Statements and Declarations

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References

[1] Beer, G. *Topologies on Closed and Closed Convex Sets*. Kluwer Academic Publishers, Dodrecht, 1993.
[2] Castaing, C., and Valadier, M. *Convex Analysis and Measurable Multifunctions*. Springer, 1977.
[3] Engelking, R. *General Topology*. PWN, Warszawa, 1977.
[4] Fabian, M., Habala, P., Hájek, P., Santalucía, V. M., Pelant, J., and Zizler, V. *Functional analysis and infinite-dimensional geometry*. Springer Science & Business Media, 2013.
[5] Ginsburg, S., and Isbell, J. Some operators on uniform spaces. *Trans. Amer. Math. Soc. 93*, 1 (1959), 145–168.
[6] Hahn, H. *Reelle Functionen I*. Leipzig, 1932.
[7] Holá, L., Holý, D., and Moors, W. B. *USCO and Quasicontinuous Mappings*, vol. 81 of *De Gruyter Studies in Mathematics*. De Gruyter, 2021.
[8] Holá, L., Holý, D., and Novotný, B. Spaces of minimal usco and minimal cusco maps as fréchet topological vector spaces. In *Advances in Topology and Their Interdisciplinary Applications*. Springer Nature Singapore, 2023, pp. 1–18.
[9] Holá, L., and Novotný, B. When is the space of minimal usco/cusco maps a topological vector space. *J. Math. Anal. Appl. 489*, 1 (sep 2020), 124125.

[10] Holá, Ľ., and Novotný, B. Fréchet subspaces of minimal usco and minimal cusco maps. *Bulletin of the Belgian Mathematical Society - Simon Stevin 29*, 5 (dec 2022).
[11] Isbell, J. Supercomplete spaces. *Pac. J. Math. 12*, 1 (1962), 287–290.
[12] Kelley, J. L. *General Topology.* Springer New York, 1975.
[13] Klein, E., and Thompson, A. *Theory of correspondences.* Wiley, New York, 1984.
[14] Michael, E. Topologies on spaces of subsets. *Trans. Amer. Math. Soc. 71* (1951), 152–182.
[15] Morita, K. Completion of hyperspaces of compact subsets and topological completion of open-closed maps. *General Topology and its Applications 4*, 3 (sep 1974), 217–233.
[16] Pompeiu, D. Sur la continuité des fonctions de variables complexes. *Annales de la Faculté des sciences de l'Université de Toulouse pour les sciences mathématiques et les sciences physiques 7*, 3 (1905), 265–315.
[17] Schaefer, H. H., and Wolff, M. P. *Topological Vector Spaces.* Springer New York, 1999.