Pseudo-asymptotically Bloch τ-periodic solutions for some neutral partial functional differential equations

AMOR REBEY[1,2], ANJA ARFA[3]

[1] Department of Mathematics,
Higher institute of Applied Mathematics and Computer Sciences of Kairouan, Kairouan University, Tunisia.
[2] Business Administration Department, College of Business Administration, Majmaah University,Majmaah, 11952, Saudi Arabia
a.rebey@mu.edu.sa
[3] Faculty of Sciences, University of Sfax, BP 1171, 3000 Sfax, Tunisia

Abstract

This paper is mainly deals with $(\mu,\nu)-$pseudo-asymptotically Bloch τ-periodic functions in Hilbert spaces. Firstly, the concept of $(\mu,\nu)-$Pseudo-asymptotically Bloch type τ periodic functions is introduced. Secondly, further properties on $(\mu,\nu)-$Pseudo-asymptotically Bloch functions are provided. Finally, using the results obtained, the existence and uniqueness of $(\mu,\nu)-$Pseudo-asymptotically Bloch type periodic solutions for a neutral partial functional differential equation are studied.

Key Words: Pseudo-asymptotically Bloch-periodic function, measure theory.
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1 Introduction

As an application in many sciences as biology, physics, engineering, the study of Bloch-periodic solutions can be considered as an important subject in the qualitative theory of differential equations. It can be seen that Bloch type periodicity has ω-periodicity and ω-anti-periodicity as special cases. In [3], the author establish a new composition theorem and a new concept of asymptotically Bloch-periodic functions. The author also investigated the existence and uniqueness of pseudo S-asymptotically Bloch type periodic mild solutions to some semilinear evolution equations. The notion of S-asymptotically Bloch type periodicity can be considered as a generalization of S or pseudo S-asymptotic ω periodicity. Recently, in [4], the author introduce a new notion of S-asymptotically Bloch type-periodic functions and S-asymptotically ω-anti periodic functions, also he investigate some fundamental properties on S-asymptotically Bloch type periodic functions. As an application he prove the existence and uniqueness of S-asymptotically Bloch type periodic mild solutions for some specific type equation that's semi-linear evolution equations in Banach spaces.

The concept of pseudo S-asymptotically Bloch type periodic solutions was also presented

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in [5], were the author prove the existence of pseudo S-asymptotically ω-anti-periodic solutions for an example of damped evolution equation, additionally, he discuss the existence of pseudo S-asymptotically Bloch type periodic solutions to nonlocal Cauchy problem. For more See also [[6],[7],[8],[9],[10], [11],[12],[13],[23]].

Motivated by the work [1], the key goal of our paper is to study the existence and uniqueness of $(\mu,\nu)-$Pseudo-asymptotically Bloch τ-periodic solutions in Hilbert spaces for the following nonlinear differential equation:

$$\frac{d}{dt}[\eta(t)-G(t,\eta(k(t)))]=A[\eta(t)-G(t,\eta(k(t)))]+B\eta(t)+F(t,\eta(k(t))),\ \ t\in\mathbb{R}, \qquad (1)$$

where $G,\ F$ and k are continuous.

In [1], the authors studied the conditions for the existence and uniqueness of (μ,ν)-pseudo almost automorphic and (μ,ν)-pseudo almost periodic solutions for some neutral partial functional differential equations in Hilbert spaces.

As an application in the physical sciences, mathematical biology and control theory, the existence and uniqueness of pseudo-almost periodic solution can be considered like one of the most attractive topics in the qualitative theory or functional differential equations. In recent years, the existence of almost periodic pseudo almost periodic solution to different kinds of differential equations have been investigated in many works, see [[14], [15], [16], [17], [18], [19], [20], [21], [22]] and the references therein. The rest of this paper is organized as follows: the second section is preliminaries including some basic definition, lemma and notations. Section 3 is based on the compositional theorem and the Banach fixed point theorem to study (μ,ν)-PAP solution of eq (1). Finally, for illustration we give an example for evolution equations which include neutral partial functional differential equations.

2 Main definitions

Let $\mathcal{B}$ denote the Lebesgue σ-field of $\mathbb{R}$ and let $\mathcal{M}$ be the set of all positive measures μ on $\mathcal{B}$ satisfying $\mu(\mathbb{R})=+\infty$ and $\mu([a,b])<\infty$, for all $a,b\in\mathbb{R}$ $(a\leq b)$.

There will be an interest to return on the fundamental notions which are necessary to introduce the work that follows.

Definition 2.1 *Let $\mu,\nu\in\mathcal{M}$. A bounded continuous function $f:\mathbb{R}\to\mathbb{H}$ is said to be $(\mu,\nu)-$Pseudo-asymptotically Bloch τ-periodic functions if*

$$\lim_{r\to\infty}\frac{1}{\nu([-r,r])}\int_{-r}^{r}\|f(s+\tau)-e^{i\rho\tau}f(s)\|d\mu(s)=0.$$

We then denote the collection of all above functions by $PSABP_{\tau,\rho}(\mathbb{R},\mathbb{H},\mu,\nu)$.

Definition 2.2 *Denoted by $BC(\mathbb{R},\mathbb{H})$ the Banach space constituted by all bounded continuous functions $f:R\to H$ with sup-norm $\|f\|_\infty=sup(\|f(t)\|),t\in\mathbb{R}$.*

Definition 2.3 *[2] For given $\tau,\rho\in\mathbb{R}$, a function $f\in BC(\mathbb{R},\mathbb{H})$ is said to be Bloch (or (ε,ρ)) type periodic if for all $t\in\mathbb{R}$, $f(t+\tau)=e^{i\tau\rho}f(t)$.*

We denote by $BP_{\tau,\rho}(\mathbb{R},\mathbb{H})$, the space of all Bloch (τ,ρ) type periodic functions from $\mathbb{R}$ to $\mathbb{H}$.

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In the rest of the work, we will need to use the following hypotheses

(M.1) Let $\mu, \nu \in \mathcal{M}$ such that

$$\limsup_{r\to+\infty} \frac{\mu([-r,r])}{\nu([-r,r])} < \infty. \tag{2}$$

(M.2) For all $\tau \in \mathbb{R}$, there exist $\beta > 0$ and a bounded interval I such that

$$\mu(\{a+\tau : \quad a \in A\}) \leq \beta\mu(A) \qquad \text{when } A \in \mathcal{B} \text{ satisfies } A \cap I = \emptyset.$$

We have the following theorem:

Theorem 2.1 *Let $f \in BC(\mathbb{R}\times\mathbb{H}, \mathbb{H}, \mu, \nu)$ satisfy the following conditions :*

(A_1) *For all $(t,x) \in \mathbb{R}\times\mathbb{H}$, $f(t+\tau, x) = e^{i\tau\rho} f(t, e^{-i\tau\rho}x)$.*

(A_2) *There exists a constant $L > 0$ such that for all $x, y \in X$ and $t \in \mathbb{R}$,*

$$||f(t,x) - f(t,y)|| \leq L||x-y||$$

*Hypothesis **(M.1)** is verified and for all B bounded of $\mathbb{H}$, F is bounded on $\mathbb{R}\times B$, then for each $\phi \in PSABP_{\tau,\rho}(\mathbb{R},\mathbb{H},\mu,\nu)$, $f(.,\phi(.)) \in PSABP_{\tau,\rho}(\mathbb{R},\mathbb{H},\mu,\nu)$.*

Proof. Let $\phi \in PSABP_{\tau,\rho}(\mathbb{R},\mathbb{H},\mu,\nu)$. By a direct calculation

$$\begin{aligned}
&\lim_{r\to+\infty} \frac{1}{\nu([-r,r])} \int_{[-r,r]} \|f(t+\tau, \phi(t+\tau)) - e^{i\rho\tau} f(t,\phi(t))\| d\mu(t) \\
= &\lim_{r\to+\infty} \frac{1}{\nu([-r,r])} \int_{[-r,r]} \|e^{i\rho\tau} f(t, e^{-i\rho\tau}\phi(t+\tau)) - e^{i\rho\tau} f(t,\phi(t))\| d\mu(t) \\
\leq &\lim_{r\to+\infty} \frac{L}{\nu([-r,r])} \int_{[-r,r]} \|e^{-i\rho\tau}\phi(t+\tau) - \phi(t)\| d\mu(t) \\
\leq &\lim_{r\to+\infty} \frac{L}{\nu([-r,r])} \int_{[-r,r]} \|\phi(t+\tau) - e^{i\rho\tau}\phi(t)\| d\mu(t) = 0
\end{aligned}$$

■

Lemma 2.1 *Let $\mu, \nu \in M$ satisfy (**M.2**). Then the spaces $PSABP_{\tau,\rho}(\mathbb{R},\mathbb{H},\mu,\nu)$ is translation invariants.*

Proof. We prove that $PSABP_{\tau,\rho}(\mathbb{R},\mathbb{H},\mu,\nu)$ is translation invariant. Let $f \in PSABP_{\tau,\rho}(\mathbb{R},\mathbb{H},\mu,\nu)$, we will show that $f_a : t \longmapsto f(t+a)$ belongs to $PSABP_{\tau,\rho}(\mathbb{R},\mathbb{H},\mu,\nu)$ for each $a \in \mathbb{R}$. Indeed, letting $\mu_a = \mu(t+a : t \in A)$ for $A \in B$ it follows from (**M.2**) that μ and μ_a are equivalent

$$\begin{aligned}
&\frac{1}{\nu([-r,r])} \int_{[-r,r]} \|f(t+a+\tau) - e^{i\rho\tau} f(t+a)\| d\mu(t) \\
= &\frac{\nu([-r-|a|, r+|a|])}{\nu([-r,r])} \frac{1}{\nu([-r-|a|, r+|a|])} \int_{[-r+a,r+a]} \|f(s+\tau) - e^{i\rho\tau} f(s)\| d\mu_a(s) \\
\leq &\frac{\nu([-r-|a|, r+|a|])}{\nu([-r,r])} \frac{cst}{\nu([-r-|a|, r+|a|])} \int_{[-r-|a|,r+|a|]} \|f(s+\tau) - e^{i\rho\tau} f(s)\| d\mu(s)
\end{aligned}$$

Since ν satisfies (**M.2**) and $f \in PSABP_{\tau,\rho}(\mathbb{R},\mathbb{H},\mu,\nu)$, we have

$$\lim_{r\to\infty} \frac{1}{\nu([-r,r)} \int_{[-r,r]} \|f_a(t+\tau)\| - e^{i\rho\tau} f_a(t)\| d\mu(t) = 0$$

Therefore, $PSABP_{\tau,\rho}(\mathbb{R},\mathbb{H},\mu,\nu)$ is translation invariant. ■

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3 Pseudo-asymptotically Bloch -periodic solution

In this section we focus on the study of the existence and uniqueness of $(\mu,\nu)-$pseudo-asymptotically Bloch τ-periodic solution of equation (1). For this, we shall assume the following hypothesis:

(H1) Let $F \in PSABP_{\tau,\rho}(\mathbb{R},\mathbb{H},\mu,\nu)$. The function F is Lipschitz with respect to the second argument uniformly in $t \in \mathbb{R}$, that is, there exist positive numbers L_F, such that for each $(t,x),(t,y) \in \mathbb{R}\times\mathbb{H}$

$$\|F(t,x)-F(t,y)\| \leq L_F\|x-y\|,$$

(H2) Let $G \in PSABP_{\tau,\rho}(\mathbb{R},\mathbb{H},\mu,\nu)$. The functions G is Lipschitz with respect to the second argument uniformly in $t \in \mathbb{R}$, that is, there exist positive numbers L_G such that for each $(t,x),(t,y) \in \mathbb{R}\times\mathbb{H}$

$$\|G(t,x)-G(t,y)\| \leq L_G\|x-y\|,$$

(H3) There exists $C \subset \mathbb{H}$, a closed subspace that reduces both A and B. We denote by P_C the orthogonal projection onto C and $Q_C = (I-P_C) = P_{\mathbb{H}\ominus C}$ the orthogonal projection onto $\mathbb{H}\ominus C$.

(H4) A, B are the infinitesimal generators of C_0-groups of bounded linear operators $(T(t))_{t\in\mathbb{R}}$, $(R(t))_{t\in\mathbb{R}}$, respectively, such that there exist $M_1, M_2, M_3, \delta_1, \delta_2, \delta_3 > 0$ with

$$\|T(t-s)P_C\| \leq M_1 e^{-\delta_1(t-s)} \;\; for\ all\ t \geq s,$$

$$\|R(t-s)Q_C\| \leq M_2 e^{-\delta_2(t-s)} \;\; for\ all\ t \geq s,$$

and

$$\|R(t-s)BQ_C\| \leq M_3 e^{-\delta_3(t-s)} \;\; for\ all\ t \geq s.$$

(H5) $R(A) \subset R(P_C) = N(Q_C)$.

(H6) $R(B) \subset R(Q_C) = N(P_C)$.

(**H7**) For $i \in \{1,2\}$, $k_i : \mathbb{R}\to\mathbb{R}$ is continuous and increasing.

(**H8**) For $i \in \{1,2\}$, there exist a functions $\lambda_i : \mathbb{R}\to\mathbb{R}^+$ continuous, such that

$$d\mu_{k_i}(s) \leq \lambda_i(s)d\mu(s), \text{ where } \mu_{k_i}(E) = \mu(k_i^{-1}(E)) \text{ for all } E \in \mathcal{B}(\mathbb{R}),$$

and

$$\limsup_{r\to+\infty} \frac{\nu([-R_i(r),R_i(r)])}{\nu([-r,r])} M(R_i(r)) < \infty,$$

where $R_i(r) = |k_i(-r)| + |k_i(r)|$ and $M(R_i(r)) = \sup_{s\in[-R_i(r),R_i(r)]} \lambda_i(s)$.

Lemma 3.1 *Assume the assumption* **(H8)** *is satisfied. If* $u \in PSABP_{\tau,\rho}(\mathbb{R},\mathbb{H},\mu,\nu)$, *then* $\eta(k_i(.)) \in PSABP_{\tau,\rho}(\mathbb{R},\mathbb{H},\mu,\nu)$ *for* $i \in \{1,2\}$.

Proof. Let $u \in PSABP_{\tau,\rho}(\mathbb{R},\mathbb{H},\mu,\nu)$, we need to prove that $[t \to \eta(k_i(t))] \in PSABP_{\tau,\rho}(\mathbb{R},\mathbb{H},\mu,\nu)$. From **(H8)** for $i \in \{1,2\}$, we have

$$\begin{aligned}
0 &\leq \frac{1}{\nu([-r,r])}\int_{[-r,r]} \|\eta(k_i(s+\tau) - e^{i\rho\tau}\eta(k_i(s)))\| d\mu(s) \\
&= \frac{1}{\nu([-r,r])}\int_{k_i([-r,r])} \|\eta(s+\tau) - e^{i\rho\tau}\eta(s)\| d\mu_{k_i}(s) \\
&\leq \frac{1}{\nu([-r,r])}\int_{[-R_i(r),R_i(r)]} \|\eta(s+\tau) - e^{i\rho\tau}\eta(s)\| d\mu_{k_i}(s) \\
&\leq \frac{1}{\nu([-r,r])}\int_{[-R_i(r),R_i(r)]} \|\eta(s+\tau) - e^{i\rho\tau}\eta(s)\| \lambda_i(s) d\mu(t). \\
&\leq \frac{M(R_i(r))}{\nu([-r,r])}\int_{[-R_i(r),R_i(r)]} \|\eta(s+\tau) - e^{i\rho\tau}\eta(s)\| d\mu(t).
\end{aligned}$$

Then we have two cases:
Step 1: If for $i \in \{1,2\}$ we have $R_i(r) \to \alpha^* < \infty$ as $r \to +\infty$, then

$$\begin{aligned}
&\frac{1}{\nu([-r,r])}\int_{[-r,r]} \|\eta(k_i(s+\tau) - e^{i\rho\tau}\eta(k_i(s)))\| d\mu(t) \\
\leq\ & \frac{M_{\alpha^*}}{\nu([-r,r])}\int_{[-\alpha^*,\alpha^*]} \|\eta(s+\tau) - e^{i\rho\tau}\eta(s)\| d\mu(t) \\
\leq\ & \frac{Cst}{\nu([-r,r])} \to 0 \text{ as } r \to +\infty.
\end{aligned}$$

Step 2: If for $i \in \{1,2\}$ we have $R_i(r) \to +\infty$ as $r \to +\infty$, then

$$\begin{aligned}
0 &\leq \frac{1}{\nu([-r,r])}\int_{[-r,r]} \|\eta(k_i(s+\tau) - e^{i\rho\tau}\eta(k_i(s))))\| d\mu(t) \\
&\leq \frac{M(R_i(r))}{\nu([-r,r])}\int_{[-R_i(r),R_i(r)]} \|\eta(s+\tau) - e^{i\rho\tau}\eta(s))\| d\mu(t) \\
&\leq M(R_i(r))\frac{\nu([-R_i(r),R_i(r)])}{\nu([-r,r])}\frac{1}{\nu([-R_i(r),R_i(r)])}\int_{[-R_i(r),R_i(r)]} \|\eta(s+\tau) - e^{i\rho\tau}\eta(s))\| d\mu(t).
\end{aligned}$$

Since $u \in PSABP_{\tau,\rho}(\mathbb{R},\mathbb{H},\mu,\nu)$ and $\limsup_{r\to+\infty} M(R_i(r))\dfrac{\nu([-R_i(r),R_i(r)])}{\nu([-r,r])} < \infty$, we have

$$\lim_{r\to+\infty}\frac{1}{\nu([-r,r])}\int_{[-r,r]} \|\eta(s+\tau) - e^{i\rho\tau}\eta(s)\| d\mu(t) = 0.$$

Therefore $[t \to \eta(k_i(t))] \in PSABP_{\tau,\rho}(\mathbb{R},\mathbb{H},\mu,\nu)$ for $i \in \{1,2\}$. Thus the result holds. ∎

Lemma 3.2 *[1] Assume the assumptions **(H1)-(H6)** are satisfied, then each solution of Eq.*(1) *can be expressed as:*

$$\begin{aligned}
\eta(t) &= G(t,\eta(k_1(t))) + \int_{-\infty}^{t} R(t-s)BQ_C G(s,\eta(k_1(s)))ds \\
&\quad + \int_{-\infty}^{t} T(t-s)P_C F(s,\eta(k_2(s)))ds + \int_{-\infty}^{t} R(t-s)Q_C F(s,\eta(k_2(s)))ds
\end{aligned}$$

Proof. In the proof, we proceed by the same reasoning as in [1] and we keep the notation of the functions $Q_C F(s,\eta(k_2(s))))$ and $P_C F(s,\eta(k_2(s))))$ as a functions in $PSABP_{\tau,\rho}(\mathbb{R},\mathbb{H},\mu,\nu)$ instead of $Q_C\varphi(s,\eta(k_2(s))))$ and $P_C\varphi(s,\eta(k_2(s))))$ in addition we eliminate the component $P_C\psi(s,\eta(k_2(s)))$ and $Q_C\psi(s,\eta(k_2(s)))$. So we get the result.
■

Lemma 3.3 *Let $\mu,\ \nu \in \mathcal{M}$ satisfy* **(M2)**. *Assume the assumptions* ***(H1)-(H8)*** *are satisfied. If $u \in PSABP_{\tau,\rho}(\mathbb{R},\mathbb{H},\mu,\nu)$, then $\Gamma u \in PSABP_{\tau,\rho}(\mathbb{R},\mathbb{H},\mu,\nu)$, where*

$$\begin{aligned}\Gamma\eta(t) &= G(t,\eta(k_1(t))) + \int_{-\infty}^{t} R(t-s)BQ_C G(s,\eta(k_1(s)))ds \\ &\quad + \int_{-\infty}^{t} T(t-s)P_C F(s,\eta(k_2(s)))ds + \int_{-\infty}^{t} R(t-s)Q_C F(s,\eta(k_2(s)))ds\end{aligned}$$

Proof. Let $u \in PSABP_{\tau,\rho}(\mathbb{R},\mathbb{H},\mu,\nu)$, we have $\Gamma : PSABP_{\tau,\rho}(\mathbb{R},\mathbb{H},\mu,\nu) \to C(\mathbb{R},\mathbb{H})$ and

$$\Gamma\eta(t) = \Gamma_1\eta(t) + \Gamma_2\eta(t) + \Gamma_3\eta(t),$$

where

$$\Gamma_1\eta(t) = \int_{-\infty}^{t} T(t-s)P_C\varphi(s,\eta(h(s)))ds + \int_{-\infty}^{t} R(t-s)Q_C\varphi(s,\eta(h(s)))ds$$

$$\Gamma_2\eta(t) = G(t,\eta(k_1(t)))$$

and

$$\Gamma_3\eta(t) = \int_{-\infty}^{t} R(t-s)BQ_C G(s,\eta(k_1(s)))ds.$$

First, from lemma 2 and theorem 1, we have $\Gamma_2 u \in PSABP_{\tau,\rho}(\mathbb{R},\mathbb{H},\mu,\nu)$.
We prove that $\Gamma_1\eta(t) \in PSABP_{\tau,\rho}(\mathbb{R},\mathbb{H},\mu,\nu)$. For that, we let $\Gamma_1\eta(t) = I_2(t) + I_3(t)$, where $I_2(t) = \int_{-\infty}^{t} T(t-s)P_C F(s)ds$ and $I_3(t) = \int_{-\infty}^{t} R(t-s)Q_C F(s)ds$. We only prove that $I_2(t) \in PSABP_{\tau,\rho}(\mathbb{R},\mathbb{H},\mu,\nu)$ since the proof for $I_3(t)$ is similar to that of $I_2(t)$.

$$\begin{aligned}
& \lim_{r\to+\infty} \frac{1}{\nu([-r,r])} \int_{[-r,r]} \|I_2(t+\tau) - e^{i\rho\tau} I_2(t)\| d\mu(t) \\
= & \lim_{r\to+\infty} \frac{1}{\nu([-r,r])} \int_{[-r,r]} \Big(\| \int_{-\infty}^{t+\tau} T(t+\tau-s)P_C F(s) - \int_{-\infty}^{t} e^{i\rho\tau} T(t-s)P_C F(s)ds \| d\mu(t) \Big) \\
= & \lim_{r\to+\infty} \frac{1}{\nu([-r,r])} \int_{[-r,r]} \Big(\| \int_{-\infty}^{t} T(t-s)P_C F(\tau+s) - \int_{-\infty}^{t} e^{i\rho\tau} T(t-s)P_C F(s)ds \| d\mu(t) \Big) \\
\leq & \lim_{r\to+\infty} \frac{1}{\nu([-r,r])} \int_{[-r,r]} \Big(\int_{-\infty}^{t} M_1 e^{-\delta_1(t-s)} \|P_C F(s+\tau) - e^{i\rho\tau} M_1 e^{-\delta_1(t-s)} P_C F(s)ds \| d\mu(t) \Big) \\
= & \lim_{r\to+\infty} \frac{1}{\nu([-r,r])} \int_{[-r,r]} \int_{-\infty}^{t} M_1 e^{-\delta_1(t-s)} \|P_C F(s+\tau) - e^{i\rho\tau} P_C F(s)\| ds d\mu(t) \\
= & \lim_{r\to+\infty} \frac{1}{\nu([-r,r])} \int_{[-r,r]} \int_{-\infty}^{t} M_1 e^{-\delta_1(t-s)} |P_C| \|F(s+\tau) - e^{i\rho\tau} F(s)\| ds d\mu(t) \\
\leq & \lim_{r\to+\infty} M_1 \int_{0}^{+\infty} e^{-\delta_1(s)} \Big(\frac{1}{\nu([-r,r])} \int_{[-r,r]} \|F(\tau+(t-s)) - e^{i\rho\tau} F(t-s)\| d\mu(t) \Big) ds
\end{aligned}$$

It's follows from lemma 2.1 and the Lebesgue dominated convergence theorem that $I_2(t) \in PSABP_{\tau,\rho}(\mathbb{R},\mathbb{H},\mu,\nu)$. Finally, Proceeding as $\Gamma_1 u$, one can show that $\Gamma_3 u \in PSABP_{\tau,\rho}(\mathbb{R},\mathbb{H},\mu,\nu)$. In view of the foregoing, it is apparent that $\Gamma u = \Gamma_1 u+\Gamma_2 u+\Gamma_3 u \in PSABP_{\tau,\rho}(\mathbb{R},\mathbb{H},\mu,\nu)$. This completes the proof. ■

Theorem 3.1 *Let $\mu,\ \nu \in \mathcal{M}$ satisfy* **(M1)** *and* **(M2)**. *Assume the assumptions **(H1)-(H8)** are satisfied. Then the Eq.(1) has a unique $(\mu,\nu)-$Pseudo-asymptotically Bloch τ-periodic solution on $\mathbb{R}$ provide that*

$$\Big[L_F\Big(\frac{M_1}{\delta_1}+\frac{M_2}{\delta_2}\Big)+\Big(L_G+\frac{M_3 L_G}{\delta_3}\Big)\Big] < 1.$$

Since $\Big[L_F\Big(\frac{M_1}{\delta_1}+\frac{M_2}{\delta_2}\Big)+\Big(L_G+\frac{M_3 L_G}{\delta_3}\Big)\Big] < 1$, then Γ is a contraction map on $PSABP_{\tau,\rho}(\mathbb{R},\mathbb{H},\mu,\nu)$. Where L_F and L_G are the Lipschitz conditions of F resp. G. Therefore, Γ has unique fixed point in $PSABP_{\tau,\rho}(\mathbb{R},\mathbb{H},\mu,\nu)$, that is, there exists unique $u \in PSABP_{\tau,\rho}(\mathbb{R},\mathbb{H},\mu,\nu)$ such that $\Gamma u = u$. Thereafter, Eq.(1) has unique (μ,ν) Pseudo-asymptotically Bloch τ-periodic solution.

4 Example

As an application for Theorem 3.1, we consider the following equation

$$\begin{cases} \dfrac{d}{dt}[x(t)-g_1(t,(x(t-r_1),y(t-r_1)))] = A_0[x(t)-g_1(t,(x(t-r_1),y(t-r_1)))] \\ \qquad\qquad +f_1(t,(x(t-r_2),y(t-r_2))),\ t\in\mathbb{R} \\ \dfrac{d}{dt}[y(t)-g_2(t,(x(t-r_1),y(t-r_1)))] = B_0[y(t)-g_2(t,(x(t-r_1),y(t-r_1)))] \\ \qquad\qquad +f_2(t,(x(t-r_2),y(t-r_2))),\ t\in\mathbb{R}, \end{cases} \tag{3}$$

where A_0 is the infinitesimal generator of a C_0-group $(T_0(t))_{t\in\mathbb{R}}$ on an Hilbert space $\mathbb{H}_1$ such that $\|T_0(t)\| \leq M_1 e^{-\delta_1 t}$ for all $t\in\mathbb{R}_+$, B_0 is the infinitesimal generator of a C_0-group $(R_0(t))_{t\in\mathbb{R}}$ on an Hilbert space $\mathbb{H}_2$ such that $\|R_0(t)\| \leq M_2 e^{-\delta_2 t}$ for all $t\in\mathbb{R}_+$, $r_1, r_2 \geq 0$, $f_1, g_1 : \mathbb{R}\times\mathbb{H}\to\mathbb{H}_1$ and $f_2, g_2 : \mathbb{R}\times\mathbb{H}\to\mathbb{H}_2$ are continuous where $\mathbb{H}=\mathbb{H}_1\times\mathbb{H}_2$.

Let $\eta(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$. Then the system (3) can be represented as

$$\frac{d}{dt}[\eta(t)-G(t,\eta(k_1(t)))] = A[\eta(t)-G(t,\eta(k_1(t)))] + B\eta(t) + F(t,\eta(k_2(t))),\quad t\in\mathbb{R}, \tag{4}$$

in the Hilbert space $\mathbb{H}$, where

$$A = \begin{pmatrix} A_0 & 0 \\ 0 & 0 \end{pmatrix},\quad B = \begin{pmatrix} 0 & 0 \\ 0 & B_0 \end{pmatrix},\ k_1(t) = t-r_1,\ k_2(t) = t-r_2,$$

$$G(t,\eta(k_1(t))) = \begin{pmatrix} g_1(t,(x(k_1(t)),y(k_1(t)))) \\ g_2(t,(x(k_1(t)),y(k_1(t)))) \end{pmatrix},$$

and

$$F(t,\eta(k_2(t))) = \begin{pmatrix} f_1(t,(x(k_2(t)),y(k_2(t)))) \\ f_2(t,(x(k_2(t)),y(k_2(t)))) \end{pmatrix},$$

where $D(A) = D(A_0) \times \mathbb{H}_2$ and $D(B) = \mathbb{H}_1 \times D(B_0)$.

If we put $C = \mathbb{H}_1 \times \{0\}$, we can see that C is a closed subspace of $\mathbb{H}$. We have $R(A) \subset \mathbb{H}_1 \times \{0\} = C$ and $R(B) \subset \{0\} \times \mathbb{H}_2 = C^{\perp}$.
We have A, B are the infinitesimal generators of C_0-groups of bounded linear operators $(T(t))_{t\in\mathbb{R}}$, $(R(t))_{t\in\mathbb{R}}$, respectively, such that $M_1, M_2, M_3, \delta_1, \delta_2, \delta_3 > 0$, with

$$\|T(t-s)P_C\| \leq M_1 e^{-\delta_1(t-s)} \;\; for\ all\ t \geq s,$$

$$\|R(t-s)Q_C\| \leq M_2 e^{-\delta_2(t-s)} \;\; for\ all\ t \geq s,$$

and

$$\|R(t-s)BQ_C\| \leq M_3 e^{-\delta_3(t-s)} \;\; for\ all\ t \geq s.$$

Since Eq.(4) is equivalent to Eq.(3), so we only need to study the existence and uniqueness of $(\mu,\nu)-$Pseudo-asymptotically Bloch τ-periodic solutions to Eq.(4). In order to study this problem, we now require that

$$F(t,\eta(k_2(t))) = \begin{pmatrix} f_1(t,(x(k_2(t)),y(k_2(t))))) \\ f_2(t,(x(k_2(t)),y(k_2(t))))) \end{pmatrix} = \begin{pmatrix} \sin(\sqrt{2}(t-r_2))x + e^{-(t-r_2)^2}y \\ \cos(t-r_2)x + e^{-(t-r_2)^2}y \end{pmatrix},$$

and

$$G(t,\eta(k_1(t))) = \begin{pmatrix} g_1(t,(x(k_1(t)),y(k_1(t)))) \\ g_2(t,(x(k_1(t)),y(k_1(t)))) \end{pmatrix} = \begin{pmatrix} \cos(\sqrt{2}(t-r_1))x + e^{-(t-r_1)^2}y \\ \sin(t-r_1)x + e^{-(t-r_1)^2}y \end{pmatrix},$$

where $f_1 : \mathbb{R} \times \mathbb{H} \to \mathbb{H}_1$ is continuous and there exist positive number L_{f_1} such that for each $(t,y), (t,z) \in \mathbb{R} \times \mathbb{H}$

$$\|f_1(t,y) - f_1(t,z)\| \leq L_{f_1}\|y-z\|,$$

$f_2 : \mathbb{R} \times \mathbb{H} \to \mathbb{H}_2$ is continuous and there exist positive number L_{f_2} such that for each $(t,y), (t,z) \in \mathbb{R} \times \mathbb{H}$

$$\|f_2(t,y) - f_2(t,z)\| \leq L_{f_2}\|y-z\|,$$

$g_1 : \mathbb{R} \times \mathbb{H} \to \mathbb{H}_1$ is continuous and there exist positive number L_{g_1} such that for each $(t,y), (t,z) \in \mathbb{R} \times \mathbb{H}$

$$\|g_1(t,y) - g_1(t,z)\| \leq L_{g_1}\|y-z\|,$$

$g_2 : \mathbb{R} \times \mathbb{H} \to \mathbb{H}_1$ is continuous and there exist positive number L_{g_2} such that for each $(t,y), (t,z) \in \mathbb{R} \times \mathbb{H}$

$$\|g_2(t,y) - g_2(t,z)\| \leq L_{g_2}\|y-z\|,$$

Now, we consider the measure ν where its Radon-Nikodym derivative of $\rho_1(t) = t^2 e^{\cos(t)}$, $t \in \mathbb{R}$, and the measure μ where its Radon-Nikodym derivative of

$$\rho_2(t) = \begin{cases} e^t & \text{if } \; t \leq 0, \\ 1 & \text{if } \; t > 0. \end{cases}$$

Since $\nu([-r,r]) = \int_{-r}^{r} \rho_1(t)dt$, $\mu([-r,r]) = \int_{-r}^{r} \rho_2(t)dt$ and $\limsup_{r\to+\infty} \dfrac{\mu([-r,r])}{\nu([-r,r])} < \infty$, then **(M1)** is true. From [18], $\mu \in \mathcal{M}$ satisfies **(M2)**. For $r > 0$, we have

$$\frac{2r^3}{3e} \leq \nu([-r,r]) \leq \frac{2er^3}{3},$$

then $\nu \in \mathcal{M}$. Furthermore $\cos(\tau + \theta) \leq 2 + \cos(\theta)$ for all $\tau \in \mathbb{R}$ and $\theta \in A$, which implies that $\nu(\tau + A) \leq e^2\nu(A)$ and view to remark 3.4 in [18], we can conclude that ν satisfies **(M2)**. Taking $\tau = \pi$ and $\rho = 1$, it's easy to show that $F(t, \eta(t))$ and $G(t, \eta(t))$ are in $PSABP_{\tau,\rho}(\mathbb{R}, \mathbb{H}, \mu, \nu)$. Since $PSABP_{\tau,\rho}(\mathbb{R}, \mathbb{H}, \mu, \nu)$ is invariant by translation, then $F(t, \eta(k_2(t)))$ and $G(t, \eta(k_1(t)))$ are also in $PSABP_{\tau,\rho}(\mathbb{R}, \mathbb{H}, \mu, \nu)$.
For $i \in \{1, 2\}$, we put $k_i(t) = t - r_i$, where $r_i \geq 0$. Then k_i is continuous and strictly nondecreasing.

Now let $u \in PSABP_{\tau,\rho}(\mathbb{R}, \mathbb{H}, \mu, \nu)$ and prove that $u \circ k$ is $\in PSABP_{\tau,\rho}(\mathbb{R}, \mathbb{H}, \mu, \nu)$.

$$\begin{aligned}
& \lim_{r\to+\infty} \frac{1}{\nu([-r,r])} \int_{-r}^{r} \|\eta(k_i(t+\tau_1)) - e^{i\rho\tau}\eta(t)\| d\mu(t) \\
= & \lim_{r\to+\infty} \frac{1}{\nu([-r,r])} \int_{-r}^{r} \|\eta(t - r_i + \tau_1) - e^{i\rho\tau}\eta(t - r_i)\| d\mu(s) \\
= & \lim_{r\to+\infty} \frac{1}{\nu([-r,r])} \int_{-r}^{r} \|\eta(t + \tau_1 - r_i) - e^{i\rho\tau}\eta(t - r_i)\| d\mu(s) \\
= & \lim_{r\to+\infty} \frac{1}{\nu([-r,r])} \int_{-r}^{r} \|\eta(t - r_i + \tau) - e^{i\rho\tau}\eta(t - r_i)\| d\mu(s) = 0
\end{aligned}$$

Now let

$$\lambda_i(t) = \begin{cases} e^{r_i} & \text{if} \quad t \leq -r_i, \\ e^{-t} & \text{if} \quad -r_i < t \leq 0, \\ 1 & \text{if} \quad t > 0. \end{cases}$$

Since for $i \in \{1, 2\}$, we have $k_i(t) = t - r_i$, then for $r > r_i \geq 0$, we have

$$R_i(r) = |k_i(r)| + |k_i(-r)| = |-r - r_i| + |r - r_i| = r + r_i + r - r_i = 2r.$$

Moreover $M(R_i(r)) = \sup_{s\in[-2r,2r]} \lambda_i(s) = e^{r_i}$ and since $\nu([-r,r]) \geq \frac{2r}{e}$, for $r > 0$, then we have

$$\begin{aligned}
\lim_{r\to+\infty} \frac{\nu([-R_i(r), R_i(r)])}{\nu([-r,r])} M(R_i(r)) &= \lim_{r\to+\infty} \frac{\nu([-2r,2r])}{\nu([-r,r])} e^{r_i} \\
&\leq \lim_{r\to+\infty} \frac{4er}{2r} e e^{r_i} \\
&= 2e^{2+r_i} < \infty.
\end{aligned}$$

This implies that the assumption **(H7-H8)** is satisfied.

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