EXISTENCE OF OPTIMAL POSITIVE SOLUTIONS FOR NONLINEAR EULER-BERNOULLI BEAM EQUATION WHOSE BOTH ENDS ARE SLIDING CLAMPED

JINGJING WANG*, CHENGHUA GAO, YANQIONG LU AND XINGYUE HE

ABSTRACT. In this paper, we consider the nonlinear boundary value problem

$$\begin{cases} y^{(4)}(x) + (k_1 + k_2)y''(x) + k_1k_2y(x) = \lambda f(x, y(x)), & x \in [0, 1], \\ y'(0) = y'(1) = y'''(0) = y'''(1) = 0, \end{cases}$$

where k_1 and k_2 are constants, λ is a parameter. Based on this, by using the fixed-point index theory in cones and upper and lower solution method, the criteria of the existence, multiplicity and nonexistence of positive solutions are established in terms of different values of λ.

1. Introduction. In this paper, we are concerned with a nonlinear Euler-Bernoulli beam equation with Neumann boundary conditions (NBVP)

$$\begin{cases} y^{(4)}(x) + (k_1 + k_2)y''(x) + k_1k_2y(x) = \lambda f(x, y(x)), & x \in [0, 1], \\ y'(0) = y'(1) = y'''(0) = y'''(1) = 0, \end{cases} \tag{1.1}$$

where k_1 and k_2 are constants, λ is a parameter, $f \in C([0, 1] \times \mathbb{R}, \mathbb{R})$. This problem is always used to describes the sliding braces at both ends of an elastic beam.

With the emergence and development of a large number of edge sciences such as electromagnetic hydrodynamics, chemistry, hydrodynamics, dynamic meteorology, ocean dynamics and groundwater dynamics, many new differential equations have appeared. Fourth-order boundary value problems (BVPs) have been used to describe the deformations of elastic beam. In particular, the elastic beam equation is also called the Euler-Bernoulli beam equation. As we all know, the following fourth-order Euler-Bernoulli beam equation problem

$$\begin{cases} y^{(4)}(x) + \eta y''(x) + \zeta y(x) = \lambda f(x, y(x)), & x \in [0, 1], \\ y(0) = y(1) = y'(0) = y'(1) = 0 \end{cases} \tag{1.2}$$

has attracted the attention of many scholars, where η, ζ and λ are parameters, it describes the deformations of elastic beams with both fixed end-point, see **[3, 25, 30, 31]**; In particular, the

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equation (1.2) with Lidstone boundary condition

$$y(0) = y(1) = y''(0) = y''(1) = 0 \tag{1.3}$$

also has received a lot of attention in the last decades, it models the stationary states of the deflection of an elastic beam with both hinged ends, see **[4, 6, 12, 18, 20, 21, 22, 23, 24]** and the references therein. In addition to the mentioned fourth-order problem under the above boundary conditions, Webb and Infante **[29]** have also obtained excellent results on the fourth-order problems with local and nonlocal boundary conditions, and extended this kind of problem to arbitrary order boundary value problems, see **[14]**.

Some of nonlinear analysis tools have been used to investigate the existence of solutions for the elastic beam equation with boundary conditions (1.2) and (1.3), such as, lower and upper solutions **[3, 23, 24, 26]**, monotone iterative technique **[2, 13, 28]**, Krasnosel'skii fixed point theorem **[11, 23, 31]**, fixed point index **[17, 18, 29]**, Leray-Schauder degree **[1]** and bifurcation theory **[20, 21, 25, 27, 30]**.

On the research of equation (1.2), the existence results can be summarized into two situations. The first case is single parameter: $\eta = \zeta = 0$, $\lambda > 0$; another situation is double parameter: $\eta, \zeta \neq 0$, $\lambda = 1$; In the case of single parameter, there are many results due to its simple structure. In particular, we found that if $\eta = \zeta = 0$, the existence of positive solutions for problem

$$y^{(4)}(x) + \eta y''(x) - \zeta y(x) = f(x, y(x)),\ x \in (0,1) \tag{1.4}$$

with boundary value condition (1.3) has been studied by Ma and Wang **[22]**. They showed the existence of positive solutions under that $f(x,y)$ is either superlinear or sublinear on y by employing the fixed point theorem of cone extension or compression. When $\eta, \zeta \neq 0$, $\lambda = 1$, Li **[18]** discussed the existence of positive solution for fourth-order boundary value problem (1.3), (1.4) with two parameters by fixed-point index theory in cones under the following assumptions:

(F1) $f : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous;

(F2) $\eta, \zeta \in \mathbf{R}$ and $\eta < 2\pi^2$, $\zeta \geq -\frac{\pi^2}{4}$, $\frac{\zeta}{\pi^4} + \frac{\eta}{\pi^2} < 1$.

For convenience, we introduce the following notations:

$$f_0 = \liminf_{y\to 0} \min_{x\in[0,1]} \frac{f(x,y)}{y},\ f_\infty = \liminf_{y\to\infty} \min_{x\in[0,1]} \frac{f(x,y)}{y},$$

$$f^0 = \limsup_{y\to 0} \max_{x\in[0,1]} \frac{f(x,y)}{y},\ f^\infty = \limsup_{y\to\infty} \max_{x\in[0,1]} \frac{f(x,y)}{y}.$$

The main results of paper **[18]** are as follows:

Theorem A. Assume (F1) and (F2) hold. Then in each of the following case hold:

(i) $f^0 < \pi^4 - \eta\pi^2 - \zeta,\ f_\infty > \pi^4 - \eta\pi^2 - \zeta;$

(ii) $f_0 > \pi^4 - \eta\pi^2 - \zeta,\ f^\infty < \pi^4 - \eta\pi^2 - \zeta,$
the problem (1.3), (1.4) has at least one positive solution.

From Theorem A, we notice that $\pi^4 - \eta\pi^2 - \zeta$ is an eigenvalue of linear boundary value corresponding to problem (1.3), (1.4), if the strict inequalities in (i) or (ii) of Theorem A are weakened to nonstrict inequalities, the existence of solution for problem (1.3), (1.4) can not be guaranteed. In addition to, they derive the following corollary from Theorem A.

Corollary B. Assume (F1) and (F2) hold. Then in each of the following cases hold:

(i) $f^0 = 0,\ f_\infty = +\infty$(superlinear case);

(ii) $f_0 = +\infty,\ f^\infty = 0$(sublinear case),
the problem (1.3), (1.4) has a positive solution.

We summarize the many studies on simply supported beams introduced earlier, from the conditions given in the article, the parameters η and ζ are satisfied the key condition $\frac{\eta}{\pi^2} + \frac{\zeta}{\pi^4} < 1$. Under this condition, we can use the same method as literature **[10, 17]** to transform the solution of the boundary value problem into the solution of an integral equation with the help of Green's function. A natural question is: when the limiting conditions of parameters η and ζ change, that is, the condition (F2) of Theorem A is broken, and more specifically, what is the result of the existence of the solution of the problem when the method described above cannot be used smoothly? Further, when the parameter λ appears, is it possible to find λ^* which is exactly the point at which the solution to the problem does and does not exist? This is the next question we have to consider, in this paper, we will discuss the problems with three parameters and overcome these difficulties.

In recently, Vrabel **[26]** was interested in establishing the existence of solution of the fourth-order differential equation which models the stationary states of the deflection of an elastic beam, namely, the ordinary differential equation

$$y^{(4)}(x) + (k_1 + k_2)y''(x) + k_1k_2y(x) = f(x, y(x)),\ x \in [0, 1] \tag{1.5}$$

subject to the Lidstone boundary condition (1.3), where $k_1 < k_2 < 0$. It is correspond to hinged ends when there is no bending moment at the ends, see Gupta **[6]**, Lazer and McKenna **[16]**. Recently, Ma **[23, 24]** are considered with the problem (1.3), (1.5) under the assumption $k_1 < 0 < k_2 < \pi^2$ and $0 < k_1 < k_2 < \pi^2$, they proved non-negative of its Green's function and established the method of lower and upper solutions for problem (1.3), (1.5).

An interesting question arises, does the equation (1.5) still have a solution when the boundary conditions are change? By using the fixed point index and the critical group, Li **[17]** discussed the existence of positive solutions to the equation (1.5) with Neumann boundary condition, where $k_1 + k_2 = -2$ and $k_1k_2 = 1$. Obviously, this problem is a special case of literature **[26]**. Same

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research see Yang **[32, 33]**, Li **[19]** and the references therein. Then, Guo **[10]** discussed the existence of positive solutions for NBVP (1.1), where $k_1 + k_2 = \eta$ and $k_1 k_2 = -\zeta$, η, ζ are positive parameters and satisfy $\frac{\zeta}{\pi^4} + \frac{\eta}{\pi^2} < 1$, $\zeta \neq -\frac{\eta^2}{4}$, $\eta > -2\pi^2$. We noticed that although the study of **[10]** expanded the work of **[17]**, it still had great limitations on parameters.

Inspired by the above literatures, we aim to investigate the problem of existence, multiplicity and nonexistence of positive solutions for NBVP (1.1), and our work is still valid for the conclusions of Ma **[23, 24]** and Vrabel **[26]**, where $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous, k_1 and k_2 are constants, λ is a positive parameter. And we make the following assumptions:

(H1) $f(x, y) > 0$ for any $x \in [0,1]$ and $y > 0$;

(H2) $f_0 = \infty$ and $f_\infty = \infty$;

(H3) $f^0 = 0$ and $f^\infty = 0$.

The main results that we will establish are as follows:

Theorem 1.1 *Assume that (H1) and (H2) hold. Then there exists $\lambda^* > 0$ such that NBVP (1.1) has at least two positive solutions for $\lambda \in (0, \lambda^*)$, at least one positive solution for $\lambda = \lambda^*$ and no positive solution for $\lambda > \lambda^*$.*

Theorem 1.2 *Assume that (H1) and (H3) hold. Then there exists $\lambda_* > 0$ such that NBVP (1.1) has at least two positive solutions for $\lambda > \lambda_*$, at least one positive solution for $\lambda = \lambda_*$ and no positive solution for $\lambda \in (0, \lambda_*)$.*

Theorem 1.3 *Let ρ_1 be the first eigenvalue of the linear boundary value problem*

$$\begin{cases} y^{(4)}(x) + (k_1 + k_2)y''(x) + k_1 k_2 y(x) = \rho y(x), & x \in [0,1], \\ y'(0) = y'(1) = y'''(0) = y'''(1) = 0, \end{cases} \tag{1.6}$$

where $\rho_1 > 0$. Then

(i) if $0 \leq f^\infty < f_0 \leq +\infty$, then NBVP (1.1) has at least one positive solution for any $\lambda \in (\frac{\rho_1}{f_0}, \frac{\rho_1}{f^\infty})$;

(ii) if $0 \leq f^0 < f_\infty \leq +\infty$, then NBVP (1.1) has at least one positive solution for any $\lambda \in (\frac{\rho_1}{f_\infty}, \frac{\rho_1}{f^0})$.

Remark 1.1 *Notice that the conditions given by the Theorem 1.3 is to guarantee of the sharp existence condition for the positive solutions of the problem (1.1). Let $k_1 + k_2 = 5$, $k_1 \cdot k_2 = 4$, consider the following fourth-order Neumann boundary value problem*

$$\begin{cases} y^{(4)}(x) + 5y''(x) + 4y(x) = \lambda f(x, y(x)), & x \in [0,1], \\ y'(0) = y'(1) = y'''(0) = y'''(1) = 0, \end{cases} \tag{1.7}$$

where $f(x,s) = s + a(s)$ for any $x \in [0,1]$,

$$a(s) = \begin{cases} \dfrac{2s}{s^2+1}, & s \in (-\infty,-1) \cup (1,+\infty), \\ \dfrac{2s^3}{s^2+1}, & s \in [-1,1]. \end{cases}$$

Clearly, $f_0 = f^\infty = 1$, it follows from Theorem 1.3 that the problem (1.7) has no positive solution. In fact, suppose that $\lambda = \rho_1 = 4$, there is a positive solution $y(x)$ of the problem (1.7), where ρ_1 is the first eigenvalue of problem (1.6). Let $\varphi_1(x) > 0$ is the eigenfunction corresponding to ρ_1, multiplying the equation of (1.7) by $\varphi_1(x)$ and integrating on [0,1], we can get

$$\rho_1 \int_0^1 \varphi_1(s)y(s)ds = \rho_1 \int_0^1 \varphi_1(s)y(s)ds + \rho_1 \int_0^1 \varphi_1(s)a(y(s))ds.$$

Hence $\rho_1 \int_0^1 \varphi_1(s)a(y(s))ds = 0$, this contradict with the fact

$$\rho_1 \int_0^1 \varphi_1(s)a(y(s))ds > 0.$$

Corollary 1.1 *Assume that $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous, $yf(x,y) \geq 0$ for any $x \in [0,1]$ and $y \in \mathbb{R}$. Then*

(i) if $0 \leq f^\infty < f_0 \leq +\infty$, the NBVP (1.1) has at least one positive solution and one negative solution for any $\lambda \in (\frac{\rho_1}{f_0}, \frac{\rho_1}{f^\infty})$;

(ii) if $0 \leq f^0 < f_\infty \leq +\infty$, the NBVP (1.1) has at least one positive solution and one negative solution for any $\lambda \in (\frac{\rho_1}{f_\infty}, \frac{\rho_1}{f^0})$.

From the previous discussion, we find that the appearance of k_1 and k_2 lead to the absence of the positivity of Green's function in NBVP (1.1), which greatly increases the complexity of the calculation of Green's function. On this basis, in the second part of this paper, we discuss the properties of Green's function in detail according to the different classification of k_1 and k_2. Including the case of $k_1 \leq k_2 < 0$, $k_1 < 0 < k_2 \leq \frac{\pi^2}{4}$ and $0 < k_1 < k_2 \leq \frac{\pi^2}{4}$, respectively. In the third part of this paper, based on the second part, the corresponding proofs of Theorem 1.1 and 1.2 are given. And Section 4 shows the results of Theorems 1.3 and Corollary 1.1. In the last part, some numerical examples are given to verify our conclusion.

2. Preliminaries. Let $X = C[0,1]$ be a Banach space, with its usual normal $\|y\| = \max\{|y(x)|, x \in [0,1]\}$ for all $y \in X$.

2.1 Green's function and its sign properties in case $k_1 \leq k_2 < 0$

From $k_1 \leq k_2 < 0$, let $k_1 = -\alpha^2$, $k_2 = -\beta^2$, where α and β are constants greater than zero that satisfy $\alpha \geq \beta$. Then, the NBVP (1.1) is transformed into the following boundary value problem

$$\begin{cases} y^{(4)}(x) - (\alpha^2+\beta^2)y''(x) + \alpha^2\beta^2 y(x) = f(x, y(x)), \quad x \in [0,1], \\ y'(0) = y'(1) = y'''(0) = y'''(1) = 0. \end{cases} \tag{2.1}$$

Define linear operator $L : D(L) \to X$ as follows

$$Ly := y^{(4)} - (\alpha^2+\beta^2)y'' + \alpha^2\beta^2 y,\ y \in D(L),$$

where $D(L) := \{y \in C^4[0,1] : y'(0) = y'(1) = y'''(0) = y'''(1) = 0\}$.

To get the Green's function $\overline{G}(x,s)$ of $Ly = 0$, we define another linear operator

$$L_1 y := y'' - \alpha^2 y,\ D(L_1) := \{y \in C^2[0,1] : y'(0) = y'(1) = 0\}.$$

It's not difficult to calculate that the Green's function of $L_1 y = 0$ is

$$G_1(t,s) = -\begin{cases} \dfrac{\cosh[\alpha(1-t)]\cosh(\alpha s)}{\alpha\sinh\alpha}, & 0 \leq s \leq t \leq 1, \\ \dfrac{\cosh[\alpha(1-s)]\cosh(\alpha t)}{\alpha\sinh\alpha}, & 0 \leq t \leq s \leq 1. \end{cases}$$

Define linear operator

$$L_2 y := y'' - \beta^2 y,\ D(L_2) := \{y \in C^2[0,1] : y'(0) = y'(1) = 0\}.$$

Then the Green's function of $L_2 y = 0$ is

$$G_2(t,s) = -\begin{cases} \dfrac{\cosh[\beta(1-t)]\cosh(\beta s)}{\beta\sinh\beta}, & 0 \leq s \leq t \leq 1, \\ \dfrac{\cosh[\beta(1-s)]\cosh(\beta t)}{\beta\sinh\beta}, & 0 \leq t \leq s \leq 1. \end{cases}$$

It's easy to verify that the Green's function of $Ly = L_2 \circ (L_1 y)$, and $Ly = 0$ is

$$\overline{G}(x,s) := \int_0^1 G_2(x,t)G_1(t,s)dt,\ (x,s) \in [0,1]\times[0,1]. \tag{2.2}$$

Notice that if $\alpha = \beta$, then the characteristic equation $\mu^4 - 2\alpha^2\mu^2 + \alpha^4 = 0$ of (2.1) has double roots $\mu_1 = \alpha$, $\mu_2 = -\alpha$. In this case, the expression of $\overline{G}(x,s)$ can not be directly obtained from (2.2).

Therefore, we divide two cases as follows:

Case 1. $\alpha = \beta > 0$

In this case,

$$y(x) = C_1\cosh(\alpha x) + C_2\sinh(\alpha x) + C_3 x\cosh(\alpha x) + C_4 x\sinh(\alpha x)$$

is the general solution of $y^{(4)}(x) - (\alpha^2+\beta^2)y''(x) + \alpha^2\beta^2 y(x) = 0$. It's easy to compute that $\varphi(x) = \frac{\alpha x\cosh(\alpha x) - \sinh(\alpha x)}{2\alpha^3}$ is the solution of initial value problem

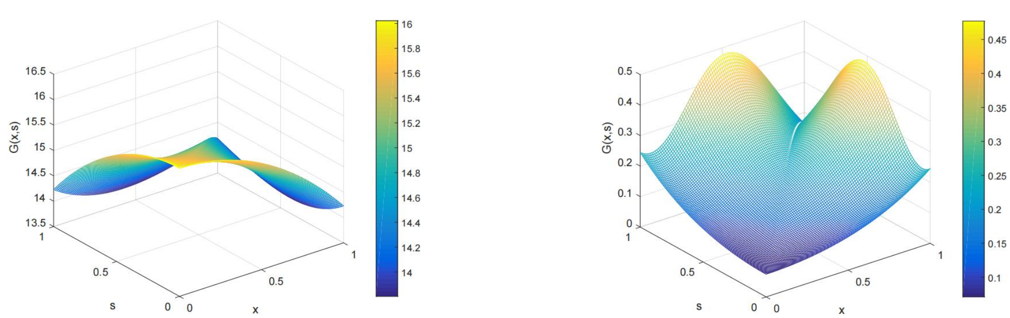
$$\begin{cases} \varphi^{(4)}(x) - 2\alpha^2\varphi''(x) + \alpha^4\varphi(x) = 0, & x \in [0,1], \\ \varphi(0) = \varphi'(1) = \varphi''(0) = 0, \varphi'''(1) = 1. \end{cases}$$

From the theory of Green's function, we can obtain the explicit expression of Green's function of (2.1) as follows

$$\overline{G}(x,s) = \begin{cases} \dfrac{\sinh\alpha\cosh[\alpha(1-s)][\cosh(\alpha x) - \alpha x\sinh(\alpha x)]}{2\alpha^3\sinh^2\alpha} \\ +\dfrac{\alpha\cosh(\alpha x)[\cosh(\alpha s + s\sinh\alpha\sinh[\alpha(1-s)]]}{2\alpha^3\sinh^2\alpha}, & s \le x, \\ \dfrac{\sinh\alpha\cosh[\alpha(1-x)][\cosh(\alpha s) - \alpha s\sinh(\alpha s)]}{2\alpha^3\sinh^2\alpha} \\ +\dfrac{\alpha\cosh(\alpha s)[\cosh(\alpha x + x\sinh\alpha\sinh[\alpha(1-x)]]}{2\alpha^3\sinh^2\alpha}, & x \le s. \end{cases} \tag{2.3}$$

In order to better characterize the properties of Green's function and verify the correctness of the results in this paper, the mathematical software *Matlab* is used to simulate the figures of Green's function under different values of α and β for reference.

Next, we first give the figure of Green's function when $\alpha = \beta$ of **Case 1**, at this time, α and β take the concrete real numbers, see illustration Fig. 1.



(1-a) $\alpha = \beta = 0.5$ in $\overline{G}(x,s)$ (1-b) $\alpha = \beta = 2$ in $\overline{G}(x,s)$

FIGURE 1. Figure of $\overline{G}(x,s)$ when $\alpha = \beta > 0$.

Case 2. $\alpha > \beta > 0$

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In this case, if $0 \le x \le s \le 1$, then

$$\begin{aligned}\overline{G}(x,s) =& \int_0^x \frac{\cosh[\beta(1-x)]\cosh(\beta t)}{\beta\sinh\beta}\frac{\cosh[\alpha(1-s)]\cosh(\alpha t)}{\alpha\sinh\alpha}dt \\ &+ \int_x^s \frac{\cosh[\beta(1-t)]\cosh(\beta x)}{\beta\sinh\beta}\frac{\cosh[\alpha(1-s)]\cosh(\alpha t)}{\alpha\sinh\alpha}dt \\ &+ \int_s^1 \frac{\cosh[\beta(1-t)]\cosh(\beta x)}{\beta\sinh\beta}\frac{\cosh[\alpha(1-t)]\cosh(\alpha s)}{\alpha\sinh\alpha}dt \\ =&\frac{1}{\alpha^2-\beta^2}\Big[\frac{-\beta\sinh\beta\cosh(\alpha x)\cosh[\alpha(1-s)]}{\alpha\beta\sinh\alpha\cosh\beta} + \frac{\alpha\sinh\alpha\cosh[\beta(1-s)]\cosh(\beta x)}{\alpha\beta\sinh\alpha\cosh\beta}\Big] \\ =&\frac{1}{\alpha^2-\beta^2}\Big[\frac{\cosh(\beta x)\cosh[\beta(1-s)]}{\beta\sinh\beta} - \frac{\cosh(\alpha x)\cosh[\alpha(1-s)]}{\alpha\sinh\alpha}\Big].\end{aligned}$$

By a similar calculation, when $0 \le s \le x \le 1$,

$$\overline{G}(x,s) = \frac{1}{\alpha^2-\beta^2}\Big[\frac{\cosh[\beta(1-x)]\cosh(\beta s)}{\beta\sinh\beta} - \frac{\cosh[\alpha(1-x)]\cosh(\alpha s)}{\alpha\sinh\alpha}\Big].$$

Thus the concrete expression of Green's function of problem (2.1) is

$$\overline{G}(x,s) = \begin{cases} \frac{1}{\alpha^2-\beta^2}\Big[\frac{\cosh[\beta(1-s)]\cosh(\beta x)}{\beta\sinh\beta} - \frac{\cosh[\alpha(1-s)]\cosh(\alpha x)}{\alpha\sinh\alpha}\Big], & 0 \le x \le s \le 1, \\ \frac{1}{\alpha^2-\beta^2}\Big[\frac{\cosh[\beta(1-x)]\cosh(\beta s)}{\beta\sinh\beta} - \frac{\cosh[\alpha(1-x)]\cosh(\alpha s)}{\alpha\sinh\alpha}\Big], & 0 \le s \le x \le 1. \end{cases} \tag{2.4}$$

Theorem 2.1 *If α, $\beta \in (0,+\infty)$ with $\alpha \ge \beta$, then the Green's function of problem (2.1) satisfies*

$$\overline{G}(x,s) > 0,\ (x,s) \in [0,1]\times[0,1].$$

Proof. According to literature **[15]**, we know

$$G_i(t,s) < 0,\ i = 1,\ 2,\ (t,s) \in [0,1]\times[0,1],$$

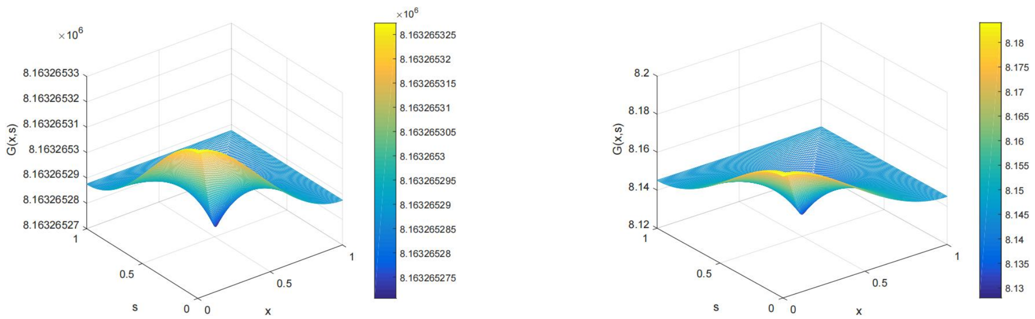
and from (2.2), we can get $\overline{G}(x,s) > 0$, $(x,s) \in [0,1]\times[0,1]$. □

Then give the figure of Green's function in **Case 2**, where α, $\beta \in (0,+\infty)$ with $\alpha \ge \beta$. In this case, α and β take the specific real numbers, as shown in Fig. 2.

2.2 Green's function and its sign properties in case $k_1 < 0 < k_2 \le \frac{\pi^2}{4}$

From $k_1 < 0 < k_2 \le \frac{\pi^2}{4}$, let $k_1 = -\alpha^2,\ k_2 = \beta^2$ and $\alpha \in (0,+\infty)$, $\beta \in (0,\frac{\pi}{2}]$, Then the NBVP (1.1) can be written as the following boundary value problem

$$\begin{cases} y^{(4)}(x) + (\beta^2-\alpha^2)y''(x) - \alpha^2\beta^2 y(x) = f(x,y(x)), & x \in [0,1], \\ y'(0) = y'(1) = y'''(0) = y'''(1) = 0. \end{cases} \tag{2.5}$$



(2-a) $\alpha = 0.7, \beta = 0.5 \times 10^{-3}$ in $\overline{G}(x,s)$ (2-b) $\alpha = 0.7, \beta = 0.5$ in $\overline{G}(x,s)$

FIGURE 2. Figure of $\overline{G}(x,s)$ when $\alpha,\ \beta \in (0, +\infty)$ with $\alpha \geq \beta$.

Define linear operator $L : D(L) \to X$ as follows

$$Ly := y^{(4)} + (\beta^2 - \alpha^2)y'' - \alpha^2\beta^2 y,\ y \in D(L),$$

where $D(L) := \{y \in C^4[0,1] : y'(0) = y'(1) = y'''(0) = y'''(1) = 0\}$.

To get the Green's function $\widetilde{G}(x,s)$ of the operator $Ly = 0$, define linear operator

$$L_1 y := y'' - \alpha^2 y,\ D(L_1) := \{y \in C^2[0,1] : y'(0) = y'(1) = 0\}.$$

It's not difficult to calculate $G_1(t,s)$ is the Green's function of $L_1 y = 0$.

Define a linear operator

$$L_3 y := y'' + \beta^2 y,\ D(L_3) := \{y \in C^2[0,1] : y'(0) = y'(1) = 0\},$$

then the Green's function of $L_3 y = 0$ is

$$G_3(t,s) = \begin{cases} \dfrac{\cos[\beta(1-t)]\cos(\beta s)}{\beta \sin \beta}, & 0 \leq s \leq t \leq 1, \\ \dfrac{\cos[\beta(1-s)]\cos(\beta t)}{\beta \sin \beta}, & 0 \leq t \leq s \leq 1. \end{cases}$$

Obviously, if $\alpha > 0$, then $G_1(t,s) < 0$, $(t,s) \in [0,1] \times [0,1]$. If $0 < \beta < \frac{\pi}{2}$, then $G_3(t,s) > 0$; If $\beta = \frac{\pi}{2}$, then $G_3(t,s) \geq 0$. Especially, $G_3(t,s) = 0$ with $t = s = 0$ or $t = s = 1$.

Hence, $Ly = L_3 \circ (L_1 y)$, the Green's function of $Ly = 0$ is

$$\widetilde{G}(x,s) := \int_0^1 G_3(x,t)G_1(t,s)dt,\ (x,s) \in [0,1] \times [0,1]. \tag{2.6}$$

Moreover, if $0 \le x \le s \le 1$, then

$$\begin{aligned}
-\widetilde{G}(x,s) =& \int_0^x \frac{\cos[\beta(1-x)]\cos(\beta t)}{\beta\sin\beta}\frac{\cosh[\alpha(1-s)]\cosh(\alpha t)}{\alpha\sinh\alpha}dt \\
&+ \int_x^s \frac{\cos[\beta(1-t)]\cos(\beta x)}{\beta\sin\beta}\frac{\cosh[\alpha(1-s)]\cosh(\alpha t)}{\alpha\sinh\alpha}dt \\
&+ \int_s^1 \frac{\cos[\beta(1-t)]\cos(\beta x)}{\beta\sinh\beta}\frac{\cosh[\alpha(1-t)]\cosh(\alpha s)}{\alpha\sinh\alpha}dt \\
=& \frac{\cosh[\alpha(1-s)]}{\alpha\beta\sin\beta\sinh\alpha}\frac{\beta\cosh(\alpha x)\sin\beta}{\alpha^2+\beta^2} + \frac{\cos(\beta x)}{\alpha\beta\sin\beta\sinh\alpha}\frac{\alpha\cos[\beta(1-s)]\sinh\alpha}{\alpha^2+\beta^2} \\
=& \frac{1}{\alpha^2+\beta^2}\Big[\frac{\cos(\beta x)\cos[\beta(1-s)]}{\beta\sin\beta} + \frac{\cosh(\alpha x)\cosh[\alpha(1-s)]}{\alpha\sinh\alpha}\Big].
\end{aligned}$$

By a similar calculation, if $0 \le s \le x \le 1$, then we can get

$$-\widetilde{G}(x,s) = \frac{1}{\alpha^2+\beta^2}\Big[\frac{\cos[\beta(1-x)]\cos(\beta s)}{\beta\sin\beta} + \frac{\cosh[\alpha(1-x)]\cosh(\alpha s)}{\alpha\sinh\alpha}\Big].$$

Thus, the concrete expression of Green's function of problem (2.5) is

$$-\widetilde{G}(x,s) = \begin{cases} \dfrac{1}{\alpha^2+\beta^2}\Big[\dfrac{\cos[\beta(1-s)]\cos(\beta x)}{\beta\sin\beta} + \dfrac{\cosh[\alpha(1-s)]\cosh(\alpha x)}{\alpha\sinh\alpha}\Big], & 0 \le x \le s \le 1, \\ \dfrac{1}{\alpha^2+\beta^2}\Big[\dfrac{\cos[\beta(1-x)]\cos(\beta s)}{\beta\sin\beta} + \dfrac{\cosh[\alpha(1-x)]\cosh(\alpha s)}{\alpha\sinh\alpha}\Big], & 0 \le s \le x \le 1. \end{cases} \tag{2.7}$$

The properties of Green's function $\widetilde{G}(x,s)$ are given as follows:

Theorem 2.2 *If $\alpha \in (0,+\infty)$, $\beta \in (0,\frac{\pi}{2}]$, then*

$$\widetilde{G}(x,s) < 0,\ (x,s) \in [0,1]\times[0,1].$$

Proof. When $\alpha \in (0,+\infty)$, $\beta \in (0,\frac{\pi}{2}]$, it can be obtained directly from literature [14] that

$$G_1(t,s) < 0,\ G_3(t,s) > 0,\ (t,s) \in [0,1]\times[0,1].$$

Combining (2.3), we know

$$\widetilde{G}(x,s) < 0,\ (x,s) \in [0,1]\times[0,1].$$

When $\alpha \in (0,+\infty)$, $\beta = \frac{\pi}{2}$, we can get $G_1(t,s) < 0$ and $G_3(x,t) \ge 0$.

In particular, $G_3(x,t) = 0$ if and only if $x = t = 0$ or $x = t = 1$. Therefore, if $x = 0$, combining this with (2.7), we can obtain

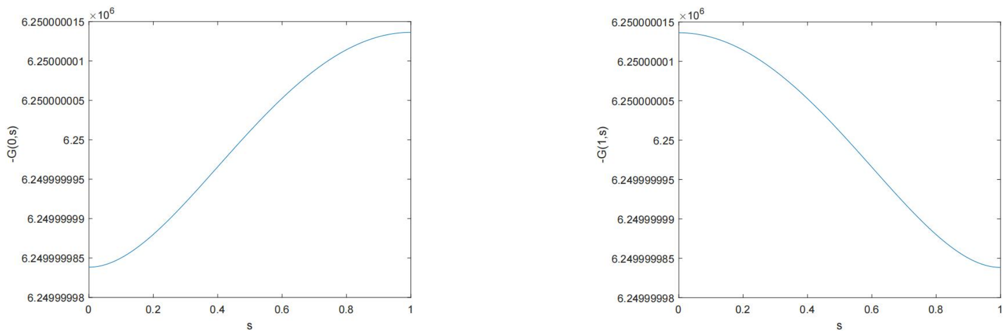
$$-\widetilde{G}(0,s) = \frac{1}{\alpha^2+\beta^2}\Big[\frac{\cos[\beta(1-s)]}{\beta\sin\beta} + \frac{\cosh[\alpha(1-s)]}{\alpha\sinh\alpha}\Big],\ 0 \le s \le 1.$$

Because $x\sin x$ is increasing on $x \in (0,\frac{\pi}{2})$, $\cos x$ is decreasing, so $\frac{\cos x}{x\sin x}$ is a decreasing function, however, $\frac{\cos x}{x\sin x}$ is positive on $x \in (0,\frac{\pi}{2})$. Apparently, $\sinh x$ and $\cosh x$ are increasing and positive on $x \in (0,+\infty)$. Therefore, $-\widetilde{G}(0,s) > 0$, $s \in [0,1]$, then we can get $\widetilde{G}(0,s) < 0$, $s \in [0,1]$;

If $x = 1$,

$$-\widetilde{G}(1,s) = \frac{1}{\alpha^2+\beta^2}\left[\frac{\cos(\beta s)}{\beta\sin\beta} + \frac{\cosh(\alpha s)}{\alpha\sinh\alpha}\right],\ 0 \le s \le 1.$$

Similar to reachable $\widetilde{G}(1,s) < 0,\ s \in [0,1]$. Figure 2.2.1 and Figure 2.2.1 are images of $-\widetilde{G}(0,s)$ and $-\widetilde{G}(1,s)$ when $\alpha = 2,\ \beta = 0.2 \times 10^{-3}$, where $s \in [0,1]$. From the images we can see that the result we want to prove is correct.



(3-a) $\alpha = 2, \beta = 0.2 \times 10^{-3}$ in $-\widetilde{G}(0,s)$ (3-b) $\alpha = 2, \beta = 0.2 \times 10^{-3}$ in $-\widetilde{G}(1,s)$

FIGURE 3. Figure of $-\widetilde{G}(0,s)$ and $-\widetilde{G}(1,s)$ when $\alpha = 2,\ \beta = 0.2 \times 10^{-3}$, where $s \in [0,1]$.

To sum up,

$$\widetilde{G}(x,s) < 0,\ (x,s) \in [0,1] \times [0,1].$$

□

Next, we give in Case $\alpha \in (0,+\infty),\ \beta \in (0,\frac{\pi}{2})$ the figures of the Green's function, when α, β take specific real numbers, see Fig. 4 (4-a,4-b). In particular, we also simulated the figures of the Green's function of case $\beta = \frac{\pi}{2}$, where α takes specific real number, see Fig. 4 (4-c,4-d).

Remark 2.1 *It is worth noting that we get $\widetilde{G}(x,s) < 0$ with the case of $k_1 < 0 < k_2 \le \frac{\pi^2}{4}$. At this point, if the problem we are studying (1.1) is transformed into the following form*

$$\begin{cases} y^{(4)}(x) + (k_1+k_2)y''(x) + k_1k_2y(x) + \lambda f(x,y(x)) = 0, & x \in [0,1], \\ y'(0) = y'(1) = y'''(0) = y'''(1) = 0, \end{cases} \tag{2.8}$$

then the results obtained in this paper still held true for the above problems.

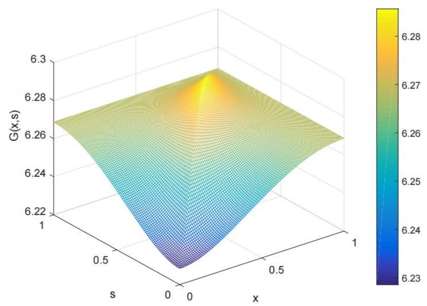
2.3 Green's function and its sign properties in case $0 < k_1 < k_2 \le \frac{\pi^2}{4}$

From $0 < k_1 < k_2 \le \frac{\pi^2}{4}$, let $k_1 = \alpha^2,\ k_2 = \beta^2$ and $0 < \alpha < \beta \le \frac{\pi}{2}$, then the NBVP (1.1) can be written as the following boundary value problem

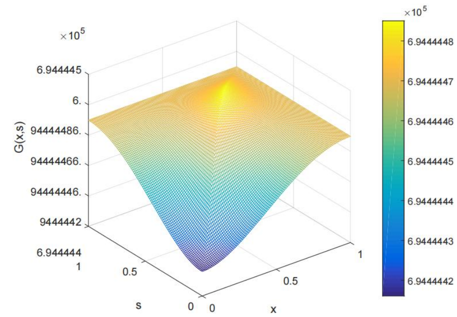
$$\begin{cases} y^{(4)}(x) + (\alpha^2+\beta^2)y''(x) + \alpha^2\beta^2 y(x) = f(x,y(x)), & x \in (0,1), \\ y'(0) = y'(1) = y'''(0) = y'''(1) = 0. \end{cases} \tag{2.9}$$

Define linear operators $L : D(L) \to X$

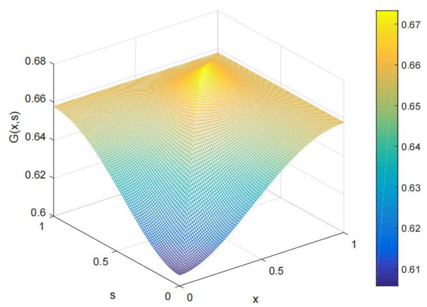
$$Ly := y^{(4)} + (\alpha^2+\beta^2)y'' + \alpha^2\beta^2 y,\ y \in D(L),$$



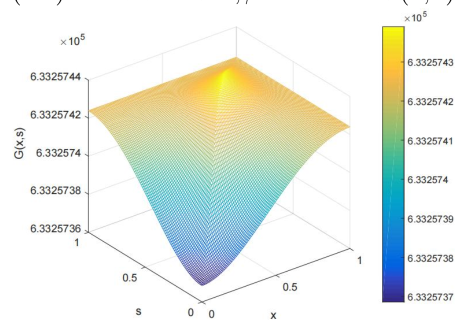
(4-a) $\alpha = 0.8, \beta = 0.5$ in $-\widetilde{G}(x,s)$



(4-b) $\alpha = 0.8 \times 10^{-3}, \beta = 1.5$ in $-\widetilde{G}(x,s)$



(4-c) $\alpha = 0.8, \beta = \frac{\pi}{2}$ in $-\widetilde{G}(x,s)$



(4-d) $\alpha = 0.8 \times 10^{-3}, \beta = \frac{\pi}{2}$

FIGURE 4. Figure of $-\widetilde{G}(x,s)$ when $\alpha \in (0,+\infty),\ \beta \in (0,\frac{\pi}{2})$ or $\beta = \frac{\pi}{2}$.

where

$$D(L) := \{y \in C^4[0,1] : y'(0) = y'(1) = y'''(0) = y'''(1) = 0\}.$$

To get the Green's function $G(x,s)$ of the operator $Ly = 0$, define another linear operator

$$L_4 y := y'' + \alpha^2 y,\ D(L_4) := \{y \in C^2[0,1] : y'(0) = y'(1) = 0\}.$$

It's not difficult to calculate the Green's function of $L_4 y = 0$ is

$$G_4(t,s) = \begin{cases} \dfrac{\cos[\alpha(1-t)]\cos(\alpha s)}{\alpha \sin\alpha}, & 0 \le s \le t \le 1, \\ \dfrac{\cos[\alpha(1-s)]\cos(\alpha t)}{\alpha \sinh\alpha}, & 0 \le t \le s \le 1. \end{cases}$$

Define the linear operator

$$L_3 y := y'' + \beta^2 y,\ D(L_3) := \{y \in C^2[0,1] : y'(0) = y'(1) = 0\}.$$

$G_3(t,s)$ is the Green's function of $L_3 y = 0$.

It is easy to verify $Ly = L_3 \circ (L_4 y)$, then the Green's function of $Ly = 0$ is

$$G(x,s) := \int_0^1 G_3(x,t) G_4(t,s)dt,\ (x,s) \in [0,1] \times [0,1]. \tag{2.10}$$

Notice that if $\alpha = \beta$, then the characteristic equation $\mu^4 + 2\alpha^2\mu^2 + \alpha^4 = 0$ of (2.5) has double roots $\mu_1 = \alpha i$, $\mu_2 = -\alpha i$. In this case, the expression of $G(x, s)$ can not be directly obtained from (2.6).

Therefore, we divide two cases as follows:

Case 3. $\alpha = \beta < \frac{\pi}{2}$

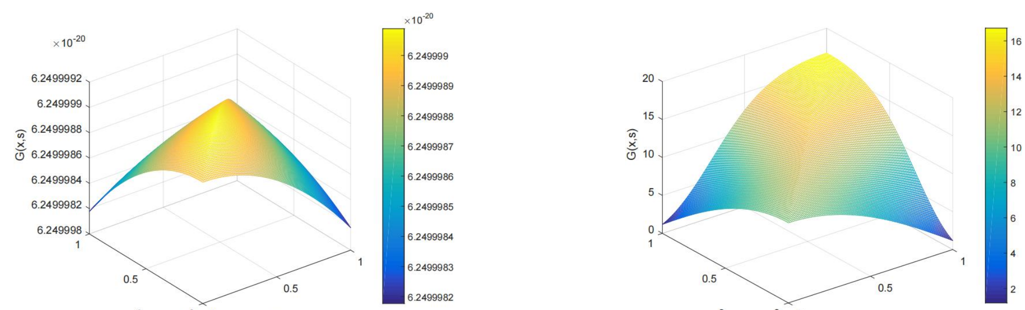
In this case, $y(x) = C_1 \cos(\alpha x) + C_2 \sin(\alpha x) + C_3 x \cos(\alpha x) + C_4 x \sin(\alpha x)$ is the general solution of $y^{(4)}(x) + (\alpha^2 + \beta^2)y''(x) + \alpha^2\beta^2 y(x) = 0$. It is easy to compute that $\varphi(x) = \frac{\sin(\alpha x) - \alpha x \cos(\alpha x)}{2\alpha^3}$ is the solution of initial value problem

$$\begin{cases} \varphi^{(4)}(x) + 2\alpha^2\varphi''(x) + \alpha^4\varphi(x) = 0, & x \in [0, 1], \\ \varphi(0) = \varphi'(1) = \varphi''(0) = 0, \varphi'''(1) = 1. \end{cases}$$

Then, we can obtain the concrete expression of Green's function of problem (2.5) as follows

$$G(x, s) = \begin{cases} \dfrac{\sin\alpha \cos[\alpha(1-s)][\cos(\alpha x) + \alpha x \sin(\alpha x)]}{2\alpha^3 \sin^2\alpha} \\ +\dfrac{\alpha \cos(\alpha x)[\cos(\alpha s - s \sin\alpha \sin[\alpha(1-s)]]}{2\alpha^3 \sin^2\alpha}, & s \leq x, \\ \dfrac{\sin\alpha \cos[\alpha(1-x)][\cos(\alpha s) + \alpha s \sin(\alpha s)]}{2\alpha^3 \sin^2\alpha} \\ +\dfrac{\alpha \cos(\alpha s)[\cos(\alpha x - x \sin\alpha \sin[\alpha(1-x)]]}{2\alpha^3 \sin^2\alpha}, & x \leq s. \end{cases} \tag{2.11}$$

Next, we first give the figure of Green's function when $\alpha = \beta$ in **Case 3**. In this case α and β take the concrete real numbers, as shown in Fig. 5.



(5-a) $\alpha = \beta = 0.5 \times 10^{-3}$ in $G(x, s)$ (5-b) $\alpha = \beta = 1.5$ in $G(x, s)$

FIGURE 5. Figure of $G(x, s)$ when $\alpha = \beta < \frac{\pi}{2}$.

In particular, if $\alpha = \beta = \frac{\pi}{2}$, then $t = s = 0$ or $t = s = 1$, $G(x, s)$ contains zero.

Case 4. $0 < \alpha < \beta < \frac{\pi}{2}$

In this case, if $0 \le x \le s \le 1$, then

$$\begin{aligned}
G(x,s) =& \int_0^x \frac{\cos[\alpha(1-s)]\cos(\alpha t)}{\alpha\sin\alpha}\frac{\cos[\beta(1-x)]\cos(\beta t)}{\beta\sin\beta}dt \\
&+ \int_x^s \frac{\cos[\alpha(1-s)]\cos(\alpha t)}{\alpha\sin\alpha}\frac{\cos[\beta(1-t)]\cos(\beta x)}{\beta\sin\beta}dt \\
&+ \int_s^1 \frac{\cos[\alpha(1-t)]\cos(\alpha s)}{\alpha\sin\alpha}\frac{\cos[\beta(1-t)]\cos(\beta x)}{\beta\sin\beta}dt \\
=&\frac{1}{\beta^2-\alpha^2}\Big[\frac{\beta\sin\beta\cos(\alpha x)\cos[\alpha(1-s)]}{\alpha\beta\sin\alpha\sin\beta} - \frac{\alpha\sin\alpha\cos[\beta(1-s)]\cos(\beta x)}{\alpha\beta\sin\alpha\sin\beta}\Big] \\
=&\frac{1}{\beta^2-\alpha^2}\Big[\frac{\cos(\alpha x)\cos[\alpha(1-s)]}{\alpha\sin\alpha} - \frac{\cos(\beta x)\cos[\beta(1-s)]}{\beta\sin\beta}\Big].
\end{aligned}$$

Similarly, if $0 \le s \le x \le 1$, then

$$G(x,s) = \frac{1}{\beta^2-\alpha^2}\Big[\frac{\cos[\alpha(1-x)]\cos(\alpha s)}{\alpha\sin\alpha} - \frac{\cos[\beta(1-x)]\cos(\beta s)}{\beta\sin\beta}\Big].$$

So the concrete expression of Green's function of problem (2.5) is

$$G(x,s) = \begin{cases} \dfrac{1}{\beta^2-\alpha^2}\Big[\dfrac{\cos[\alpha(1-s)]\cos(\alpha x)}{\alpha\sin\alpha} - \dfrac{\cos[\beta(1-s)]\cos(\beta x)}{\beta\sin\beta}\Big], & 0 \le x \le s \le 1, \\ \dfrac{1}{\beta^2-\alpha^2}\Big[\dfrac{\cos[\alpha(1-x)]\cos(\alpha s)}{\alpha\sin\alpha} - \dfrac{\cos[\beta(1-x)]\cos(\beta s)}{\beta\sin\beta}\Big], & 0 \le s \le x \le 1. \end{cases} \tag{2.12}$$

The properties of Green's function $G(x,s)$ are given as follows:

Theorem 2.3 *If $0 < \alpha < \beta \le \frac{\pi}{2}$, then*

$$G(x,s) > 0,\ (x,s) \in [0,1]\times[0,1].$$

Proof. According to literature **[15]**, we know that

$$G_i(t,s) > 0,\ i = 3,\ 4,\ (t,s) \in [0,1]\times[0,1].$$

Combining (2.6), we can obtain

$$G(x,s) > 0,\ (x,s) \in [0,1]\times[0,1].$$

If $0 < \alpha < \beta = \frac{\pi}{2}$, then we can get $G_4(t,s) > 0$ and $G_3(x,t) \ge 0$. Especially, $G_3(x,t) = 0$ if and only if $x = t = 0$ or $x = t = 1$. Therefore, when $x = 0$, by combining (2.12), we can obtain

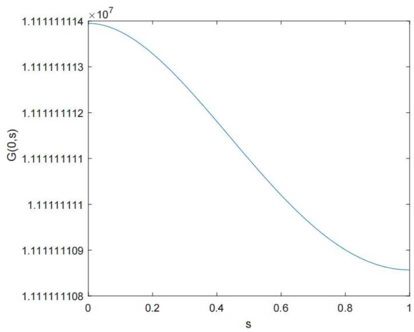
$$G(0,s) = \frac{1}{\beta^2-\alpha^2}\Big[\frac{\cos[\alpha(1-s)]}{\alpha\sin\alpha} - \frac{\cos[\beta(1-s)]}{\beta\sin\beta}\Big],\ 0 \le s \le 1.$$

Because $x\sin x$ is increasing on $x \in (0,\frac{\pi}{2})$, $\cos x$ is decreasing, so $\frac{\cos x}{x\sin x}$ is a decreasing function. Therefore $G(0,s) > 0$, $s \in [0,1]$;

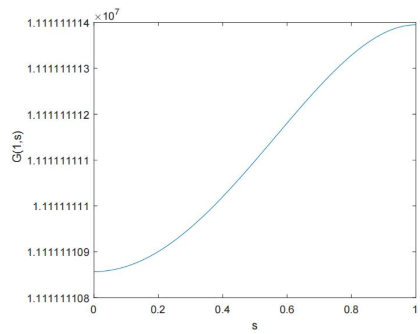
When $x = 1$,

$$G(1,s) = \frac{1}{\beta^2 - \alpha^2}\left[\frac{\cos(\alpha s)}{\alpha \sin\alpha} - \frac{\cos(\beta s)}{\beta \sin\beta}\right],\ 0 \le s \le 1.$$

Similarly, we can get $G(1,s) > 0,\ s \in [0,1]$. Figure (6-a) and Figure (6-b) are images of $G(0,s)$ and $G(1,s)$ when $\alpha = 0.2 \times 10^{-3},\ \beta = 1.5$, where $s \in [0,1]$. From the images we can see that the result we want to prove is correct.



(6-a) $\alpha = 0.2 \times 10^{-3}$ $\beta = 1.5$ in $G(0,s)$



(6-b) $\alpha = 0.2 \times 10^{-3}$ $\beta = 1.5$ in $G(1,s)$

FIGURE 6. Figure of $G(0,s)$ and $G(1,s)$ when $\alpha = 0.2 \times 10^{-3},\ \beta = 1.5$, where $s \in [0,1]$.

To sum up,

$$G(x,s) > 0,\ (x,s) \in [0,1] \times [0,1].$$

□

The figure of Green's function at $0 < \alpha < \beta < \frac{\pi}{2}$ in **Case 4** is given. At this point, α and β take the concrete real numbers, as is shown in Fig. 7 (7-a,7-b). In particular, the figure of Green's function at $0 < \alpha < \beta = \frac{\pi}{2}$ is also simulated, as shown in Fig. 7 (7-c,7-d).

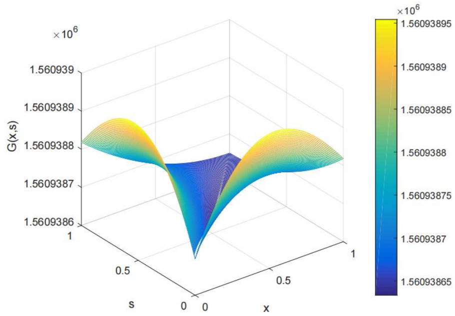
Remark 2.2 *It should be noted that in the three cases discussed in this section, if the parameter $k_1 = 0$ or $k_2 = 0$, the operator Ly has eigenvalue $\lambda_0 = 0$ and $Ly = 0$ has nontrivial solution $y \equiv C\,(C \neq 0)$. Therefore, according to the Fredholm alternative theorem, there is no solution to the problem (1.1), so the parameters in this paper meet the requirement that $k_1 k_2 \neq 0$ are always valid. In particular, the parameter $k_1 = k_2 = \frac{\pi^2}{4}$, if $t = s = 0$ or $t = s = 1$, then $G(x,s)$ contains zero.*

2.4 Some lemmas

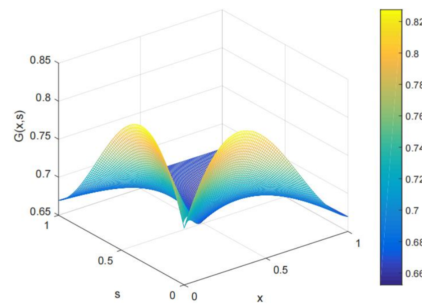
Based on the sign of Green's function of NBVP (1.1), without loss of generality, we discuss the case of $k_1 \le k_2 < 0$.

Obviously, $y(x)$ is a solution of the problem

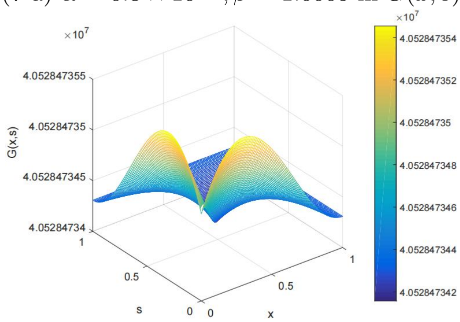
$$\begin{cases} y^{(4)}(x) + (k_1 + k_2)y''(x) + k_1 k_2 y(x) = h(x), & x \in [0,1], \\ y'(0) = y'(1) = y'''(0) = y'''(1) = 0, \end{cases}$$



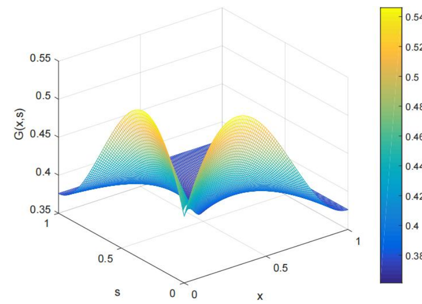
(7-a) $\alpha = 0.8 \times 10^{-3}, \beta = 1.0005$ in $G(x,s)$



(7-b) $\alpha = 0.8, \beta = 1.5$ in $G(x,s)$



(7-c) $\alpha = 0.1 \times 10^{-3}, \beta = \frac{\pi}{2}$ in $G(x,s)$



(7-d) $\alpha = 1, \beta = \frac{\pi}{2}$ in $G(x,s)$

FIGURE 7. Figure of $G(x,s)$ when $0 < \alpha < \beta < \frac{\pi}{2}$ and $0 < \alpha < \beta = \frac{\pi}{2}$.

then

$$y(x) = \int_0^1 \overline{G}(x,s)h(s)ds, \ x \in [0,1],$$

where $\overline{G}(x,s)$ is given by (2.4).

From Theorem 2.3, there exist $0 < m < M$ such that

$$m = \min_{x,s\in[0,1]} \overline{G}(x,s), \ M = \max_{x,s\in[0,1]} \overline{G}(x,s).$$

Consider 4-dimensional Banach space

$$E = \{y \in C^4[0,1] : y'(0) = y'(1) = y'''(0) = y'''(1) = 0\}$$

with the norm $\|y\| = \max_{0\le x\le 1} |y(x)|$ for all $y \in E$ and the cone P in E given by

$$P = \{y \in E, y(x) \ge 0, y(x) \ge \frac{m}{M}\|y\|\}.$$

For $u, v \in E$, we write $u \le v$ if $u(x) \le v(x)$ for any $x \in [0,1]$. For any $r > 0$, let $B_r = \{y \in E : \|y\| < r\}$ and $\partial B_r = \{y \in E : \|y\| = r\}$. We denote θ is the zero element in E.

Lemma 2.1 *Define operators $K, \mathbf{f}, A : E \to E$, by*

$$(Ky)(x) = \int_0^1 \overline{G}(x,s)y(s)ds, y \in E, x \in [0,1]; \tag{2.13}$$

$$(\mathbf{f}y)(x) = f(x, y(x)), y \in E, x \in [0,1];$$

$$A = K\mathbf{f}. \tag{2.14}$$

Then $K(P) \subset P$, $A(P) \subset P$ and $K : E \to E$, $A : E \to E$ are completely continuous.

Proof. By the definitions of m and M, it follows that

$$Ay(x) = \int_0^1 \overline{G}(x,s)f(s,y(s))ds \geq m\int_0^1 f(s,y(s))ds, x \in [0,1],$$

$$Ay(x) = \int_0^1 \overline{G}(x,s)f(s,y(s))ds \leq M\int_0^1 f(s,y(s))ds, x \in [0,1].$$

Accordingly,

$$Ay(x) \geq \frac{m}{M}\max_{x\in[0,1]} Ay(x) = \frac{m}{M}\|Ay\|.$$

So $A(P) \subset P$, by using the Arzelà-Ascoli theorem, A is a completely continuous operator.

By using similar method it yields that $K(P) \subset P$ and $K : E \to E$ is completely continuous. □

It is evident that $y \in P$ is a fixed point of the operator λA if and only if y is a solution of NBVP (1.1). K defined by (2.13) is an important operator in our later discussion. We present some properties of it as follows.

Lemma 2.2 *The spectral radius $r(K) > 0$ and there exist $\xi \in E$ with $\xi > 0$ on [0,1] such that $K\xi = r(K)\xi$ and $\int_0^1 \xi(s)ds = \frac{1}{r(K)}$. Moreover, $\rho_1 = \frac{1}{r(K)}$ is the first positive eigenvalue of the linear NBVP (1.1) and*

$$\int_0^1 (Ky)(s)\xi(s)ds = \frac{1}{\rho_1}\int_0^1 y(s)\xi(s)ds,\ \forall y \in E. \tag{2.15}$$

Proof. Define the cone $P_0 = \{y \in E : y(x) \geq 0, \forall\ x \in [0,1]\}$. Then the cone P_0 is normal and has nonempty interiors int P_0. It is clear that P_0 is also a total cone of E, that is, $E = \overline{P_0 - P_0}$, which means that the set $P_0 - P_0 = \{u - v : u, v \in P_0\}$ is dense in E. It follows from Lemma 2.1 that K is strongly positive, that is, $K(y) \in intP_0$ for $y \in P_0\backslash\{\theta\}$. Obviously, $K(P_0) \subseteq P_0$. By the Krein-Rutman theorem ([**5**], Theorem 19.3; [**34**], Theorem 7.C), the spectral radius $r(K) > 0$ and there exists $\xi_0 \in E$ with $\xi_0 > 0$ on [0,1] such that $K\xi_0 = r(K)\xi_0$. Let $\xi = \frac{\xi_0}{r(K)\int_0^1 \xi_0(s)ds}$. Obviously, $\xi > 0$ on [0,1], $K\xi = r(K)\xi$ and $\int_0^1 \xi(s)ds = \frac{1}{r(K)}$.

Notice that $K\xi = r(K)\xi$ is equivalent to the following NBVP

$$\begin{cases} \xi^{(4)}(x) + (k_1 + k_2)\xi''(x) + k_1k_2\xi(x) = \dfrac{1}{r(K)}\xi(x), & x \in [0,1], \\ \xi'(0) = \xi'(1) = \xi'''(0) = \xi'''(1) = 0, \end{cases}$$

we can obtain that $\rho_1 = \dfrac{1}{r(K)}$ is an eigenvalue of the linear NBVP (1.6). From the strong positivity of K, we know that there exist $\eta \in P_0$ and a constant $c > 0$ such that $cK\eta \geq \eta$ on [0,1]. Then ρ_1 is the first positive eigenvalue of the linear problem (1.6).

Since $\xi'(0) = \xi'(1) = \xi'''(0) = \xi'''(1) = 0$, that we have

$$\begin{aligned}\rho_1 \int_0^1 (Ky)(s)\xi(s)ds &= \int_0^1 (Ky)(s)\{\xi^{(4)}(s) + (k_1+k_2)\xi''(s) + k_1k_2\xi(s)\}ds \\ &= \int_0^1 \xi(s)(Ky)^{(4)}(s)ds + (k_1+k_2)\int_0^1 \xi(s)(Ky)''(s)ds \\ &+ k_1k_2\int_0^1 \xi(s)(Ky)(s)ds \\ &= \int_0^1 y(s)\xi(s)ds.\end{aligned}$$

Then (2.15) holds. □

The proof of the main theorems are based on the fixed point index theory. The following three well-known theorem in **[5, 7, 34]** are needed in our argument.

Lemma 2.3 *Let E be a Banach space and $X \subset E$ be a cone in E. Assume that Ω is a bounded open subset of E. Suppose that $A : X\cap\overline{\Omega} \to X$ is a completely continuous operator. If $\inf\limits_{x\in X\cap\partial\Omega} \|Ax\| > 0$ and $\mu Ax \neq x$ for $x \in X \cap \partial\Omega, \mu \geq 1$, then the fixed point index $i(A, X \cap \Omega, X) = 0$.*

Lemma 2.4 *Let E be a Banach space and $X \subset E$ be a cone in E. Assume that Ω is a bounded open subset of E. Suppose that $A : X\cap\overline{\Omega} \to X$ is a completely continuous operator. If there exist $x_0 \in X\backslash\{\theta\}$ such that $x - Ax \neq \mu x_0$ for all $x \in X \cap \partial\Omega$ and $\mu \geq 0$, then the fixed point index $i(A, X \cap \Omega, X) = 0$.*

Lemma 2.5 *Let E be a Banach space and $X \subset E$ be a cone in E. Assume that Ω is a bounded open subset of E with $\theta \in \Omega$. Suppose that $A : X \cap \overline{\Omega} \to X$ is a completely continuous operator. If $Ax \neq \mu x$ for all $x \in X \cap \partial\Omega$ and $\mu \geq 1$, then the fixed point index $i(A, X \cap \Omega, X) = 1$.*

3. Proofs of Theorems 1.1 and 1.2. Firstly, we introduce the following notations:

$\Phi = \{(\lambda, y) : \lambda > 0,\ y$ is a positive solution of NBVP (1.1)$\}$;

$\Lambda = \{\lambda > 0 :$ there exist y such that $(\lambda, y) \in \Phi\}$;

$\lambda^* = \sup\Lambda,\ \lambda_* = \inf\Lambda$.

Lemma 3.1 *Assume that $f_0 = \infty$. Then $\Phi \neq \emptyset$.*

Proof. Let $R > 0$ is a fixed number. Then we can choose $\lambda_0 > 0$ small enough such that $\lambda_0(\sup_{y\in P\cap\overline{B}_R} \|Ay\|) < R$. It is easy to see that

$$\lambda_0 Ay \neq \mu y,\ \forall y \in P \cap \partial B_R, \mu \geq 1.$$

By Lemma 2.5, it follows that

$$i(\lambda_0 A, P\cap B_R, P) = 1. \tag{3.1}$$

From $f_0 = \infty$, it follows that there exists $r \in (0, R)$ such that

$$f(x, y) \geq \frac{\rho_1}{\lambda_0} y,\ \forall y \in [0, r], x \in [0, 1], \tag{3.2}$$

where $\rho_1 > 0$ is given in Lemma 2.2. We may suppose that $\lambda_0 A$ has no fixed point on $P \cap \partial B_r$. Otherwise, the proof is finished. Now we shall prove

$$y \neq \lambda_0 Ay + \mu\xi,\ \forall P \cap \partial B_r, \mu \geq 0, \tag{3.3}$$

where ξ is given in Lemma 2.2. Suppose on the contrary that there exist $y_1 \in P\cap\partial B_r$ and $\mu_1 \geq 0$ such that $y_1 = \lambda_0 Ay_1 + \mu_1\xi$. Then $\mu_1 > 0$. Multiplying the equality $y_1 = \lambda_0 Ay_1 + \mu_1\xi$ by ξ and integrating on [0,1], by using (2.8) and (3.2), it follows that

$$\begin{aligned}
\int_0^1 y_1(s)\xi(s)ds &= \int_0^1 (\lambda_0 Ay_1)(s)\xi(s)ds + \mu_1 \int_0^1 \xi^2(s)ds \\
&= \frac{\lambda_0}{\rho_1}\int_0^1 f(s, y_1(s))\xi(s)ds + \mu_1 \int_0^1 \xi^2(s)ds \\
&\geq \int_0^1 y_1(s)\xi(s)ds + \mu_1 \int_0^1 \xi^2(s)ds,
\end{aligned}$$

which contradicts $\xi > 0$ on [0,1]. Thus, (3.3) holds. It follows from Lemma 2.4 that

$$i(\lambda_0 A, P \cap B_r, P) = 0. \tag{3.4}$$

According to the additivity of the fixed point index, it follows from (3.1) and (3.4) that

$$i(\lambda_0 A, P \cap (B_R\backslash\overline{B}_r), P) = i(\lambda_0 A, P\cap B_R, P) - i(\lambda_0 A, P\cap B_r, P) = 1,$$

which implies that the nonlinear operator $\lambda_0 A$ has one fixed point $y_0 \in P \cap (B_R\backslash\overline{B}_r)$.

Therefore, $(\lambda_0, y_0) \in \Phi$, i.e. $\Phi \neq \emptyset$. □

Lemma 3.2 *Assume that (H1) and (H2) hold. Then Λ is a bounded set.*

Proof. Let $(\lambda, y) \in \Phi$. It follows from (H1) and (H2) that there exists $C > 0$ such that $f(x, y) \geq Cy$ for all $y \geq 0$ and $x \in [0, 1]$. By Lemma 2.1, we obtain that $K^{-1} : E \to E$ is the inverse mapping of K, and $(K^{-1}y)(x) = \lambda(\mathbf{f}y)(x)$ for $x \in [0, 1]$. Since $\lambda K\mathbf{f}y = y$, we assume that $y(x_0) = \|y\| = \max_{x\in[0,1]} |y(x)|$ for $x_0 \in [0, 1]$. Then,

$$\|K^{-1}\|\|y\| \geq \|K^{-1}y\| \geq |(K^{-1}y)(x_0)| = \lambda f(x_0, y(x_0)) \geq \lambda C\|y\|,$$

where $\|K^{-1}\| = \sup_{\|y\|=1} \|K^{-1}y\|$. By Matrix theory([**8**], Vol.2, Page105, Theorem 13, [**9**], Page87, Theorem 6), $\|K^{-1}\| = \rho_1$. Thus, $\lambda \leq \rho_1 C^{-1}$. □

Lemma 3.3 *Assume that (H1) and (H2) hold. Then* $(0, \lambda^*) \subset \Lambda$. *Moreover, for any* $\lambda \in (0, \lambda^*)$, *NBVP (1.1) has at least two positive solutions.*

Proof. For any fixed $\lambda \in (0, \lambda^*)$, by the definition of λ^*, there exists $\lambda_2 \in \Lambda$ such that $\lambda < \lambda_2 \leq \lambda^*$ and $(\lambda_2, y_2) \in \Phi$. Fixed $R < \min_{x\in[0,1]} y_2(x)$. From the proof of Lemma 3.1, we see that there exist $\lambda_1 < \lambda, R > r$ and $y_1 \in P \cap (B_R \backslash \overline{B}_r)$ such that $(\lambda_1, y_1) \in \Phi$. It is easy to see that $0 < y_1(x) < y_2(x)$, $x \in [0, 1]$. Then by (H1), we have

$$y_1^{(4)}(x) + (k_1 + k_2)y_1''(x) + k_1k_2y_1(x) = \lambda_1 f(x, y_1(x)) < \lambda f(x, y_1(x)),\ x \in [0, 1]$$

and

$$y_2^{(4)}(x) + (k_1 + k_2)y_2''(x) + k_1k_2y_2(x) = \lambda_2 f(x, y_2(x)) > \lambda f(x, y_2(x)),\ x \in [0, 1]. \tag{3.5}$$

Now, we consider the modified boundary value problem

$$\begin{cases} y^{(4)}(x) + (k_1 + k_2)y''(x) + k_1k_2y(x) = \lambda f_1(x, y(x)), \quad x \in [0, 1], \\ y'(0) = y'(1) = y'''(0) = y'''(1) = 0, \end{cases} \tag{3.6}$$

where

$$f_1(x, y) = \begin{cases} f(x, y_1(x)), & y(x) < y_1(x), \\ f(x, y(x)), & y_1(x) \leq y(x) \leq y_2(x), \\ f(x, y_2(x)), & y(x) > y_2(x). \end{cases}$$

Clearly, the function λf_1 is bounded for $x \in [0, 1]$. Define the operator $A_1 : E \to E$ by

$$(A_1y)(x) = \int_0^1 \overline{G}(x, s) f_1(s, y(s))ds,\ x \in [0, 1].$$

Then $A_1 : P \to P$ is completely continuous and $y(x)$ is a solution of (3.6) if and only if $y = y(x) \in E$ is a fixed point of operator λA_1. It is easy to see that there exists $r_0 > \|y_2\|$ such that $\|\lambda A_1 y\| < r_0$ for any $y \in E$. It follows from Lemma 2.5 that

$$i(\lambda A_1, P \cap B_{r_0}, P) = 1. \tag{3.7}$$

Choose

$$U = \{y \in P : y_1(x) \leq y(x) \leq y_2(x), \forall x \in [0, 1]\}.$$

We claim that if $y \in P$ is a fixed point of operator λA_1, then $y \in U$. We first prove that $y(x) \leq y_2(x)$ on [0,1]. Suppose on the contrary that there exist some $x \in [0, 1]$ such that $y(x) > y_2(x)$. Thus, there exists $x_0 \in [0, 1]$ such that

$$y(x_0) - y_2(x_0) = \max_{x\in[0,1]}\{y(x) - y_2(x)\} > 0.$$

Consequently, we have

$$\begin{aligned}&(y-y_2)^{(4)}(x_0)+(k_1+k_2)(y-y_2)''(x_0)+k_1k_2(y-y_2)(x_0)\\=&\lambda f_1(x_0,y(x_0))-\lambda f(x_0,y_2(x_0))\\<&\lambda f(x_0,y(x_0))-\lambda f(x_0,y_2(x_0))=0.\end{aligned}$$

It's easy verify that there exists $h\in C[0,1]$ satisfying $h(x)\leq 0$, $h(x)\not\equiv 0$ on any subinterval and $h(x_0)\neq 0$ such that the linear boundary value problem

$$\begin{cases}(y-y_2)^{(4)}(x)+(k_1+k_2)(y-y_2)''(x)+k_1k_2(y-y_2)(x)=h(x), & x\in[0,1],\\(y-y_2)'(0)=(y-y_2)'(1)=(y-y_2)'''(0)=(y-y_2)'''(1)=0\end{cases}$$

has a unique solution

$$(y-y_2)(x)=\int_0^1\overline{G}(x,s)h(s)ds\leq 0.$$

Especially, $(y-y_2)(x_0)=\int_0^1\overline{G}(x_0,s)h(s)ds\leq 0$. This is a contradiction. It follows that $y(x)\leq y_2(x)$ on [0,1].

Using the similar methods, we can prove that $y(x)\geq y_1(x)$ on [0,1]. By virtue of the claim, the excision property of the fixed point index and (3.7), we can obtain

$$i(\lambda A_1,U,P)=i(\lambda A_1,P\cap B_{r_0},P)=1.$$

From the definition of A_1, we know that $A_1=A$ on $\overline{U}$. Then,

$$i(\lambda A,U,P)=1. \tag{3.8}$$

Hence, the nonlinear operator λA has at least one fixed-point $v_1\in U$. That is, $v_1(x)$ is a positive solution of NBVP (1.1). This means $(\lambda,v_1)\in\Phi$ and $(0,\lambda^*)\subset\Lambda$.

Now, we find the second positive solution of NBVP (1.6). By $f_\infty=\infty$ and the continuity of $f(x,y)$, there exists $C>0$ such that

$$f(x,y)\geq 2\rho_1\lambda^{-1}y-C,\ \forall y\geq 0, x\in[0,1]. \tag{3.9}$$

Set $\Omega=\{y\in P: y=\lambda Ay+\tau\xi \text{ for some } \tau\geq 0\}$, where ξ is given in Lemma 2.3. We claim that Ω is bounded in E. In fact, for any $y\in\Omega$, there exists $\tau\geq 0$ such that $y=\lambda Ay+\tau\xi\geq\lambda Ay$. Then, by (3.9), we have

$$y(x)\geq 2\rho_1(Ky)(x)-\lambda C(Kv_0)(x),\ x\in[0,1],$$

where $v_0(x)\equiv 1$. Multiplying the above inequality by $\xi(x)$ and integrating on [0,1], it follows from

Lemma 2.3 that

$$\begin{aligned}\int_0^1 y(s)\xi(s)ds &\geq 2\rho_1 \int_0^1 (Ky)(s)\xi(s)ds - \lambda C \int_0^1 (Kv_0)(s)\xi(s)ds \\ &= 2\int_0^1 y(s)\xi(s)ds - \lambda C.\end{aligned}$$

This implies that $\int_0^1 y(s)\xi(s)ds \leq \lambda C$. Let $\delta = \min_{x\in[0,1]}\{\xi(x)\} > 0$. Thus, $\|y\| \leq \lambda\delta^{-1}C$. Then we can conclude that Ω is bounded in E. Therefore there exists $R_1 > \max\{\|y_2\|, \lambda^*\delta^{-1}C\}$ such that

$$y \neq \lambda Ay + \tau\xi, \ \forall y \in P \cap \partial B_{R_1}, \tau \geq 0.$$

This together with Lemma 2.4 implies that

$$i(\lambda A, P \cap B_{R_1}, P) = 0. \tag{3.10}$$

Using a similar argument as in deriving (3.4), we have

$$i(\lambda A, P \cap B_{r_1}, P) = 0,$$

where $0 < r_1 < \min_{x\in[0,1]}\{y_1(x)\}$. Then according to the additivity of the fixed point index, by (3.8) and (3.10), we deduce that

$$\begin{aligned}&i(\lambda A, P \cap (B_{R_1} \backslash (\overline{U} \cup \overline{B}_{r_1})), P) \\ =&i(\lambda A, P \cap B_{R_1}, P) - i(\lambda A, \overline{U}, P) - i(\lambda A, P \cap B_{r_1}, P) = -1,\end{aligned}$$

which implies that the nonlinear operator λA has at least one fixed point $v_2 \in P \cap (B_{R_1} \backslash (\overline{U} \cup \overline{B}_{r_1}))$. Therefore, NBVP (1.1) has another positive solution. □

Lemma 3.4 *Assume that (H1) and (H2) hold. Then* $\Lambda = (0, \lambda^*]$.

Proof. In view of Lemma 3.3, it suffices to prove that $\lambda^* \in \Lambda$. By the definition of λ^*, we can choose $\{\lambda_n\} \subset \Lambda$ with $\lambda_n \geq \frac{\lambda^*}{2} (n = 1, 2, \cdots)$ such that $\lambda_n \to \lambda^*$ as $n \to \infty$. By Lemma 3.3, we can choose $y_n \subset P \backslash \{\theta\}$ such that $(\lambda_n, y_n) \in \Phi$. Then from (H2) and the continuity of f, we know that there exist $c > \frac{2\rho_1}{\lambda^*}$ and $d > 0$ such that $f(x, y) \geq cy - d$ for any $y \geq 0$ and $x \in [0, 1]$. Then

$$y_n(x) = (\lambda_n Ay_n)(x) \geq \frac{c\lambda^*}{2}(Ky_n)(x) - \frac{d\lambda^*}{2}(Kv_0)(x), \ x \in [0, 1],$$

where $v_0(x) \equiv 1$. Multiplying the above inequality by $\xi(x)$ and integrating on [0,1], it follows from Lemma 2.2 that

$$\begin{aligned}\int_0^1 y_n(s)\xi(s)ds &\geq \frac{c\lambda^*}{2}\int_0^1 (Ky_n)(s)\xi(s)ds - \frac{d\lambda^*}{2}\int_0^1 (Kv_0)(s)\xi(s)ds \\ &= \frac{c\lambda^*}{2\rho_1}\int_0^1 (y_n)(s)\xi(s)ds - \frac{d\lambda^*}{2}.\end{aligned} \tag{3.11}$$

Now we show that $\{y_n\}$ is bounded. Suppose on the contrary that there exists a subsequence of $\{y_n\}$ (still denoted by $\{y_n\}$) such that $y_n(x_0) \to +\infty$ as $n \to \infty$ for some $x_0 \in [0,1]$. By (3.11), we have

$$y_n(x_0)\xi(x_0)(\frac{c\lambda^*}{2\rho_1} - 1) \leq (\frac{c\lambda^*}{2\rho_1} - 1)\int_0^1 y_n(s)\xi(s)ds \leq \frac{d\lambda^*}{2},$$

which contradicts $y_n(x_0) \to +\infty$ as $n \to \infty$. Hence, $\{y_n\}$ is bounded. Since E is a Banach space and A is a compact operator, there exist a subsequence of $\{y_n\}$ (still denoted by $\{y_n\}$) and $y^* \in P$ such that $y_n \to y^*$ as $n \to \infty$. By $y_n = \lambda_n A y_n$, $n \to \infty$, we obtain that $y^* = \lambda^* A y^*$. Since $y_n(x) = \lambda_n \int_0^1 G(x,s)f(s,y_n(s))ds \geq \frac{\lambda^* m}{2} f(x, y_n(x))$ for all $x \in [0,1]$, then by $\{y_n\} \subset P\backslash\{\theta\}$ we know that $1 \geq \frac{\lambda^*}{2} \cdot \frac{f(x,y_n(x))}{y_n(x)}$ for all $x \in [0,1]$. It follows from $f_\infty = \infty$ that $y^* \in P\backslash\{\theta\}$. So, $\lambda^* \in \Lambda$. □

Proof of Theorem 1.1. Theorem 1.1 really follows from Lemmas 3.1-3.4.

Lemma 3.5 *Assume that (H1) and (H3) hold. Then NBVP (1.1) has at least one positive solution for λ large enough and has no positive solution for λ small enough.*

Proof. Let $r > 0$ be fixed. From (H1) and the definition of cone P, it follows that there exists $C > 0$ such that $f(x, y(x)) \geq C$ for all $x \in [0,1]$ and $y \in P \cap \partial B_r$. Then for any fixed $\lambda > \frac{r}{mC}$ and $y \in P \cap \partial B_r$, one has

$$\lambda(Ay)(x) = \lambda \int_0^1 \overline{G}(x,s)f(s,y(s))ds \geq \lambda mC > r, \ x \in [0,1].$$

This gives that $\inf_{y \in P\cap\partial B_r} \|\lambda Ay\| > 0$ and $\mu\lambda Ay \neq y$ for $y \in P \cap \partial B_r, \mu \geq 1$. By Lemma 2.4, it follows that

$$i(\lambda A, P \cap B_r, P) = 0. \tag{3.12}$$

From $f^\infty = 0$, there exists $R > r$ such that

$$f(x,y) \leq \frac{1}{2\lambda M} y, \ \forall\, y \in [\frac{m}{M}R, \infty), x \in [0,1].$$

Then for $y \in P \cap \partial B_R$, by the definition of cone P, one has

$$\min_{x\in[0,1]}\{y(x)\} \geq \frac{m}{M}\|y\| = \frac{m}{M}R,$$

and so

$$\lambda(Ay)(x) = \lambda \int_0^1 \overline{G}(x,s)f(s,y(s))ds \leq \lambda M \frac{1}{2\lambda M}\|y\| < R, x \in [0,1].$$

It follows from Lemma 2.5 that

$$i(\lambda A, P \cap B_R, P) = 1. \tag{3.13}$$

According to the additivity of the fixed point index, by (3.12) and (3.13), we have

$$i(\lambda A, P \cap (B_R \backslash \overline{B}_r), P) = i(\lambda A, P \cap B_R, P) - i(\lambda A, P \cap B_r, P) = 1,$$

which implies that the nonlinear operator λA has at least one fixed point $y \in P \cap (B_R \backslash \overline{B}_r)$. Therefore, NBVP (1.1) has at least one positive solution.

Next, we prove the nonexistence result. From (H3) and the continuity of $f(x, y)$ with respect to y, there exists $C_1 > 0$ such that $f(x, y) \le C_1 y$ for any $x \in [0, 1]$ and $y \ge 0$. Assume that NBVP (1.1) has one positive solution $y(x)$ for λ small enough such that $\lambda M C_1 < 1$. Then

$$\|y\| = \|\lambda \int_0^1 \overline{G}(x, s) f(s, y(s)) ds\| \le \lambda M C_1 \int_0^1 y(s) ds \le \lambda M C_1 \|y\| < \|y\|,$$

which is a contradiction. □

Lemma 3.6 *Assume that (H1) and (H3) hold. Then $0 < \lambda_* < \infty$ and $(\lambda_*, +\infty) \subset \Lambda$. Moreover, for any $\lambda \in (\lambda_*, +\infty)$, NBVP (1.1) has at least two positive solutions.*

Proof. By virtue of Lemma 3.5, we can easily obtain that $0 < \lambda_* < \infty$. For any fixed $\lambda \in (\lambda_*, +\infty)$, we prove that $\lambda \in \Lambda$. By the definition of λ_*, there exists $\lambda_1 \in \Lambda$ such that $\lambda_* \le \lambda_1 < \lambda$ and $(\lambda_1, y_1) \in \Phi$. Let $r > \frac{M}{m}\|y_1\|$ be fixed. From the proof of Lemma 3.5, we see that there exist $\lambda_2 > \lambda, R > r$ and $y_2 \in P \cap (B_R \backslash \overline{B}_r)$ such that $(\lambda_2, y_2) \in \Phi$. By the definition of cone P, it is easy to see that $0 < y_1(x) < y_2(x)$ for all $x \in [0, 1]$. Define

$$V = \{y \in P : y_1(x) \le y(x) \le y_2(x), \forall x \in [0, 1]\}.$$

An argument similar to the one used in deriving (3.8) in the proof of Lemma 3.3 yields

$$i(\lambda A, V, P) = 1. \tag{3.14}$$

Hence, the nonlinear operator λA has at least fixed point $v_1 \in V$. Then $v_1(x)$ is a positive solution of NBVP (1.1). This gives $(\lambda, v_1) \in \Phi$ and $(\lambda_*, +\infty) \subset \Lambda$.

Now we find the second positive solution of NBVP (1.1). From $f^0 = 0$, there exists $0 < r_0 < \min\limits_{x \in [0,1]} \{y(x)\}$ such that

$$f(x, y) \le \frac{1}{2\lambda M} y, \ \forall y \in [0, r_0], x \in [0, 1].$$

Then for $y \in P \cap \partial B_{r_0}$, we have

$$\lambda (Ay)(x) = \lambda \int_0^1 \overline{G}(x, s) f(s, y(s)) ds \le \lambda M \frac{1}{2\lambda M} \|y\| < r_0, x \in [0, 1].$$

It follows from Lemma 2.5 that

$$i(\lambda A, P \cap B_{r_0}, P) = 1. \tag{3.15}$$

Using a similar argument as in deriving (3.13), we have

$$i(\lambda A, P \cap B_{R_0}, P) = 1, \tag{3.16}$$

where $R_0 > \|y_2\|$. According to the fixed point index, by (3.14)-(3.16), we have

$$\begin{aligned}&i(\lambda A, P\cap(\overline{V}\cup\overline{B}_{r_0}), P)\\=&i(\lambda A, P\cap B_{R_0}, P)-i(\lambda A, V, P)-i(\lambda A, P\cap B_{r_0}, P)=-1,\end{aligned}$$

which implies that the nonlinear operator λA has at least one fixed point $v_2\in P\cap(B_{r_0}\backslash(\overline{V}\cup\overline{B}_{r_0}))$. Thus, NBVP (1.1) has another positive solution. □

Lemma 3.7 *Assume that (H1) and (H3) hold. Then* $\Lambda=[\lambda_*, +\infty)$.

Proof. The proof is similar to Lemma 3.4, so we omit it here. □

Proof of Theorem 1.2. Theorem 1.2 directly follows from Lemmas 3.5-3.7.

Remark 3.1 *The results of Theorem 1.1 and 1.2 are satisfied for the case of* $0<k_1<k_2\le\frac{\pi^2}{4}$. *Besides, the conclusion of Theorem 1.1 and 1.2 for* $k_1<0<k_2\le\frac{\pi^2}{4}$ *also apply when NBVP (1.1) is converted into the problem (2.8) as Remark 2.1. On account of the proof is similar to Theorems 1.1 and 1.2, so we omit it here.*

4. Proofs of Theorem 1.3 and Corollary 1.1.

Proof of Theorem 1.3. (i) Fix $\lambda\in(\frac{\rho_1}{f_0},\frac{\rho_1}{f^\infty})$. Then $f_0>\frac{\rho_1}{\lambda}$ and $f^\infty<\frac{\rho_1}{\lambda}$. By $f_0>\frac{\rho_1}{\lambda}$, there exists $r_1>0$ such that

$$f(x,y)\ge\frac{\rho_1}{\lambda}y,\ \forall\, y\in[0,r_1], x\in[0,1].$$

Suppose that λA has no fixed point on $P\cap\partial B_{r_1}$. Otherwise, the proof of (i) is finished. From (3.4), we have

$$i(\lambda A, P\cap B_{r_1}, P)=0. \tag{4.1}$$

On the other hand, by $f^\infty<\frac{\rho_1}{\lambda}$ and the continuity of $f(x,y)$, there exist $C>0$ and $\sigma\in(0,1)$ such that

$$f(x,y)\le\frac{\rho_1\sigma}{\lambda}y+C,\ \forall\, y\in[0,+\infty), x\in[0,1]. \tag{4.2}$$

Define

$$W=\{y\in P: y=s\lambda Ay \text{ for some } s\in[0,1]\}.$$

Now we show that W is bounded in E. For any $y\in W$, then there exists $s\in[0,1]$ such that $y=s\lambda Ay$. By (4.2), we have $y=s\lambda Ay\le\rho_1\sigma Ky+\lambda CKv_0$, where $v_0(x)\equiv1, x\in[0,1]$. Thus

$$(I-K_1)y\le CKv_0, \tag{4.3}$$

where $K_1=\rho_1\sigma K$ and I is the identity operator. Since $r(K_1)=\rho_1\sigma r(K)<1$, the inverse operator $(I-K_1)^{-1}$ exists and is given by $(I-K_1)^{-1}=I+K_1+K_1^2+\cdots$. This together with $K_1(P)\subset P$ yields that $(I-K_1)^{-1}(P)\subset P$. Now, from (4.3), we have $y\le(I-K_1)^{-1}CKv_0$. Hence, W is bounded. Then there exists $R_1 > \max\{r_1, \sup_{y\in W} \|y\|\}$ such that

$$y \neq s\lambda Ay,\ \forall y \in P \cap \partial B_{R_1}, s \in [0,1].$$

This and Lemma 2.5 imply that $i(\lambda A, P \cap B_{R_1}, P) = 1$. Taking (4.1) into account, we have

$$i(\lambda A, P \cap (B_{R_1} \backslash \overline{B}_{r_1}), P) = 1,$$

which means that λA has at least one fixed point in $P \cap (B_{R_1} \backslash \overline{B}_{r_1})$. That is, NBVP (1.1) has at least one positive solution.

(ii) Fix $\lambda \in (\frac{\rho_1}{f_\infty}, \frac{\rho_1}{f^0})$. Then $f^0 < \frac{\rho_1}{\lambda}$ and $f_\infty > \frac{\rho_1}{\lambda}$. By $f^0 < \frac{\rho_1}{\lambda}$, there exist $\varepsilon \in (0,1)$ and $r_2 > 0$ such that

$$f(x,y) \le \frac{\rho_1}{\lambda}(1-\varepsilon)y,\ \forall y \in [0, r_2], x \in [0,1]. \tag{4.4}$$

Now we prove

$$\lambda Ay \neq \mu y,\ y \in P \cap \partial B_{r_2}, \mu \ge 1. \tag{4.5}$$

If (4.5) holds, then there exist $\mu_0 \ge 1$ and $y_0 \in P \cap \partial B_{r_2}$ such that $\lambda A y_0 = \mu_0 y_0$. Then, by (4.4), we have

$$\begin{aligned} y_0(x) \le& \lambda(Ay_0)(x) \le \lambda \int_0^1 \overline{G}(x,s) f(s, y_0(s)) ds \\ \le& \rho_1(1-\varepsilon) \int_0^1 \overline{G}(x,s) y_0(s) ds, x \in [0,1]. \end{aligned}$$

This gives $\rho_1(1-\varepsilon)Ky_0 \ge y_0$. Multiplying this inequation by ξ and integrating on [0,1], it follows from (2.9) that

$$(1-\varepsilon)\int_0^1 y_0(s)\xi(s)ds = \rho_1(1-\varepsilon)\int_0^1 (Ky_0)(s)\xi(s)ds \ge \int_0^1 y_0(s)\xi(s)ds.$$

This together with $\int_0^1 y_0(s)\xi(s)ds > 0$ implies that $\varepsilon \le 0$, which contradicts the choice of ε, and so (4.5) holds. It follows from Lemma 2.5 that

$$i(\lambda A, P \cap B_{r_2}, P) = 1. \tag{4.6}$$

By $f_\infty > \frac{\rho_1}{\lambda}$, using a similar argument, we have $i(\lambda A, P \cap (B_{R_2} \backslash \overline{B}_{r_2}), P) = -1$, which implies that λA has at least one fixed point in $P \cap (B_{R_2} \backslash \overline{B}_{r_2})$. Therefore, NBVP (1.1) has at least one positive solution. □

Proof of Corollary 1.1. (i) Fix $\lambda \in (\frac{\rho_1}{f_0}, \frac{\rho_1}{f^\infty})$. In view of the fact that $yf(x,y) \ge 0$ for any $x \in [0,1]$ and $y \in \mathbf{R}$, we know that $A(P) \subset P$. It follows from Theorem 1.3 that NBVP (1.1) has at least one positive solution.

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Set $f_2(x,y) = -f(x,-y), \forall (x,y) \in [0,1] \times \overline{R}$. Define operators $\mathbf{f}_2, A_2 : E \to E$, respectively, by

$$(\mathbf{f}_2 y)(x) = f_2(x, y(x));$$

$$A_2 = K\mathbf{f}_2.$$

Obviously, $A_2(P) \subset P$. From the proof of Theorem 1.3, it is easy to see that λA_2 has at least one fixed point $y_0 \in P\backslash\{\theta\}$. Then, $\lambda A(-y_0) = \lambda K\mathbf{f}(-y_0) = \lambda K(-\mathbf{f}_2)y_0 = -(\lambda A_2)y_0 = -y_0$. That is, $\lambda A(-y_0) = -y_0$. Hence, NBVP (1.1) has at least one negative solution.

The proof of (ii) is similar and omitted. □

Remark 4.1 *Note that the results of Theorem 1.3 and Corollary 1.1 are satisfied for the cases of $0 < k_1 < k_2 \leq \frac{\pi^2}{4}$. In particular, the conclusion of Theorem 1.3 and Corollary 1.1 for $k_1 < 0 < k_2 \leq \frac{\pi^2}{4}$ also apply when NBVP (1.1) is converted into the problem (2.8) as Remark 2.1. On account of the proof is similar to Theorem 1.3 and Corollary 1.1, so we omit it here.*

5. Some examples and branching figures of solutions.

In this part, in order to verify the correctness of the main conclusions are obtained, we give some corresponding numerical examples and figures of the solution branches, so as to judge the number of solutions more directly.

Example 5.1 *Let $k_1 + k_2 = 5$, $k_1 \cdot k_2 = 4$, consider the following fourth-order Neumann boundary value problems*

$$\begin{cases} y^{(4)}(x) + 5y''(x) + 4y(x) = \lambda f(x, y(x)), & x \in [0,1], \\ y'(0) = y'(1) = y'''(0) = y'''(1) = 0, \end{cases} \tag{5.1}$$

where

$$f(x,y) = \begin{cases} 4800 \min\left\{\frac{y^2}{2\sqrt{x(1-x)}}, \sqrt{y}\right\}, & y \in [0,1], \\ 2400, & y \in (1, 24], \\ \frac{4800}{\sqrt{24}} \min\left\{\frac{\sqrt{y}}{2\sqrt{x(1-x)}}, (y-13)^2\right\}, & y \in (24, +\infty) \end{cases}$$

for any $x \in [0,1]$.

Next, we verify the contents of the condition (H2). Apparently, we get $f_0 = f_\infty = \infty$, it's follows from Theorem 1.1, then there exists $\lambda^* > 0$ such that problem (5.1) has at least two positive solutions for $\lambda \in (0, \lambda^*)$, at least one positive solution for $\lambda = \lambda^*$ and no positive solution for $\lambda > \lambda^*$. For an overview of the branching of the solution, see Figure (8-a).

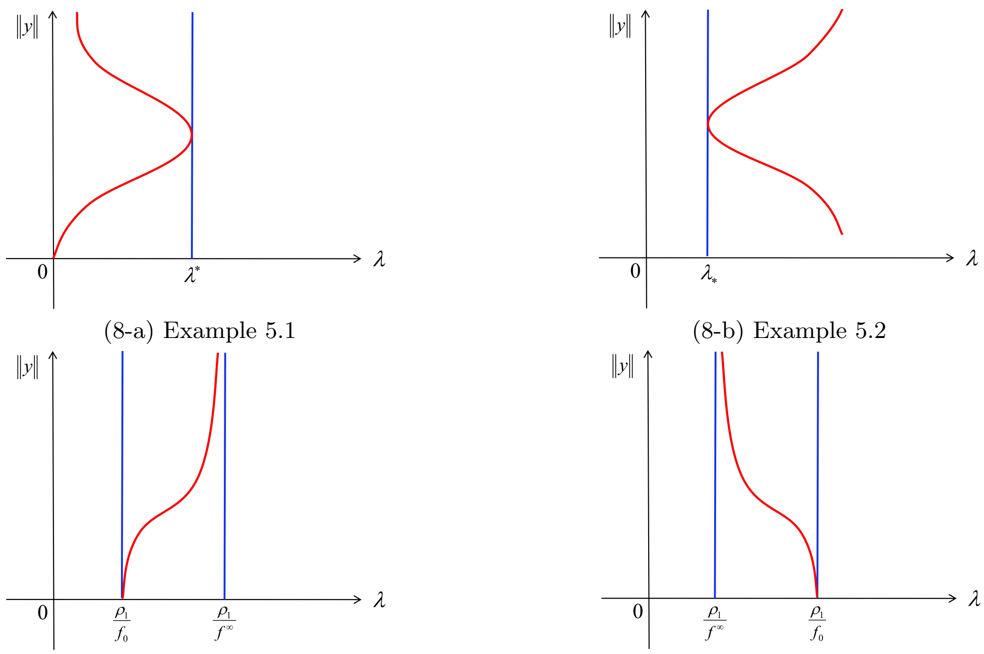
Example 5.2 *Consider the following fourth-order elastic beam boundary value problems*

$$\begin{cases} y^{(4)}(x) + \frac{\pi^2+9}{9} y''(x) + \frac{\pi^2}{9} y(x) = \lambda f(x, y(x)), & x \in [0,1], \\ y'(0) = y'(1) = y'''(0) = y'''(1) = 0 \end{cases} \tag{5.2}$$

with Neumann boundary condition, where

$$f(x,y)=\begin{cases}[\ln(y+e-1)^2+\sin^2 1](e^x+\sin\frac{\pi x}{2}), & y\in[0,1],\\ (\frac{2}{y}+\sin 2)(e^x+\sin\frac{\pi x}{2}), & y\in(1,\infty)\end{cases}$$

for any $x\in[0,1]$, *and* $k_1+k_2=\frac{\pi^2+9}{9}$, $k_1\cdot k_2=\frac{\pi^2}{9}$. *Clearly,* $f^0=f^\infty=0$, *satisfy the condition of Theorem 1.2, then there exists* $\lambda_*>0$ *such that problem (5.2) has at least two positive solutions for* $\lambda>\lambda_*$, *at least one positive solution for* $\lambda=\lambda_*$ *and no positive solution for* $\lambda\in(0,\lambda_*)$. *For an overview of the branching of the solution, see Figure (8-b).*



(8-a) Example 5.1 (8-b) Example 5.2
(8-c) Example 5.3 (8-d) Example 5.3

FIGURE 8. Figure of example 5.1-5.3.

Example 5.3 *Let* $k_1+k_2=\frac{4\pi^2-\pi^4}{4}$, $k_1\cdot k_2=-\frac{\pi^6}{4}$, *consider the following fourth-order Neumann boundary value problem*

$$\begin{cases} y^{(4)}(x)+\frac{4\pi^2-\pi^4}{4}y''(x)-\frac{\pi^6}{4}y(x)+\lambda f(x,y(x))=0, & x\in[0,1],\\ y'(0)=y'(1)=y'''(0)=y'''(1)=0,\end{cases} \tag{5.3}$$

where

$$f(x,y)=\begin{cases}\ln(y+1)+|x-1|, & y\in[0,1],\\ y^2|x-1|+\log_{2(x+1)}y+y\ln 2, & y\in(1,+\infty)\end{cases}$$

for any $x\in[0,1]$, *and obviously, we can get* $f(x,y)$ *is a continuous function. Further, we get* $f_0=f^0=\infty$, $f^\infty=0$ *and* $f_\infty=\ln 2$. *By the calculation, we can easily verify that there is a positive solution* $y(x)$ *of the problem (5.3), where* ρ_1 *is the first eigenvalue of boundary value* *problem*

$$\begin{cases} y^{(4)}(x) + \frac{4\pi^2-\pi^4}{4}y''(x) - \frac{\pi^6}{4}y(x) + \rho_1 y(x) = 0, \quad x \in [0,1], \\ y'(0) = y'(1) = y'''(0) = y'''(1) = 0. \end{cases}$$

Clearly, we get $0 \leq f^{\infty} < f_0 \leq +\infty$*, it follows from Theorem 1.3 that the problem (5.3) has at least one positive solution for any* $\lambda \in (\frac{\rho_1}{f_0}, \frac{\rho_1}{f^{\infty}})$*; and when* $0 \leq f^0 < f_{\infty} \leq +\infty$*, then problem (5.3) has at least one positive solution for any* $\lambda \in (\frac{\rho_1}{f_{\infty}}, \frac{\rho_1}{f^0})$*. For an overview of the branching of the solution, see Figure (8-c) and Figure (8-d).*

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Department of Mathematics, Northwest Normal University, Lanzhou 730070, People's Republic of China
Email address: **WJJ950712@163.com**

Department of Mathematics, Northwest Normal University, Lanzhou 730070, People's Republic of China
Email address: **gaokuguo@163.com**

Department of Mathematics, Northwest Normal University, Lanzhou 730070, People's Republic of China
Email address: **linmu8610@163.com**

Department of Mathematics, Northwest Normal University, Lanzhou 730070, People's Republic of China
Email address: **hett199527@163.com**

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