On countable tightness type properties of spaces of quasicontinuous functions

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Abstract

In this paper we get characterizations countable tightness, countable fan-tightness and countable strong fan-tightness of spaces of quasicontinuous functions from an open Whyburn regular space X into the discrete two-point space $\{0,1\}$ with the topology of pointwise convergence through properties of X determined by selection principles. These properties (e.g. $S_1(\mathcal{K},\mathcal{K})$, $\mathcal{K}_\Omega$-Lindelöfness, $S_1(\mathcal{K}_\Omega,\mathcal{K}_\Omega)$) were defined by M. Scheepers and studied in theory of selection principles in the class of metric spaces.

For any uncountable cardinal number κ, we get a functional characterization of κ-Lusin spaces in class of separable metrizable spaces through tightness of compact subsets of a space of quasicontinuous real-valued functions with the topology of pointwise convergence.

Keywords: quasicontinuous function, Lusin space, open Whyburn space, tightness, fan-tightness, strong fan-tightness, selection principle, Fréchet-Urysohn
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1. Introduction

A study of some convergence properties in function spaces is an important task of general topology. The general question in the theory of function spaces is to characterize topological properties of a space of functions on a topological space X.

In C_p-theory it have been obtained interested results on cardinal properties of first-countability, Fréchet-Urysohn properties, tightness [1, 7, 8, 22, 23, 30] of a space $C_p(X,\mathbb{R})$ of continuous real-valued functions on a Tychonoff space X with the topology of pointwise convergence.

Archangel'skii-Pytkeev theorem [1] is a nice result about tightness of function spaces: $t(C_p(X,\mathbb{R})) = \sup\{l(X^n) : n \in \mathbb{N}\}$. Thus, $C_p(X,\mathbb{R})$ has countable tightness if and only if X^n is Lindelöf for each $n \in \mathbb{N}$.

The following result on countable fan tightness of function spaces $C_p(X,\mathbb{R})$ is shown by A.V. Archangel'skii [3]: $C_p(X,\mathbb{R})$ has countable fan tightness if and only if X^n is a Menger space for each $n \in \mathbb{N}$ (i.e. X has the property $S_{fin}(\Omega,\Omega)$).

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In [30], M. Sakai is shown that $C_p(X,\mathbb{R})$ has countable strong fan-tightness if and only if X has the property $S_1(\Omega,\Omega)$.

In papers [17, 19, 20], tightness, fan tightness and strong fan-tightness of a space of continuous functions with a set-open (e.g. compact-open) topology were investigated. In [18], we study tightness type properties of spaces of Baire-one functions with the topology of pointwise convergence.

In this paper, we continue to study countable tightness, countable fan-tightness and countable strong fan-tightness of spaces of quasicontinuous functions with the topology of pointwise convergence.

A function $f: X \to Y$ is *quasicontinuous* at x if for any open set V containing $f(x)$ and any U open containing x, there exists a nonempty open set $W \subseteq U$ such that $f(W) \subseteq V$. It is *quasicontinuous* if it is quasicontinuous at every point. Call a set semi-open (or quasi-open) if it is contained in the closure of its interior. Then $f: X \to Y$ is quasicontinuous if and only if the inverse of every open set is quasi-open.

Quasicontinuous functions were studied in many papers, see for examples [5, 24, 25, 26, 27, 28].

Levine [12] studied quasicontinuous maps under the name of semi-continuity using the terminology of semi-open sets. A function $f: X \to Y$ is called *semi-continuous* if $f^{-1}(V)$ is semi-open in X for every open set V of Y. A map $f: X \to \mathbb{R}$ is quasicontinuous if and only if f is semi-continuous [12].

Let X and Y be Hausdorff topological spaces, $Q_p(X,Y) = (Q(X,Y), \tau_p)$ be the space of all quasicontinuous functions on X with values in Y and τ_p be the pointwise convergence topology.

2. Preliminaries

All spaces under consideration are assumed to be regular. A subset U of a topological space X is called a *regular open set* or an *open domain* if $U = Int\overline{U}$ holds. A subset F of a topological space X is called a *regular closed* set or a *closed domain* if $F = \overline{IntF}$ holds.

A set A is called *minimally bounded* with respect to the topology τ in a topological space (X,τ) if $\overline{IntA} \supseteq A$ and $Int\overline{A} \subseteq A$ ([4], p.101). Clearly this means A is semi-open and $X \setminus A$ is semi-open. In the case of *open* sets, minimal boundedness coincides with regular openness.

Note that if U is a minimally bounded (e.g. regular open) set of X such that U is not dense subset in X and $B \subset \overline{U} \setminus U$ then there is a quasicontinuous function $f: X \to \mathbb{R}$ such that $f(U \cup B) = 0$ and $f(X \setminus (U \cup B)) = 1$ (see Lemma 4.2 in [29]).

Let us recall some properties of a topological space X.

(1) A space X is *Fréchet-Urysohn* provided that for every $A \subset X$ and $x \in \overline{A}$ there exists a sequence in A converging to x.

(2) A space X has *countable tightness* at a point x (denoted $t(x,X) = \omega$) if $x \in \overline{A}$, then $x \in \overline{B}$ for some countable $B \subseteq A$. A space X has countable tightness (denoted $t(X) = \omega$) if $t(x,X) = \omega$ for every $x \in X$.

(3) A space X has *countable fan-tightness* at a point x (denoted $vet(x,X)=\omega$) if for any countable family $\{A_n : n\in\omega\}$ of subsets of X satisfying $x\in\bigcap_{n\in\omega}\overline{A_n}$ it is possible to select finite sets $K_n\subset A_n$ in such a way that $x\in\overline{\bigcup_{n\in\omega}K_n}$. A space X has countable fan-tightness (denoted $vet(X)=\omega$) if $vet(x,X)=\omega$ for every $x\in X$.

(4) A space X is said to have *countable strong fan-tightness* at a point x (denoted $vet_1(x,X)=\omega$) if for each countable family $\{A_n : n\in\omega\}$ of subsets of X such that $x\in\bigcap_{n\in\omega}\overline{A_n}$, there exist $a_i\in A_i$ such that $x\in\overline{\{a_i : i\in\omega\}}$. A space X has countable strong fan-tightness (denoted $vet_1(X)=\omega$) if $vet_1(x,X)=\omega$ for every $x\in X$.

(5) A space X is said to be *open Whyburn* if for every open set $A\subset X$ and every $x\in\overline{A}\setminus A$ there is an open set $B\subseteq A$ such that $\overline{B}\setminus A=\{x\}$ [16].

Note that the class of open Whyburn spaces is quite wide; for example, it includes all first countable regular spaces [16] and, therefore, all metrizable spaces.

Let X be a Tychonoff topological space, $C(X,\mathbb{R})$ be the space of all continuous functions on X with values in $\mathbb{R}$ and τ_p be the pointwise convergence topology. Denote by $C_p(X,\mathbb{R})$ the topological space $(C(X,\mathbb{R}),\tau_p)$.

A real-valued function f on a space X is a *Baire-one function* (or a *function of the first Baire class*) if f is a pointwise limit of a sequence of continuous functions on X.

The symbol $\mathbf{0}$ stands for the constant function to 0. A basic open neighborhood of $\mathbf{0}$ in $\mathbb{R}^X$ is of the form $[F,(-\epsilon,\epsilon)]=\{f\in\mathbb{R}^X : f(F)\subset(-\epsilon,\epsilon)\}$, where $F\in[X]^{<\omega}$ and $\epsilon>0$.

Let us recall that a cover $\mathcal{U}$ of a set X is called

- an *ω-cover* if each finite set $F\subseteq X$ is contained in some $U\in\mathcal{U}$;
- a *γ-cover* if for any $x\in X$ the set $\{U\in\mathcal{U} : x\notin U\}$ is finite.

In this paper $\mathcal{A}$ and $\mathcal{B}$ will be collections of the following covers of a space X:

$\mathcal{O}^s$: the collection of all semi-open covers of X.

Ω : the collection of open ω-covers of X.

$\mathcal{K}$: the collection $\mathcal{U}$ of open subsets of X such that $X=\bigcup\{\overline{U} : U\in\mathcal{U}\}$.

Ω^s : the collection of minimally bounded ω-covers of X.

Γ^s : the collection of minimally bounded γ-covers of X.

$\mathcal{K}_\Omega$ is the set of $\mathcal{U}$ in $\mathcal{K}$ such that no element of $\mathcal{U}$ is dense in X, and for each finite set $F\subseteq X$, there is a $U\in\mathcal{U}$ such that $F\subseteq\overline{U}$.

$\mathcal{K}_\Gamma$ is the set of $\mathcal{U}$ in $\mathcal{K}$ such that no element of $\mathcal{U}$ is dense in X, and $\{\overline{U} : U\in\mathcal{U}\}$ is a γ-cover of X.

Definition 2.1. Let $\mathcal{P}$ be a collection of covers of X. A space is *$\mathcal{P}$-Lindelöf* if each element of $\mathcal{P}$ has a countable subset in $\mathcal{P}$.

Definition 2.2. ([11]) A Hausdorff space X is called a *Lusin space* (*in the sense of Kunen*) if

(a) Every nowhere dense set in X is countable;
(b) X has at most countably many isolated points;
(c) X is uncountable.

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If X is an uncountable Hausdorff space then X is $\mathcal{O}^s$-Lindelöf (semi-Lindelöf) if and only if X is a Lusin space (Corollary 2.5 in [21]).

If X is a Lusin space, X is hereditarily Lindelöf (Lemma 1.2 in [11]). Hence, if X is a regular Lusin space then X is perfect normal (3.8.A.(b) in [6]).

If X is a Lusin space, so is every uncountable subspace (Lemma 1.1 in [11]).

Many topological properties are defined or characterized in terms of the following classical selection principles (see [31]). Let $\mathcal{A}$ and $\mathcal{B}$ be sets consisting of families of subsets of an infinite set X. Then:

$S_1(\mathcal{A}, \mathcal{B})$ is the selection hypothesis: for each sequence $\{A_n : n \in \mathbb{N}\}$ of elements of $\mathcal{A}$ there is a sequence $\{b_n\}_{n\in\mathbb{N}}$ such that for each n, $b_n \in A_n$, and $\{b_n : n \in \mathbb{N}\} \in \mathcal{B}$.

$S_{fin}(\mathcal{A}, \mathcal{B})$ is the selection hypothesis: for each sequence $\{A_n : n \in \mathbb{N}\}$ of elements of $\mathcal{A}$ there is a sequence $\{B_n\}_{n\in\mathbb{N}}$ of finite sets such that for each n, $B_n \subseteq A_n$, and $\bigcup_{n\in\mathbb{N}} B_n \in \mathcal{B}$.

In [32], M. Scheepers investigated combinatoric properties (e.g. $S_1(\mathcal{K}, \mathcal{K})$, $\mathcal{K}_\Omega$-Lindelöfness, $S_1(\mathcal{K}_\Omega, \mathcal{K}_\Omega)$, $S_{fin}(\mathcal{K}, \mathcal{K})$) in the class of separable metric spaces. Unexpectedly, it turned out that these properties are characterized by countable tightness type properties of spaces of quasicontinuous functions. Observe that every T_2 countable space X satisfies all these properties and therefore Theorems 3.1, 4.1 and 4.3 are true for arbitrary countable spaces.

For other notation and terminology almost without exceptions we follow the Engelking's book [6].

3. Countable tightness

Lemma 3.1. *Every uncountable open Whyburn $\mathcal{K}$-Lindelöf space is a Lusin space.*

Proof. (1) Claim that every nowhere dense set in X is countable.

Since the closure of a nowhere dense subset in X is a nowhere dense set, we can consider only closed nowhere dense sets in X.

Assume that A is an uncountable closed nowhere dense set in X. Since X is open Whyburn, for every point $a \in A$ there is a regular open set $O_a \subseteq X \setminus A$ such that $\overline{O_a} \setminus (X \setminus A) = \{a\}$.

Consider the family $\gamma = \{O(x) : x \in X\}$ of open sets of X where $O(x) = O_x$ for $x \in A$ and $O(x)$ is an open neighborhood of x such that $\overline{O(x)} \cap A = \emptyset$ for $x \notin A$. Then $\gamma \in \mathcal{K}$, but $\gamma' \notin \mathcal{K}$ for any countable subfamily $\gamma' \subset \gamma$.

(2) Claim that X has at most countably many isolated points.

Assume that X has uncountable many isolated points D.

Consider the set $W = Int\overline{D}$. Since X is open Whyburn, for every point $d \in W \setminus D$ there is an open subset $O_d \subseteq D$ such that $\overline{O_d} \setminus D = \{d\}$.

(a) Suppose that for every point $d \in W \setminus D$ there is a neighborhood V_d of d such that $|O_d \cap V_d| \leq \omega$. Let $W_d = O_d \cap V_d$. Then $\overline{W_d} \setminus D = \{d\}$, $W_d \subset D$ and $|W_d| \leq \omega$.

Consider the open family $\mu = \{\{d\} : d \in D\} \cup \{W_d : d \in W \setminus D\} \cup (X \setminus \overline{D})$. Note that $\mu \in \mathcal{K}$, but $\mu' \notin \mathcal{K}$ for any countable subfamily $\gamma' \subset \gamma$.

(b) Suppose that there is a point $d \in W \setminus D$ such that $|O_d \cap V_d| > \omega$ for every neighborhood V_d of d. Let $O_d = O_1 \cup O_2$ such that $O_1 \cap O_2 = \emptyset$ and $|O_i| > \omega$ for $i = 1, 2$.

Let $d \in \overline{O_2}$. Then, we consider the open family $\sigma = \{\{x\} : x \in O_1\} \cup \{O_2, X \setminus \overline{O_d}\}$. Note that $\sigma \in \mathcal{K}$, but $\sigma' \notin \mathcal{K}$ for any countable subfamily $\sigma' \subset \sigma$.

□

Remark 3.2. In [32], M. Scheepers proved that a separable metrizable space X is Lusin if and only if it is $\mathcal{K}$-Lindelöf. Note that every Lusin space is hereditarily Lindelöf (Lemma 1.2 in [11]), hence, every Lusin space is $\mathcal{K}$-Lindelöf.

Corollary 3.3. An uncountable open Whyburn space X is $\mathcal{K}$-Lindelöf if and only if it is a Lusin space.

It is well known that f is of the first Baire class if and only if $f^{-1}(U)$ is a countable unions of zero sets for every open $U \subseteq \mathbb{R}$ (see Exercise 3.A.1 in [34]).

Proposition 3.4. *Let X be a Lusin space. Then every real-valued quasicontinuous function is of the first Baire class.*

Proof. Let X be a Lusin space. Then X is a perfect normal space and, hence, any open set is a countable unions of zero sets. It remains to note that any semi-open set in X is a unions of open set and (countable set of points) countable nowhere dense subset of X. □

Corollary 3.5. Let X be a Lusin space and f be a real-valued quasicontinuous function. Then the set $D_f = \{x \in X : f \text{ is discontinuous in } x\}$ is countable.

Proof. Fix an open countable basis $\{V_n\}$ for $\mathbb{R}$. We then have

$x \in D_f \Leftrightarrow \exists\, n\ [x \in f^{-1}(V_n) \setminus Int(f^{-1}(V_n))]$, i.e.,

$D_f = \bigcup\{f^{-1}(V_n) \setminus Int(f^{-1}(V_n)) : n \in \mathbb{N}\}$. It remains to note that any set $f^{-1}(V_n) \setminus Int(f^{-1}(V_n))$ is countable. □

The space of all quasicontinuous functions from X into the discrete space $\mathbb{D} = \{0, 1\}$ is denote by $Q_p(X, \mathbb{D})$.

Theorem 3.6. *For an uncountable open Whyburn space (X, τ) the following statements are equivalent:*

1. *X is Ω^s-Lindelöf;*
2. *X is $\mathcal{K}_\Omega$-Lindelöf;*
3. $t(\mathbf{0}, Q_p(X, \mathbb{R})) = \omega$;
4. $t(f, Q_p(X, \mathbb{R})) = \omega$ *for every* $f \in C(X, \mathbb{R})$;
5. $t(Q_p(X, \mathbb{D})) = \omega$.

Proof. (1) $\Rightarrow$ (2) and (4) $\Rightarrow$ (3). It is trivial.

(2) $\Rightarrow$ (1). It is enough to prove that for any $\mathcal{U} = \{U_\alpha\}_{\alpha \in A} \in \Omega^s$ there exists $\mathcal{V} = \{V_\beta\}_{\beta \in B} \in \mathcal{K}_\Omega$ such that for any $\beta \in B$ there is $\alpha \in A$ such that $\overline{V_\beta} \subseteq U_\alpha$. We can

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denote this as $\mathcal{V} \succ \mathcal{U}$. Let F be a finite subset of X. Then there is U_α such that $F \subseteq U_\alpha$. Since X is an open Whyburn space, there is an open set V_F such that $F \subseteq \overline{V_F} \subseteq U_\alpha$. Let $\mathcal{V} = \{V_F : F \in [X]^{<\omega}\}$.

(2) $\Rightarrow$ (3). Assume that $\mathbf{0} \in \overline{\{f_\alpha : \alpha \in A\}}$ where $|A| > \omega$. Let $n \in \mathbb{N}$ and $\mathcal{V}_n = \{V_{\alpha,n} = f_\alpha^{-1}((-\frac{1}{n}, \frac{1}{n})) : \alpha \in A\}$. Then, $\mathcal{V}_n$ is a semi-open ω-cover of X. Since X is open Whyburn, for every $V \in \mathcal{V}_n$ and a finite subset F of V there is an open subset $W_{F,V}$ in X such that $F \subseteq \overline{W_{F,V}} \subseteq V$. Then $\mathcal{W} = \{W_{F,V} : V \in \mathcal{V}, F \in [V]^{<\omega}\} \in \mathcal{K}_\Omega$ and $\mathcal{W} \succ \mathcal{V}_n$. Since X is $\mathcal{K}_\Omega$-Lindelöf, there is a countable subfamily $\mathcal{W}' = \{W_{F_i, V_{\alpha_i,n}} : i \in \mathbb{N}\}$ of $\mathcal{W}$ such that $\mathcal{W}' \in \mathcal{K}_\Omega$. It follows that $\mathcal{V}'_n = \{V_{\alpha_i,n} : i \in \mathbb{N}\}$ is a countable subfamily of $\mathcal{V}_n$. Denote by $F_n = \{f_{\alpha_i} : i \in \mathbb{N}\}$. Thus, for every $n \in \mathbb{N}$, $\mathcal{V}'_n \in \Omega^s$ which implies $\mathbf{0} \in \overline{\bigcup\{F_n : n \in \mathbb{N}\}}$.

(3) $\Rightarrow$ (1). Let $\{U_\alpha\}_{\alpha \in A} \in \Omega^s$. Consider the quasicontinuous function $f_\alpha : X \to \{0,1\}$ such that $f_\alpha(U_\alpha) = 0$ and $f_\alpha(X \setminus U_\alpha) = 1$ for each $\alpha \in A$. Then $\mathbf{0} \in \overline{\{f_\alpha : \alpha \in A\}}$. Since $t(\mathbf{0}, Q(X,\mathbb{R})) = \omega$, there is $B \subset A$ such that $|B| = \omega$ and $\mathbf{0} \in \overline{\{f_\alpha : \alpha \in B\}}$. It follows that $\{U_\alpha : \alpha \in B\} \in \Omega^s$.

(3) $\Rightarrow$ (4). Note that for any space X and maps $f, g : X \to \mathbb{R}$ such that f is continuous and g is quasicontinuous, the map $f + g : X \to \mathbb{R}$ defined by $(f+g)(x) = f(x) + g(x)$ is quasicontinuous (Proposition 5.4 in [10]). Thus, the mapping $h_f : Q_p(X,\mathbb{R}) \to Q_p(X,\mathbb{R})$ such that $h_f(g) = f + g$ for every $g \in Q_p(X,\mathbb{R})$ is a homeomorphism for any $f \in C_p(X,\mathbb{R})$. It follows that (3) implies (4).

(2) $\Rightarrow$ (5). Let $f \in Q_p(X,\mathbb{D})$. Note that $f^{-1}(\{d\})$ is a semi-open set in X for every $d \in \mathbb{D}$. The set $D_f = \{x \in X : f$ is discontinuous in $x\}$ is a nowhere dense subset of X. Since X is Lusin, the set $\overline{D_f}$ is countable.

Consider the new topology τ_f, the base of which forms the family $\tau \cup \{\{d\} : d \in \overline{D_f}\}$. Let $id : (X, \tau_f) \to (X, \tau)$ be the identity mapping from (X, τ_f) onto (X, τ).

It's easy to check that if $g \in Q_p((X,\tau),\mathbb{D})$ then $g \circ id \in Q_p((X,\tau_f),\mathbb{D})$.

Claim that (X, τ_f) is $\mathcal{K}_\Omega$-Lindelöf. To do this, we will prove two facts for $\mathcal{K}_\Omega$-Lindelöf spaces.

(a) *If X is $\mathcal{K}_\Omega$-Lindelöf and G is an open subset of X then G is $\mathcal{K}_\Omega$-Lindelöf.*

By Lemma 3.1, X is a Lusin space and, hence, X is a perfect normal space. Thus the set $X \setminus G$ is G_δ. Let $X \setminus G = \bigcap W_i$ where $W_{i+1} \subset W_i$ and $W_i \in \tau$ for each $i \in \mathbb{N}$. Consider $\mathcal{V} = \{V_\alpha : \alpha \in A\} \in \mathcal{K}_\Omega$ where $\mathcal{K}_\Omega$ in the subspace G. Note that V_α is not dense in G for each $V_\alpha \in \mathcal{V}$. Since X is regular, there is an open set O_α in X such that $\overline{O_\alpha} \subset G \setminus \overline{V_\alpha}$.

Let $\mathcal{O}_i = \{V_{\alpha,i} = V_\alpha \cup (W_i \setminus \overline{O_\alpha}) : \alpha \in A\}$. Note that $V_{\alpha,i}$ is not dense in (X, τ) for each $\alpha \in A$. Then $\mathcal{O}_i \in \mathcal{K}_\Omega$ in the space (X,τ). Then, there exists $\mathcal{O}'_i = \{V_{\alpha_j} \cup (W_i \setminus \overline{O_{\alpha_j}}) : j \in \mathbb{N}\}$ such that $\mathcal{O}'_i \in \mathcal{K}_\Omega$ in the space (X,τ).

Let $\mathcal{V}' = \{V_{\alpha_j(i)} : i, j \in \mathbb{N}\}$. Remain note that $\mathcal{V}' \in \mathcal{K}_\Omega$ where $\mathcal{K}_\Omega$ in the subspace G. If $F \in [G]^{<\omega}$ then there is $i' \in \mathbb{N}$ such that $F \cap W_{i'} = \emptyset$. Hence, there is j' such that $F \subseteq \overline{V_{\alpha_{j'}(i')}}$.

(b) *If X is an open $\mathcal{K}_\Omega$-Lindelöf subspace of $X \cup S$ where S is countable then $X \cup S$ is $\mathcal{K}_\Omega$-Lindelöf.*

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We can assume that $X\cap S=\emptyset$ otherwise we can consider $S'=S\setminus X$. Let $S=\{s_n : n\in\mathbb{N}\}$. Consider $\mathcal{V}=\{V_\alpha:\alpha\in A\}\in\mathcal{K}_\Omega$ where $\mathcal{K}_\Omega$ in the space $X\cup S$.

Let $\mathcal{V}_n=\{V_\alpha\in\mathcal{V}:\{s_1,...,s_n\}\subseteq\overline{V_\alpha}$ and $V_\alpha\cap X\neq\emptyset\}$.

(1) Assume that for any $n\in\mathbb{N}$ there is $k(n)>n$ and $V_{\alpha(k(n))}\in\mathcal{V}_{k(n)}$ such that $X\subset\overline{V_{\alpha(k(n))}}$. Then $\{V_{\alpha(k(n))}:n\in\mathbb{N}\}\in\mathcal{K}_\Omega$.

(2) Otherwise there is $n'\in\mathbb{N}$ such that for any $k>n'$ and $V_\alpha\in\mathcal{V}_k$ the set $X\setminus\overline{V_\alpha}$ is not empty.

Thus $\mathcal{U}_n=\{X\cap V_\alpha : V_\alpha\in\mathcal{V}_n\}\in\mathcal{K}_\Omega$ in the space X for every $n>k$. Then, there is $\mathcal{U}'_n=\{X\cap V_{\alpha_i} : i\in\mathbb{N}\}\in\mathcal{K}_\Omega$ in the space X for every $n>k$. Note that $P_n=\{V_{\alpha_i}\in\mathcal{U}'_n : i\in\mathbb{N}\}\in\mathcal{K}_\Omega$ in the space $X\cup\{s_1,...,s_n\}$. Let $P=\bigcup P_n$. Then P is countable, $P\subset\mathcal{V}$ and $P\in\mathcal{K}_\Omega$ in the space $X\cup S$.

By the fact (a), the subspace $X\setminus\overline{D_f}$ is $\mathcal{K}_\Omega$-Lindelöf.

By the fact (b), the space (X,τ_f) is $\mathcal{K}_\Omega$-Lindelöf.

Assume that $f\in\overline{\{f_\alpha:\alpha\in A\}}$ where $F=\{f_\alpha:\alpha\in A\}\subset Q_p((X,\tau),\mathbb{D})$ and $|A|>\omega$. Then $f\circ id\in\overline{\{f_\alpha\circ id:\alpha\in A\}}$ where $\{f_\alpha\circ id:\alpha\in A\}\subset Q_p((X,\tau_f),\mathbb{D})$. Note that (2) implies (4), (X,τ_f) is $\mathcal{K}_\Omega$-Lindelöf and $f\circ id\in C((X,\tau_f),\mathbb{R})$. Then, by (4), there is a countable set $B\subset A$ such that $f\circ id\in\overline{\{f_{\alpha_i}\circ id:\alpha_i\in B\}}$. It follows that $f\in\overline{\{f_{\alpha_i}:\alpha_i\in B\}}$.

(5) $\Rightarrow$ (1). Similar to the implication (3) $\Rightarrow$ (1).

□

In particular, we get the following corollary in class of metrizable spaces.

Corollary 3.7. A metrizable space X is $\mathcal{K}_\Omega$-Lindelöf if, and only if, $t(Q_p(X,\mathbb{D}))=\omega$.

4. Countable strong fan-tightness and countable fan-tightness

Theorem 4.1. *For an uncountable open Whyburn space X the following statements are equivalent:*

1. *X satisfy $S_1(\Omega^s,\Omega^s)$;*
2. *X satisfy $S_1(\mathcal{K}_\Omega,\mathcal{K}_\Omega)$;*
3. *$vet_1(\mathbf{0},Q_p(X,\mathbb{R}))=\omega$;*
4. *$vet_1(f,Q_p(X,\mathbb{R}))=\omega$ for every $f\in C(X,\mathbb{R})$;*
5. *$vet_1(Q_p(X,\mathbb{D}))=\omega$.*

Proof. (1) $\Rightarrow$ (2) and (4) $\Rightarrow$ (3). It is trivial.

(2) $\Rightarrow$ (1). Let $\mathcal{U}_i=\{U^i_\alpha\}_{\alpha\in A_i}\in\Omega^s$ for each $i\in\mathbb{N}$. In Theorem 3.6 ((2) $\Rightarrow$ (1)), we proved for any $\mathcal{U}=\{U_\alpha\}_{\alpha\in A}\in\Omega^s$ there exists $\mathcal{V}=\{V_\beta\}_{\beta\in B}\in\mathcal{K}_\Omega$ such that for any $\beta\in B$ there is $\alpha\in A$ such that $\overline{V_\beta}\subseteq U_\alpha$, i.e. $\mathcal{V}\succ\mathcal{U}$. Thus, for every $i\in\mathbb{N}$ there is $\mathcal{V}_i\in\mathcal{K}_\Omega$ such that $\mathcal{V}_i\succ\mathcal{U}_i$. By (2), there is $V^i_{\beta_i}\in\mathcal{V}_i$ for each $i\in\mathbb{N}$ such that $\{V^i_{\beta_i}:i\in\mathbb{N}\}\in\mathcal{K}_\Omega$. For every β_i there is α_i such that $\overline{V^i_{\beta_i}}\subset U^i_{\alpha_i}$. It follows that $\{U^i_{\alpha_i}:i\in\mathbb{N}\}\in\Omega^s$.

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(3) $\Rightarrow$ (1). Let $\mathcal{U}_n = \{U^n_\alpha\}_{\alpha\in A_n} \in \Omega^s$ for each $n \in \mathbb{N}$. Consider the quasicontinuous function $f_{\alpha,n} : X \to \{0,1\}$ such that $f_{\alpha,n}(U^n_\alpha) = 0$ and $f_{\alpha,n}(X \setminus U^n_\alpha) = 1$ for each $\alpha \in A_n$ and $n \in \mathbb{N}$. Then $\mathbf{0} \in \overline{\{f_{\alpha,n} : \alpha \in A_n\}}$ for each $n \in \mathbb{N}$. Since $vet_1(\mathbf{0}, Q_p(X,\mathbb{R})) = \omega$, there is $f_{\alpha_n,n} \in \{f_{\alpha,n} : \alpha \in A_n\}$ for each $n \in \mathbb{N}$ such that $\mathbf{0} \in \overline{\{f_{\alpha_n,n} : n \in \mathbb{N}\}}$. It follows that $\{U^n_{\alpha_n} : n \in \mathbb{N}\} \in \Omega^s$.

(1) $\Rightarrow$ (3). Let X has the property $S_1(\Omega^s, \Omega^s)$. Then X is Ω^s-Lindelöf and, by Theorem 3.6, X is $\mathcal{K}_\Omega$-Lindelöf. Consider a countable family $\{A_n : n \in \mathbb{N}\}$ of subsets of $Q_p(X,\mathbb{R})$ such that $\mathbf{0} \in \bigcap_{n\in\mathbb{N}} \overline{A_n}$. For every $n \in \mathbb{N}$ we consider $\mathcal{V}_n = \{V_{n,i,f} = f^{-1}((-\frac{1}{i}, \frac{1}{i})) : i \in \mathbb{N}$ and $i \geq n, f \in A_n\}$. Since $\mathbf{0} \in \overline{A_n}$, the family $\mathcal{V}_n$ is a semi-open ω-cover of X.

Since X is an open Whyburn regular space, there is $\mathcal{U}_n \in \mathcal{K}_\Omega$ such that $\mathcal{U}_n \succ \mathcal{V}_n$ for each $n \in \mathbb{N}$. By implication ((1) $\Rightarrow$ (2)), for each $n \in \mathbb{N}$ there is $U_{n,\beta_n} \in \mathcal{U}_n$ such that $\{U_{n,\beta_n} : n \in \mathbb{N}\} \in \mathcal{K}_\Omega$. For each $n \in \mathbb{N}$ there are i_n and f_n such that $\overline{U_{n,\beta_n}} \subseteq V_{n,i_n,f_n}$. Hence, $\{V_{n,i_n,f_n} : n \in \mathbb{N}\}$ is an ω-cover of X. Then, we consider the set $\{f_n : n \in \mathbb{N}\}$.

(1) $f_n \in A_n$ for each $n \in \mathbb{N}$.

(2) $\mathbf{0} \in \overline{\{f_n : n \in \mathbb{N}\}}$.

Let $K \in [X]^{<\omega}$ and $\epsilon > 0$ and $[K, \epsilon] = \{f \in Q_p(X,\mathbb{R}) : f(K) \subset (-\epsilon, \epsilon)\}$.

Then, there is n' such that $\frac{1}{i_{n'}} < \epsilon$ and $K \subseteq V_{n',i_{n'},f_{n'}}$. It implies that $f_{n'} \in [K, \epsilon]$.

(3) $\Rightarrow$ (4). Similarly ((3) $\Rightarrow$ (4)) in Theorem 3.6.

(2) $\Rightarrow$ (5). Let $f \in Q_p(X,\mathbb{D})$. Note that $f^{-1}(\{d\})$ is a semi-open set in X for every $d \in \mathbb{D}$. Thus, D_f is countable nowhere dense subset of X. Since X is Lusin, the set $\overline{D_f}$ is countable.

Similarly the proof of ((2) $\Rightarrow$ (5)) in Theorem 3.6, we consider the new topology τ_f, the base of which forms the family $\tau \cup \{\{d\} : d \in \overline{D_f}\}$.

It's easy to check (almost the same as in Theorem 3.6) that the space (X, τ_f) has the property $S_1(\mathcal{K}_\Omega, \mathcal{K}_\Omega)$ and $f \in C((X, \tau_f), \mathbb{R})$. Then, by (4), $vet_1(f, Q_p(X, \{0,1\}) = \omega$.

(5) $\Rightarrow$ (1). Similar to the implication (3) $\Rightarrow$ (1).

□

Corollary 4.2. A metrizable space X is $S_1(\mathcal{K}_\Omega, \mathcal{K}_\Omega)$ if, and only if, $vet_1(Q_p(X,\mathbb{D})) = \omega$.

Similar to the proof of Theorem 4.1, we can prove the following theorem.

Theorem 4.3. *For an uncountable open Whyburn space X the following statements are equivalent:*

1. *X satisfy $S_{fin}(\Omega^s, \Omega^s)$;*
2. *X satisfy $S_{fin}(\mathcal{K}_\Omega, \mathcal{K}_\Omega)$;*
3. *$vet(\mathbf{0}, Q_p(X,\mathbb{R})) = \omega$;*
4. *$vet(f, Q_p(X,\mathbb{R})) = \omega$ for every $f \in C(X,\mathbb{R})$;*
5. *$vet(Q_p(X,\mathbb{D})) = \omega$.*

Corollary 4.4. A metrizable space X is $S_{fin}(\mathcal{K}_\Omega, \mathcal{K}_\Omega)$ if, and only if, $vet(Q_p(X,\mathbb{D})) = \omega$.

5. Tightness of compact subsets

Let κ be an unfinite cardinal number. Let $\{X_\lambda : \lambda \in A\}$ be a family of topological spaces. Let $X = \prod_{\lambda\in A} X_\lambda$ be the Cartesian product with the Tychonoff topology. Take a point $p = (p_\lambda)_{\lambda\in A} \in X$. For each $x = (x_\lambda)_{\lambda\in A} \in X$, let $Supp(x) = \{\lambda \in A : x_\lambda \neq p_\lambda\}$. Then the subspace $\Sigma_\kappa(p) = \{x \in X : |Supp(x)| \leq \kappa\}$ of X is called a Σ_κ *-product* of $\{X_\lambda : \lambda \in A\}$ about p (p is called the base point).

In ([9], Proposition 1), A.P. Kombarov and V.I. Malykhin proved that

($\bullet$) if $t(\prod\limits_{i=1}^{n} X_{\alpha_i}) \leq \kappa$ for every $n \in \mathbb{N}$ and a finite family $\alpha_1, ..., \alpha_n \in A$ then $t(\Sigma_\kappa(p)) \leq \kappa$.

Suppose that κ is a cardinal number. A separable metrizable space X is a *κ-Lusin set* if $|X| \geq \kappa$ and, for every meager set M, we have $|X \cap M| < \kappa$. Usually, $\aleph_1$-Lusin sets and 2^ω-Lusin sets are called Lusin sets and $\mathfrak{c}$-Lusin sets, respectively. Every Lusin set is also $\mathfrak{c}$-Lusin. Moreover, if Continuum Hypothesis (CH) holds, then every $\mathfrak{c}$-Lusin set is also a Lusin set. However, it is consistent that these notions are not equivalent. Indeed, e.g., under Martin's Axiom (MA) and the failure of CH there are $\mathfrak{c}$-Lusin sets on $\mathbb{R}$ which are not Lusin [14].

If the axiom of choice holds, then every cardinal κ has a successor, denoted κ^+, where $\kappa^+ > \kappa$ and there are no cardinals between κ and its successor.

Theorem 5.1. *Let κ be an uncountable cardinal number. A separable metrizable space X of cardinality $\geq \kappa$ is a κ-Lusin set if and only if $t(K) < \kappa$ for every compact subset K of $Q_p(X, \mathbb{R})$.*

Proof. ($\Rightarrow$). Let A be a countable dense subset of a κ-Lusin space X. Note that if $g, f \in Q_p(X, \mathbb{R})$ and $g(x) = f(x)$ for every $x \in A$ then $\{x \in X : g(x) \neq f(x)\} \subseteq D_g \cup D_f$ where D_h is a set of discontinuous points of a function h. Since X is κ-Lusin, $|D_g \cup D_f| < \kappa$ and we get that $|\{x \in X : g(x) \neq f(x)\}| < \kappa$.

Let K be a compact subset of $Q_p(X, \mathbb{R})$. Consider the projection function $p = \pi_A : Q_p(X, \mathbb{R}) \to \mathbb{R}^A$, i.e., $p(f) = f \mid A$ for every $f \in Q_p(X, \mathbb{R})$. Since $\mathbb{R}^A$ is metrizable, the set $p(K)$ is a metrizable compact space. Let $z \in p(K)$. Then $S_z = p^{-1}(z) := \{f \in Q_p(X, \mathbb{R}) : f|A = z\}$ is closed in $Q_p(X, \mathbb{R})$. Let $\tilde{z} \in S_z$. Then $S_z \subset \Sigma_\kappa(\tilde{z})$ where $\Sigma_\kappa(\tilde{z}) := \{h \in \mathbb{R}^X : |\{x \in X : h(x) \neq \tilde{z}(x)\}| < \kappa\}$. By ($\bullet$), $t(\Sigma_\kappa(\tilde{z})) < \kappa$. It follows that $t(S_z \cap K) < \kappa$ for every $z \in p(K)$ and $K = \bigcup\{S_z \cap K : z \in p(K)\}$. By Theorem 6 in [2] (If $f : X \to Y$ is a continuous closed mapping then $t(X) \leq \sup\{t(Y), t(f^{-1}(y)) : y \in Y\}$), we get that $t(K) < \kappa$.

($\Leftarrow$). Assume that $t(K) < \kappa$ for every compact subset K of $Q_p(X, \mathbb{R})$ and X is not κ-Lusin. Then there exists a closed nowhere dense subset A of X such that $|A| \geq \kappa$.

Let $B \subset A$. Then, there is $f_B : X \to \mathbb{D}$ be a quasicontinuous function such that $f_B(B) = 1$ and $f_B(A \setminus B) = 0$.

Indeed, let O be an open set in X such that $\overline{O} \setminus O \supseteq A$ and $X \setminus \overline{O} \neq \emptyset$.

Then $f_B(x) = 1$ for $x \in B \cup (\overline{O} \setminus A)$ and $f_B(x) = 0$ for other $x \in X$.

Note that $f_{B'}|(X \setminus A) = f_{B''}|(X \setminus A)$ for any $B', B'' \subset A$.

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It is clear that $K = \{f_B : B \subset A\}$ is homeomorphic to the compact space 2^A. But, $t(2^A) = t(K) \geq \kappa$, it is a contradiction.

□

Corollary 5.2. A uncountable separable metrizable space X is Lusin if, and only if, $t(K) = \omega$ for every compact subset K of $Q_p(X, \mathbb{R})$.

Corollary 5.3. If $Q_p(X, \mathbb{R})$ is homeomorphic to $Q_p(Y, \mathbb{R})$ where X is κ-Lusin, then Y is κ-Lusin, too.

6. Examples

In [16], it is proved that if X is a metric space then $Q_p(X, \mathbb{R})$ is Fréchet-Urysohn at the point $\mathbf{0}$ if, and only if, X is countable. The following example shows that for a countable tightness and even for a countable strong fan-tightness of the space $Q_p(X, \mathbb{R})$, a space X can be uncountable.

Given some special axioms, one can show that there are uncountable separable metrizable space X such that $t(Q_p(X, \mathbb{D})) = \omega$. In particular: The axiom ($\diamond$) asserts that there is a sequence $(S_\alpha : \alpha < \omega_1)$ such that

(1) For each α, $S_\alpha \subset \alpha$, and

(2) For every subset A of ω_1, the set $\{\alpha < \omega_1 : A \cap \alpha = S_\alpha\}$ is stationary.

It is well known that the axiom ($\diamond$) is consistent relative to the consistency of classical mathematics and implies but is not equivalent to the Continuum Hypothesis.

Example 6.1. ($\diamond$) *There exists a Lusin space X such that $vet_1(\mathbf{0}, Q_p(X, \mathbb{R})) = \omega$.*

In ([32], Theorem 5), M. Scheepers constructed an example of a uncountable separable metrizable space X which has the property $S_1(\mathcal{K}_\Omega, \mathcal{K}_\Omega)$. By Theorem 4.1, we get an example with the required properties.

Example 6.2. ($\diamond$) *There exists a Lusin space X such that $t(Q_p(X, \mathbb{R})) > \omega$.*

In ([33], see ref.[2] in [32]), W. Just proved that if there is any Lusin set at all, then there is a Lusin set which is not $\mathcal{K}_\Omega$-Lindelöf. By Theorem 3.6, we get an example with the required properties.

Example 6.3. ($MA + \neg CH$) *For each cardinal $\kappa \leq 2^\omega$ with $cf(\kappa) > \omega$ there is a separable metric space X such that $t(K) \leq \kappa$ for each compact subset K of $Q_p(X, \mathbb{R})$ and $t(C) > \omega$ for some compact subset C of $Q_p(X, \mathbb{R})$.*

Under Martin's Axiom (MA) and the failure of CH for each cardinal $\kappa \leq 2^\omega$ with $cf(\kappa) > \omega$ there are κ-Lusin sets in $\mathbb{R}$ which are not Lusin [14]. By Theorem 5.1, we get an example with the required properties.

7. Remark

The idea of defining a new topology τ_f for a quasicontinuous function f, which we use in Theorems 3.6 and 4.1, can be easily used for the Fréchet-Urysohn property of space $Q_p(X,\mathbb{D})$. Thus, combining the results of the article [16], we obtain the following theorem.

Theorem 7.1. *For an uncountable open Whyburn space X the following statements are equivalent:*

1. *X satisfy $S_1(\Omega^s,\Gamma^s)$;*
2. *X satisfy $S_1(\mathcal{K}_\Omega,\mathcal{K}_\Gamma)$;*
3. *$Q_p(X,\mathbb{R})$ is Fréchet-Urysohn at the point $\mathbf{0}$;*
4. *$Q_p(X,\mathbb{R})$ is Fréchet-Urysohn at the point f for every $f\in C(X,\mathbb{R})$;*
5. *$Q_p(X,\mathbb{D})$ is Fréchet-Urysohn.*

By Theorem 3.11 in [16], Theorem 7.1, Theorem 4.1 in [29] and Theorem 4.6 in [10] we get the following result.

Corollary 7.2. Let X and Y be nontrivial metrizable spaces. Then the following are equivalent:

1. X is countable;
2. $Q_p(X,\mathbb{D})$ is Fréchet-Urysohn;
3. $Q_p(X,Y)$ is Fréchet-Urysohn;
4. $Q_p(X,Y)$ is first countable;
5. $Q_p(X,Y)$ is metrizable.

In [32], it is proved that a metrizable space X is Lusin if, and only if, it is $\mathcal{K}$-Lindelöf. Obviously, $\mathcal{K}_\Omega$-Lindelöfness implies $\mathcal{K}$-Lindelöfness of space. Let us note however that Kunen (Theorem 0.0. in [11]) has shown that under $(MA+\neg CH)$ there are no Lusin spaces at all.

Thus, in the class of metrizable spaces we get the following result.

Corollary 7.3. $(MA+\neg CH)$ Let X and Y be nontrivial metrizable spaces. Then the following are equivalent:

1. X is countable;
2. $Q_p(X,\mathbb{D})$ is metrizable;
3. $Q_p(X,Y)$ is Fréchet-Urysohn;
4. $Q_p(X,Y)$ is first countable;
5. $Q_p(X,Y)$ is metrizable;
6. $Q_p(X,Y)$ has countable tightness;
7. $Q_p(X,Y)$ has countable fan-tightness;
8. $Q_p(X,Y)$ has countable strong fan-tightness.

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8. Open questions

Question 1. Could it be that some $Q_p(X,\mathbb{D})$ has countable tightness (countable fan-tightness, countable strong fan-tightness, is Fréchet-Urysohn) but none $Q_p(X,\mathbb{R})$ has this property?

In ([32], Problem 3), M. Scheepers asks: *Could it be that some Lusin set is $\mathcal{K}_\Omega$-Lindelöf, but none has property $S_1(\mathcal{K}_\Omega,\mathcal{K}_\Omega)$?*

This question can be divided into two sub-questions in a functional context.

Question 2. Is there a T_2-space X such that $Q_p(X,\mathbb{D})$ has countable tightness but none $Q_p(X,\mathbb{D})$ has countable fan-tightness?

Question 3. Is there a T_2-space X such that $Q_p(X,\mathbb{D})$ has countable fan-tightness but none $Q_p(X,\mathbb{D})$ has countable strong fan-tightness?

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