# ON WEIGHTED SIMPLICIAL HOMOLOGY 

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#### Abstract

We develop a framework for computing the homology of weighted simplicial complexes with coefficients in a discrete valuation ring. A weighted simplicial complex, $(X, v)$, introduced by Dawson [Cah. Topol. Géom. Différ. Catég. 31 (1990), pp. 229-243], is a simplicial complex, $X$, together with an integer-valued function, $v$, assigning weights to simplices, such that the weight of any of faces are monotonously increasing. In addition, weighted homology, $H_{n}^{v}(X)$, features a new boundary operator, $\partial_{n}^{v}$. In difference to Dawson, our approach is centered at a natural homomorphism $\theta$ of weighted chain complexes. The key object is $H_{n}^{v}(X / \theta)$, the weighted homology of a quotient of chain complexes induced by $\theta$, appearing in a long exact sequence linking weighted homologies with different weights. We shall construct bases for the kernel and image of the weighted boundary map, identifying $n$-simplices as either $\kappa_{n}$ - or $\mu_{n}$-vertices. Long exact sequences of weighted homology groups and the bases, allow us to prove a structure theorem for the weighted simplicial homology with coefficients in a ring of formal power series $R=\mathbb{F}[[\pi]]$, where $\mathbb{F}$ is a field. Relative to simplicial homology new torsion arises and we shall show that the torsion modules are connected to a pairing between distinguished $\kappa_{n}$ and $\mu_{n+1}$ simplices.


## 1. Introduction

Topology aside, the concept of simplicial complexes is of central importance in a variety of fields including data analysis and biology. Many real world data-sets exhibit a simplicial structure [13, 15, 12] and indeed have been organized as such $[4,20,9]$. While the arising simplicial complexes can straightforwardly be studied via topological data analysis (TDA) [25, 5, 4, 21], a prevalent feature of data-sets is the presence of additional simplex-specific data [8].

Dawson introduced in 1990 [6] the concept of a weighted simplicial complex as a simplicial complex equipped with a function $v: X \rightarrow R$, mapping simplices to elements of a ring $R$, such that for simplices $\sigma, \tau \in X$ with $\sigma \subseteq \tau$, we have $v(\sigma) \mid v(\tau)$. Dawson focused on establishing the Eilenberg-Steenrod axioms based on a weighted version of the Mayer-Vietoris sequence and provided a category-theory centered treatment. The key difference between standard and weighted simplicial complexes lies in the weighted boundary operator that incorporates the weight-function $v$

$$
d_{n}^{v}(\sigma)=\sum_{i=0}^{n} \frac{v(\sigma)}{v\left(\hat{\sigma}_{i}\right)} \cdot(-1)^{i} \hat{\sigma}_{i},
$$

where $\sigma$ is a $n$-simplex and $\hat{\sigma}_{i}$ denotes the $i$-th face of $\sigma$. By assumption $v\left(\hat{\sigma}_{i}\right) \mid v(\sigma)$, whence $d_{n}^{v}$ is a well-defined boundary map.

Subsequent contributions of Ren et al. [16] were more application focused, where an extension of Dawson's framework to a persistent homology of weighted simplicial complexes was presented,

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followed by [24], where weighted Laplacians were introduced. In [17, 14], weighted simplicial homology has been expanded to the theory of weighted sheaves over posets to study the (co)homology of Artin groups with coefficients in certain local systems. In particular, the weight function $v(\sigma)$ in their setup is given by the Poincaré series of the parabolic Coxeter subgroup generated by $\sigma$ (see [17], Section 2.4).

Bura et al. [1] studied the homology of certain weighted simplicial complexes with coefficients in discrete valuation rings, arising from the intersections of loops of a pair of RNA secondary structures [3]. [1] connected weighted simplicial homology with simplicial homology via short exact sequences and certain chain maps $\theta$. These chain maps originated from the inflation map defined in [1] that allowed to compute the first weighted homology group.

To illustrate how weighted complexes naturally arise and reflecting on [6] and [1], we shall have a closer look at research collaboration networks. These exhibit a simplicial complex structure as follows: researchers are considered vertices, and a $n$-simplex in-between $n+1$ researchers appears if those researchers appeared together as authors on a paper (by themselves or among others), see Fig. 1.

However, important features cannot be expressed via the simplicial structure alone, as, for instance, the citation number of a simplex. i.e. the number of citations the $n+1$ authors appeared on together. For each simplex, this integer constitutes a weight and, by construction, the weight of a face of a simplex is larger than or equal to its weight. The weight of a face, however, does not necessarily divide the weight of its simplex and as a result the weighted homology theory put forward by $[6,16]$ is not immediately applicable. To incorporate this type of integer-valued weights, arising in a plethora of real-world data, we follow [1] and work with homology with coefficients in discrete valuation rings.

Definition 1. A weighted simplicial complex is a pair $(X, \omega)$ consisting of a simplicial complex $X$ and a non-negative integer function $\omega: X \rightarrow \mathbb{N}$ satisfying

$$
\sigma \subseteq \tau \Longrightarrow \omega(\sigma) \geq \omega(\tau)
$$

for simplices $\sigma, \tau \in X$.

Definition 2. Let $R$ be an integral domain with $\pi \in R \backslash\{0\}$. The weight function $v$ induced by a weighted simplicial complex $(X, \omega)$ is given by setting $v(\sigma)=\pi^{\omega(\sigma)}$. In the following, we also denote the weighted simplicial complex as $(X, v)$.

Definition 3. The weighted chain complex $C_{n}(X, R)$ is the free $R$-module generated by all $n$-simplices of $X$. The weighted boundary map $\partial_{n}^{v}: C_{n}(X, R) \rightarrow C_{n-1}(X, R)$ is given by

$$
\partial_{n}^{v}(\sigma)=\sum_{i=0}^{n} \frac{v\left(\hat{\sigma}_{i}\right)}{v(\sigma)} \cdot(-1)^{i} \hat{\sigma}_{i}=\sum_{i=0}^{n} \pi^{\omega\left(\hat{\sigma}_{i}\right)-\omega(\sigma)} \cdot(-1)^{i} \hat{\sigma}_{i} .
$$

The weighted homology $H_{n}^{v}(X)$ of $(X, v)$ is then the $R$-module $H_{n}^{v}(X)=\operatorname{ker} \partial_{n}^{v} / \operatorname{Im} \partial_{n+1}^{v}$.
Clearly, $\partial_{n}^{v}$ is well-defined, as the weight satisfies $\omega\left(\hat{\sigma}_{i}\right) \geq \omega(\sigma)$ and the coefficients $\frac{v\left(\hat{\sigma}_{i}\right)}{v(\sigma)}$ are always in $R$. Our weighted boundary map $\partial_{n}^{v}$ can be viewed as taking the reciprocal of the coefficients in

FIGURE 1. Weighted simplicial complex, $(X, \omega)$, of a research collaboration network composed by filled (gray) and empty (white) triangles. Suppose $A, B, C, D$ represent four authors that have not appeared as co-authors on any papers, however, $\{A, B, C\}$ or $\{A, C, D\}$ have written papers together. Suppose that $\{A, B, C\}$ has been cited twice, while $\{A, C, D\}$ has been cited once, i.e., $\omega(A B C)=2$ and $\omega(A C D)=1$. Furthermore, any pair appears as authors on some paper, such that the respective citation numbers are given by $\omega(A B)=3, \omega(B C)=4, \omega(A C)=$ $5, \omega(C D)=6, \omega(A D)=7, \omega(B D)=8$. Furthermore, suppose each individual author has been cited 100 times. Then the first simplicial homology group of the complex is given by $H_{1}(X) \cong \mathbb{Z}$ and the first weighted homology group is given by $H_{1}^{v}(X) \cong R \oplus R /(\pi) \oplus R /\left(\pi^{4}\right)$. The free submodule of $H_{1}^{v}(X)$ satisfies $\operatorname{rnk} H_{1}^{v}(X)=\operatorname{rnk} H_{1}(X)$. Note that the differences in citation numbers between pairs of simplices $A B, A B C$ and $A C, A C D$ are one and four, which correspond to the $R /(\pi)$ and $R /\left(\pi^{4}\right)$ direct summands of the torsion, respectively. of $H_{n}^{v}\left(X^{n}\right)$ and the set of $\mu_{n}$-simplices forms a basis of $\partial_{n}^{v}(X)$. We show that the coefficients of $\hat{\beta}_{\kappa_{n}}$
can be reduced to $\mathbb{F}$, which in turn, using Nakayama's Lemma, facilitates the efficient computation of weighted homology modules [2, 7]. We are then in position to prove the structure theorem for the weighted simplicial homology. Specifically, we shall prove that the rank of the weighted simplicial homology equals that of the simplicial homology with coefficients in $R$, and provide a combinatorial interpretation for the torsion of weighted homology. We show that there exists a pairing between $\kappa_{n}{ }^{-}$ and $\mu_{n+1}$-simplices of dimension $n$ and $(n+1)$, such that the torsion modules stem from primary ideals determined by the difference of weights of each respective pair.

We finally present a case study, where we apply the structure theorem to RNA bi-structures [1]. This produces a different, short proof for the weighted homology of the loop complex of an RNA bi-secondary structure [1].

The paper is organized as follows: in Section 2, we show that $\theta: H_{n}(X) \rightarrow H_{n}^{v}(X)$ is injective if and only if $H_{n}(X)$ has no torsion and establish a long exact sequence for weighted homologies having different weights. In Section 3, we construct the $\kappa_{n^{-}}$and $\mu_{n}$-basis for the kernel and image of the weighted boundary map, $\partial_{n}^{v}$. In Section 4 we prove the structure theorem for weighted homology and in Section 5, we apply our results to RNA bi-structures.

## 2. First properties of weighted homology

Definition 4. Given two simplicial complexes $(X, v)$ and $\left(X, v^{\prime}\right)$, we call $v^{\prime} \preceq v$ if $v^{\prime}(\sigma) \mid v(\sigma)$ for any $\sigma \in X$. A mapping $\theta_{n}^{v^{\prime}, v}: C_{n}(X, R) \rightarrow C_{n}(X, R)$ between their weighted chain complexes is given by linearly extending $\theta_{n}^{\nu^{\prime}, v}(\sigma)=\frac{v(\sigma)}{v^{\prime}(\sigma)} \sigma$.

Similar notions have been proposed in different contexts: the map $\theta$ that is specifically constructed to connect the first simplicial and weighted chain groups in the context of RNA bi-secondary structures [1],
and the diagonal map $\Delta$ between chain complexes arising from a weighted sheaf with weights being the Poincaré polynomials associated with the parabolic Coxeter subgroups (see [17] eq. (4.7)). Both the map $\theta$ and the diagonal map $\Delta$ can be viewed as special cases of $\theta_{n}^{v^{\prime}, v}$, where the weight $v^{\prime}$ is constant $1_{R}$. Our $\theta_{n}^{v^{\prime}, v}$-map generalizes these concepts to a weighted simplicial complex with two different weights, and will be the focus of this paper.

The condition $v^{\prime} \preceq v$ guarantees that $\frac{v(\sigma)}{v^{\prime}(\sigma)} \in R$ and thus $\theta_{n}^{v^{\prime}, v}$ is well-defined. By abuse of notation we shall write $\theta_{n}=\theta_{n}^{v^{\prime}, v}$.

Lemma 1. Given two simplicial complexes $(X, v)$ and $\left(X, v^{\prime}\right)$ with $v^{\prime} \preceq v$, we set $\theta_{n}: C_{n}(X, R) \rightarrow$ $C_{n}(X, R), \theta_{n}(\sigma)=\frac{v(\sigma)}{\nu^{\prime}(\sigma)} \sigma$. Then we have the commutative diagram


Proof. Clearly,

$$
\theta_{n-1}^{v^{\prime}, v} \circ \partial_{n}^{v^{\prime}}(\sigma)=\sum_{i=0}^{n}(-1)^{i} \frac{v\left(\hat{\sigma}_{i}\right)}{v^{\prime}\left(\hat{\sigma}_{i}\right)} \cdot \frac{v^{\prime}\left(\hat{\sigma}_{i}\right)}{v^{\prime}(\sigma)} \cdot \hat{\sigma}_{i}=\sum_{i=0}^{n}(-1)^{i} \frac{v(\sigma)}{v^{\prime}(\sigma)} \frac{v\left(\hat{\sigma}_{i}\right)}{v(\sigma)} \hat{\sigma}_{i}=\partial_{n}^{v} \circ \theta_{n}^{v^{\prime}, v}(\sigma) .
$$

Since the $\theta_{n}$ are chain maps, they induce homomorphisms
Lemma 2. The chain maps $\theta_{n}$ induce natural homomorphisms

$$
\bar{\theta}_{n}: H_{n}^{v^{\prime}}(X) \longrightarrow H_{n}^{v}(X), \quad \bar{\theta}_{n}\left(\sum_{j} a_{j} \sigma_{j}+\operatorname{Im} \partial_{n+1}^{v^{\prime}}\right)=\theta_{n}\left(\sum_{j} a_{j} \sigma_{j}\right)+\operatorname{Im} \partial_{n+1}^{v} .
$$

Both Lemmas 1 and 2 extend similar results on the inflation map in case of RNA bi-structures (see [1], Lemma 3) to arbitrary weighted simplicial complexes with weights $v^{\prime} \preceq v$. The next proposition is straightforward to verify:

Proposition 1. Let $\left(A, v_{A}\right),\left(A, v_{A}^{\prime}\right),\left(B, v_{B}\right),\left(B, v_{B}^{\prime}\right),\left(C, v_{C}\right)$ and $\left(C, v_{C}^{\prime}\right)$ be weighted simplicial complexes with weights $v_{A}^{\prime} \preceq v_{A}, v_{B}^{\prime} \preceq v_{B}$ and $v_{C}^{\prime} \preceq v_{C}$, respectively. Let $\theta_{n, A}=\theta_{n}^{v_{A}^{\prime}, v_{A}}, \theta_{n, B}=\theta_{n}^{v_{B}^{\prime}, v_{B}}$ and $\theta_{n, C}=\theta_{n}^{v_{C}^{\prime}, v_{C}}$ be $\theta$-maps. Suppose we have two short exact sequences of chain complexes such that $i \circ \theta_{n, A}=\theta_{n, B} \circ i^{\prime}$ and $j \circ \theta_{n, B}=\theta_{n, C} \circ j^{\prime}$, i.e. we have the commutative diagram of short exact sequences


Then we have the commutative diagram of long exact homology sequences

and in particular $\bar{\theta}_{n-1, A} \circ \delta_{n}^{\prime}=\delta_{n} \circ \bar{\theta}_{n, C}$.
Each simplicial complex can be equipped with a constant weight by setting $v^{\prime}(\sigma)=1_{R}$ (the multiplicative identity of $R$ ) for any $\sigma \in X$. Accordingly, we obtain the chain map $\theta_{n}: C_{n}(X) \rightarrow C_{n}(X, R)$ given by $\theta_{n}^{v^{\prime}, v}(\sigma)=v(\sigma) \sigma$, and the induced homomorphism $\bar{\theta}_{n}: H_{n}(X) \rightarrow H_{n}^{v}(X)$ between the simplicial homology and the weighted homology.

Theorem 1. Let $(X, v)$ be a weighted complex with coefficients in $R=\mathbb{Q}[[\pi]]$. Then the following assertions are equivalent:
(a) $\bar{\theta}_{n}$ induces the short exact sequence

$$
0 \longrightarrow H_{n}(X) \xrightarrow{\bar{\theta}_{n}} H_{n}^{v}(X)
$$

(b) $H_{n}(X)$ has no torsion.

Proof. $(a) \Rightarrow(b)$ : we show that if $H_{n}(X)$ has torsion, then $\bar{\theta}_{n}$ is not injective. Suppose there exists some nontrivial $\sum_{i} a_{i} \sigma_{i}+\operatorname{Im} \partial_{n+1}$ such that $q\left(\sum_{i} a_{i} \sigma_{i}+\operatorname{Im} \partial_{n+1}\right)=0$. Then $q\left(\sum_{i} a_{i} \sigma_{i}\right)=\partial_{n+1}\left(\sum_{j} z_{j} \tau_{j}\right)$ is equivalent to $\sum_{i} v\left(\sigma_{i}\right) a_{i} \sigma_{i}=\partial_{n+1}^{v}\left(\sum_{j} \frac{z_{j}}{q} v\left(\tau_{j}\right) \tau_{j}\right)$. Consequently,

$$
\bar{\theta}_{n}\left(\sum_{i} a_{i} \sigma_{i}+\operatorname{Im} \partial_{n+1}\right)=\theta_{n}\left(\sum_{i} a_{i} \sigma_{i}\right)+\operatorname{Im} \partial_{n+1}^{v}=0 .
$$

$(b) \Rightarrow(a):$ let $\sum_{i} a_{i} \sigma_{i}+\operatorname{Im} \partial_{n+1} \in H_{n}(X)$, where $a_{i} \in \mathbb{Z}$ and $\sigma_{i} \in C_{n}(X)$.
Claim. Suppose $\theta_{n}\left(\sum_{i \in I} a_{i} \sigma_{i}\right)=\partial_{n+1}^{v}(z)$, then we have for $q_{j} \in \mathbb{Q}$ :

$$
\theta_{n}\left(\sum_{i \in I} a_{i} \sigma_{i}\right)=\partial_{n+1}^{v}\left(\theta_{n+1}\left(\sum_{j \in J} q_{j} \tau_{j}\right)\right) .
$$

To prove the Claim, let $z=\sum_{j \in J} b_{j} \tau_{j}$, where $z \in C_{n+1}(X, R)$. We compute

$$
\partial_{n+1}^{v}(z)=\sum_{j, k} b_{j}(-1)^{k} \frac{v\left(\hat{\tau}_{j, k}\right)}{v\left(\tau_{j}\right)} \hat{\tau}_{j, k}=\sum_{i}\left[\sum_{\sigma_{i} \subset \tau_{j}} c_{i, j} b_{j} \frac{v\left(\sigma_{i}\right)}{v\left(\tau_{j}\right)}\right] \sigma_{i} .
$$

Then

$$
\begin{equation*}
\sum_{h} a_{h} v\left(\sigma_{h}\right) \sigma_{h}=\sum_{h}\left[\sum_{\sigma_{h} \subset \tau_{j}} c_{h, j} b_{j} \frac{v\left(\sigma_{h}\right)}{v\left(\tau_{j}\right)}\right] \sigma_{h} \tag{1}
\end{equation*}
$$

where $\left\{\sigma_{h}\right\}$ is the set of faces of the set of simplices $\left\{\tau_{j}\right\}$ and $a_{h}=0$ for $h \notin I$. We write $v\left(\tau_{j}\right)=\pi^{m_{j}}$ and $b_{j}=\sum_{n} x_{j, n} \pi^{n}$, where $x_{j, n} \in \mathbb{Q}$ and reformulate eq. (1) via power series

$$
\sum_{h} a_{h} \sigma_{h}=\sum_{h}\left[\sum_{\sigma_{h} \subset \tau_{j}} \sum_{n} c_{h, j} x_{j, n} \pi^{n-m_{j}}\right] \sigma_{h} .
$$

Eq. (2) implies that $r_{j}=x_{j, m_{j}} \pi^{m_{j}}$ has the property

$$
\theta_{n}\left(\sum_{i} a_{i} \sigma_{i}\right)=\partial_{n+1}^{v}\left(\sum_{j} b_{j} \tau_{j}\right)=\partial_{n+1}^{v}\left(\sum_{j} r_{j} \tau_{j}\right)
$$

By construction, any $r_{j} \equiv 0 \bmod \pi^{m_{j}}$ which implies

$$
\partial_{n+1}^{v}\left(\sum_{j} r_{j} \tau_{j}\right)=\partial_{n+1}^{v}\left(\theta_{n+1}\left(\sum_{j} x_{j, m_{j}} \tau_{j}\right)\right)
$$

and setting $q_{j}=x_{j, m_{j}}$ the Claim follows.
Consequently $\zeta=\sum_{j} q_{j} \tau_{j}$ has the property $\theta_{n}\left(\sum_{i} a_{i} \sigma_{i}\right)=\left(\partial_{n+1}^{v} \circ \theta_{n+1}\right)(\zeta)$. Let $q$ denote the smallest common multiple of the denominators of the $q_{j}$. Then $q \cdot \zeta$ has integer coefficients and we have

$$
\left(\partial_{n+1}^{v} \circ \theta_{n+1}\right)(q \cdot \zeta)=\theta_{n}\left(q \cdot \sum_{i} a_{i} \sigma_{i}\right) .
$$

In view of $\partial_{n+1}^{v} \circ \theta_{n+1}=\theta_{n} \circ \partial_{n+1}$, we derive

$$
\theta_{n}\left(q \cdot \sum_{i} a_{i} \sigma_{i}\right)=\partial_{n+1}^{v} \circ \theta_{n+1}(q \cdot \zeta)=\theta_{n} \circ \partial_{n+1}(q \cdot \zeta) .
$$

Since $\theta_{n}: C_{n}(X) \rightarrow C_{n}(X, R)$ is injective on $n$-chains, this implies $q \cdot \sum_{i} a_{i} \sigma_{i}=\partial_{n+1}(q \cdot \zeta)$, i.e. $q \cdot \sum_{i} a_{i} \sigma_{i}$ is a boundary in $H_{n}(X)$.

By construction, $q \cdot\left(\sum_{i} a_{i} \sigma_{i}+\operatorname{Im} \partial_{n+1}\right)=q \cdot \sum_{i} a_{i} \sigma_{i}+\operatorname{Im} \partial_{n+1}=0+\operatorname{Im} \partial_{n+1}$, whence $q \cdot\left(\sum_{i} a_{i} \sigma_{i}+\right.$ $\left.\operatorname{Im} \partial_{n+1}\right)=0$. Since $H_{n}(X)$ has no torsion this implies $\sum_{i} a_{i} \sigma_{i}+\operatorname{Im} \partial_{n+1}=0$, i.e. $\sum_{i} a_{i} \sigma_{i}$ is a boundary and thus trivial in $H_{n}(X)$ and the proof of the theorem is complete.

Clearly, $\theta_{n}^{v^{\prime}, v}\left(C_{n}(X, R)\right) \subset C_{n}(X, R)$ and denoting the quotient module by $C_{n}\left(X / \theta^{v^{\prime}, v}\right)=C_{n}(X, R) / \theta_{n}^{v^{\prime}, v}\left(C_{n}(X, R)\right)$ we have the following commutative diagram


We shall write $\theta_{n}$ instead of $\theta_{n}^{v^{\prime}, v}$, and $C_{n}(X / \theta)$ instead of $C_{n}\left(X / \theta^{v^{\prime}, v}\right)$.
Let $H_{n}^{v}(X / \theta)$ denote the homology with respect to the chain complex $\left\{C_{n}(X / \theta), \partial_{n}^{v}\right\}_{n}$.
Theorem 2. (a) Let $\left(X, v^{\prime}\right)$ and $(X, v)$ be weighted complexes with coefficients in an integral domain $R$. If $v^{\prime} \preceq v$, then we have the long exact homology sequence
$\longrightarrow H_{n+1}^{v}(X / \theta) \xrightarrow{\delta_{n+1}^{v}} H_{n}^{v^{\prime}}(X) \xrightarrow{\theta_{n}} H_{n}^{v}(X) \xrightarrow{j} H_{n}^{v}(X / \theta) \xrightarrow{\delta_{n}^{v}} H_{n-1}^{v^{\prime}}(X) \longrightarrow$
(b) Suppose $R=\mathbb{F}[[\pi]]$, where $\mathbb{F}$ is a field and $\nu^{\prime}(\sigma)=1_{R}$ for any $\sigma$, i.e., $H_{n}^{\nu^{\prime}}(X) \cong H_{n}(X, R)$. Then the long sequence splits into the exact sequences

$$
0 \longrightarrow H_{n}(X, R) \xrightarrow{\theta_{n}} H_{n}^{v}(X) \xrightarrow{j} H_{n}^{v}(X / \theta) \longrightarrow 0 .
$$

Our long exact sequence (4) is a weighted analogue of the long exact sequence for a pair, and the quotient $H_{n}^{v}(X / \theta)$ is a weighted analogue of the relative homology. A similar long exact sequence ([17], eq. (4.9)) has also been derived for the weighted complex associated to a weighted sheaf, where $\Delta_{*}$ and $H_{k}\left(L_{*}\right)$ in their notations play analogous roles as $\bar{\theta}_{n}$ and $H_{n}^{\nu}(X / \theta)$, respectively. We also point out that our exact sequence (4) enhances the long exact sequence (4.9) in [17] by allowing to link weighted homologies with two different weights $v$ and $v^{\prime}$.

Comparing with the previous results on weighted homology $[6,17,16,1]$, we believe that our proofs for the injectivity of $\bar{\theta}_{n}$ in Theorems 1 and 2 are original and utilize the technique involving formal power series comparison in $R=\mathbb{F}[[\pi]]$.

Proof. In view of the commutative diagram of eq. (3), statement (a) is a standard result from homological algebra.

Claim. We have the short exact sequence $0 \longrightarrow H_{n}(X, R) \xrightarrow{\theta_{n}} H_{n}^{\nu}(X)$.
We first observe that, if $\theta_{n}\left(\sum_{i \in I} a_{i} \sigma_{i}\right)=\partial_{n+1}^{v}(z)$, then

$$
\theta_{n}\left(\sum_{i \in I} a_{i} \sigma_{i}\right)=\partial_{n+1}^{v}\left(\theta_{n+1}\left(\sum_{j \in J} r_{j} \tau_{j}\right)\right),
$$

where $r_{j} \in R$. Let $z=\sum_{j \in J} b_{j} \tau_{j}, z \in C_{n+1}(X, R)$. Then

$$
\begin{equation*}
\sum_{h} a_{h} v\left(\sigma_{h}\right) \sigma_{h}=\sum_{h}\left[\sum_{\sigma_{h} \subset \tau_{j}} c_{h, j} b_{j} \frac{v\left(\sigma_{h}\right)}{v\left(\tau_{j}\right)}\right] \sigma_{h} \tag{5}
\end{equation*}
$$

where $\left\{\sigma_{h}\right\}$ is the set of faces of the set of simplices $\left\{\tau_{j}\right\}$ and $a_{h}=0$ for $h \notin I$. We write $v\left(\tau_{j}\right)=\pi^{m_{j}}$, $a_{h}=\sum_{n} y_{h, n} \pi^{n}$ and $b_{j}=\sum_{n} x_{j, n} \pi^{n}$, where $x_{j, n} \in \mathbb{F}$. Rewriting eq. (5) via power series we obtain

$$
\begin{equation*}
\sum_{h}\left[\sum_{n} y_{h, n} \pi^{n}\right] \sigma_{h}=\sum_{h}\left[\sum_{\sigma_{h} \subset \tau_{j}} \sum_{n} c_{h, j} x_{j, n} \pi^{n-m_{j}}\right] \sigma_{h} \tag{6}
\end{equation*}
$$

and eq. (6) implies that $r_{j}=\sum_{n \geq m_{j}} x_{j, n} \pi^{m_{j}}$ has the property

$$
\theta_{n}\left(\sum_{i} a_{i} \sigma_{i}\right)=\partial_{n+1}^{v}\left(\sum_{j} b_{j} \tau_{j}\right)=\partial_{n+1}^{v}\left(\sum_{j} r_{j} \tau_{j}\right)
$$

Furthermore by construction, for any $r_{j}$ holds $r_{j} \equiv 0 \bmod \pi^{m_{j}}$ i.e., $r_{j}=r_{j}^{\prime} \pi^{m_{j}}$, whence

$$
\partial_{n+1}^{v}\left(\sum_{j} r_{j} \tau_{j}\right)=\partial_{n+1}^{v}\left(\theta_{n+1}\left(\sum_{j} r_{j}^{\prime} \tau_{j}\right)\right)
$$

As a result we obtain the equality of $n$-chains with coefficients in $R$ :

$$
\theta_{n}\left(\sum_{i} a_{i} \sigma_{i}\right)=\partial_{n+1}^{v}\left(\sum_{j} b_{j} \tau_{j}\right)=\partial_{n+1}^{v}\left(\theta_{n+1}\left(\sum_{j} r_{j}^{\prime} \tau_{j}\right)\right)=\theta_{n} \circ \partial_{n+1}\left(\sum_{j} r_{j}^{\prime} \tau_{j}\right) .
$$

Consequently we derive $\sum_{i} a_{i} \sigma_{i}=\partial_{n+1}\left(\sum_{j} r_{j}^{\prime} \tau_{j}\right)$, i.e., $\sum_{i} a_{i} \sigma_{i}$ is a boundary in $H_{n}(X, R)$.

As a result the connecting homomorphisms, $\delta_{n+1}^{v}$, are trivial, whence the long exact sequence splits into the exact sequences

$$
0 \longrightarrow H_{n}(X, R) \xrightarrow{\theta_{n}} H_{n}^{v}(X) \xrightarrow{j} H_{n}^{v}(X / \theta) \longrightarrow 0 .
$$

Corollary 1. We have the exact sequence

$$
0 \longrightarrow H_{n}\left(X^{n}, R\right) \xrightarrow{\theta_{n}} H_{n}^{v}\left(X^{n}\right) \xrightarrow{j} H_{n}^{v}\left(X^{n} / \theta\right) \longrightarrow 0,
$$

where $X^{n}$ denotes the n-skeleton of $X$.

## 3. Some combinatorics

Lemma 3. Let $(X, v)$ be a weighted complex with coefficients in $R=\mathbb{F}[[\pi]]$. Then we have the short exact sequence of $R$-modules

$$
\begin{equation*}
0 \longrightarrow \pi H_{n}(X, R) \longrightarrow H_{n}(X, R) \xrightarrow{\bar{\rho}} H_{n}(X, \mathbb{F}) \longrightarrow 0 \tag{7}
\end{equation*}
$$

where the homomorphism $\bar{\rho}$ is induced by $\rho$, which maps a formal power series $r \in R$ to its constant term $\bar{r}$.

Proof. We first show $\operatorname{Ker}(\bar{\rho}) \subset \pi H_{n}(X, R)$. Suppose $\bar{\rho}\left(\sum_{j} r_{j} \tau_{j}+\operatorname{Im} \partial_{n+1}\right)=\sum_{j} \bar{r}_{j} \tau_{j}+\operatorname{Im} \bar{\partial}_{n+1}=$ 0 . Then there exists some $\sum_{h} \bar{a}_{h} \mu_{h} \in C_{n+1}(X, \mathbb{F})$, producing the equality of $n$-chains $\sum_{j} \bar{r}_{j} \tau_{j}-$ $\bar{\partial}_{n+1}\left(\sum_{h} \bar{a}_{h} \mu_{h}\right)=\sum_{j} \bar{x}_{j} \tau_{j}=0$, where each coefficient, $\bar{x}_{j}=0$. Clearly

$$
\sum_{j} r_{j} \tau_{j}+\operatorname{Im} \partial_{n+1}=\left[\sum_{j} r_{j} \tau_{j}-\partial_{n+1, R}\left(\sum_{h} a_{h} \mu_{h}\right)\right]+\operatorname{Im} \partial_{n+1} \in \pi H_{n}(X, R)
$$

from which $\operatorname{Ker}(\bar{\rho}) \subset \pi H_{n}(X, R)$ follows. It remains to observe $\pi H_{n}(X, R) \subset \operatorname{Ker}(\bar{\rho})$, which is immediate.

Remark. While $H_{n}(X, \mathbb{F})$ is free as an $\mathbb{F}$-module, $H_{n}(X, \mathbb{F})$ is not a free $R$-module. In fact, by Lemma 3, we can derive that, as an $R$-module, $H_{n}(X, \mathbb{F})$ is full torsion and is composed of $m$ copies of $R /(\pi)$, where $m=\operatorname{rnk} H_{n}(X, R)$. Accordingly, the short exact sequence (7) is not split exact.

Theorem 3. Let $(X, v)$ be a weighted complex with coefficients in $R=\mathbb{F}[[\pi]]$, and $X_{n}$ denote the set of $n$-simplices in $X$. Then for each $n$, there exists a subset of $n$-simplices, $K \subsetneq X_{n}$, such that the following holds
(i) let $M=X_{n} \backslash K$, then $\left\{\partial_{n}^{v}(\mu)\right\}_{\mu \in M}$ is a basis of $\partial_{n}^{v}\left(C_{n}(X, R)\right)$,
(ii) there exists a distinguished basis $\hat{\mathfrak{B}}_{K}^{v}$ of $H_{n}^{v}\left(X^{n}\right)$ that can be indexed by $K$, i.e., $\hat{\mathfrak{B}}_{K}^{v}=\left\{\hat{\beta}_{\kappa} \mid \kappa \in K\right\}$, and we refer to $\hat{\mathfrak{B}}_{K}^{v}$ as a $K$-basis,
(iii) each $\hat{\beta}_{\kappa} \in \hat{\mathfrak{B}}_{K}^{v}$ contains a unique, distinguished simplex $\kappa \in K$, having coefficient one, and

$$
\hat{\beta}_{\kappa}=\sum_{\mu_{\ell} \in M} r_{\ell} \mu_{\ell}+\kappa, \quad \text { where } r_{\ell} \text { are monomials satisfying } \operatorname{deg} v\left(\mu_{\ell}\right)=\operatorname{deg} r_{\ell} v(\kappa)
$$

(vi) let $\theta_{n}\left(\beta_{\kappa}\right)=v(\kappa) \hat{\beta}_{\kappa}$, then $\mathfrak{B}_{K}^{v}=\left\{\beta_{\kappa} \mid \kappa \in K\right\}$ is a basis of $H_{n}\left(X^{n}, R\right)$,
(v) let $\gamma_{\kappa}=\bar{\rho}\left(\beta_{\kappa}\right)$, then $\left\{\gamma_{\kappa} \mid \kappa \in K\right\}$ is a basis of $H_{n}\left(X^{n}, \mathbb{F}\right)$ and $H_{n}\left(X^{n}, R\right)$.

Furthermore, any $K$ and $K^{\prime}$ satisfying the above properties have the same cardinality.
Proof. We construct $M$ and $\hat{\mathfrak{B}}_{K}^{v}$ recursively via the following procedure: set $M_{0}=\varnothing$ and $S_{0}=\{\sigma \mid$ $\left.\sigma \in C_{n}(X, R)\right\}$. Label the simplices $\sigma_{i}$ arbitrary and examine them one by one, producing recursively the sequence $\left(M_{i}, S_{i}\right)$, where $M_{1}=M_{0} \cup\left\{\mu_{1} \mid \mu_{1}=\sigma_{1}\right\}$ and $S_{1}=S_{0} \backslash\left\{\sigma_{1}\right\}$, i.e. we remove $\sigma_{1}$ from $S_{0}$, relabel as $\mu_{1}$ and add to $M_{0}=\varnothing$.

Having constructed ( $M_{m}, S_{m}$ ) we proceed by examining $\sigma_{m+1}$. We set $S_{m+1}=S_{m} \backslash\left\{\sigma_{m+1}\right\}$ and given the equation

$$
\begin{equation*}
\partial_{n}^{v}\left(\sum_{\ell} r_{\ell} \mu_{\ell}+r_{m+1} \sigma_{m+1}\right)=0 \tag{8}
\end{equation*}
$$

distinguish two scenarios. In case there exists no nontrivial solution of $r_{\ell}, r_{m+1} \in R$, we set $M_{m+1}=$ $M_{m} \cup\left\{\mu_{n+1}=\sigma_{m+1}\right\}$. Otherwise, clearing the gcd of $r_{\ell}$ and $r_{m+1}$, we either have $\sigma_{m+1}$ has coefficient one or some $\mu_{\ell}$ does. In the former case we set $M_{m+1}=M_{m}$ and in the latter

$$
M_{m+1}=\left(M_{m} \backslash\left\{\mu_{\ell}\right\}\right) \cup\left\{\mu_{m+1}=\sigma_{m+1}\right\}, \quad S_{m+1}=S_{m} \backslash\left\{\sigma_{m+1}\right\} .
$$

Accordingly we either add a new $\mu$-simplex or replace a previously added $\mu$-simplex, while step by step examining all $n$-simplices. In this process we have $\partial_{n}^{v}\left(M_{m}\right) \subset \partial_{n}^{v}\left(M_{m+1}\right)$, since a $\mu$-simplex replaced in $M_{m}$ is by construction a linear combination of $M_{m+1} \mu$-simplices.

The procedure terminates in case of $S_{t}=\varnothing$ and all simplices have been examined. $M_{t}$ is by construction a basis of $\partial_{n}^{v}\left(C_{n}(X, R)\right)$ inducing the bipartition into the set of $\mu$-simplices, $M$, and the complimentary set of $\kappa$-simplices, $K$. Since any $\partial_{n}^{v}\left(C_{n}(X, R)\right.$ )-basis has the same size, any $K$ and $K^{\prime}$ satisfying the properties have the same cardinality.

For each $\kappa$ there exist unique coefficients $r_{\ell} \in R$, such that $\hat{\beta}_{\kappa}=\sum r_{\ell} \mu_{\ell}+\kappa$ is a $H_{n}^{v}\left(X^{n}\right)$-cycle and the $\hat{\beta}_{\kappa}$-cycles are linearly independent: $0=\sum_{\kappa} \lambda_{\kappa} \hat{\beta}_{\kappa}$ implies $\lambda_{\kappa}=0$ for all $\kappa$, since the simplex $\kappa$ appears uniquely in $\hat{\beta}_{\kappa}$.

Claim 1. $\hat{\mathfrak{B}}_{K}^{v}=\left\{\hat{\boldsymbol{\beta}}_{\kappa} \mid \kappa \in K\right\}$ is a basis of $H_{n}^{v}\left(X^{n}\right)$.
Let $c=\sum_{h} a_{h} \sigma_{h}$ be a $H_{n}^{v}\left(X^{n}\right)$-cycle. By construction, $c$ contains at least one $\kappa$-simplex. We prove by induction on the number of distinct $\kappa$-simplices contained in $c$ that $c=\sum_{\kappa} \lambda_{\kappa} \hat{\beta}_{\kappa}$. In case of the induction basis $c$ contains exactly one $\kappa$-simplex, $\kappa_{0}$. Then $c$ contains the summand $r_{\kappa_{0}} \kappa_{0}$ and exclusively $\mu$-simplices, otherwise. Clearly, $c-r_{\kappa_{0}} \cdot \hat{\beta}_{\kappa_{0}}=c^{\prime}$ is a cycle containing only $\mu$-simplices which is, by construction, trivial, whence $c=r_{\kappa_{0}} \cdot \hat{\beta}_{\kappa_{0}}$. For the induction step assume $c$ contains $(m+1)$ $\kappa$ simplices, $\kappa_{1}, \ldots \kappa_{m+1}$. Suppose $c$ has the summand $r_{\kappa_{m+1}}$. Then $c-r_{\kappa_{m+1}} \hat{\beta}_{\kappa_{m+1}}$ is a cycle that contains exactly $m \kappa$-simplices since $\hat{\beta}_{\kappa_{m+1}}$ contains, besides $\kappa_{m+1}$, only $\mu$-simplices. By induction hypothesis we then have $c-r_{\kappa_{m+1}} \hat{\beta}_{\kappa_{m+1}}=\sum_{i=1}^{m} r_{\kappa_{i}} \hat{\beta}_{i}$ and Claim 1 follows.

Claim 2. For each $\hat{\beta}_{\kappa}=\sum r_{\ell} \mu_{\ell}+\kappa$, there exist monomials $r_{\ell}$ satisfying $\operatorname{deg} v\left(\mu_{\ell}\right)=\operatorname{deg}\left(r_{\ell} v(\kappa)\right)$ for any $\ell$.

As a $H_{n}^{v}\left(X^{n}\right)$-cycle, $\hat{\beta}_{\kappa}$ satisfies $\partial_{n}^{v}\left(\hat{\beta}_{\kappa}\right)=\partial_{n}^{v}\left(\sum r_{\ell} \mu_{\ell}+\kappa\right)=0$. For any $\hat{\beta}_{\kappa}$-face $\sigma$, we derive

$$
\sum_{\sigma \subset \mu_{\ell}} c_{\ell} \frac{r_{\ell}}{v\left(\mu_{\ell}\right)}+c_{\kappa} \frac{1}{v(\kappa)}=0, \text { for } \sigma \subset \kappa, \quad \sum_{\sigma \subset \mu_{\ell}} c_{\ell} \frac{r_{\ell}}{v\left(\mu_{\ell}\right)}=0, \text { for } \sigma \not \subset \kappa
$$

where $c_{\ell}$ and $c_{\kappa}$ are $\pm 1$. We write $v\left(\mu_{\ell}\right)=\pi^{\omega\left(\mu_{\ell}\right)}, v(\kappa)=\pi^{\omega(\kappa)}$ and $r_{\ell}=\sum_{n} x_{\ell, n} \pi^{n}$, where $x_{\ell, n} \in \mathbb{F}$. Rewriting the equations we obtain

$$
\begin{aligned}
\sum_{n} \sum_{\sigma \subset \mu_{\ell}} c_{\ell} x_{\ell, n} \pi^{n-\omega\left(\mu_{\ell}\right)}+c_{\kappa} \pi^{-\omega(\kappa)} & =0 & & \text { for } \sigma \subset \kappa \\
\sum_{n} \sum_{\sigma \subset \mu_{\ell}} c_{\ell} x_{\ell, n} \pi^{n-\omega\left(\mu_{\ell}\right)} & =0 & & \text { for } \sigma \not \subset \kappa
\end{aligned}
$$

In particular, taking $\left[\pi^{-\omega(\kappa)}\right]$-terms, we derive

$$
\begin{aligned}
\sum_{\sigma \subset \mu_{\ell}} c_{\ell} x_{\ell, \omega\left(\mu_{\ell}\right)-\omega(\kappa)}+c_{\kappa}=0 & \text { for } \sigma \subset \kappa \\
\sum_{\sigma \subset \mu_{\ell}} c_{\ell} x_{\ell, \omega\left(\mu_{\ell}\right)-\omega(\kappa)}=0 & \text { for } \sigma \not \subset \kappa .
\end{aligned}
$$

Let $m_{\ell}=x_{\ell, \omega\left(\mu_{\ell}\right)-\omega(\kappa)} \pi^{\omega\left(\mu_{\ell}\right)-\omega(\kappa)}$ be the monomials obtained by taking $\left[\pi^{\omega\left(\mu_{\ell}\right)-\omega(\kappa)}\right]$-terms of $r_{\ell}$. Then $\hat{\beta}_{\kappa}^{\prime}=\sum m_{\ell} \mu_{\ell}+\kappa$ is by construction a $H_{n}^{v}\left(X^{n}\right)$-cycle, and therefore $\hat{\beta}_{\kappa}^{\prime}=\hat{\beta}_{\kappa}$ since $\hat{\beta}_{\kappa}$ is unique. Accordingly, $r_{\ell}=m_{\ell}$, i.e., $r_{\ell}$ are monomials satisfying $\operatorname{deg} v\left(\mu_{\ell}\right)=\operatorname{deg}\left(r_{\ell} v(\kappa)\right)$.

Claim 3. $\mathfrak{B}_{K}^{v}=\left\{\beta_{\kappa} \mid \kappa \in K\right\}$ is a basis of $H_{n}\left(X^{n}, R\right)$, and $\left\{\gamma_{\kappa} \mid \kappa \in K\right\}$ is a basis of $H_{n}\left(X^{n}, \mathbb{F}\right)$ and $H_{n}\left(X^{n}, R\right)$.

By definition, $\beta_{\kappa}=\theta_{n}^{-1}\left(v(\kappa) \hat{\beta}_{\kappa}\right)=\sum \frac{r_{\ell \nu}(\kappa)}{v\left(\mu_{\ell}\right)} \mu_{\ell}+\kappa$. Since $r_{\ell}$ satisfy $\operatorname{deg} v\left(\mu_{\ell}\right)=\operatorname{deg}\left(r_{\ell} v(\kappa)\right)$ by Claim 2, $\beta_{\kappa}$ is well-defined. Note that $\sum_{i} \lambda_{i} \beta_{i}=0$ implies $0=\sum_{i} \lambda_{i} \theta_{n}\left(\beta_{i}\right)=\sum_{i} \lambda_{i} v(\kappa) \hat{\beta}_{i}$ and hence $\lambda_{i} v(\kappa)=0$ for all $i$, from which $\lambda_{i}=0$ follows.

To prove $\left\{\beta_{\kappa} \mid \kappa \in K\right\}$ generates $H_{n}\left(X^{n}, R\right)$, we observe that $\kappa$ retains coefficient one in $\beta_{\kappa}$. In view of this we proceed as in Claim 1 by induction on the number of distinct $\kappa$-edges contained in a $H_{n}\left(X^{n}, R\right)$-cycle.

Analogously we can show, using Lemma 3, that $\left\{\bar{\rho}\left(\beta_{\kappa}\right) \mid \kappa \in K\right\}$ is a basis of $H_{n}\left(X^{n}, \mathbb{F}\right)$, observing that $\kappa$ appears exclusively in $\bar{\rho}\left(\beta_{\kappa}\right)$ having coefficient one. Lemma 3 and Nakayama's Lemma ${ }^{1}$ imply that $\left\{\bar{\rho}\left(\beta_{\kappa}\right) \mid \kappa \in K\right\}$ is also a basis of $H_{n}\left(X^{n}, R\right)$, whence Claim 3.

Therefore $\hat{\mathfrak{B}}_{K}^{v}=\left\{\hat{\boldsymbol{\beta}}_{\kappa} \mid \kappa \in K\right\}$ is a basis of $H_{n}^{v}\left(X^{n}\right)$ satisfying $(i)-(v)$ and the proof is complete.
Remark. (a) The $K$-bases of $H_{n}^{\nu}\left(X^{n}\right),\left\{\hat{\beta}_{\kappa} \mid \kappa \in K\right\}$, depend on $\mathbb{F}$, since $\mathbb{F}$ factors into whether or not eq. (8) has a nontrivial solution in $R=\mathbb{F}[[\pi]]$.
(b) The above proof can be generalized to the case where $R$ is a ring of polynomials over a field, i.e. $R=\mathbb{F}[\pi]$.

Corollary 2. Let $\hat{\mathfrak{B}}_{K}^{v}$ be a $K$-basis of $H_{n}^{v}\left(X^{n}\right)$. Then

$$
H_{n}^{v}\left(X^{n} / \theta\right) \cong \bigoplus_{\kappa \in K} R /(v(\kappa)) .
$$

Proof. The projection $p: C_{n}(X, R) \rightarrow \oplus_{\sigma} R / v(\sigma)$, given by $\sum_{i} a_{i} \sigma \mapsto \sum_{i}\left(a_{i}+v(\sigma)\right) \sigma$ has kernel $\theta_{n}\left(C_{n}(X, R)\right)$ and consequently $C_{n}(X, R) / \theta_{n}\left(C_{n}(X, R)\right) \cong \oplus_{\sigma} R / v(\sigma)$. Since $v(\kappa) \hat{\beta}_{\kappa}=\theta_{n}\left(\beta_{\kappa}\right)$, each

[^0]$\hat{\beta}_{\kappa}$ generates a cyclic $H_{n}^{v}\left(X^{n} / \theta\right)$ submodule isomorphic to $R /(v(\kappa))$, from which the Corollary follows.

## 4. The main theorem

Lemma 4. Let $(X, v)$ be a weighted complex with coefficients in $R=\mathbb{F}[[\pi]]$. Given $v$, we consider the sequence of weight functions $\left(v_{0}, v_{1}, \ldots, v_{t}=v\right)$ defined by $v_{r}(\sigma)=v(\sigma)$ for $\operatorname{dim}(\sigma) \leq r$ and $v_{r}(\sigma)=1$, otherwise. Then there exist the exact sequences

$$
\begin{gathered}
0 \longrightarrow H_{n}(X, R) \xrightarrow{\bar{\eta}_{n}^{n}} H_{n}^{v_{n}}(X) \xrightarrow{j} \oplus_{\kappa} R /(v(\kappa)) \longrightarrow 0 \\
0 \longrightarrow \oplus_{\mu} R /(v(\mu)) \longrightarrow H_{n-1}^{v_{n-1}}(X) \xrightarrow{\bar{\eta}_{n-1}^{n}} H_{n-1}^{v_{n}}(X) \longrightarrow 0
\end{gathered}
$$

where $\bar{\eta}_{n}^{r}$ is induced from

$$
\eta_{n}^{r}(\sigma)=\theta_{n}^{v_{r-1}, v_{r}}(\sigma)=\frac{v_{r}(\sigma)}{v_{r-1}(\sigma)} \sigma= \begin{cases}v(\sigma) \sigma & \text { if } \operatorname{dim}(\sigma)=r \\ \sigma & \text { otherwise }\end{cases}
$$

Proof. By construction of $v_{n}$, the quotient $C_{\ell}(X, R) / \eta_{\ell}^{n}\left(C_{\ell}(X, R)\right)$ is only nontrivial for $\ell=n$, in which case $C_{n}(X, R) / \eta_{n}^{n}\left(C_{n}(X, R)\right) \cong \oplus_{\sigma} R /(v(\sigma))$, where the summation is over the set of all $n$-simplices. Consequently, the boundary maps $\bar{\partial}_{n}^{v_{n}}$ and $\bar{\partial}_{n+1}^{v_{n}}$ are trivial, whence

$$
H_{\ell}^{v_{n}}\left(X / \eta_{\ell}^{n}\right) \cong \begin{cases}\oplus_{\sigma} R /(v(\sigma)) & \text { for } \ell=n \\ 0 & \text { for } \ell \neq n\end{cases}
$$

The long homology sequence of Theorem 2 then becomes the five term exact sequence

$$
0 \longrightarrow H_{n}^{v_{n-1}}(X) \xrightarrow{\bar{\eta}_{n}^{n}} H_{n}^{v_{n}}(X) \xrightarrow{j} H_{n}^{v_{n}}\left(X / \eta^{n}\right) \xrightarrow{\delta_{n}^{v_{n}}} H_{n-1}^{v_{n-1}}(X) \xrightarrow{\bar{\eta}_{n-1}^{n}} H_{n-1}^{v_{n}}(X) \longrightarrow 0
$$

where $H_{n}^{v_{n-1}}(X)=H_{n}(X, R)$, since all $v_{n-1}$-weights of $n$ - and $(n+1)$-simplices are one. By exactness at $H_{n}^{v_{n}}\left(X / \eta_{n}^{n}\right) \cong \oplus_{\sigma} R /(v(\sigma))$ and $H_{n-1}^{v_{n-1}}(X)$, we have $\operatorname{Im} j=\operatorname{Ker} \delta_{n}^{v_{n}}$ and $\operatorname{Im} \delta_{n}^{v_{n}}=\operatorname{Ker} \bar{\eta}_{n-1}^{n}$. Since $\left.\eta_{n-1}^{n}\right|_{C_{n-1}(X, R)}=\mathrm{id}$ and id $\circ \partial_{n}^{v_{n-1}}=\partial_{n}^{v_{n}} \circ \eta_{n}^{n}$, we have

$$
\operatorname{Im} \partial_{n}^{v_{n}} / \operatorname{Im} \partial_{n}^{v_{n-1}}=\oplus_{\mu}\left\langle\partial_{n}^{v_{n}}(\mu)\right\rangle /\left\langle v(\mu) \partial_{n}^{v_{n}}(\mu)\right\rangle \cong \oplus_{\mu} R /(v(\mu))
$$

and the sequence

$$
0 \longrightarrow \bigoplus_{\mu} R /(v(\mu)) \xrightarrow{\bar{\partial}_{n}^{v_{n}}} \operatorname{Ker} \partial_{n-1}^{v_{n}} / \operatorname{Im} \partial_{n}^{v_{n-1}} \xrightarrow{\text { proj }} H_{n-1}^{v_{n}}(X) \longrightarrow 0
$$

is exact. Since $\operatorname{Ker} \partial_{n-1}^{v_{n}} / \operatorname{Im} \partial_{n}^{v_{n-1}}=\operatorname{Ker} \partial_{n-1}^{v_{n-1}} / \operatorname{Im} \partial_{n}^{v_{n-1}}=H_{n-1}^{v_{n-1}}(X)$ this provides an interpretation of Ker $\bar{\eta}_{n-1}^{n}$, via

$$
0 \longrightarrow \oplus_{\mu} R /(v(\mu)) \longrightarrow H_{n-1}^{v_{n-1}}(X)^{\bar{\eta}_{n-1}^{n}=\text { proj }^{v_{n}}} H_{n-1}^{v_{n}}(X) \longrightarrow 0 .
$$

$v$ bipartitions the set of $n$-simplices into $\kappa$ - and $\mu$-simplices and using the exactness at $H_{n}^{v_{n}}\left(X / \eta_{n}^{n}\right) \cong$ $\oplus_{\sigma} R /(v(\sigma))$, we obtain

$$
0 \longrightarrow H_{n}(X, R) \xrightarrow{\bar{\eta}_{n}^{n}} H_{n}^{v_{n}}(X) \xrightarrow{j} \oplus_{\kappa} R /(v(\kappa)) \longrightarrow 0 .
$$

Theorem 4. Let $(X, v)$ be a weighted complex with coefficients in $R=\mathbb{F}[[\pi]]$. Let $F_{n}^{v}$ and $T_{n}^{v}$ denote the free and the torsion submodules of $H_{n}^{\nu}(X)$. Then there exists an exact sequence

$$
0 \longrightarrow H_{n}(X, R) \xrightarrow{\bar{\theta}_{n}} F_{n}^{v} \xrightarrow{j} \bigoplus_{s=1}^{q} R /\left(v\left(\kappa_{s}^{n}\right)\right) \longrightarrow 0,
$$

where $\left\{\kappa_{1}^{n}, \ldots, \kappa_{q}^{n}\right\} \cup \dot{\cup}\left\{\kappa_{q+1}^{n}, \ldots, \kappa_{p}^{n}\right\}=K$ is a distinguished bipartition of the $\kappa$-simplices of dimension n. Furthermore, $\operatorname{rnk}_{R}\left(H_{n}^{v}(X)\right)=\operatorname{rnk}_{\mathbb{F}}\left(H_{n}(X, \mathbb{F})\right)$ and

$$
T_{n}^{v} \cong \bigoplus_{s=q+1}^{p} R /\left(v\left(\kappa_{s}^{n}\right) / v\left(\mu_{\alpha(s)}^{n+1}\right)\right),
$$

where $\alpha \in S_{p-q}$ establishes a pairing between $\kappa_{s}^{n}$ - and $\mu_{\alpha(s)}^{n+1}$-simplices of dimension $n$ and $(n+1)$, respectively.

Proof. By the general structure theorem of finitely generated modules over pids, we have $H_{n}^{v}(X) \cong F_{n}^{v} \oplus$ $T_{n}^{v}$. Furthermore, we have $H_{n}^{v}\left(X^{n}\right) \cong \mathfrak{F}_{n}^{v} \oplus \mathfrak{T}_{n}^{v}$, where $\mathfrak{F}_{n}^{v} \cong{ }_{\phi} F_{n}^{v}, \phi(f)=f+\operatorname{Im} \partial_{n+1}^{v}$ and $\mathfrak{T}_{n}^{v} / \operatorname{Im} \partial_{n+1}^{v} \cong$ $T_{n}^{\nu}$. This follows from the diagram


Here $\mathfrak{T}_{n}^{v}=\operatorname{Ker}(p)$ and since $F_{n}^{v}$ is free, it is projective and we have $H_{n}^{v}\left(X^{n}\right)=\mathfrak{T}_{n}^{v} \oplus \mathfrak{F}_{n}^{v}$ with $\mathfrak{F}_{n}^{v} / \operatorname{Im} \partial_{n+1}^{v} \cong F_{n}^{v}$. Finally, by construction, we have $\operatorname{rnk}\left(\operatorname{Im} \partial_{n+1}^{v}\right)=\operatorname{rnk}\left(\mathfrak{T}_{n}^{v}\right)$.

Let $\varphi^{n}=\eta^{n+1} \circ \eta^{n}$, we note that $\varphi_{n}^{n}=\theta_{n}$ since both maps coincide on $n$ - and $(n+1)$-simplices.
Claim 1. We have the exact sequence

$$
0 \longrightarrow H_{n}(X, R) \xrightarrow{\bar{\theta}_{n}} H_{n}^{v}(X) \xrightarrow{j}\left(\bigoplus_{\kappa^{n}} R /\left(v\left(\kappa^{n}\right)\right)\right) /\left(\oplus_{\mu^{n+1}} R /\left(v\left(\mu_{n+1}\right)\right)\right) \longrightarrow 0 .
$$

By Theorem 2 we have the long exact sequence of homology groups


By construction $\bar{\varphi}_{n}^{n}=\bar{\theta}_{n}, H_{n}^{v_{n-1}}(X) \cong H_{n}(X, R), H_{n}^{v_{n+1}}(X)=H_{n}^{v}(X)$ and $H_{n-1}^{v_{n+1}}\left(X / \varphi^{n}\right)=0$. In view of $C_{n+1}\left(X / \varphi^{n}\right) \cong \bigoplus_{\tau^{n+1}} R /\left(v\left(\tau^{n+1}\right)\right)$, where the direct sum ranges over all $(n+1)$-simplices, $\tau^{n+1}$, we have

$$
\bar{\partial}_{n+1}^{v_{n+1}}\left(C_{n+1}\left(X / \varphi^{n}\right)\right) \cong \bigoplus_{\mu^{n+1}} R /\left(v\left(\mu^{n+1}\right)\right),
$$

where the summation ranges over all $\mu^{n+1}$-simplices which form a basis of $\partial_{n+1}^{v_{n+1}}(X)$. Since $C_{n-1}\left(X / \varphi^{n}\right)=$
0 we obtain $\bar{\partial}_{n}^{v_{n+1}}: C_{n}\left(X / \varphi^{n}\right) \rightarrow 0$, where $C_{n}\left(X / \varphi^{n}\right) \cong \bigoplus_{\sigma^{n}} R /\left(v\left(\sigma^{n}\right)\right)$. Using $\bar{\partial}_{n}^{v_{n+1}} \circ \bar{\partial}_{n+1}^{v_{n+1}}=0$, we derive

$$
\begin{equation*}
H_{n}^{v_{n+1}}\left(X / \varphi^{n}\right) \cong\left(\bigoplus_{\kappa^{n}} R /\left(v\left(\kappa_{n}\right)\right) / \bigoplus_{\mu^{n+1}} R /\left(v\left(\mu^{n+1}\right)\right)\right) \oplus\left(\bigoplus_{\mu^{n}} R /\left(v\left(\mu^{n}\right)\right)\right) . \tag{10}
\end{equation*}
$$

By Lemma 4 we have the exact sequence

$$
0 \longrightarrow \bigoplus_{\mu^{n}} R /\left(v\left(\mu^{n}\right)\right) \xrightarrow{\bar{\partial}_{n}^{v_{n+1}}} H_{n-1}^{v_{n-1}}(X) \xrightarrow{\bar{\varphi}_{n}^{n}} H_{n-1}^{v_{n+1}}(X) \longrightarrow 0
$$

which combined with exactness of eq. (9) at $H_{n}^{v_{n+1}}\left(X / \varphi^{n}\right)$ and eq. (10) gives rise to the exact sequence of Claim 1:

$$
0 \longrightarrow H_{n}(X, R) \xrightarrow{\bar{\theta}_{n}} H_{n}^{v}(X) \xrightarrow{j}\left(\oplus_{\kappa^{n}} R /\left(v\left(\kappa^{n}\right)\right) /\left(\oplus_{\mu^{n+1}} R /\left(v\left(\mu^{n+1}\right)\right)\right) \longrightarrow 0\right.
$$

and Claim 1 follows.
We proceed by dissecting the exact sequence of Claim 1 into the free and torsion modules.
Claim 2. We have the exact sequence

$$
0 \longrightarrow H_{n}(X, R) \xrightarrow{\bar{\theta}_{n}} F_{n}^{v}(X) \xrightarrow{j} \oplus_{s=1}^{q} R /\left(v\left(\kappa_{s}^{n}\right)\right) \longrightarrow 0,
$$

and $\operatorname{rnk}_{R}\left(H_{n}^{v}(X)\right)=\operatorname{rnk}_{\mathbb{F}}\left(H_{n}(X, \mathbb{F})\right)$.
In view of $\theta_{n}\left(H_{n}(X, R)\right) \subset F_{n}^{v}(X)$ and Theorem 2, we have

$$
0 \longrightarrow H_{n}(X, R) \xrightarrow{\bar{\theta}_{n}} F_{n}^{v}(X) .
$$

Furthermore, by Theorem 2 and Corollary 2,

$$
0 \longrightarrow H_{n}\left(X^{n}, R\right) \xrightarrow{\theta_{n}} H_{n}^{v}\left(X^{n}\right) \xrightarrow{j} \oplus_{\kappa^{n}} R /\left(v\left(\kappa^{n}\right)\right) \longrightarrow 0 .
$$

By restriction, $j$ induces the surjective homomorphism $\left.j\right|_{\mathfrak{F}_{n}}: \mathfrak{F}_{n}^{v} \rightarrow \bigoplus_{s=1}^{q} R /\left(v\left(\kappa_{s}^{n}\right)\right)$ and


Since $\bigoplus_{s=1}^{q} R /\left(v\left(\kappa_{s}^{n}\right)\right)$ is full torsion, the exact sequence implies $\operatorname{rnk}_{R}\left(H_{n}^{v}(X)\right)=\operatorname{rnk}_{R}\left(H_{n}(X, R)\right)$. Combing with $\operatorname{rnk}_{\mathbb{F}}\left(H_{n}(X, \mathbb{F})\right)=\operatorname{rnk}_{R}\left(H_{n}(X, R)\right)$ derived by Lemma 3, we have $\operatorname{rnk}_{R}\left(H_{n}^{v}(X)\right)=$ $\operatorname{rnk}_{\mathbb{F}}\left(H_{n}(X, \mathbb{F})\right)$, whence Claim 2.

Claim 3. We have

$$
T_{n}^{v}(X) \cong \bigoplus_{s=q+1}^{|K|} R /\left(v\left(\kappa_{s}^{n}\right) /\left(v\left(\mu_{\alpha(s)}^{n+1}\right)\right)\right.
$$

We consider the homomorphism embedding $\operatorname{Im} \partial_{n+1}^{v}$ into $\mathfrak{T}_{n}^{v}$. Since $R$ is pid, there exists a $\mathfrak{T}_{n}^{v}$-basis, $\mathfrak{B}_{1}=\left\{\hat{\mathfrak{t}}_{q+1}, \ldots \hat{\mathfrak{t}}_{p}\right\}$ and a $\operatorname{Im} \partial_{n+1}^{v}$-basis $\mathfrak{B}_{0}=\left\{x_{s} \cdot \hat{\mathfrak{t}}_{s} \mid s=q+1, \ldots, p\right\}$, where $x_{s} \in R$ represent the invariant factors.

Claim 3 follows from two observations that put these bases into context with Corollary 1 and Corollary 2. First, since $\varphi_{n}^{n}$ elevates $n$ - as well as $(n+1)$-simplices to their $v$-weight, we have

$$
\operatorname{Im} \partial_{n+1}^{v} / \theta_{n}\left(\operatorname{Im} \partial_{n+1}\right)=\bar{\partial}_{n+1}^{v_{n+1}}\left(C_{n+1}\left(X / \varphi^{n}\right)\right) \cong \bigoplus_{s=q+1}^{p} R /\left(v\left(\mu_{s}^{n+1}\right)\right) .
$$

Secondly, using $H_{n}^{v}\left(X^{n}\right) \cong \mathfrak{T}_{n}^{v} \oplus \mathfrak{F}_{n}^{v}$ and the commutative diagram

we arrive at

$$
\mathfrak{T}_{n}^{v}(X) / \theta_{n}\left(\operatorname{Im} \partial_{n+1}\right) \cong \bigoplus_{s=q+1}^{p} R /\left(v\left(\kappa_{s}^{n}\right)\right) .
$$

In order to see how the $\kappa_{s}^{n}$ and $\mu_{s}^{n+1}$ align, we consider the commutative diagram

where we extend $\psi\left(\mathfrak{t}_{s}\right)=x_{s} \cdot \mathfrak{t}_{s}$ linearly to an $R$-module homomorphism $\psi$. Choosing the $\mathfrak{T}_{n}^{\nu}{ }^{-}$ and $\operatorname{Im} \partial_{n+1}^{v}$-bases $\mathfrak{B}_{1}$ and $\mathfrak{B}_{0}$, respectively, we have $\mathfrak{T}_{n}^{v}(X) / \theta_{n}\left(\operatorname{Im} \partial_{n+1}\right) \cong \sum_{s}\left\langle\mathfrak{t}_{s}+\theta_{n}\left(\operatorname{Im} \partial_{n+1}\right)\right\rangle$ as well as $\operatorname{Im} \partial_{n+1}^{v} / \theta_{n}\left(\operatorname{Im} \partial_{n+1}\right) \cong \sum_{s}\left\langle x_{s} \mathfrak{t}_{s}+\theta_{n}\left(\operatorname{Im} \partial_{n+1}\right)\right\rangle$. Since $R$ is a discrete valuation ring, $\left\langle\mathfrak{t}_{s}+\theta_{n}\left(\operatorname{Im} \partial_{n+1}\right)\right\rangle$ and $\left\langle x_{s} \mathfrak{t}_{s}+\theta_{n}\left(\operatorname{Im} \partial_{n+1}\right)\right\rangle$ are primary modules and as such indecomposable, whence for each $q+1 \leq s \leq p$

$$
\left\langle\mathfrak{t}_{s}+\theta_{n}\left(\operatorname{Im} \partial_{n+1}\right)\right\rangle \cong R /\left(v\left(\kappa_{s_{1}}^{n}\right)\right) \quad \text { and } \quad\left\langle x_{s} \mathfrak{t}_{s}+\theta_{n}\left(\operatorname{Im} \partial_{n+1}\right)\right\rangle \cong R /\left(v\left(\mu_{s_{2}}^{n+1}\right)\right) .
$$

By the commutativity of the right square,

$$
\bar{\psi}\left(R /\left(v\left(\kappa_{s_{1}}^{n}\right)\right)\right) \cong \bar{\psi}\left(\left\langle\mathfrak{t}_{s}+\theta_{n}\left(\operatorname{Im} \partial_{n+1}\right)\right)=\left\langle x_{s} \mathfrak{t}_{s}+\theta_{n}\left(\operatorname{Im} \partial_{n+1}\right)\right\rangle \cong R /\left(v\left(\mu_{s_{2}}^{n+1}\right)\right) .\right.
$$

Thus there exists some permutation $\alpha$ that pairs $\kappa_{s_{1}}^{n}$ with $\mu_{s_{2}}^{n+1}$ with $\alpha\left(s_{1}\right)=s_{2}$ such that

$$
\bar{\psi}\left(R /\left(v\left(\kappa_{s_{1}}^{n}\right)\right)\right) \cong R /\left(v\left(\mu_{s_{2}}^{n+1}\right)\right),
$$

and as a result we arrive at

$$
\begin{aligned}
\mathfrak{T}_{n}^{v}(X) / \operatorname{Im} \partial_{n+1}^{v} & \cong\left[\mathfrak{T}_{n}^{v}(X) / \theta_{n}\left(\operatorname{Im} \partial_{n+1}\right)\right] /\left[\operatorname{Im} \partial_{n+1}^{v} / \theta_{n}\left(\operatorname{Im} \partial_{n+1}\right)\right] \\
& \cong \bigoplus_{s=q+1}^{p}\left[R /\left(v\left(\kappa_{s}^{n}\right)\right)\right] /\left[R /\left(v\left(\mu_{\alpha(s)}^{n+1}\right)\right)\right] \\
& \cong \bigoplus_{s=q+1}^{p} R /\left(v\left(\kappa_{s}^{n}\right) / v\left(\mu_{\alpha(s)}^{n+1}\right)\right) .
\end{aligned}
$$

Remark. In view of the structure theorem, let us revisit the weighted simplicial complex $(X, v)$ depicted in Figure 1. Based on Theorem 3, we compute the $K$-basis of $H_{1}^{v}\left(X^{1}\right)$ given by $\hat{\mathfrak{B}}_{K}^{v}=$ $\left\{\hat{\beta}_{A C}, \hat{\beta}_{C B}, \hat{\beta}_{B A}\right\}$ with $K=\{A C, C B, B A\}$, where

$$
\begin{aligned}
& \hat{\beta}_{A C}=A C+\pi C D+\pi^{2} D A \\
& \hat{\beta}_{C B}=C B+\pi^{4} B D+\pi^{2} D C \\
& \hat{\beta}_{B A}=B A+\pi^{4} A D+\pi^{5} D B .
\end{aligned}
$$

The $\mu^{2}$-simplices are given by $A B C, A C D$ and thus $\partial_{2}^{v}(X)=\left\{\partial_{2}^{v}(A B C), \partial_{2}^{v}(A C D)\right\}$. By Theorem 4 , we derive a partition $K=\{B A, A C\} \cup \dot{\cup}\{C B\}$ and a pairing $\alpha:\{B A, A C\} \rightarrow\{A B C, A C D\}$ with $\alpha(B A)=$ $A B C, \alpha(A C)=A C D$. Then the torsion of the first weighted homology $H_{1}^{v}(X)$ is given by

$$
T_{1}^{v} \cong R /\left(\pi^{\omega(B A)-\omega(A B C)}\right) \bigoplus R /\left(\pi^{\omega(A C)-\omega(A C D)}\right) \cong R /(\pi) \oplus R /\left(\pi^{4}\right) .
$$

Since rnk $H_{1}^{v}(X)=\operatorname{rnk} H_{1}(X, R)=1$, we obtain $H_{1}^{\nu}(X) \cong R \oplus R /(\pi) \oplus R /\left(\pi^{4}\right)$.
We refer the readers to [2, 7] for a fast algorithm and its accompanying software implementation for computing the bijection $\alpha$ and the weighted homology.

## 5. Case study: RNA bi-structures

RNA is a biomolecule that folds into a helical configuration of its sequence by forming base pairs. The most prominent class of coarse-grained structures are the RNA secondary structures [22, 19, 18]. An RNA structure can be represented as a diagram, a labeled graph over the vertex set $\{1, \ldots, n\}$, whose vertices are arranged in a horizontal line and arcs are drawn in the upper half-plane, see Fig. 2. A vertex corresponds to a nucleotide in the primary sequence, and an arc, denoted by $(i, j)$, represents the base pairing between the $i$-th and $j$-th nucleotides in the RNA structure. The backbone of a diagram is the sequence of consecutive integers $(1, \ldots, n)$ together with the edges $\{\{i, i+1\} \mid 1 \leq i \leq n-1\}$. We shall distinguish the backbone edge $\{i, i+1\}$ from the arc $(i, i+1)$, which we refer to as a 1 -arc. Two
arcs $\left(i_{1}, j_{1}\right)$ and $\left(i_{2}, j_{2}\right)$ are crossing if $i_{1}<i_{2}<j_{1}<j_{2}$. An RNA secondary structure is defined as a diagram satisfying (1) it does not contain any 1 -arcs, (2) any two arcs are non-crossing, (3) any two arcs do not have a common vertex $[22,19]$.


FigURE 2. LHS: a planar RNA secondary structure with base pairs between nucleotides, and a loop $L$ is shaded. RHS: its diagram representation on the set of vertices $\{1, \ldots, 91\}$. The loop $L$ (shaded) is the set of vertices $\{9,10,11,12,27,28,46,47,48,65,66,67,68,82,83\}$ covered by the arc $(9,83)$.

In an RNA secondary structure, a vertex $k$ is covered by an arc $(i, j)$ if $i \leq k \leq j$ and there exists no other arc $(p, q)$ such that $i<p<k<q<j$. A loop is the set of vertices covered by an arc $(i, j)$, see Fig. 2 RHS. In particular, the exterior loop is given by the set of vertices covered by an artificial rainbow arc connecting the first and last vertices. A secondary structure can be uniquely decomposed into loops and the free energy of a structure is calculated as the sum of the energy of its individual loops [26].

A bi-structure $(S, T)$ is a pair of secondary structures $S$ and $T$ over the same backbone [10]. We also represent a bi-structure as a diagram on a horizontal backbone with the $S$-arcs drawn in the upper and the $T$-arcs drawn in the lower half plane, see Fig. 3 LHS. Recall that an $S$-arc $(i, j)$ and a $T$-arc $(k, l)$ are crossing if $i<k<j<l$ or $k<i<l<j$. Crossing induces an equivalence relation on arcs for which nontrivial equivalence classes are called crossing components.

The loop complex, $K(S, T)$, is the nerve formed by $S$-loops and $T$-loops of a bi-structure $(S, T)$. Specifically, let $\mathscr{L}$ be the collection of $S$-loops and $T$-loops in a bi-structure $(S, T)$. A subset $\left\{L_{0}, L_{1}, \ldots, L_{d}\right\} \subset \mathscr{L}$ is a $d$-simplex of $\mathscr{L}$ if the set intersection $\cap_{k=0}^{d} L_{k} \neq \varnothing$. Let $K_{d}(\mathscr{L})$ be the set of all $d$-simplices of $\mathscr{L}$. The loop complex, $K(S, T)$ is a simplicial complex given by $K(S, T)=\bigcup_{d=0}^{\infty} K_{d}(\mathscr{L}) \subseteq 2^{\mathscr{L}}$. The loop complex $X=K(S, T)$ can be augmented by assigning an integer-valued weight $\omega(\sigma)$ to each simplex $\sigma$ of $X$, where the weight $\omega$ encodes the cardinality of intersections of loops in the simplex, see Fig. 3 RHS. Clearly, the weight function satisfies $\sigma \subseteq \tau \Longrightarrow \omega(\sigma) \geq \omega(\tau)$. We call $(X, \omega)$ the weighted loop complex of a bi-structure $(S, T)$, and $H_{n}^{v}(X)$ the weighted homology for the loop complex of $(S, T)$ [1].
[1] computed the weighted homology for the loop complex of RNA bi-structures. In particular, [1] showed that the weighted simplicial complex of an arbitrary bi-structure can be transformed via Whitehead moves [23] to a complex, which does not contain any 3-simplices or 2-simplices having weight greater than 1 . Referring to such complexes as lean, the following holds:


Figure 3. LHS: A bi-structure $(S, T)$ with $S$-loops $A, B, C$ and $T$-loops $D, E$, where $A$ and $E$ are exterior loops covered by artificial rainbow arcs (dashed), and $\operatorname{arcs}(1,11)$ and $(6,17)$ form a crossing component. The loops are given by $A=\{1,11,12,13,14,15,16,17,18\}$, $B=\{1,2,3,5,6,7,8,9,10,11\}, C=\{3,4,5\}, D=\{6,7,8,9,10,11,12,13,14,15,16,17\}$ and $E=\{1,2,3,4,5,6,17,18\}$. RHS: its corresponding loop complex given by $K(S, T)=$ $\{A, B, C, D, E, A B, A D, A E, B C, B D, B E, C E, D E, A B D, A B E, A D E, B D E, B C E\}$. The weights assigned to simplices in the loop complex encode the size of intersections of loops in the simplex. While $\omega(A)=9, \omega(B)=10, \omega(C)=3, \omega(D)=12, \omega(E)=8$, the weights of 1-simplices are given by $\omega(A B)=\omega(B C)=\omega(D E)=2, \omega(A D)=7, \omega(A E)=3, \omega(B D)=6, \omega(B E)=$ $5, \omega(C E)=3$ and the weights of 2-simplices are $\omega(A B D)=\omega(A B E)=\omega(A D E)=\omega(B D E)=$ $1, \omega(B C E)=2$.

Theorem 5. [1] Let $(X, v)$ be a lean, weighted loop complex of a bi-structure $(S, T)$, where $v(\sigma)=$ $\pi^{\omega(\sigma)}$ is given by the size of the intersection of loops. Let $\hat{\mathfrak{B}}_{K}^{v}$ be $K$-basis of $H_{1}^{v}\left(X^{1}\right)$ and $M=X_{1} \backslash K=$ $\left\{\partial_{1}^{v}\left(\mu_{s}\right)\right\}$ be a basis of $\partial_{1}^{v}(X)$, given by Theorem 3. Then

$$
\begin{aligned}
& H_{2}^{v}(X) \cong R^{C} \\
& H_{1}^{v}(X) \cong \oplus_{\kappa \in K} R /\left(\pi^{\omega(\kappa)-1}\right) \\
& H_{0}^{v}(X) \cong R \oplus \bigoplus_{\mu_{\alpha(s)} \in M} R /\left(\pi^{\omega\left(v_{s}\right)-\omega\left(\mu_{\alpha(s)}\right)}\right),
\end{aligned}
$$

where $C$ denotes the number of crossing components in $(S, T), v_{s}$ is a 0 -simplex of $X$ and the pairing $\left(v_{s}, \mu_{\alpha(s)}\right)$ between 0 -simplices $v_{s}$ and 1 -simplices $\mu_{\alpha(s)} \in M$ is given by Theorem 4.

This result can be derived from our structure theorem as follows:
Proof. For simplicial homology with integer coefficients, [3] proved that the loop complex $X$ of a bi-structure satisfies $H_{2}(X)=\mathbb{Z}^{C}, H_{1}(X)=0$ and $H_{0}(X)=\mathbb{Z}$. Combing with $\operatorname{rnk}_{R}\left(H_{n}^{v}(X)\right)=$ $\operatorname{rnk}_{\mathbb{F}}\left(H_{n}(X, \mathbb{F})\right)$ by Theorem 4, we have $\operatorname{rnk} H_{2}^{v}(X)=C, \operatorname{rnk} H_{1}^{v}(X)=0$ and $\operatorname{rnk} H_{0}^{v}(X)=1$.

Since the lean complex $X$ contains no 3 -simplices, $H_{2}^{v}(X)$ is free, whence $H_{2}^{v}(X) \cong R^{C}$.
Recall that $X_{2}$ denotes the set of 2-simplices in $X$. Since $\operatorname{rnk} H_{1}^{v}(X)=0$, Theorem 4 shows there exists a bijection $p$ between the set $K$ of 1 -simplices $\kappa$ and the set $X_{2}$ of 2 -simplices $\delta$, i.e., the pairings $\left(\kappa_{i}, \delta_{p(i)}\right)$ for each $\kappa_{i} \in K$. Since each 2-simplex in a lean complex has weight one, i.e. $v(\boldsymbol{\delta})=\pi$, we have $\frac{v\left(\kappa_{i}\right)}{v\left(\delta_{p(i)}\right)}=\pi^{\omega\left(\kappa_{i}\right)-1}$. Theorem 4 establishes that $H_{1}^{v}(X) \cong \bigoplus_{\kappa \in K} R /\left(\pi^{\omega(\kappa)-1}\right)$.

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[^0]:    ${ }^{1}$ Let $M$ be a finitely generated module over a local ring $R$ with maximal ideal $m$. Then every minimal set of generators of $M$ is obtained from the lifting of some basis of $M / \mathrm{mM}$.

