A NOTE ON L-POSITIVELY LIMITED SETS IN DUAL BANACH LATTICES

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ABSTRACT. Following the concept of L-sets in dual Banach spaces, and the class of positively limited sets in Banach lattices, the notion of L-positively limited sets is introduced. The connection between L-positively limited sets, relatively weakly compact sets, and weak*-sequentially compact sets is discussed. Moreover, using positively limited completely continuous operators, some operator characterizations of Banach lattices with the L-positively limited property are obtained. In particular, some new results of the positive Gelfand-Phillips property, positive DP* property, and dual positive Schur property are investigated. Finally, the notion of weak positive Gelfand-Phillips property is defined, and using the class of almost positively limited completely continuous operators, an operator characterization of Banach lattices with order continuous norm is provided.

1. Notation and preliminaries

Throughout this paper E, F are Banach lattices, X, Y are Banach spaces, and $E^+ = \{x \in E : x \geq 0\}$ is the positive cone of E. B_X is the closed unit ball of X. The lattice operations are weakly sequentially continuous in E, if for every weakly null sequence $(x_n) \subset E$, $|x_n| \xrightarrow{w} 0$. Also, the lattice operations are weak* sequentially continuous in E^*, if for every weak*-null sequence $(x_n^*) \subset E^*$, $|x_n^*| \xrightarrow{w^*} 0$ [1, 13].

A norm bounded subset $C \subset X$ is *limited* (resp. *Dunford-Pettis)*, if every weak*-null (resp. weakly null) sequence $(x_n^*) \subset X^*$ converges uniformly to zero on C; that is, $\sup_{x\in C} |x_n^*(x)| \to 0$. If each limited set in X is relatively compact, then X has the *Gelfand-Phillips (GP)* property. Each separable Banach space has the GP property. If every Dunford-Pettis subset of X is relatively compact, then X has the *relatively compact Dunford-Pettis property* (abb. *$DP_{rc}P$*). Each reflexive space has the $DP_{rc}P$ [8, 9, 11].

If $C \subseteq X^*$ is a bounded set, and every weakly null sequence $(x_n) \subset X$ converges uniformly to zero on C, then C is called an *L-set.* By the equality $\sup_{x^* \in B_{X^*}} |x^*(x_n)| = \|x_n\|$, for every sequence (x_n) in X, it follows that B_{X^*} is an L-set if and only if every weakly null sequence in X is norm null or X has the Schur property. Using the class of L-sets Banach spaces not containing ℓ_1 are characterized, and several consequences concerning limited and Dunford-Pettis

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sets are obtained. A Banach space X has the *reciprocal Dunford-Pettis property (RDP)* if every completely continuous operator on X is weakly compact. It was proved that X has the RDP property if and only if every L-set in X^* is relatively weakly compact [10, 12].

Later, the classes of L-limited sets and limited completely continuous operators on Banach spaces were defined. If every limited weakly null sequence (x_n) in X converges uniformly to zero on $C \subseteq X^*$, then C is called an L-limited set. A Banach space X has the GP property if and only if B_{X^*} is an L-limited set. Each L-set is an L-limited set. However, for each Banach space X with the GP property, and without the Schur property such as c_0, B_{X^*} is an L-limited set, which is not an L-set. Relatively weakly compact sets are L-limited sets, and if the converse is valid, then X has the L-limited property. It was proved that X has the L-limited property if and only if each limited completely continuous operator from X into ℓ_∞ is weakly compact. Each weakly compact operator is lcc [14, 15].

Recently, the classes of positively limited sets, and positively limited completely continuous operators on Banach lattices were introduced. A bounded set $C \subset E$ is positively limited, if each positive weak*-null sequence in E^* converges uniformly to zero on C. Each limited set is positively limited, however B_{ℓ_∞} is a positively limited set which cannot be limited. If each positively limited set in E is relatively compact, then we say that E has the positive GP property. Each Banach lattice with the positive GP property has the GP property. The converse is false. Banach lattices $C[0,1]$ and c have the GP property, but they fail to have the positive GP property. E has the positive GP property if and only if each positively limited completely continuous operator from E into ℓ_∞ is weakly compact. Each limited completely continuous operator is a positively limited completely continuous operator. But the converse is false. Consider the identity operator Id_c. Also, E has the positive DP* property, if each relatively weakly compact set in E is positively limited. It is proved that E has the positive DP* property, if and only if each positively limited completely continuous operator from E into c_0 is completely continuous. It is clear that the DP* property implies the positive DP* property. The converse is not valid. Consider, $L^1[0,1]$ [6, 7].

In the following items, all the concepts mentioned above and needed in the present paper are collected.

(1) A Banach lattice E has the:
- *Schur property*, if each weakly null sequence in E is norm null [1].
- *positive Schur property*, if each positive weakly null sequence in E is norm null [16].
- *DP* property* if each relatively weakly compact set in E is limited [9].
- *dual positive Schur property* if each positive weak*-null sequence in E^* is norm null [18].
- *positive Grothendieck property*, if each positive weak*-null sequence in E^* is weakly null [18].

- *Interpolation property (I)*, if for all sequences (x_n) and (y_m) in E such that $x_n \leq y_m$, for all m, n, there is an element $u \in E$ satisfying $x_n \leq u \leq y_m$, for all m, n [13].

(2) An operator $T : E \to X$ is called
- *completely continuous or Dunford-Pettis*, if for every weakly null sequence $(x_n) \subset E$, $\|Tx_n\| \to 0$ [1].
- *almost Dunford-Pettis*, if for every disjoint weakly null sequence $(x_n) \subset E$, $\|Tx_n\| \to 0$ [2].
- *limited completely continuous (lcc)*, if for every weakly null limited sequence $(x_n) \subset E$, $\|Tx_n\| \to 0$ [15].
- *positively limited completely continuous (plcc)*, if for every weakly null positively limited sequence $(x_n) \subset E$, $\|Tx_n\| \to 0$ [7].
- *order weakly compact*, if for every order bounded disjoint sequence $(x_n) \subset E$, $\|Tx_n\| \to 0$ [13].

Motivated by the above works, and using the class of positively limited sets in Banach lattices, the present paper is organized as follows.

In section 2, the class of L-positively limited sets in dual Banach lattices is introduced. Some results of the properties GP, positive GP, DP*, positive DP*, Dunford-Pettis, and dual positive Schur in Banach lattices are investigated. The connection between L-positively limited sets, L-limited sets, and L-sets are obtained.

Section 3 is concerned with the connection between the classes of plcc operators, the weakly compact operators, and weak*-sequentially compact operators. In that section, with respect to plcc operators, some operator characterizations of Banach lattices with the L-positively limited property are discussed. In particular, a new characterization of Grothendieck Banach lattices is obtained. It is proved that a Banach lattice is Grothendieck if and only if it has the L-limited property.

In the last section, with respect to positively limited weakly null sequences with positive terms, the notion of weak positive GP property is introduced. It is proved that a Banach lattice has the weak positive GP property if and only if it has order continuous norm. Finally, using the class of almost plcc operators, and almost Dunford-Pettis operators, provide an operator characterization of Banach lattices with order continuous norm, and the positive DP* property.

2. L-positively limited sets

Let us define the class of L-positively limited sets.

Definition 2.1. A bounded subset $B \subset E^*$ is an L-positively limited set if for every weakly null and positively limited sequence (x_n) of E, $\sup_{f\in B} |f(x_n)| \to 0$.

It is easy to see that $B \subset E^*$ is an L-positively limited set if and only if for each sequence (f_n) in B, and each weakly null positively limited sequence (x_n) of E, $f_n(x_n) \to 0$.

Theorem 2.2. *If each L-positively limited set in E^* is relatively compact, then E has the dual positive Schur property.*

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Proof. From [6, Theorem 2.11 & Theorem 3.6], each positive operator $T : E \to c_0$ takes positively limited subsets of E to positively limited subsets of c_0 which are relatively compact. Hence by [7], T is plcc. Similar to [14, Theorem 2.2 (c)], we can prove that each positive weak*-null sequence in E^* is an L-positively limited set. Hence if each L-positively limited set in E^* is relatively compact, then each positive weak*-null sequence in E^* is norm null, and so E has the dual positive Schur property. □

For the converse of above theorem, note that E has the dual positive Schur property if and only if B_E is a positively limited set. Hence in the dual of a Banach lattice with the dual positive Schur property, the classes of L-positively limited sets, and L-sets of E^* coincide. Also, each L-set in E^* is relatively compact if and only if E does not contain a copy of ℓ_1 [10]. ℓ_∞, and generally, $C(K)$ spaces have the dual positive Schur property, and also containing a copy of ℓ_1. Hence there is an L-positively limited (L-set) set in their dual which is not relatively compact.

It is noteworthy that if each L- limited set in E^* is relatively compact, then each weak*-null sequence in E^* is norm null, and by the Josefson–Nissenzweig theorem [9], E must be finite dimensional.

Proposition 2.3. *For a Banach lattice E, the following are equivalent:*

(a) *for each $f \in (E^*)^+, [-f, f]$ is an L-positively limited set,*
(b) *for every weakly null and positively limited sequence (x_n) of E, $|x_n| \xrightarrow{w} 0$.*

Proof. It follows immediately from the equality $\sup_{g\in[-f,f]} |g(x_n)| = f(|x_n|)$ □

Each AM-space E has weakly sequentially continuous lattice operations [13, Proposition 2.1.11] and so each order interval in E^* is an L-positively limited set.

Theorem 2.4. *Suppose that E has weakly sequentially continuous lattice operations. Then the solid hull of an L-positively limited subset of E^* is likewise an L-positively limited set.*

Proof. Assume by a way of contradiction that $A \subset E$ is an L-positively limited set, and $Sol(A)$ is not an L-positively limited set. Then there exist a sequence (f_n) in $Sol(A)$, a weakly null and positively limited sequence (x_n) of E and an $\epsilon > 0$ such that $|f_n(x_n)| \geq \epsilon$ for all n. For each n there exists $g_n \in A$, such that $|f_n| \leq |g_n|$. Since E has the weakly sequentially continuous lattice operations, the sequence $(|x_n|)$ is weakly null and positively limited [6]. However, $\epsilon \leq |f_n(x_n)| \leq |f_n|(|x_n|) \leq |g_n|(|x_n|)$. Since $|g_n|(|x_n|) = \sup_{|y|\leq|x_n|} |g_n(y)|$, for every n there is a sequence (y_n) in E with $|y_n| \leq |x_n|$ and $|g_n(y_n)| > \epsilon$. The sequence (y_n) is weakly null and positively limited. To see this, note that solid hull of a positively limited sequence (x_n), is positively limited. Also for each $f \in E^*$, $|f(y_n)| \leq |g|(|y_n|) \leq |g|(|x_n|) \to 0$. Since A is an L-positively limited set, $g_n(y_n) \to 0$ which is impossible. Hence, $Sol(A)$ is an L-positively limited set. □

Solid hull of an L-positively limited set in E^* is not an L-positively limited set, necessarily. The Rademacher sequence (r_n) in $L^1[0, 1]$ is weakly null and positively limited (by the positive DP* property), but $|r_n| = 1$ for all n. By

Proposition 2.3, there is an element $f \in L^\infty[0,1]$ such that $Sol\{f\} = [-f, f]$ is not L-positively limited, while $\{f\}$ is L-positively limited in $L^\infty[0,1]$.

Proposition 2.5. *Let E be a Banach lattice. Then B_{E^*} is an L-positively limited set if and only if E has the positive GP property.*

Proof. From [6, Theorem 3.6], a Banach lattice E has the positive GP property if and only if each weakly null and positively limited sequence in E is norm null. Hence the desired conclusion follows from the equality $\sup_{x^* \in B_{E^*}} |x^*(x_n)| = \|x_n\|$, for every sequence (x_n) in E. □

The following example shows that the class of L-positively limited sets is generally larger than L-sets and smaller than L-limited sets. It is proved that Banach lattice E has the Schur property if and only if E has the positive GP and positive DP* properties if and only if B_{E^*} is an L-set [6, Proposition 3.11].

Example 2.6.
(a) Each non-discrete Banach lattice with order continuous norm such as $L^1[0,1]$ has the GP property, but it does not have the positive GP property. That is, $B_{L^\infty[0,1]}$ is an L-limited set while it is not an L-positively limited set, see [6, Theorem 3.6] and [17, Theorem 4.5].
(b) Each Banach lattice with the positive GP property failing the positive DP* property such as c_0 cannot have the Schur property. That is, B_{ℓ_1} is an L-positively limited set while it is not an L-set.

The set of all plcc (resp. completely continuous) operators from E into X is denoted by $\mathcal{L}_{plcc}(E,X)$ (resp. $\mathcal{L}_{cc}(E,X)$). In [7], it is proved that a Banach lattice E has the positive DP* property, if and only if $\mathcal{L}_{plcc}(E,c_0) = \mathcal{L}_{cc}(E,c_0)$. The following theorem also characterizes the positive DP* property. Note that an operator $T : E \to F$ is plcc if and only if $T^*(B_{F^*})$ is L-positively limited.

Theorem 2.7. *A Banach lattice E has the positive DP* property if and only if each L-positively limited set in E^* is an L-set.*

Proof. If E has the positive DP* property, then every weakly null sequence in E is positively limited, and so every L-positively limited set in E^* is an L-set.
Conversely, it is enough to show that $\mathcal{L}_{plcc}(E,c_0) = \mathcal{L}_{cc}(E,c_0)$. If $T : E \to c_0$ is plcc, then $T^*(B_{\ell_1})$ is an L-positively limited set, and so it is an L-set. Thus the operator $T : E \to c_0$ is completely continuous. □

It follows from the definition that a Banach lattice E has the positive DP* property if and only if each positive weak*-null sequence in E^* is an L-set.

Corollary 2.8. *For a Banach lattice E the following assertions are equivalent:*
(a) *E has the positive DP* property,*
(b) *each positive operator $T : E \to F$ is completely continuous, for each Banach lattice F with the positive GP property.*

Proof. $(a) \Rightarrow (b)$ First, note that positive operators take positively limited sets to positively limited ones [6, Theorem 2.11]. If F has the positive GP property, then each positive operator $T : E \to F$ is plcc. Then $T^*(B_{F^*})$ is an L-positively limited set, and by Theorem 2.7 it is an L-set. Therefore T is completely continuous.

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$(b) \Rightarrow (a)$ If E does not have the positive DP* property, then there is a positive weak*-null sequence (x_n^*) in E^* which is not an L-set. Hence there is a weakly null sequence (x_n) in E, and some $\epsilon > 0$ with $\epsilon < |(x_n^*(x_n))|$, for all n. The positive operator $T : E \to c_0$ defined by $T(x) = (x_n^*(x))$ for all $x \in E$ is not completely continuous and this is a contradiction. □

Note that in Corollary 2.8, the positivity of the operator T cannot be removed.

Example 2.9. Although, $L^1[0,1]$ has the positive DP* property, and c_0 has the positive GP property, but an operator $T : L^1[0,1] \to c_0$ be defined as

$$Tf = \left(\int_0^1 f(t) r_n(t) \mathrm{d}t \right) \quad \text{for all } f \in L^1[0,1],$$

where $r_n(t)$ is the nth Rademacher function on $[0,1]$ is not completely continuous. Indeed, $(r_n(t))_{n=1}^{\infty}$ is weakly null in $L^1[0,1]$, but $\|Tr_n\| = 1,\ n \in \mathbb{N}$.

Theorem 2.10. *If E^* has weak* sequentially continuous lattice operations, then each L-limited set in E^* is an L-positively limited set.*

Proof. Just notice that if E^* has the weak* sequentially continuous lattice operations, then each positively limited set in E is a limited set [6, Theorem 2.5]. This implies that, each L-limited set in E^* is an L-positively limited set. □

The converse of Theorem 2.10 is false. Consider ℓ_∞. By the DP* property, each L-limited set in ℓ_∞^* is an L-set, and so it is an L-positively limited set. However, ℓ_∞^* does not have the weak* sequentially continuous lattice operations.

The following theorem shows that the converse of Theorem 2.10 holds, in Banach lattices with order continuous norm.

Theorem 2.11. *If each L-limited set in E^* is an L-positively limited set, and also E has order continuous norm, then E^* has weak* sequentially continuous lattice operations. In particular, E has the positive GP property.*

Proof. If E has order continuous norm, then it has the GP property, and so B_{E^*} is an L-limited set. By hypothesis B_{E^*} is an L-positively limited set, and so E has the positive GP property. Thus E^* has the weak* sequentially continuous lattice operations.

As an another proof, from [6, Theorem 2.5] it is enough to show that each order interval in E is limited; or equivalently each operator $T : E \to c_0$ is AM-compact. If $T : E \to c_0$ is an operator, then T is lcc, and so $T^*(B_{\ell_1})$ is an L-limited set. By hypothesis, $T^*(B_{\ell_1})$ is an L-positively limited set, and then T is plcc. Since E has order continuous norm, by [7, Corollary 3.10] the operator $T : E \to c_0$ is AM-compact. □

In general, the class of L-positively limited sets, relatively weakly compact sets, and weak* sequentially compact set in E^* are unrelated.

Example 2.12. Let E be a Banach lattice. Then

(a) A relatively weakly compact set in E^* is not an necessarily L-positively limited and vice versa. Indeed, for each reflexive and non-discrete Banach lattice E such as Hilbert space $L^2(-\pi,\pi)$, B_{E^*} is relatively weakly compact, while is not L-positively limited. Also, for each non-reflexive Banach lattice E with the positive GP property such as c_0, B_{E^*} is L-positively limited, which is not relatively weakly compact.

(b) A weak* sequentially compact set in E^* is not an necessarily L-positively limited, and vice versa. Indeed, for each separable Banach lattice E without the positive GP property such as $L^1[0,1]$, B_{E^*} is a weak* sequentially compact set, which is not an L-positively limited set. Also, $B_{l_\infty[0,2\pi]}$ is an L-positively limited set (it is an L-set, by the Schur property of $\ell_1[0,2\pi]$), however it is not weak* sequentially compact. The sequence $g_n(t)=\sin nt$, for each $t\in[0,2\pi]$ and $n\in\mathbb{N}$, is not weak* sequentially compact in $l_\infty[0,2\pi]$. If (g_n) has a weak* convergent subsequence (g_{n_k}), then $\lim_{n_k} g_{n_k}(t)$ must be exist for each $t\in[0,2\pi]$, which is impossible.

3. PLCC OPERATORS AND L-POSITIVELY LIMITED PROPERTY

In this section, some relations between L-positively limited sets, relatively weakly compact sets, and w^*-sequentially compact sets in the dual of Banach lattices are considered. Moreover by plcc operators Banach lattices with the L-positively limited property are characterized.

A Banach lattice E has the:

- L_w*-positively limited property*, if every L-positively limited subset of E^* is relatively weakly compact.
- L^w*-positively limited property*, if every relatively weakly compact subset of E^* is L-positively limited.

Theorem 3.1. *Each Banach lattice E with the dual positive Schur property has the L_w-positively limited property.*

Proof. If E has the dual positive Schur property, then by [18, Proposition 2.1], E^* has an order continuous norm. Hence by [3, Theorem 3.1], each L-set in E^* is relatively weakly compact. On the other hand the classes of L-sets and L-positively limited sets in E^* are the same. Thus L-positively limited sets in E^* are relatively weakly compact. Therefore E has the L_w-positively limited property. □

The converse of Theorem 3.1 is not valid. Every reflexive Banach lattice has the L_w-positively limited property, and also it has order continuous norm. However, by [18, Proposition 2.1], it cannot have the dual positive Schur property.

Theorem 3.2. *For a Banach lattice E, the following are equivalent:*

(a) *E has the L_w-positively limited property,*
(b) *For each Banach space Y, $\mathcal{L}_{plcc}(E,Y)\subset\mathcal{L}_w(E,Y)$,*
(c) *$\mathcal{L}_{plcc}(E,\ell_\infty)\subset\mathcal{L}_w(E,\ell_\infty)$.*

Proof. $(a)\Rightarrow(b)$ Suppose that E has the L_w-positively limited property and $T:E\to Y$ is plcc. Thus $T^*(B_{Y^*})$ is an L-positively limited set. By hypothesis, it is relatively weakly compact, and so T is a weakly compact operator.
$(b)\Rightarrow(c)$ It is obvious.

$(c) \Rightarrow (a)$ If E does not have the L_w-positively limited property, there exists an L-positively limited set subset A of E^* that is not relatively weakly compact. So there is a sequence $(x_n^*) \subset A$ with no weakly convergent subsequence. Now we show that the operator $T : E \to \ell_\infty$ by

$$Tx = (\langle x, x_n^* \rangle) \ , \ x \in E$$

is plcc, but it is not weakly compact. As $(x_n^*) \subset A$ is an L-positively limited sequence, for every weakly null and positively limited sequence (x_m) in E, we have

$$\|Tx_m\| = \sup_n |\langle x_m, x_n^* \rangle| \to 0.$$

Hence, T is a plcc operator. However $T^*(e_n^*) = x_n^* \ , \ n \in \mathbb{N}$. Hence T^* is not a weakly compact operator and neither is T. This finishes the proof. □

Although each weakly compact operator is lcc, but the identity operator on each reflexive non-discrete Banach lattice such as $Id_{L^2(-\pi,\pi)}$ is a weakly compact operator, which is not plcc. $L^2(-\pi, \pi)$ has the L_w-positively limited property.

Corollary 3.3. *A Banach lattice with the positive GP property has the L_w-positively limited property if and only if it is reflexive.*

Proof. If a Banach lattice E has the positive GP property, then the identity operator on E is plcc and so is weakly compact, thanks to the L_w-positively limited property of E. Hence E is reflexive. □

If E is a Grothendieck Banach lattice, then E^* has an order continuous norm, and so E has the RDP [13, Theorems 3.7.10 & 5.3.13].

Theorem 3.4. *If a Banach lattice E has the L_w-positively limited property, then E^* has an order continuous norm, and so E has the RDP.*

Proof. It is evident that every L-set in E^* is an L-positively limited set. If E is a Banach lattice E with the L_w-positively limited property, then every L-set in E^* is relatively weakly compact. By [3, Theorem 3.1], E^* has an order continuous norm, and so it has the RDP [13, Theorem 3.7.10]. □

The converse of Theorem 3.4 is false. For example c_0 has the RDP, and also its dual $c_0^* = \ell_1$ has an order continuous norm. However, c_0 does not have L_w-positively limited property.

Theorem 3.5. *Each Grothendieck Banach lattice E has the L_w-positively limited property.*

Proof. Every order weakly compact operator on a Grothendieck space is weakly compact, see [13, Theorem 5.3.13]. From Theorem 3.2, it is enough to show that every plcc operator T on E is order weakly compact. Let (x_n) be a disjoint sequence of $[0, x]$, and $x \in E^+$. Then (x_n) is a weakly null, and positively limited sequence in E. Since T is plcc, $\|Tx_n\| \to 0$. This implies that, T is order weakly compact. Hence T is weakly compact, and so E has the L_w-positively limited property. □

The converse of Theorem 3.5 is not valid. From Theorem 3.1, the Banach lattices $C[0,1], c$ have the L_w-positively limited property, while they are not Grothendieck. $C[0,1], c$ are AM-spaces with a unit, without the L-limited property. From [13, Corollary 2.5.17], each AM-space with a unit, and the interpolation property (I) is a Grothendieck space, and so it has the L-limited property.

Theorem 3.6. *Suppose that E has the L_w-positively limited property. If at least one of the following cases holds, then E is Grothendieck.*

(1) *E^* has weakly sequentially continuous lattice operations.*
(2) *E has the interpolation property (I).*

Proof. (1). If E^* has weakly sequentially continuous lattice operations, then each positively limited set in E is limited, and so each L-limited set in E^* is an L-positively limited set. If E has the L_w-positively limited property, then it has the L-limited property. We show that E is Grothendieck. By the GP property of c_0, and the L-limited property of E, $\mathcal{L}_{lcc}(E, c_0) = \mathcal{L}(E, c_0) = \mathcal{L}_w(E, c_0)$ [14, 15]. Therefore [13, Theorem 5.3.10] implies that E is Grothendieck.

(2). If E has the L_w-positively limited property, then E^* has order continuous norm. On the other hand , each positive operator $T : E \to c_0$ is plcc, and by the L_w-positively limited property, it is weakly compact. Hence E has the positive Grothendieck property [18]. Hence by the the interpolation property (I), and [13, Theorem 5.3.13], E is Grothendieck. □

The following theorem provides an interesting characterization of Grothendieck Banach lattices.

Theorem 3.7. *A Banach lattice E is Grothendieck if and only if E has the L-limited property.*

Proof. Similar to Theorem 3.5, each Grothendieck Banach lattice E has the L-limited property. Just notice that each order bounded disjoint sequence in a Grothendieck Banach lattice is limited [3, Theorem 2.5]. On the other hand, in each Banach lattice E with the L-limited property, $\mathcal{L}_{lcc}(E, c_0) = \mathcal{L}(E, c_0) = \mathcal{L}_w(E, c_0)$. By [13, Theorem 5.3.10], E is Grothendieck. □

In the following, we show that the L_w-positively limited property is carried by every positively complemented sublattice. A closed subspace Y of E is positively complemented if there is an onto positive projection $P : E \to Y$. Note that ℓ_∞ has the L-positively limited property, while c_0 as its closed sublattice does not have this property.

Theorem 3.8. *If a Banach lattice E has the L_w-positively limited property, then every positively complemented sublattice F of E has the L_w-positively limited property too.*

Proof. Note that the image of each positively limited set under a positive operator is positively limited [6]. Consider a positively complemented sublattice F of E and a positive projection map $P : E \to F$. Suppose $T : F \to \ell_\infty$ is a plcc operator. Then for each weakly null and positively limited sequence (x_n) in E, (Px_n) is weakly null and positively limited in F, and so $\|TPx_n\| \to 0$. Hence

$TP : E \to \ell_\infty$ is also plcc. Also, E has the L_w-positively limited property and then by Theorem 3.2, TP is weakly compact. Hence T is weakly compact. □

Now, the results about the L^w-positively limited property will be considered.

One easily verifies that, E has the L^w-positively limited property if and only if for every weakly null and positively limited sequence $(x_n) \in E$ and each weakly null sequence (x_n^*) in E^*, $x_n^*(x_n) \to 0$.

Theorem 3.9. *Banach lattices with the Dunford-Pettis property have the L^w-positively limited property. The converse holds in a Banach lattice with the positive DP* property.*

Proof. Note that E has the Dunford-Pettis property if and only if for all weakly null sequences $(x_n) \subset E$ and (x_n^*) in E^*, $x_n^*(x_n) \to 0$. Hence each Banach lattice with the Dunford-Pettis property has the L^w-positively limited property.

If E has the L^w-positively limited property, then every weakly null and positively limited sequence $(x_n) \subset E$ is Dunford-Pettis. By the positive DP* property, every weakly null sequence $(x_n) \subset E$ is Dunford-Pettis. This implies that E the Dunford-Pettis property. □

Each reflexive space $\ell_p (1 < p < \infty)$ has the L^w-positively limited property, while cannot have the Dunford-Pettis property.

Theorem 3.10. *Each Banach lattice with the positive GP property has the L^w-positively limited property. The converse holds in Banach lattices with the DP$_{rc}$P.*

Proof. For each Banach lattice E with the positive GP property, B_{E^*} is L-positively limited set. So E has the L^w-positively limited property.

For the converse, note that E has the L^w-positively limited property if and only if for every weakly null and positively limited sequence $(x_n) \subset E$ is Dunford-Pettis. Hence in each Banach lattice with the DP$_{rc}$P, and L^w-positively limited property, each weakly null and positively limited sequence $(x_n) \subset E$ is weakly null and Dunford-Pettis, and so it is norm null [8]. Therefore E has the positive GP property. □

The converse of Theorem 3.10 is false. In fact, AM-spaces with a unit are Banach lattices with the L^w-positively limited property, and without the positive GP property.

Theorem 3.11. *For a Banach lattice E, the following are equivalent:*

(a) *E has the L^w-positively limited property,*
(b) *For each Banach space Y, $\mathcal{L}_w(E, Y) \subset \mathcal{L}_{plcc}(E, Y)$,*
(c) *$\mathcal{L}_w(E, c_0) \subset \mathcal{L}_{plcc}(E, c_0)$.*

Proof. $(a) \Rightarrow (b) \Rightarrow (c)$ Obvious.

$(c) \Rightarrow (a)$ If E does not have the L^w-positively limited property, then there is a weakly null and positively limited sequence $(x_n) \subset E$, a weakly null sequence $(x_n^*) \subset E^*$, and an $\epsilon > 0$ such that $|x_n^*(x_n)| > \epsilon$ for all n. Now the operator $T : E \to c_0$ defined by

$$Tx = (\langle x, x_n^* \rangle), \ x \in E$$

is weakly compact, but it is not plcc. The sequence $(x_n) \subset E$ is weakly null and positively limited, however $\|Tx_n\| \nrightarrow 0$, and the proof is completed. $\square$

The identity operator on each non-reflexive Banach lattice with the positive GP property such as Id_{c_0} is plcc, which is not weakly compact. c_0 has the L^w-positively limited property.

Each discrete reflexive Banach lattice E has L^w-positively limited property and L_w-positively limited property; that is, $\mathcal{L}_{plcc}(E,Y) = \mathcal{L}_w(E,Y) = \mathcal{L}(E,Y)$ for all Banach space Y.

Corollary 3.12. *For a Banach lattice E, the following are equivalent:*

(a) *E has the L^w-positively limited property and L_w-positively limited property,*
(b) *for each Banach space Y, $\mathcal{L}_{plcc}(E,Y) = \mathcal{L}_w(E,Y)$,*
(c) *$\mathcal{L}_{plcc}(E,\ell_\infty) = \mathcal{L}_w(E,\ell_\infty)$.*

In the rest of this section, the relation between L-positively limited sets, and weak* sequentially compact sets is considered. A Banach lattice E has the:

- *L_{w^*}-positively limited property*, if every L-positively limited subset of E^* is weak* sequentially compact.
- *L^{w^*}-positively limited property*, if every weak* sequentially compact subset of E^* is L-positively limited.

It follows easily from the definition that E has the L^{w^*}-positively limited property if and only if for every weakly null and positively limited sequence $(x_n) \subset E$, and each weak* null sequence (x_n^*) in E^*, $x_n^*(x_n) \to 0$. It is clear that each separable Banach lattice with the L^{w^*}-positively limited property has the positive GP property. The following theorem is proved similar to Theorem 3.9 and Theorem 3.10.

Theorem 3.13. *(a). Each Banach lattice with the DP* property has the L^{w^*}-positively limited property. The converse holds in a Banach lattice with the positive DP* property.*

(b). Each Banach lattice with the positive GP property has the L^{w^*}-positively limited property. The converse holds in a Banach lattice with the GP property

Each reflexive space $\ell_p (1 < p < \infty)$ has the L^{w^*}-positively limited property, while it cannot have the DP* property. In particular, Theorem 3.13 shows that in that each Banach lattice with the L^{w^*}-positively limited property has the DP* property if and only if it has the positive DP*.

Also, note that ℓ_∞ has the DP* property and so it has the L^{w^*}-positively limited property, however it does not have the positive GP property. In particular, in each Banach lattice with the L^{w^*}-positively limited property two properties GP, and positive GP are the same.

The following theorem gives an operator characterization of the L^{w^*}-positively limited property. We say that an operator $T : X \to Y$ is weak* sequentially compact operator (w*sc operator) if $T^*(B_{Y^*})$ is a weak* sequentially compact set in X^*. The class of w*sc operators from X to Y is denoted by $\mathcal{L}_{w^*sc}(X,Y)$.

Theorem 3.14. *For a Banach lattice E, the following are equivalent:*

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(a) *E has the L^{w^*}-positively limited property,*
(b) *For each Banach space Y, $\mathcal{L}_{w^*sc}(E,Y) \subset \mathcal{L}_{plcc}(E,Y)$,*
(c) *$\mathcal{L}_{w^*sc}(E,c_0) \subset \mathcal{L}_{plcc}(E,c_0)$.*

Proof. $(a) \Rightarrow (b)$ Suppose that E has the L^{w^*}-positively limited property, and $T : E \to Y$ is w*sc. Thus $T^*(B_{Y^*})$ is a weak* sequentially compact set in E^*. By hypothesis, $T^*(B_{Y^*})$ is L-positively limited, and so T is a plcc operator.
$(b) \Rightarrow (c)$ It is obvious.
$(c) \Rightarrow (a)$ If E does not have the L^{w^*}-positively limited property, then there exists a weakly null and positively limited sequence $(x_n) \subset E$, a weak* null sequence (x_n^*) in E^*, and an $\epsilon > 0$ such that $|x_n^*(x_n)| > \epsilon$ for all n. Now the operator $T : E \to c_0$ defined by

$$Tx = (\langle x, x_n^* \rangle)\ ,\ x \in E$$

is w*sc (since $T^*(e_n^*) = x_n^*\ ,\ n \in N$), which is not plcc. As $(x_n) \subset E$ is a weakly null and positively limited sequence, but $\|Tx_n\| \nrightarrow 0$, and the proof is completed. □

Note that the identity operator on ℓ_1 is plcc, however it is not a w*sc operator.

If E has the L^{w^*}-positively limited property, then it has the L^w-positively limited property. The converse is not true. A separable Banach lattice without the positive GP property such as $L^1[0,1]$ cannot have the L^{w^*}-positively limited property. Then $B_{L^\infty[0,1]}$ is a weak*-sequentially compact set , which is not L-positively limited. However, $L^1[0,1]$ has the Dunford-Pettis property, and so it has the L_w-positively limited property.

The following theorem which is proved similar to Theorem 3.14, characterizes Banach lattices with the L_{w^*}-positively limited property. Each separable Banach lattice has the L_{w^*}-positively limited property.

Theorem 3.15. *For a Banach lattice E, the following are equivalent:*

(a) *E has the L_{w^*}-positively limited property,*
(b) *For each Banach space Y, $\mathcal{L}_{plcc}(E,Y) \subset \mathcal{L}_{w^*sc}(E,Y)$,*
(c) *$\mathcal{L}_{plcc}(E,\ell_\infty) \subset \mathcal{L}_{w^*sc}(E,\ell_\infty)$.*

Note that each weakly compact set is E^* is weak*-sequentially compact. Hence, if E has the L_w-positively limited property, then it has the L_{w^*}-positively limited property. However, separable spaces $\ell_1, L^1[0,1]$ have the the L_{w^*}-positively limited property, which do not have the the L_w-positively limited property.

Corollary 3.16. *For a Banach lattice E, the following are equivalent:*

(a) *E has the L_{w^*}-positively limited property and L^{w^*}-positively limited property,*
(b) *For each Banach space Y, $\mathcal{L}_{plcc}(E,Y) = \mathcal{L}_{w^*sc}(E,Y)$,*
(c) *$\mathcal{L}_{plcc}(E,\ell_\infty) = \mathcal{L}_{w^*sc}(E,\ell_\infty)$.*

Separable Banach lattices E with the positive GP property such as $\ell_p(1 \leq p < \infty)$ and c_0 have two properties L_{w^*}-positively limited property and L^{w^*}-positively limited property and so $\mathcal{L}_{plcc}(E,Y) = \mathcal{L}_{w^*sc}(E,Y) = \mathcal{L}(E,Y)$ for all Banach space Y.

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4. WEAK POSITIVE GP PROPERTY

In this section the concept of weak positive GP property is considered. Also, with respect to the class of almost positively limited completely continuous operators, Banach lattices with order continuous norm are characterized.

Definition 4.1. A Banach lattice E has the *weak positive GP property* if each weakly null and positively limited sequence with the positive terms in E is norm null.

Each Banach lattice with the positive GP property has the weak positive GP property, but the converse is false. Banach lattice $L^1[0,1]$ has the weak positive GP property (since it has the positive Schur property [16]), however it does not have the positive GP property. Similar to [5, Theorem 4.4], it can be proved that E has the weak positive GP property if and only if E has order continuous norm. Hence each discrete Banach lattice with the weak positive GP property, has the positive GP property too.

Theorem 4.2. *For a Banach lattice E, these are equivalent:*

(a) *E has the weak positive GP property,*
(b) *each disjoint weakly null positively limited sequence (x_n) in E is norm null.*

Proof. $(a) \Rightarrow (b)$ For each disjoint positively limited weakly null sequence (x_n), the sequence $(|x_n|)$ is weakly null, and positively limited in E. Thus, $\|x_n\| = \||x_n|\| \to 0$.
$(b) \Rightarrow (a)$ Assume by way of contradiction that E does not have the weak positive GP property. Then there exists a (x_n) is a weakly null positively limited positive sequence in E such that $\|x_n\| \nrightarrow 0$ By [13, Corollary 2.3.5], there is a disjoint positive subsequence (x_{n_k}) of (x_n) such that $\|x_{n_k}\| \nrightarrow 0$. Note that, (x_{n_k}) is a disjoint positively limited weakly null sequence and by hypothesis it must be norm null. This is a contradiction, and it proves that E has the weak positive GP property. □

Similar to [5, Theorem 4.3], we can prove the following proposition.

Proposition 4.3. *For a σ-Dedekind complete Banach lattice E, the following assertions are equivalent:*

(a) *E has the weak positive GP property,*
(b) *each positively limited weakly null disjoint sequence (x_n) in E is norm null,*
(c) *each almost limited weakly null disjoint sequence (x_n) in E is norm null,*
(d) *each limited weakly null disjoint sequence (x_n) in E is norm null.*

Note that the σ-Dedekind completeness of E cannot be removed. Indeed, c has the GP property, and so each limited weakly null sequence in c is norm null, while c does not have the weak positive GP property.

Corollary 4.4. *Suppose that E is discrete σ-Dedekind complete, or E^* has the weak* sequentially continuous lattice operations. Then these are equivalent:*

(a) *E has the positive GP property,*
(b) *E has the weak positive GP property,*
(c) *E has the GP property.*

Proof. $(a) \Rightarrow (b) \Rightarrow (c)$ Obvious.
$(c) \Rightarrow (a)$ Suppose that E has the GP property. If E^* has the weak* sequentially continuous lattice operations, then each positively limited set in E is limited, and so E has the positive GP property.
Each σ-Dedekind complete Banach lattice with the GP property has order continuous norm [17]. Also, discrete Banach lattice with order continuous norm has the positive GP property. □

Definition 4.5. A bounded operator $T : E \to X$ is almost positively limited completely continuous (*abbr.* aplcc) if for every weakly null and positively limited sequence with the positive terms $(x_n) \subset E$, $\|Tx_n\| \to 0$. The class of all aplcc operators from E to Y is denoted by $\mathcal{L}_{aplcc}(E, X)$.

It is clear that E has the weak positive GP property if and only if the identity operator Id_E is aplcc.

Remark 4.6. Each plcc operator on a Banach lattice is aplcc. However the identity operator on each Banach lattice with the weak positive GP property, and without the positive GP property such as $L^1[0, 1]$ is aplcc, which is not plcc.

Theorem 4.7. *Every aplcc operator $T : E \to X$ is an lcc operator.*

Proof. Let (x_n) be an arbitrary order bounded disjoint sequence in E. Then (x_n) is a weakly null and positively limited sequence in E. From [18], the sequences $(|x_n|)$, (x_n^+), and (x_n^-) are weakly null, and by [6, Theorem 3.10] they are positively limited in E_+. By hypothesis, $\|Tx_n\| \leq \|Tx_n^+\| + \|Tx_n^-\| \to 0$. This implies that T is order weakly compact. Therefore by [1, Theorem 5.58], T admits a factorization through a Banach lattice F with order continuous norm. Since each Banach lattice with order continuous norm has the GP property, it follows from [15, Theorem 2.2] that T is lcc. □

The converse of Theorem 4.7 is false. Consider, Id_c, and $Id_{C[0,1]}$. In general, the identity operator on each Banach lattice with GP property, and without order continuous norm is an lcc operator which is not aplcc. If E^* has the weak* sequentially continuous lattice operations, then for each Banach space Y, each operator $T : E \to Y$ is lcc if and only if it is aplcc.

Theorem 4.8. *For a Banach lattice E the following statements are equivalent:*

(a) *E has the weak positive GP property.*
(b) $\mathcal{L}_{aplcc}(E, Y) = \mathcal{L}(E, Y)$, *for each Banach space Y,*
(c) $\mathcal{L}_{aplcc}(E, \ell_\infty) = \mathcal{L}(E, \ell_\infty)$.

Proof. It suffices to prove that $(c) \Rightarrow (a)$ Assume to the contrary that E does not have the weak positive GP property. Then there exists a weakly null and positively limited sequence (x_n) in E_+ such that $\|x_n\| = 1$ for all n. Choose a normalized sequence (x_n^*) in E^* such that $|\langle x_n, x_n^* \rangle| = 1$ for all n. The operator $T : E \to \ell_\infty$ defined by

$$Tx = (\langle x, x_n^* \rangle) \ , \ x \in E$$

is not aplcc. Since (x_n) is a weakly null and positively limited sequence in E_+, however $\|Tx_n\| \geq 1$ for all $n \in \mathbb{N}$. This leads to a contradiction. □

A bounded operator $T : E \to Y$ is almost Dunford-Pettis if and only if for every positive weakly null sequence $(x_n) \subset E$, $\|Tx_n\| \to 0$ [2]. Each almost Dunford-Pettis operator is aplcc, while the converse is false. Consider, Id_{c_0}. The class of almost Dunford-Pettis operators from E into X is denoted by $\mathcal{L}_{acc}(E, X)$.

The following theorem provides a characterization of the positive DP* property. In [6] it is proved that E has the positive DP* property if and only if for every weakly null sequence (x_n) in E_+, and every positive weak* null sequence (x_n^*) in E^*, $x_n^*(x_n) \to 0$.

Theorem 4.9. *Let E be a Banach lattice. Then the following are equivalent:*

(a) *E has the positive DP* property,*
(b) *$\mathcal{L}_{aplcc}(E, Y) = \mathcal{L}_{acc}(E, Y)$, for each Banach space Y,*
(c) *$\mathcal{L}_{aplcc}(E, c_0) = \mathcal{L}_{acc}(E, c_0)$.*

Proof. $(a) \Rightarrow (b)$ Each almost Dunford-Pettis operator is aplcc. For the converse, assume that, T is aplcc and $(x_n) \subset E$ is a positive weakly null sequence. By the positive DP* property of E, $(x_n) \subset E$ is positively limited in E_+. Hence, $\|Tx_n\| \to 0$. Hence T is almost Dunford-Pettis.
$(b) \Rightarrow (c)$ It is clear.
$(c) \Rightarrow (a)$ We show that for every positive weakly sequence $(x_n) \subset E$, and positive weak* null sequence (x_n^*) in E^*, $x_n^*(x_n) \to 0$. Consider the positive operator $T : E \to c_0$ defined by

$$Tx = (\langle x, x_n^* \rangle) \ , \ x \in E.$$

By [6], T is a plcc operator. By hypothesis (c), T is almost Dunford-Pettis. Then $\|Tx_n\| \to 0$ for all n, and so $x_n^*(x_n) \to 0$. It proves that E has the positive DP* property. □

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