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Results on extended Branciari generalized b -distance spaces, and applications to fractional integrals and nonlinear matrix equations

Received ..; revised ..; accepted ..

Abstract: We introduce notions of extended Branciari generalized b -distance space and implicit relation on it, and derive new fixed point results based on a new implicit contractive condition. Following this, we demonstrate the weak well-posed property and generalized Ulam-Hyers stability in the underlying space. We use these findings to obtain the existence of solution of a fractional integral equation, which is an equivalent form of Riesz-Caputo fractional differential equation with antiperiodic boundary values. In addition, the solution of nonlinear matrix equations is discussed. All concepts, outcomes, and applications are illustrated by appropriate examples.

Keywords: Fixed point, extended Branciari b -distance space, implicit relation, Riesz-Caputo derivative, anti-periodic boundary conditions, nonlinear matrix equation

MSC: Primary 47H10; Secondary 34A08, 26D10, 15A24, 65H05

1 Introduction

We denote the set of real numbers by \mathbb{R} , $\mathbb{R}_+ = [0, +\infty)$, the set of natural numbers is denoted by \mathbb{N} , and $\mathbb{N}^* := \mathbb{N} \cup \{0\}$.

In the process of rapid development of fixed point theory, many new spaces have emerged. The study of these new kinds of spaces was an interesting topic among the mathematical research community. A very interesting notion of b -metric space as a generalization of metric spaces was introduced by Bakhtin [3] [later used by Czerwinski in [6, 7]]. Branciari [4] developed the notion of Branciari distance space via substituting the triangle inequality with the quadrilateral inequality, while Kamran et al. [14] introduced the notion of extended b -metric space.

Recently, Abdeljawad et al. [1] have defined the notion of extended Branciari b -distance space by combining the notions of extended b -metric and Branciari distance.

Definition 1.1. [1] Let $\mathcal{X} \neq \emptyset$ be a set and $\omega : \mathcal{X}^2 \rightarrow \mathbb{R}_+ \setminus (0, 1)$. A function $d_{eb} : \mathcal{X}^2 \rightarrow \mathbb{R}_+$ is said to be an extended Branciari b -metric (d_{eb} -metric, for short) if it satisfies:

$$(ebb1) d_{eb}(x, y) = 0 \text{ if and only if } x = y,$$

$$(ebb2) d_{eb}(x, y) = d_{eb}(y, x),$$

$$(ebb3) d_{eb}(x, y) \leq \omega(x, y)[d_{eb}(x, z) + d_{eb}(z, \varrho) + d_{eb}(\varrho, y)]$$

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for all $x, y \in \mathcal{X}$, $z \neq \varrho \in \mathcal{X} \setminus \{x, y\}$. Then the pair (\mathcal{X}, d_{eb}) is an extended Branciari b -distance space (EB bDS, in short). If $\omega(x, y) = b = \text{const}$, (\mathcal{X}, d_{eb}) is called a Branciari b -distance space.

Most recently, Li and Cui [13] introduced the concept of rectangular G -metric space which generalizes the notion of rectangular metric space and G -metric space as follows:

Definition 1.2. Let $\mathcal{X} \neq \emptyset$ be a set and let $g_m : \mathcal{X}^3 \rightarrow \mathbb{R}_+$ be a function satisfying, for all $x, y, z \in \mathcal{X}$:

- (RGB1) $g_m(x, y, z) = 0$ if and only if $x = y = z$,
- (RGB2) $g_m(x, x, y) > 0$ with $y \neq x$,
- (RGB3) $g_m(x, x, y) \leq g_m(x, y, z)$ for all $z \in \mathcal{X}$ with $y \neq z$,
- (RGB4) $g_m(x, y, z) = g_m(x, z, y) = g_m(z, x, y) = \dots$ (symmetry in all three variables),
- (RGB5) there exist a real number $b \geq 1$ such that $g_m(x, y, z) \leq b[g_m(x, z, z) + g_m(z, h, h) + g_m(h, y, z)]$ for all $z \neq h \in \mathcal{X} \setminus \{x, y, z\}$.

The pair (\mathcal{X}, G) is said to be a rectangular G_b -metric space (RG $_b$ MS, in short).

By integrating the concept of RG $_b$ MS with EBbDS, we introduce the idea of extended Branciari generalized b -distance space as follows:

Definition 1.3. Let $\mathcal{X} \neq \emptyset$ be a set and $\omega : \mathcal{X}^3 \rightarrow \mathbb{R}_+ \setminus (0, 1)$. A function $g_b : \mathcal{X}^3 \rightarrow \mathbb{R}_+$ is said to be an extended Branciari generalized b -distance (g_b -metric, in brief) if it satisfies, for all $x, y, z \in \mathcal{X}$:

- (gbb1) $g_b(x, y, z) = 0$ if and only if $x = y = z$,
- (gbb2) $g_b(x, x, y) > 0$ with $y \neq x$,
- (gbb3) $g_b(x, x, y) \leq g_b(x, y, z)$ for all $z \in \mathcal{X}$ with $y \neq z$,
- (gbb4) $g_b(x, y, z) = g_b(x, z, y) = g_b(z, x, y) = \dots$ (symmetry in all three variables),
- (gbb5) $g_b(x, y, z) \leq \omega(x, y, z)[g_b(x, z, z) + g_b(z, h, h) + g_b(h, y, z)]$ for all distinct $z, h \in \mathcal{X} \setminus \{x, y, z\}$.

The triplet $(\mathcal{X}, g_b, \omega)$ is then called an extended Branciari generalized b -distance space (EBgbDS, in brief).

Remark 1.4. It is worth mentioning that if $\omega(x, y, z) = b = \text{const}$ for $b \geq 1$, then we obtain the notion of rectangular G_b -metric spaces.

Example 1. Let $\mathcal{X} = \mathbb{R}$ and define

$$g_b(x, y, z) = (|x - y| + |y - z| + |x - z|)^r, \quad r \geq 1,$$

with $\omega(x, y, z) = 5x + 5y + 5z + 3$, then it is clear that $(\mathcal{X}, g_b, \omega)$ is an EBgbDS. We observe that g_b satisfy (gbb1)–(gbb4); let us verify that g_b satisfy (gbb5). For distinct $z, h \in \mathcal{X} \setminus \{x, y, z\}$, we have

$$\begin{aligned} g_b(x, y, z) &= (|x - y| + |y - z| + |x - z|)^2 \\ &= (|x - z| + |z - y| + |y - z| + |x - z| + |z - z|)^2 \\ &= (|x - z| + |z - h| + |h - y| + |y - z| + |x - z| + |z - h| + |h - z|)^2 \\ &= (|x - z| + |x - z| + |z - h| + |z - h| + |h - y| + |y - z| + |h - z|)^2 \\ &\leq 3 \left[(|x - z| + |x - z|)^2 + (|z - h| + |z - h|)^2 \right. \\ &\quad \left. + (|h - y| + |y - z| + |h - z|)^2 \right] \\ &\leq (5x + 5y + 5z + 3) \left[(|x - z| + |x - z|)^2 + (|z - h| + |z - h|)^2 \right. \\ &\quad \left. + (|h - y| + |y - z| + |h - z|)^2 \right] \\ &= \omega(x, y, z)[g_b(x, z, z) + g_b(z, h, h) + g_b(h, y, z)]. \end{aligned}$$

Thus, $(\mathcal{X}, g_b, \omega)$ is an EBgbDS with $\omega(x, y, z) = 5x + 5y + 5z + 3$.

Definition 1.5. Let $(\mathcal{X}, g_b, \omega)$ be an EBgbDS and $\{z_n\}$ be a sequence in \mathcal{X} . Then $\{z_n\}$ is said to be

- (i) g_b -Cauchy if for every $\delta > 0$, there is an n_1 such that for every $l, m, n \geq n_1$, $g_b(z_l, z_m, z_n) < \delta$.
- (ii) g_b -convergent to z in \mathcal{X} if for every $\delta > 0$, there is an n_1 such that for every $m, n \geq n_1$, $g_b(z_m, z_n, z) < \delta$.

Definition 1.6. An EBgbDS $(\mathcal{X}, g_b, \omega)$ is said to be complete, if every Cauchy sequence in it is convergent.

Definition 1.7. An EBgbDS $(\mathcal{X}, g_b, \omega)$ is referred as symmetric, if $g_b(x, z, z) = g_b(x, x, z)$ for all $x, z \in \mathcal{X}$.

Remark 1.8. In an EBgbDS $(\mathcal{X}, g_b, \omega)$, the uniqueness of limit of every convergent sequence can be guaranteed only if g_b is continuous.

2 \mathcal{M}_b -implicit relation

Following [20], we introduce the following.

Denote by Ψ_ω the collection of all functions $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfying the following conditions:

- (i) ψ is increasing and $\psi(0) = 0$;
- (ii) For a set $\mathcal{X} \neq \emptyset$, there exists $\omega : \mathcal{X}^3 \rightarrow \mathbb{R}_+ \setminus (0, 1)$ such that $\sum_{n=1}^{\infty} \psi^n(t) \prod_{j=1}^n \omega(x_l, x_m, x_n) < \infty$, for $t > 0$, $x_i \in \mathcal{X}, \forall i \in N$, $m, n \in N$, where ψ^n denotes the n -th iterate.

It is clear that the class Ψ_ω is not empty and that $\psi(x) < x$ for $\psi \in \Psi_\omega$ and each $x \in \mathbb{R}_+$.

Example 2. Let (\mathcal{X}, e_b) be an EBgbDS, where $\mathcal{X} = [1, \infty)$ and $\omega(x, y, z) = 1 + \frac{3}{1+\ln(z+x+y)}$. Define the mapping $\psi(x) = \frac{\lambda x}{4}$, where $0 < \lambda < 1$. Note that $1 + \frac{3}{1+\ln(z+x+y)} \leq 4$. Thus

$$\psi^n(x) \prod_{j=1}^n \omega(z_i, z_m, z_n) \leq \frac{\lambda^n x}{4^n} \cdot 4^n = \lambda^n x.$$

Therefore, $\sum_{n=1}^{\infty} \psi^n(x) \prod_{j=1}^n \omega(z_i, z_m, z_n) < \infty$ and hence $\psi \in \Psi_\omega$.

We start by introducing a modified implicit relation, following [2, 19].

Let \mathcal{M} be the collection of all mappings $\mathcal{M} : \mathbb{R}_+^5 \rightarrow \mathbb{R}$ satisfying the following conditions:

- (\mathcal{M}_1) $\mathcal{M}(cx, z, x, y) \leq 0$ for all $x, z, y \geq 0$ and some $c \geq 1$, implies that there exists $\psi \in \Psi_\omega$ such that $cx \leq \psi(z)$;
- (\mathcal{M}_2) If $\mathcal{M}(cx, z, x, 0, x) \leq 0$ and $\mathcal{M}(cz, x, z, 0, z) \leq 0$, then $x = 0$ and $z = 0$.

Example 3. Let $\mathcal{M}(\varrho_1, \varrho_2, \varrho_3, \varrho_4, \varrho_5) = \varrho_1 - a \max\{\varrho_2, \varrho_3, \varrho_4\} + b\varrho_5$, $0 < b$ and $0 \leq a < 1$.

(\mathcal{M}_1) Let $x, z, y \geq 0$, $c \geq 1$ and $\mathcal{M}(cx, z, x, y) = cx - a \max\{z, z, x\} + by \leq 0$.

Now if $z < x$ then we get $cx \leq ax - by < ax$ which gives $x < x$ as $b > 0$ and $a < 1$, a contradiction.

Therefore, we get $cx < \psi(z)$ where $\psi(z) = az$ so that $\psi(z) < z$.

(\mathcal{M}_2) $\mathcal{M}(cx, z, x, 0, x) = cx - a \max\{z, x, 0\} + bx \leq 0$. Now if $z < x$, then we get $cx + bx \leq ax$ which gives $(c + b) \leq a$, a contradiction. Therefore, we get

$$(c + b)x \leq az. \quad (2.1)$$

Similarly $\mathcal{M}(cz, x, z, 0, z) = cz - a \max\{x, z, 0\} + bz \leq 0$ gives

$$(c + b)z \leq ax. \quad (2.2)$$

Combining (2.1) and (2.2), we get $x = 0$ and $z = 0$.

3 Main Results

We introduce \mathcal{M}_w -implicit type mappings on an EBgbDS.



Definition 3.1. Let $(\mathcal{X}, g_b, \omega)$ be an EBgbDS and $\mathcal{T} : \mathcal{X} \rightarrow \mathcal{X}$ be a mapping. We say that \mathcal{T} is an \mathcal{M}_w -implicit type mapping, if there exists $\mathcal{M} \in \mathfrak{M}$ such that for $x, y, z \in \mathcal{X}$,

$$\mathcal{M} \left(\begin{array}{l} \omega(x, y, z)g_b(\mathcal{T}x, \mathcal{T}y, \mathcal{T}z), g_b(x, \mathcal{T}x, \mathcal{T}y), \\ g_b(x, y, \mathcal{T}y), g_b(y, \mathcal{T}y, \mathcal{T}z), g_b(x, \mathcal{T}y, \mathcal{T}z) \end{array} \right) \leq 0. \quad (3.1)$$

For $\mathcal{T} : \mathcal{X} \rightarrow \mathcal{X}$ denote $Fix(\mathcal{T}) := \{x \in \mathcal{X} : \mathcal{T}x = x\}$.

Theorem 3.2. Let $(\mathcal{X}, g_b, \omega)$ be a complete EBgbDS with $\omega : \mathcal{X}^3 \rightarrow \mathbb{R}_+ \setminus (0, 1)$ and $\mathcal{T} : \mathcal{X} \rightarrow \mathcal{X}$ be an \mathcal{M}_w -implicit type mapping for some $\mathcal{M} \in \mathfrak{M}$. Then $Fix(\mathcal{T})$ is a singleton set, if \mathcal{T} is continuous.

Proof. Let $x_0 \in \mathcal{X}$ and define a sequence $\{x_n\} \subset \mathcal{X}$ by $x_n = \mathcal{T}x_{n-1}$ for all $n \in \mathbb{N}$. If $x_n = x_{n+1}$ for some $n \in \mathbb{N}^*$ then we obtain the conclusion. So assume that $x_n \neq x_{n-1}$ for all $n \in \mathbb{N}$. Using (3.1) with $x = x_{n-1}$, $y = x_n$ and $z = x_{n+1}$, we have

$$\mathcal{M} \left(\begin{array}{l} \omega(x_{n-1}, x_n, x_{n+1})g_b(\mathcal{T}x_{n-1}, \mathcal{T}x_n, \mathcal{T}x_{n+1}), g_b(x_{n-1}, \mathcal{T}x_{n-1}, \mathcal{T}x_n), \\ g_b(x_{n-1}, x_n, \mathcal{T}x_n), g_b(x_n, \mathcal{T}x_n, \mathcal{T}x_{n+1}), g_b(x_{n-1}, \mathcal{T}x_n, \mathcal{T}x_{n+1}) \end{array} \right) \leq 0,$$

that is,

$$\mathcal{M} \left(\begin{array}{l} \omega(x_{n-1}, x_n, x_{n+1})g_b(x_n, x_{n+1}, x_{n+2}), g_b(x_{n-1}, x_n, x_{n+1}), \\ g_b(x_{n-1}, x_n, x_{n+1}), g_b(x_n, x_{n+1}, x_{n+2}), g_b(x_{n-1}, x_{n+1}, x_{n+2}) \end{array} \right) \leq 0.$$

Following (\mathcal{M}_1) , there is a $\psi \in \Psi_\omega$ such that

$$\omega(x_{n-1}, x_n, x_{n+1})g_b(x_n, x_{n+1}, x_{n+2}) \leq \psi(g_b(x_{n-1}, x_n, x_{n+1})), \text{ for all } n \in \mathbb{N},$$

and so

$$g_b(x_n, x_{n+1}, x_{n+2}) \leq \psi(g_b(x_{n-1}, x_n, x_{n+1})).$$

With successive use of (\mathcal{M}_1) , we can have

$$g_b(x_n, x_{n+1}, x_{n+2}) \leq \psi^n(g_b(x_0, x_1, x_2)), \text{ for all } n \in \mathbb{N},$$

$$g_b(x_n, x_{n+1}, x_{n+1}) \leq \psi^n(g_b(x_0, x_1, x_2)), \text{ for all } n \in \mathbb{N}$$

and

$$g_b(x_n, x_n, x_{n+1}) \leq \psi^n(g_b(x_0, x_1, x_2)), \text{ for all } n \in \mathbb{N}.$$

Next we prove that $\{x_n\}$ is a g_b -Cauchy sequence. Take $m > n > l$; then by (gb5), we have

$$\begin{aligned} & g_b(x_n, x_m, x_l) \\ & \leq \omega(x_n, x_m, x_l)[g_b(x_n, x_{n+1}, x_{n+1}) + g_b(x_{n+1}, x_{n+2}, x_{n+2}) \\ & \quad + g_b(x_{n+2}, x_m, x_l)] \\ & \leq \omega(x_n, x_m, x_l)\psi^n g_b(x_0, x_1, x_2) + \omega(x_n, x_m, x_l)\psi^{n+1} g_b(x_0, x_1, x_2) + \\ & \quad \omega(x_n, x_m, x_l)g_b(x_{n+2}, x_m, x_l) \\ \\ & \leq \omega(x_n, x_m, x_l)\psi^n(g_b(x_0, x_1, x_2)) + \omega(x_n, x_m, x_l)\psi^{n+1}(g_b(x_0, x_1, x_2)) + \\ & \quad \omega(x_n, x_m, x_l)\omega(x_{n+2}, x_m, x_l)[g_b(x_{n+2}, x_{n+3}, x_{n+3}) + g_b(x_{n+3}, x_{n+4}, x_{n+4})] \\ & \quad g_b(x_{n+4}, x_m, x_l)] \\ & \leq \omega(x_n, x_m, x_l)\psi^n(g_b(x_0, x_1, x_2)) + \omega(x_n, x_m, x_l)\psi^{n+1}(g_b(x_0, x_1, x_2)) + \\ & \quad \omega(x_n, x_m, x_l)\omega(x_{n+2}, x_m, x_l)\psi^{n+2}(g_b(x_0, x_1, x_2)) + \omega(x_n, x_m, x_l) \\ & \quad \omega(x_{n+2}, x_m, x_l)\psi^{n+3}(g_b(x_0, x_1, x_2)) \\ & \quad + \omega(x_n, x_m, x_l)\omega(x_{n+2}, x_m, x_l)g_b(x_{n+4}, x_m, x_l) \end{aligned}$$

$$\begin{aligned}
& \vdots \\
& \leq \omega(x_n, x_m, x_l) \psi^n(g_b(x_0, x_1, x_2)) + \omega(x_n, x_m, x_l) \psi^{n+1}(g_b(x_0, x_1, x_2)) + \\
& \quad \omega(x_n, x_m, x_l) \omega(x_{n+2}, x_m, x_l) \psi^{n+2}(g_b(x_0, x_1, x_2)) + \omega(x_n, x_m, x_l) \\
& \quad \omega(x_{n+2}, x_m, x_l) \psi^{n+3}(g_b(x_0, x_1, x_2)) + \dots + \omega(x_n, x_m, x_l) \omega(x_{n+2}, x_m, x_l) \\
& \quad \omega(x_{n+4}, x_m, x_l) \dots \omega(x_{l-2}, x_l, x_l) \psi^{l-2}(g_b(x_0, x_1, x_2)) + \omega(x_n, x_m, x_l) \\
& \quad \omega(x_{n+2}, x_m, x_l) \omega(x_{n+4}, x_m, x_l) \dots \omega(x_{l-2}, x_l, x_l) \psi^{l-1}(g_b(x_0, x_1, x_2)) \\
& \leq \omega(x_n, x_m, x_l) \psi^n(g_b(x_0, x_1, x_2)) \\
& \quad + \omega(x_n, x_m, x_l) \omega(x_{n+1}, x_m, x_l) \psi^{n+1}(g_b(x_0, x_1, x_2)) \\
& \quad + \omega(x_n, x_m, x_l) \omega(x_{n+1}, x_m, x_l) \omega(x_{n+2}, x_m, x_l) \psi^{n+2}(g_b(x_0, x_1, x_2)) \\
& \quad + \omega(x_n, x_m, x_l) \omega(x_{n+1}, x_m, x_l) \omega(x_{n+2}, x_m, x_l) \omega(x_{n+3}, x_m, x_l) \\
& \quad \times \psi^{n+3}(g_b(x_0, x_1, x_2)) \\
& \quad + \dots + \omega(x_n, x_m, x_l) \omega(x_{n+1}, x_m, x_l) \omega(x_{n+2}, x_m, x_l) \omega(x_{n+3}, x_m, x_l) \\
& \quad \omega(x_{n+4}, x_m, x_l) \dots \omega(x_{l-2}, x_l, x_l) \psi^{l-2}(g_b(x_0, x_1, x_2)) + \omega(x_n, x_m, x_l) \\
& \quad \omega(x_{n+1}, x_m, x_l) \omega(x_{n+2}, x_m, x_l) \omega(x_{n+3}, x_m, x_l) \omega(x_{n+4}, x_m, x_l) \dots \\
& \quad \omega(x_{l-2}, x_l, x_l) \omega(x_{l-1}, x_l, x_l) \psi^{l-1}(g_b(x_0, x_1, x_2)) \\
& \leq \sum_{j=n}^{m-1} \psi^i(g_b(x_0, x_1, x_2)) \prod_{j=n}^i \omega(x_j, x_m, x_l) \\
& = \sum_{j=n}^{m-1} \psi^i(g_b(x_0, x_1, x_2)) \prod_{j=n}^i \omega(x_j, x_m, x_l) \\
& \quad - \sum_{j=1}^{n-1} \psi^i(g_b(x_0, x_1, x_2)) \prod_{j=n}^i \omega(x_j, x_m, x_l) \\
& \rightarrow 0 \text{ as } n, m \rightarrow \infty
\end{aligned}$$

since $\psi \in \Psi_\omega$, and hence the sequence $\{x_n\}$ is a g_b -Cauchy sequence.

Using completeness of $(\mathcal{X}, g_b, \omega)$, there exists a point $x \in \mathcal{X}$ such that $x_n \rightarrow x$ as $n, m \rightarrow \infty$, that is,

$$\lim_{n,m \rightarrow \infty} g_b(x_n, x_m, x) = 0.$$

Using (gb5), we have

$$g_b(x, \mathcal{T}x, \mathcal{T}x) \leq \omega(x, \mathcal{T}x, \mathcal{T}x) \left[\begin{array}{c} g_b(x, x_n, x_n) + g_b(x_n, x_{n+1}, x_{n+1}) \\ + g_b(x_{n+1}, \mathcal{T}x, \mathcal{T}x) \end{array} \right].$$

Since \mathcal{T} is continuous, taking $n \rightarrow \infty$, we obtain $g_b(x, \mathcal{T}x, \mathcal{T}x) = 0$, that is, $\mathcal{T}x = x$.

Finally, we claim that $Fix(\mathcal{T})$ is a singleton set. To prove it, suppose that there exists $x \neq x_0 \in Fix(\mathcal{T})$. Using (3.1)

$$\mathcal{M} \left(\begin{array}{c} \omega(x, x_0, x_0) g_b(\mathcal{T}x, \mathcal{T}x_0, \mathcal{T}x_0), g_b(x, \mathcal{T}x, \mathcal{T}x_0), \\ g_b(x, x_0, \mathcal{T}x_0), g_b(x_0, \mathcal{T}x_0, \mathcal{T}x_0), g_b(x, \mathcal{T}x_0, \mathcal{T}x_0) \end{array} \right) \leq 0,$$

i.e.

$$\mathcal{M}(\omega(x, x_0, x_0) g_b(x, x_0, x_0), g_b(x, x, x_0), g_b(x, x_0, x_0), 0, g_b(x, x_0, x_0))) \leq 0. \quad (3.2)$$

Again using \mathcal{M}_w -implicit condition of \mathcal{T} , we get

$$\mathcal{M} \left(\begin{array}{c} \omega(x_0, x, x) g_b(\mathcal{T}x_0, \mathcal{T}x, \mathcal{T}x), g_b(x_0, \mathcal{T}x_0, \mathcal{T}x), \\ g_b(x_0, x, \mathcal{T}x), g_b(x, \mathcal{T}x, \mathcal{T}x), g_b(x_0, \mathcal{T}x, \mathcal{T}x) \end{array} \right) \leq 0,$$



i.e.

$$\mathcal{M}(\omega(x_0, x, x)g_b(x_0, x, x), g_b(x_0, x_0, x), g_b(x_0, x, x), 0, g_b(x_0, x, x)) \leq 0. \quad (3.3)$$

It follows from (\mathcal{M}_2) with (3.2)–(3.3) that $g_b(x_0, x, x) = 0$ and $g_b(x, x_0, x_0) = 0$, subsequently we obtain $x = x_0$. \square

Example 4. Let $\mathcal{X} = \mathbb{R}$. We define $g_b : \mathcal{X}^3 \rightarrow \mathbb{R}$ and $\omega : \mathcal{X}^3 \rightarrow [1, \infty]$ as

$$g_b(x, y, z) = (\max\{x, y\} - z)^2$$

and

$$\omega(x, y, z) = \max\{x, y\} + z + 3.$$

It is easily observed that $(\mathcal{X}, g_b, \omega)$ is an EBgbDS.

Let the mapping $\mathcal{T} : \mathcal{X} \rightarrow \mathcal{X}$ be defined by $\mathcal{T}(x) = \frac{x}{3}$ and $\mathcal{M} : \mathbb{R}_+^5 \rightarrow \mathbb{R}$ is considered to be

$$\mathcal{M}(\varrho_1, \varrho_2, \varrho_3, \varrho_4, \varrho_5) = \varrho_1 - a \max\{\varrho_2, \varrho_3, \varrho_4\} + b\varrho_5,$$

$0 < b$ and $0 \leq a < 1$. It is evident from Example 3 that $\mathcal{M} \in \mathfrak{M}$.

Let $z, x, y \in \mathcal{X}$. Then the inequality (3.1) has the form

$$\begin{aligned} \omega(x, y, z)[g_b(\mathcal{T}x, \mathcal{T}y, \mathcal{T}z)] &\leq a \max\{g_b(x, \mathcal{T}x, \mathcal{T}y), g_b(x, y, \mathcal{T}y), g_b(y, \mathcal{T}y, \mathcal{T}z)\} \\ &\quad - bg_b(x, \mathcal{T}y, \mathcal{T}z), \end{aligned}$$

that is,

$$\begin{aligned} &(\max\{x, y\} + z + 3)(\max\{\mathcal{T}x, \mathcal{T}y\} - \mathcal{T}z)^2 \\ &\leq a \max\{(\max\{x, \mathcal{T}x\} - \mathcal{T}y)^2, (\max\{x, y\} - \mathcal{T}z)^2, (\max\{y, \mathcal{T}y\} - \mathcal{T}z)^2\} \\ &\quad - b(\max\{x, \mathcal{T}y\} - \mathcal{T}z)^2 \end{aligned}$$

i.e.

$$\begin{aligned} &(\max\{x, y\} + z + 3)(\max\{\frac{x}{3}, \frac{y}{3}\} - \frac{z}{3})^2 \\ &\leq a \max\{(\max\{x, \frac{x}{3}\} - \frac{y}{3})^2, (\max\{x, y\} - \frac{z}{3})^2, (\max\{y, \frac{y}{3}\} - \frac{z}{3})^2\} \\ &\quad - b(\max\{x, \frac{y}{3}\} - \frac{z}{3})^2. \end{aligned}$$

Case 1: Let $x > y > z$ or $z > x > y$ or $x > z > y$ then

$$(x + z + 3)(\frac{x}{3} - \frac{z}{3})^2 \leq a \max\{(x - \frac{y}{3})^2, (x - \frac{z}{3})^2, (y - \frac{z}{3})^2\} - b(x - \frac{z}{3})^2$$

i.e.

$$(x + z + 3)(\frac{x}{3} - \frac{z}{3})^2 \leq a(x - \frac{z}{3})^2 - b(x - \frac{z}{3})^2$$

or

$$(\frac{x}{3} - \frac{z}{3})^2 \leq (a - b)(x - \frac{z}{3})^2.$$

It can be easily seen that the above equality holds for suitably chosen values of a and b .

Case 2: Let $y > x > z$ or $z > y > x$ or $y > z > x$; then

$$(y + z + 3)(\frac{y}{3} - \frac{z}{3})^2 \leq a \max\{(x - \frac{y}{3})^2, (y - \frac{z}{3})^2, (y - \frac{z}{3})^2\} - b(\max\{x, \frac{y}{3}\} - \frac{z}{3})^2,$$

i.e.,

$$(x + z + 3)(\frac{y}{3} - \frac{z}{3})^2 \leq (a - b)(y - \frac{y}{3})^2$$

or

$$(\frac{y}{3} - \frac{z}{3})^2 \leq (a - b)(y - \frac{y}{3})^2.$$

It can be easily seen that the above equality holds for suitably chosen values of a and b .

All the assumptions of Theorem 3.2 are fulfilled and $\text{Fix}(\mathcal{T}) = \{0\}$.

Corollary 3.3. *Let all the conditions of Theorem 3.2 hold, except that (3.1) is replaced by any of the following*

(1)

$$\begin{aligned}\omega(x, y, z)[g_b(\mathcal{T}x, \mathcal{T}y, \mathcal{T}z)] &\leq a \max\{g_b(x, \mathcal{T}x, \mathcal{T}y), g_b(x, y, \mathcal{T}y), g_b(y, \mathcal{T}y, \mathcal{T}z)\} \\ &\quad - bg_b(x, \mathcal{T}y, \mathcal{T}z),\end{aligned}$$

or

(2)

$$\begin{aligned}\omega(x, y, z)g_b(\mathcal{T}x, \mathcal{T}y, \mathcal{T}z) &\leq ag_b(x, \mathcal{T}x, \mathcal{T}y) + bg_b(x, y, \mathcal{T}y) + c \frac{g_b(y, \mathcal{T}y, \mathcal{T}z)g_b(x, \mathcal{T}y, \mathcal{T}z)}{1 + g_b(x, \mathcal{T}y, \mathcal{T}z)}. \quad (3.4)\end{aligned}$$

Then $\text{Fix}(\mathcal{T})$ is a singleton.

4 Generalized w -Ulam-Hyers stability

We introduce a generalized ω -Ulam-Hyers stability in context of EBgbDS as an expansion of discussion from [16].

Definition 4.1. Let $(\mathcal{X}, g_b, \omega)$ be a complete EBgbDS and $\mathcal{T} : \mathcal{X} \rightarrow \mathcal{X}$ be a mapping. We say that fixed point equation (FPE)

$$x = \mathcal{T}x, \quad x \in \mathcal{X} \quad (4.1)$$

is generalized ω -UH stable (ω -UHS), if there exists an increasing function $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, continuous at 0, with $\phi(0) = 0$, such that for each $\varepsilon > 0$ and an ε -solution $y \in \mathcal{X}$, that is,

$$g_b(y, \mathcal{T}y, \mathcal{T}y) \leq \varepsilon,$$

there exists a solution $x^* \in \mathcal{X}$ of (4.1) such that

$$g_b(y, x^*, x^*) \leq \phi(\omega(y, x^*, x^*)\varepsilon). \quad (4.2)$$

If $\phi(z) = \alpha z$ for all $z \in \mathbb{R}_+$, where $\alpha > 0$, then we say that FPE (4.1) has a generalized ω -UHS in context of EBgbDS.

Theorem 4.2. *Let $(\mathcal{X}, g_b, \omega)$ be a complete EBgbDS with $\omega : \mathcal{X}^3 \rightarrow \mathbb{R}_+ \setminus (0, 1)$ and $\mathcal{T} : \mathcal{X} \rightarrow \mathcal{X}$ be a continuous mapping satisfying (3.4). Then the FPE (4.1) is generalized ω -UHS.*

Proof. According to Theorem 3.2, there is a unique solution $x^* \in \mathcal{X}$ of the FPE (4.1), i.e., $\mathcal{T}x^* = x^*$ holds, with $g_b(x^*, \mathcal{T}x^*, \mathcal{T}x^*) = 0$. Let $\epsilon > 0$ and $\rho^* \in \mathcal{X}$ be an ϵ -solution of (4.1), that is,

$$g_b(\rho^*, \mathcal{T}\rho^*, \mathcal{T}\rho^*) \leq \epsilon.$$

Since $g_b(x^*, \mathcal{T}x^*, \mathcal{T}x^*) = g_b(x^*, x^*, x^*) = 0 \leq \epsilon$, x^* and ρ^* are ϵ -solutions. As $\omega(x^*, \rho^*, \rho^*) \geq 1$, then

$$\begin{aligned}g_b(x^*, \rho^*, \rho^*) &\leq \omega(x^*, \rho^*, \rho^*)[g_b(x^*, \mathcal{T}x^*, \mathcal{T}x^*) + g_b(\mathcal{T}x^*, \mathcal{T}\rho^*, \mathcal{T}\rho^*) + g_b(\mathcal{T}\rho^*, \rho^*, \rho^*)] \\ &\leq \omega(x^*, \rho^*, \rho^*)g_b(\mathcal{T}x^*, \mathcal{T}\rho^*, \mathcal{T}\rho^*) + \epsilon\omega(x^*, \rho^*, \rho^*).\end{aligned} \quad (4.3)$$



From the contractive condition (3.4) for \mathcal{T} , we get

$$\begin{aligned} & \omega(x^*, \rho^*, \rho^*) g_b(\mathcal{T}x^*, \mathcal{T}\rho^*, \mathcal{T}\rho^*) \\ & \leq ag_b(x^*, \mathcal{T}x^*, \mathcal{T}\rho^*) + bg_b(x^*, \rho^*, \mathcal{T}\rho^*) + c\frac{g_b(\rho^*, \mathcal{T}\rho^*, \mathcal{T}\rho^*)g_b(x^*, \mathcal{T}\rho^*, \mathcal{T}\rho^*)}{1+g_b(x^*, \mathcal{T}\rho^*, \mathcal{T}\rho^*)} \\ & \leq (a+b)g_b(x^*, \rho^*, \mathcal{T}\rho^*) + c\epsilon \\ & \leq (a+b)\omega(x^*, \rho^*, \mathcal{T}\rho^*)g_b(\mathcal{T}x^*, \mathcal{T}\rho^*, \mathcal{T}\rho^*) + c\epsilon \end{aligned}$$

which gives

$$g_b(\mathcal{T}x^*, \mathcal{T}\rho^*, \mathcal{T}\rho^*) \leq \frac{c\epsilon}{\omega(x^*, \rho^*, \rho^*) - (a+b)\omega(x^*, \rho^*, \mathcal{T}\rho^*)}.$$

Therefore, from (4.3), we obtain

$$\begin{aligned} g_b(x^*, \rho^*, \rho^*) & \leq \frac{c\epsilon\omega(x^*, \rho^*, \rho^*)}{\omega(x^*, \rho^*, \rho^*) - (a+b)\omega(x^*, \rho^*, \mathcal{T}\rho^*)} + \epsilon\omega(x^*, \rho^*, \rho^*) \\ & = \epsilon\omega(x^*, \rho^*, \rho^*) \left[1 + \frac{c}{\omega(x^*, \rho^*, \rho^*) - (a+b)\omega(x^*, \rho^*, \mathcal{T}\rho^*)} \right] \end{aligned}$$

i.e.,

$$g_b(x^*, \rho^*, \rho^*) \leq \phi(\omega(x^*, \rho^*, \rho^*))\epsilon \quad \text{as } 1 + \frac{c}{\omega(x^*, \rho^*, \rho^*) - (a+b)\omega(x^*, \rho^*, \mathcal{T}\rho^*)} > 0.$$

As a result, the inequality (4.2) holds, and the FPE (4.1) is generalized ω -UHS. \square

5 Weak well-posed property

We expand the idea of weak well-posedness property (wwp) established in [5] in the context of EBgbDS in this section. One can refer to [17, 18] for more details.

Definition 5.1. Let $(\mathcal{X}, g_b, \omega)$ be a complete EBgbDS and $\mathcal{T} : \mathcal{X} \rightarrow \mathcal{X}$ be a mapping. We say that the fixed point problem (fpp) of \mathcal{T} is wwp if the following conditions hold:

1. $\text{Fix}(\mathcal{T}) = \{x^*\}$ is a singleton set in \mathcal{X} ;
2. for any sequence $\{x_p\}$ in \mathcal{X} with $\lim_{p \rightarrow \infty} g_b(x_p, \mathcal{T}(x_p), \mathcal{T}(x_p)) = 0$ and $\lim_{p,r \rightarrow \infty} g_b(\mathcal{T}(x_p), \mathcal{T}(x_r), \mathcal{T}(x_r)) = 0$, one has $\lim_{p \rightarrow \infty} g_b(x_p, x^*, x^*) = 0$.

We add the following additional condition for functions $\mathcal{M} \in \mathfrak{M}$ and refer to the corresponding set as \mathfrak{M}' in order to ensure the wwp of a mapping \mathcal{T} :

(\mathcal{M}_3) for all $x, z, y > 0$, $c \geq 1$, $\mathcal{M}(cx, y, z, 0, z) \leq 0$ implies that there exists $\psi \in \Psi_\omega$ such that $x \leq \psi(z)$.

Theorem 5.2. Let $(\mathcal{X}, g_b, \omega)$ be a complete symmetric EBgbDS with $\omega : \mathcal{X}^3 \rightarrow \mathbb{R}_+ \setminus (0, 1)$ and $\mathcal{T} : \mathcal{X} \rightarrow \mathcal{X}$ be a continuous and \mathcal{M}_w -implicit mapping for $\mathcal{M} \in \mathfrak{M}$ and $\psi \in \Psi_\omega$ such that $\lim_{n \rightarrow \infty} g_b(x_n, \mathcal{T}x_n, \mathcal{T}x_n) = 0$,

$\lim_{n,m \rightarrow \infty} g_b(\mathcal{T}x_n, \mathcal{T}x_m, \mathcal{T}x_m) = 0$ and x^* is a fixed point of \mathcal{T} . Then the fpp of \mathcal{T} is wwp, provided $\mathcal{M} \in \mathfrak{G}'$ is continuous.

Proof. Let $\{x_n\}$ be a sequence in \mathcal{X} such that $\lim_{n \rightarrow \infty} g_b(x_n, \mathcal{T}(x_n), \mathcal{T}(x_n)) = 0$ and $\lim_{n,m \rightarrow \infty} g_b(\mathcal{T}x_n, \mathcal{T}x_m, \mathcal{T}x_m) = 0$, for $m > n$. We obtain from (gbb2) that

$$g_b(x_n, x^*, x^*) \leq \omega(x_n, x^*, x^*) \left[\begin{array}{l} g_b(x_n, \mathcal{T}x_m, \mathcal{T}x_m) \\ + g_b(\mathcal{T}x_m, \mathcal{T}x_n, \mathcal{T}x_n) \\ + g_b(\mathcal{T}x_n, x^*, x^*) \end{array} \right].$$

Taking the limit as $n, m \rightarrow \infty$, we get

$$\lim_{n \rightarrow \infty} g_b(x_n, x^*, x^*) \leq \lim_{n \rightarrow \infty} \omega(x_n, x^*, x^*)[g_b(x_n, \mathcal{T}x_m, \mathcal{T}x_m) + g_b(\mathcal{T}x_n, x^*, x^*)]. \quad (5.1)$$

We may assume that there exists a subsequence $\{\mathcal{T}x_{n_k}\}$ of $\{\mathcal{T}x_n\}$ containing different entries. Otherwise, there exists $x_0 \in \mathcal{X}$ and $n_1 \in \mathbb{N}$ such that $\mathcal{T}x_n = x_0$ for $n \geq n_1$. Since $\lim_{n \rightarrow \infty} g_b(x_n, \mathcal{T}x_n, \mathcal{T}x_n) = 0$, we get

$\lim_{n \rightarrow \infty} g_b(x_n, x_0, x_0) = 0$. If $x_0 \neq x^*$, then $x_0 \neq \mathcal{T}x_0$ due to uniqueness of the fixed point of \mathcal{T} . For $n \geq n_1$, we obtain $x_0 = \mathcal{T}x_n \neq \mathcal{T}x_0$.

Since \mathcal{T} is an \mathcal{M}_w -implicit mapping, we get

$$\mathcal{M} \left(\begin{array}{l} \omega(x_n, x_0, x_0)g_b(\mathcal{T}x_n, \mathcal{T}x_0, \mathcal{T}x_0), g_b(x_n, \mathcal{T}x_n, \mathcal{T}x_0), \\ g_b(x_n, x_0, \mathcal{T}x_0), g_b(x_0, \mathcal{T}x_0, \mathcal{T}x_0), g_b(x_n, \mathcal{T}x_0, \mathcal{T}x_0) \end{array} \right) \leq 0,$$

i.e.,

$$\mathcal{M} \left(\begin{array}{l} \omega(x_n, x_0, x_0)g_b(x_0, \mathcal{T}x_0, \mathcal{T}x_0), g_b(x_n, x_0, \mathcal{T}x_0), \\ g_b(x_n, x_0, \mathcal{T}x_0), g_b(x_0, \mathcal{T}x_0, \mathcal{T}x_0), g_b(x_n, \mathcal{T}x_0, \mathcal{T}x_0) \end{array} \right) \leq 0.$$

(\mathcal{M}_1) implies that there exists $\psi \in \Psi_\omega$ such that

$$\omega(x_n, x_0, x_0)g_b(x_0, \mathcal{T}x_0, \mathcal{T}x_0) \leq \psi(g_b(x_n, x_0, \mathcal{T}x_0)) \text{ for all } n \in \mathbb{N}.$$

So we get

$$\begin{aligned} g_b(x_0, \mathcal{T}x_0, \mathcal{T}x_0) &\leq \omega(x_n, x_0, x_0)g_b(x_0, \mathcal{T}x_0, \mathcal{T}x_0) \\ &\leq \psi(g_b(x_n, x_0, \mathcal{T}x_0)) < g_b(x_n, x_0, \mathcal{T}x_0). \end{aligned}$$

This yields $g_b(x_0, \mathcal{T}x_0, \mathcal{T}x_0) < g_b(x_0, x_0, \mathcal{T}x_0)$ when $n \rightarrow \infty$. Due to symmetric property of EBgbDS, $g_b(x_0, \mathcal{T}x_0, \mathcal{T}x_0) < g_b(x_0, x_0, \mathcal{T}x_0)$ is a contradiction. Hence, there exist $m, q, n > n_0$ ($m > q > n$) such that $\mathcal{T}x_m \neq \mathcal{T}x_q \neq \mathcal{T}x_n \neq x_n$. Then

$$\begin{aligned} g_b(x_n, \mathcal{T}x_m, \mathcal{T}x_m) &\leq \omega(x_n, \mathcal{T}x_m, \mathcal{T}x_m) \left[\begin{array}{l} g_b(x_n, \mathcal{T}x_n, \mathcal{T}x_n) \\ +g_b(\mathcal{T}x_n, \mathcal{T}x_q, \mathcal{T}x_q)+ \\ g_b(\mathcal{T}x_q, \mathcal{T}x_m, \mathcal{T}x_m) \end{array} \right] \\ &\rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

Therefore, from (5.1),

$$\lim_{n \rightarrow \infty} g_b(x_n, x^*, x^*) \leq \lim_{n \rightarrow \infty} \omega(x_n, x^*, x^*)g_b(\mathcal{T}x_n, x^*, x^*). \quad (5.2)$$

Again using \mathcal{M}_w -implicit condition of \mathcal{T} , we get

$$\mathcal{M} \left(\begin{array}{l} \omega(x_n, x^*, x^*)g_b(\mathcal{T}x_n, \mathcal{T}x^*, \mathcal{T}x^*), g_b(x_n, \mathcal{T}x_n, \mathcal{T}x^*), \\ g_b(x_n, x^*, \mathcal{T}x^*), g_b(x^*, \mathcal{T}x^*, \mathcal{T}x^*), g_b(x_n, \mathcal{T}x^*, \mathcal{T}x^*) \end{array} \right) \leq 0,$$

i.e.,

$$\mathcal{M} \left(\begin{array}{l} \omega(x_n, x^*, x^*)g_b(\mathcal{T}x_n, x^*, x^*), \\ g_b(x_n, \mathcal{T}x_n, x^*), g_b(x_n, x^*, x^*), 0, g_b(x_n, x^*, x^*) \end{array} \right) \leq 0.$$

Letting $n \rightarrow \infty$, by the continuity of \mathcal{M} ,

$$\mathcal{M} \left(\begin{array}{l} \lim_{n \rightarrow \infty} \omega(x_n, x^*, x^*)g_b(\mathcal{T}x_n, x^*, x^*), \lim_{n \rightarrow \infty} g_b(x_n, \mathcal{T}x_n, x^*), \\ \lim_{n \rightarrow \infty} g_b(x_n, x^*, x^*), 0, \lim_{n \rightarrow \infty} g_b(x_n, x^*, x^*) \end{array} \right) \leq 0.$$

(\mathcal{M}_3) implies that there exists $\psi \in \Psi_\omega$ such that

$$\lim_{n \rightarrow \infty} \omega(x_n, x^*, x^*)g_b(\mathcal{T}x_n, x^*, x^*) \leq \psi(\lim_{n \rightarrow \infty} g_b(x_n, x^*, x^*)).$$

Therefore, from (5.2)

$$\begin{aligned} \lim_{n \rightarrow \infty} g_b(x_n, x^*, x^*) &\leq \lim_{n \rightarrow \infty} \omega(x_n, x^*, x^*)g_b(\mathcal{T}x_n, x^*, x^*) \\ &\leq \psi(\lim_{n \rightarrow \infty} g_b(x_n, x^*, x^*)) < \lim_{n \rightarrow \infty} g_b(x_n, x^*, x^*), \end{aligned}$$

a contradiction. Therefore, $\lim_{n \rightarrow \infty} g_b(x_n, x^*, x^*) = 0$. \square



6 Application to fractional integral equations

In this part of the article, we investigate the existence of solutions of fractional integral equations, which are derived from fractional differential equations. One can read the paper [8] for different type of work on fractional integral equation. In particular, we consider a new anti-periodic boundary value problem (APBVP) for the Riesz-Caputo fractional differential equations of the form

$${}_{0}^{RC}\mathcal{D}_{\ell}^{\zeta}x(\kappa) = \Phi(\kappa, x(\kappa)), \quad \zeta \in (1, 2], \quad 0 \leq \kappa \leq \ell \quad (6.1)$$

$$\alpha x(0) + \beta x(\ell) = 0, \quad \gamma x'(0) + \delta x'(\ell) = 0, \quad (6.2)$$

where ${}_{0}^{RC}\mathcal{D}_{\ell}^{\zeta}$ is the Riesz-Caputo derivative, $\Phi : [0, \ell] \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function, and $\alpha, \beta, \gamma, \delta$ are nonnegative constants with $\alpha > \beta, \gamma > \delta$.

In order to do this, we derive a Banach fixed point result in EBgbDS.

Theorem 6.1. Let $(\mathcal{X}, g_b, \omega)$ be a complete EBgbDS with $\omega : \mathcal{X}^3 \rightarrow \mathbb{R}_+ \setminus (0, 1)$ and $\mathcal{T} : \mathcal{X} \rightarrow \mathcal{X}$ be a ω -contraction mapping, that is, there exists $k \in (0, 1)$ such that

$$\omega(x, y, z)g_b(\mathcal{T}x, \mathcal{T}y, \mathcal{T}z) \leq kg_b(x, y, z). \quad (6.3)$$

Then $\text{Fix}(\mathcal{T})$ is a singleton set.

Proof. The proof can be made on the lines of proof of Theorem 3.2. □

Definition 6.2. [12] Let $\zeta > 0$. The left and right Riemann-Liouville fractional integral of a function $x \in C[0, \ell]$ of order ζ are defined as follows:

$$\begin{aligned} {}_0\mathcal{I}_{\ell}^{\zeta}x(\kappa) &= \frac{1}{\Gamma(\zeta)} \int_0^{\kappa} (\kappa - y)^{\zeta-1} x(y) dy, \quad \kappa \in [0, \ell]. \\ {}_{\ell}\mathcal{I}_{\kappa}^{\zeta}x(\kappa) &= \frac{1}{\Gamma(\zeta)} \int_{\kappa}^{\ell} (y - \kappa)^{\zeta-1} x(y) dy, \quad \kappa \in [0, \ell]. \end{aligned}$$

Definition 6.3. Let $\zeta > 0$. The Riesz fractional integral of a function $x \in C[0, \ell]$ of ζ is defined as:

$${}_0\mathcal{I}_{\ell}^{\zeta}x(\kappa) = \frac{1}{2} \left({}_0\mathcal{I}_{\ell}^{\zeta}x(\kappa) + {}_{\ell}\mathcal{I}_{\kappa}^{\zeta}x(\kappa) \right). \quad (6.4)$$

Definition 6.4. [12] Let $\zeta \in (m, m+1]$, $m \in \mathbb{N}$. The left and right Caputo fractional derivative of a function $x \in C^{m+1}[0, \ell]$ of order ζ are defined as:

$$\begin{aligned} {}_0^C\mathcal{D}_{\kappa}^{\zeta}x(\kappa) &= \frac{1}{\Gamma(m+1-\zeta)} \int_0^{\kappa} (\kappa - s)^{m-\zeta} x^{(m+1)}(s) ds = \left({}_0\mathcal{I}_{\kappa}^{m+1-\zeta} \mathcal{D}^{m+1} \right) x(\kappa) \\ {}_{\kappa}^C\mathcal{D}_{\ell}^{\zeta}x(\kappa) &= \frac{(-1)^{m+1}}{\Gamma(m+1-\zeta)} \int_{\kappa}^{\ell} (s - \kappa)^{m-\zeta} x^{(m+1)}(s) ds = (-1)^{m+1} \left({}_{\ell}\mathcal{I}_{\kappa}^{m+1-\zeta} \mathcal{D}^{m+1} \right) x(\kappa) \end{aligned}$$

where \mathcal{D} is the ordinary differential operator.

Definition 6.5. Let $\zeta \in (m, m+1]$, $m \in \mathbb{N}$. The Riesz-Caputo fractional derivative ${}^{RC}_0\mathcal{D}^\zeta x$ of order ζ of a function $x \in C^{m+1}[0, \ell]$ is defined by

$$\begin{aligned} {}^{RC}_0\mathcal{D}_\ell^\zeta x(\kappa) &= \frac{1}{\Gamma(m+1-\zeta)} \int_0^\ell |\kappa-s|^{m-\zeta} x^{(m+1)}(s) ds \\ &= \frac{1}{2} \left({}^C_0\mathcal{D}_\kappa^\zeta x(\kappa) + (-1)^{m+1} {}_\kappa\mathcal{D}_\ell^\zeta x(\kappa) \right) \\ &= \frac{1}{2} \left(({}^C\mathcal{I}_0^{m+1-\zeta} \mathcal{D}^{m+1}) x(\kappa) + (-1)^{m+1} ({}_\ell\mathcal{I}^{m+1-\zeta} \mathcal{D}^{m+1}) x(\kappa) \right). \end{aligned}$$

Lemma 6.6. [12] Let $x \in C^m[0, \ell]$ and $\zeta \in (m, m+1]$. Then we have the following relations

$$\begin{aligned} {}_0\mathcal{I}_\ell^{\zeta RC} \mathcal{D}_\kappa^\zeta x(\kappa) &= x(\kappa) - \sum_{j=0}^{m-1} \frac{x^{(j)}(\alpha)}{j!} (\kappa-\alpha)^j, \\ {}_\ell\mathcal{I}_\kappa^{\zeta RC} \mathcal{D}_\ell^\zeta x(\kappa) &= x(\kappa) - \sum_{j=0}^{m-1} \frac{(-1)^j x^{(j)}(\alpha)}{j!} (\alpha-\kappa)^j. \end{aligned}$$

When $\zeta \in (1, 2]$ and $x(\kappa) \in C^2(0, \ell)$, we have

$${}_0\mathcal{I}_\ell^{\zeta RC} \mathcal{D}_\ell^\zeta x(\kappa) = x(\kappa) - \frac{1}{2}[x(0) + x(\ell)] - \frac{1}{2}x'(0)\kappa + \frac{1}{2}x'(\ell)(\ell-\kappa). \quad (6.5)$$

Lemma 6.7. Suppose that $\hbar \in \mathcal{X} := C([0, \ell], \mathbb{R})$ and $x \in C^2([0, \ell])$. Then the fractional APBVP of order $(1, 2]$

$${}^{RC}_0\mathcal{D}_\ell^\zeta x(\kappa) = \hbar(\kappa), \quad \zeta \in (1, 2], \quad 0 \leq \kappa \leq \ell \quad (6.6)$$

$$\alpha x(0) + \beta x(\ell) = 0, \quad \gamma x'(0) + \delta x'(\ell) = 0, \quad (6.7)$$

is equivalent to integral equation of the form

$$\begin{aligned} x(\kappa) &= \frac{(-\delta + \gamma\alpha + \gamma\beta)\ell + (\delta - \gamma)(\alpha + \beta)\kappa}{(\alpha + \beta)(\gamma + \delta)\Gamma(\zeta - 1)} \int_0^\ell (\ell-y)^{\zeta-2} \hbar(y) dy \\ &\quad - \frac{(\alpha - \beta)}{(\alpha + \beta)\Gamma(\zeta)} \int_0^\ell (\ell-y)^{\zeta-1} \hbar(y) dy \\ &\quad + \frac{1}{\Gamma(\zeta)} \int_0^\kappa (\kappa-y)^{\zeta-1} \hbar(y) dy + \frac{1}{\Gamma(\zeta)} \int_\kappa^\ell (y-\kappa)^{\zeta-1} \hbar(y) dy. \end{aligned} \quad (6.8)$$

Proof. From (6.1) and (6.5), we conclude that

$$\begin{aligned} x(\kappa) &= \frac{1}{2}[x(0) + x(\ell)] + \frac{1}{2}x'(0)\kappa + \frac{1}{2}x'(\ell)(\ell-\kappa) + {}_0\mathcal{I}_\ell^\zeta \hbar(\kappa) \\ &= \frac{1}{2}[x(0) + x(\ell)] + \frac{1}{2}x'(0)\kappa + \frac{1}{2}x'(\ell)(\ell-\kappa) \\ &\quad + \frac{1}{\Gamma(\zeta)} \int_0^\kappa (\kappa-y)^{\zeta-1} \hbar(y) dy + \frac{1}{\Gamma(\zeta)} \int_\kappa^\ell (y-\kappa)^{\zeta-1} \hbar(y) dy. \end{aligned} \quad (6.9)$$

Then

$$\begin{aligned} x'(\kappa) &= \frac{1}{2}[x'(0) + x'(\ell)] + \frac{1}{\Gamma(\zeta-1)} \int_0^\kappa (\kappa-y)^{\zeta-2} \hbar(y) dy \\ &\quad + \frac{1}{\Gamma(\zeta-1)} \int_\kappa^\ell (y-\kappa)^{\zeta-2} \hbar(y) dy. \end{aligned} \quad (6.10)$$

Applying APBVP (6.2), we have

$$\begin{aligned}
x(0) &= \frac{-2\beta\delta\ell}{(\alpha+\beta)(\gamma+\delta)\Gamma(\zeta-1)} \int_0^\ell (\ell-y)^{\zeta-2} \bar{h}(y) dy \\
&\quad + \frac{2\beta}{(\alpha+\beta)\Gamma(\zeta)} \int_0^\ell (\ell-\zeta)^{\zeta-1} \bar{h}(y) dy, \\
x(\ell) &= \frac{-2\alpha\delta\ell}{(\alpha+\beta)(\gamma+\delta)\Gamma(\zeta-1)} \int_0^\ell (\ell-y)^{\zeta-2} \bar{h}(y) dy \\
&\quad + \frac{2\alpha}{(\alpha+\beta)\Gamma(\zeta)} \int_0^\ell (\ell-\zeta)^{\zeta-1} \bar{h}(y) dy, \\
x'(0) &= \frac{2\delta}{(\gamma+\delta)\Gamma(\zeta-1)} \int_0^\ell (\ell-y)^{\zeta-2} \bar{h}(y) dy, \\
x'(\ell) &= \frac{-2\gamma}{(\gamma+\delta)\Gamma(\zeta-1)} \int_0^\ell (\ell-y)^{\zeta-2} \bar{h}(y) dy.
\end{aligned}$$

When we plug in the quantities that we obtained from $x(0)$ to $x'(\ell)$ into (6.9), we get (6.8). □

Now, we have the potential to consider the appropriate conditions for the solution of our APBVP to exist and be unique. Let $C[0, \ell]$ be the space of continuous functions x defined on $[0, \ell]$ with norm $\|x\| = \sup_{y \in [0, \ell]} |x(y)|$.

Theorem 6.8. *Let $\Phi : [0, \ell] \times \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function. Assume that there exist nonnegative real numbers $\lambda, r \geq 1$ such that for all $(y, \eta), (y, \eta') \in \mathbb{R}^2$, such that*

(A1)

$$|\Phi(y, \eta) - \Phi(y, \eta')| \leq \lambda \frac{|\eta - \eta'|}{(5|\eta| + 5|\eta'| + 3)^{1/r}}, \forall y \in [0, \ell], \eta, \eta' \in \mathbb{R}.$$

(A2)

$$\left[\left(\frac{(-\delta + \gamma\alpha + \gamma\beta)\ell^\zeta}{(\alpha+\beta)(\gamma+\delta)\Gamma(\zeta)} + \frac{(\alpha-\beta)\ell^\zeta}{(\alpha+\beta)\Gamma(\zeta+1)} + \frac{2\ell^\zeta}{\Gamma(\zeta+1)} \right) \lambda \right]^r < 1.$$

Then the problem (6.1) has a unique solution on $[0, \ell]$.

Proof. We convert the fractional AVBVP (6.1)-(6.2) into integral equation using the operator $\mathcal{T} : \mathcal{X} \rightarrow \mathcal{X}$ of the form

$$\begin{aligned}
\mathcal{T}x(\kappa) &= \frac{(-\delta + \gamma\alpha + \gamma\beta)\ell + (\delta - \gamma)(\alpha + \beta)\kappa}{(\alpha+\beta)(\gamma+\delta)\Gamma(\zeta-1)} \int_0^\ell (\ell-y)^{\zeta-2} \Phi(y, x(y)) dy \\
&\quad - \frac{(\alpha-\beta)}{(\alpha+\beta)\Gamma(\zeta)} \int_0^\ell (\ell-y)^{\zeta-1} \Phi(y, x(y)) dy \\
&\quad + \frac{1}{\Gamma(\zeta)} \int_0^\kappa (\kappa-y)^{\zeta-1} \Phi(y, x(y)) dy + \frac{1}{\Gamma(\zeta)} \int_\kappa^\ell (y-\kappa)^{\zeta-1} \Phi(y, x(y)) dy. \tag{6.11}
\end{aligned}$$

Due to continuity of Φ on \mathcal{X} , \mathcal{T} is continuous.

For $x, \hat{x} \in \mathcal{X}$ and for each $\kappa \in [0, \ell]$, we have

$$\begin{aligned}
& |\mathcal{T}x(\kappa) - \mathcal{T}\hat{x}(\kappa)| \\
& \leq \frac{(-\delta + \gamma\alpha + \gamma\beta)\ell + (\delta - \gamma)(\alpha + \beta)\kappa}{(\alpha + \beta)(\gamma + \delta)\Gamma(\zeta - 1)} \\
& \quad \times \int_0^\ell (\ell - y)^{\zeta-2} |\Phi(y, x(y)) - \Phi(y, \hat{x}(y))| dy \\
& \quad + \frac{(\alpha - \beta)}{(\alpha + \beta)\Gamma(\zeta)} \int_0^\ell (\ell - y)^{\zeta-1} |\Phi(y, x(y)) - \Phi(y, \hat{x}(y))| dy \\
& \quad + \frac{1}{\Gamma(\zeta)} \int_0^\kappa (\kappa - y)^{\zeta-1} |\Phi(y, x(y)) - \Phi(y, \hat{x}(y))| dy \\
& \quad + \frac{1}{\Gamma(\zeta)} \int_\kappa^\ell (y - \kappa)^{\zeta-1} |\Phi(y, x(y)) - \Phi(y, \hat{x}(y))| dy \\
& \leq \frac{(-\delta + \gamma\alpha + \gamma\beta)\ell^\zeta}{(\alpha + \beta)(\gamma + \delta)\Gamma(\zeta)} \left[\frac{\lambda \|x - \hat{x}\|}{(5|x| + 5|\hat{x}| + 3)^{1/r}} \right] \\
& \quad + \frac{(\alpha - \beta)\ell^\zeta}{(\alpha + \beta)\Gamma(\zeta + 1)} \left[\frac{\lambda \|x - \hat{x}\|}{(5|x| + 5|\hat{x}| + 3)^{1/r}} \right] \\
& \quad + \frac{2\ell^\zeta}{\Gamma(\zeta + 1)} \left[\frac{\lambda \|x - \hat{x}\|}{(5|x| + 5|\hat{x}| + 3)^{1/r}} \right] \\
& \leq \left[\frac{(-\delta + \gamma\alpha + \gamma\beta)\ell^\zeta}{(\alpha + \beta)(\gamma + \delta)\Gamma(\zeta)} + \frac{(\alpha - \beta)\ell^\zeta}{(\alpha + \beta)\Gamma(\zeta + 1)} + \frac{2\ell^\zeta}{\Gamma(\zeta + 1)} \right] \\
& \quad \times \left[\frac{\lambda \|x - \hat{x}\|}{(5|x| + 5|\hat{x}| + 3)^{1/r}} \right].
\end{aligned}$$

Set $\xi := \frac{(-\delta + \gamma\alpha + \gamma\beta)\ell^\zeta}{(\alpha + \beta)(\gamma + \delta)\Gamma(\zeta)} + \frac{(\alpha - \beta)\ell^\zeta}{(\alpha + \beta)\Gamma(\zeta + 1)} + \frac{2\ell^\zeta}{\Gamma(\zeta + 1)}$, then

$$|\mathcal{T}x(\kappa) - \mathcal{T}\hat{x}(\kappa)| \leq \xi \left[\frac{\lambda \|x - \hat{x}\|}{(5|x| + 5|\hat{x}| + 3)^{1/r}} \right].$$

Similarly

$$|\mathcal{T}z(\kappa) - \mathcal{T}\hat{z}(\kappa)| \leq \xi \left[\frac{\lambda \|z - \hat{z}\|}{(5|z| + 5|\hat{z}| + 3)^{1/r}} \right],$$

and

$$|\mathcal{T}\vartheta(\kappa) - \mathcal{T}\hat{\vartheta}(\kappa)| \leq \xi \left[\frac{\lambda \|\vartheta - \hat{\vartheta}\|}{(5|\vartheta| + 5|\hat{\vartheta}| + 3)^{1/r}} \right].$$

Define EBgbDS (\mathcal{X}, g_b, w) on \mathcal{X}^3 as

$$g_b(x, y, z) = (|x - y| + |y - z| + |x - z|)^r, \quad r \geq 1$$

with $\omega(x, y, z) = 5x + 5y + 5z + 3$. Then

$$\begin{aligned}
g_b(\mathcal{T}x, \mathcal{T}y, \mathcal{T}z) &= (|\mathcal{T}x - \mathcal{T}y| + |\mathcal{T}y - \mathcal{T}z| + |\mathcal{T}x - \mathcal{T}z|)^r \\
&\leq \left(\left[\frac{\xi \lambda \|x - y\|}{(5|x| + 5|y| + 3)^{1/r}} \right] + \left[\frac{\xi \lambda \|y - z\|}{(5|y| + 5|z| + 3)^{1/r}} \right] + \left[\frac{\xi \lambda \|x - z\|}{(5|z| + 5|x| + 3)^{1/r}} \right] \right)^r.
\end{aligned}$$



Consequently

$$\omega(x, y, z)g_b(\mathcal{T}x, \mathcal{T}y, \mathcal{T}z) \leq k(\|x - y\| + \|y - z\| + \|x - z\|)^r, \quad k = (\xi\lambda)^r < 1.$$

Hence, following Theorem 6.1, \mathcal{T} has a unique fixed point, which is a unique solution to the equations (6.1)-(6.2). \square

Remark 6.9. Lemma 6.7 and Theorem 6.8 generalize the work that is discussed in [15]. Also we have used the generalized Banach fixed point theorem 6.1 in EBgDS to prove these results.

Example 5. Consider the fractional APBVP

$$\begin{aligned} {}_0^{RC}\mathcal{D}_1^{\frac{3}{2}}x(\kappa) &= \frac{|x(\kappa)|}{(9+e^\kappa)(3+2|x(\kappa)|)}, \quad 0 \leq \kappa \leq 1, \\ 2x(0) + x(1) &= 0, \quad 4x'(0) + 3x'(1) = 0. \end{aligned} \tag{6.12}$$

Here, $\Phi(\kappa, x(\kappa)) = \frac{|x(\kappa)|}{(9+e^\kappa)(3+2|x(\kappa)|)}$, $\ell = 1$, $\zeta = \frac{3}{2}$, and $\alpha = 2, \beta = 1, \gamma = 4, \delta = 3$. For $r = 2$, we have

$$\begin{aligned} |\Phi(\kappa, x(\kappa)) - \Phi(\kappa, z(\kappa))| &\leq \left| \frac{|x(\kappa)|}{(9+e^\kappa)(3+2|x(\kappa)|)} - \frac{|z(\kappa)|}{(9+e^\kappa)(3+2|z(\kappa)|)} \right| \\ &\leq \frac{3}{(9+e^\kappa)} \left| \frac{|x| - |z|}{6|x| + 6|z| + 4|x||z| + 9} \right| \\ &\leq \frac{3}{10} \frac{\|x - z\|}{(5|x| + 5|z| + 3)^{1/2}}, \end{aligned}$$

thus the assumption (A1) holds true with $\lambda = \frac{3}{10}$. In addition $(\xi\lambda)^2 \approx 0.63358342 < 1$. Therefore, the FBVP (6.12) has a solution by Theorem 6.8.

Example 6. Consider the fractional APBVP

$$\begin{aligned} {}_0^{RC}\mathcal{D}_1^{\frac{5}{3}}x(\kappa) &= \frac{e^{-\kappa}}{(2+e^{2\pi})} \sin\left(\frac{|x(\kappa)|}{5+|x(\kappa)|}\right), \quad 0 \leq \kappa \leq 1, \\ \frac{2}{3}x(0) + \frac{1}{3}x(1) &= 0, \quad \frac{5}{2}x'(0) + \frac{3}{2}x'(1) = 0. \end{aligned} \tag{6.13}$$

Here, $\Phi(\kappa, x(\kappa)) = \frac{e^{-\kappa}}{(2+e^{2\pi})} \sin\left(\frac{|x(\kappa)|}{5+|x(\kappa)|}\right)$, $\ell = 1$, $\zeta = \frac{5}{3}$, and $\alpha = 2/3, \beta = 1/3, \gamma = 5/2, \delta = 3/2$. For $r = 2$, we have

$$\begin{aligned} &|\Phi(\kappa, x(\kappa)) - \Phi(\kappa, z(\kappa))| \\ &\leq \left| \frac{e^{-\kappa}}{(2+e^{2\pi})} \sin\left(\frac{|x(\kappa)|}{5+|x(\kappa)|}\right) - \frac{e^{-\kappa}}{(2+e^{2\pi})} \sin\left(\frac{|z(\kappa)|}{5+|z(\kappa)|}\right) \right| \\ &\leq \frac{1}{(2+e^{2\pi})} \left| \frac{|x| - |z|}{5|x| + 5|z| + |x||z| + 25} \right| \\ &\leq \frac{1}{3} \frac{\|x - z\|}{(5|x| + 5|z| + 3)^{1/2}}, \end{aligned}$$

thus the assumption (A1) holds true with $\lambda = \frac{1}{3}$. In addition $(\xi\lambda)^2 \approx 0.33016226441 < 1$. Therefore, the FBVP (6.12) has a solution by Theorem 6.8.

7 Application to nonlinear matrix equations

Denote by $s(\mathcal{K})$ any singular value of a Hermitian $n \times n$ matrix \mathcal{K} over \mathbb{C} , and by $s^+(\mathcal{K}) = \|\mathcal{K}\|$ its trace norm. We will use the standard partial order on the set $\mathcal{H}(n)$ of all $n \times n$ hermitian matrices, given by

$\mathcal{K} \succeq \mathcal{L}$ if and only if $\mathcal{K} - \mathcal{L}$ is a positively defined semi-definite matrix. The set of all positively defined matrices from $\mathcal{H}(n)$ will be denoted as $\mathcal{P}(n)$.

In [22], Shil and Nashine studied the NME of the form $\mathcal{K} = \mathcal{G} + \sum_{i=1}^k \mathcal{D}_i^* \varrho(\mathcal{K}) \mathcal{D}_i$ using the spectral norm of a matrix, and applied a generalized contraction condition in metric spaces; they also tested numerically its approximate solutions. The related work in solving NMEs using fixed point results can be referred in [9–11, 21–23]. We find the positive definite solution (PDS) of the aforementioned NME in the context of EBgbDS, and two numerical examples validate the result.

Theorem 7.1. Consider the matrix equation

$$\mathcal{K} = \mathcal{G} + \sum_{j=1}^k \mathcal{D}_j^* \varrho(\mathcal{K}) \mathcal{D}_j, \quad (7.1)$$

where $\mathcal{G} \in \mathcal{P}(n)$, \mathcal{D}_i are any $n \times n$ matrices, and the operator $\varrho: \mathcal{P}(n) \rightarrow \mathcal{P}(n)$ is continuous in the trace norm. Let, for some $M, N_1 \in \mathbb{R}$, and for any $\mathcal{K} \in \mathcal{P}(n)$ with $\|\mathcal{K}\| \leq M$, $s(\varrho(\mathcal{K})) \leq N_1$ holds for all singular values of $\varrho(\mathcal{K})$.

Assume that:

1. $\|\mathcal{G}\| \leq M - NN_1n$, where $\sum_{j=1}^k \|\mathcal{D}_j^*\| \|\mathcal{D}_j\| = N$;
2. for any $\mathcal{Z} \in \mathcal{P}(n)$ with $\|\mathcal{Z}\| \leq M$, $\sum_{j=1}^k \mathcal{D}_j^* \varrho(\mathcal{Z}) \mathcal{D}_j \succeq O$ holds;
3. for some $\mathcal{Z} \in \mathcal{P}(n)$ with $\|\mathcal{Z}\| \leq M$, $\mathcal{Z} \preceq \mathcal{G} + \sum_{j=1}^k \mathcal{D}_j^* \varrho(\mathcal{Z}) \mathcal{D}_j$ holds.
4. there exist $b > 0$, $0 \leq a < 1$ such that

$$(2nNN_1)^2 \leq \frac{1}{(\max\{s^+(\mathcal{K}), s^+(\mathcal{L})\} + s^+(\mathcal{J}) + 3)} \Upsilon(\mathcal{K}, \mathcal{L}, \mathcal{J}) \quad (7.2)$$

holds, where

$$\begin{aligned} & \Upsilon(\mathcal{K}, \mathcal{L}, \mathcal{J}) \\ &= a \max \left\{ \begin{array}{l} \left(\max\{s^+(\mathcal{K}), |s^+(\mathcal{G} + \sum_{j=1}^k \mathcal{D}_j^* \varrho(\mathcal{K}) \mathcal{D}_j)|\} - |s^+(\mathcal{G} + \sum_{j=1}^k \mathcal{D}_j^* \varrho(\mathcal{L}) \mathcal{D}_j)| \right)^2, \\ \left(\max\{s^+(\mathcal{K}), s^+(\mathcal{L})\} - |s^+(\mathcal{G} + \sum_{j=1}^k \mathcal{D}_j^* \varrho(\mathcal{L}) \mathcal{D}_j)| \right)^2, \\ \left(\max\{s^+(\mathcal{L}), |s^+(\mathcal{G} + \sum_{j=1}^k \mathcal{D}_j^* \varrho(\mathcal{L}) \mathcal{D}_j)|\} - |s^+(\mathcal{G} + \sum_{j=1}^k \mathcal{D}_j^* \varrho(\mathcal{J}) \mathcal{D}_j)| \right)^2 \end{array} \right\} \\ &\quad - b \left(\max\{s^+(\mathcal{K}), |s^+(\mathcal{G} + \sum_{j=1}^k \mathcal{D}_j^* \varrho(\mathcal{L}) \mathcal{D}_j)|\} - |s^+(\mathcal{G} + \sum_{j=1}^k \mathcal{D}_j^* \varrho(\mathcal{J}) \mathcal{D}_j)| \right)^2, \end{aligned} \quad (7.3)$$

for all $\mathcal{K}, \mathcal{L}, \mathcal{J} \in \mathcal{P}(n)$ with $\|\mathcal{K}\|, \|\mathcal{L}\|, \|\mathcal{J}\| \leq M$ and

$$\sum_{j=1}^k \mathcal{D}_j^* \varrho(\mathcal{K}) \mathcal{D}_j \neq \sum_{j=1}^k \mathcal{D}_j^* \varrho(\mathcal{L}) \mathcal{D}_j \neq \sum_{j=1}^k \mathcal{D}_j^* \varrho(\mathcal{J}) \mathcal{D}_j.$$

Then the equation (7.1) has a unique solution $\widehat{\mathcal{K}} \in \mathcal{P}(n)$ with $\|\widehat{\mathcal{K}}\| \leq M$. Further, the solution can be obtained as the limit of the iterative sequence $\{\mathcal{K}_n\}$, where for $j \geq 0$,

$$\mathcal{K}_{j+1} = \mathcal{G} + \sum_{j=1}^k \mathcal{D}_j^* \varrho(\mathcal{K}_j) \mathcal{D}_j \quad (7.4)$$

and \mathcal{K}_0 is an arbitrary element of $\mathcal{P}(n)$ satisfying $\|\mathcal{K}_0\| \leq M$.

Proof. Denote $\mathcal{X} := \{\mathcal{K} \in \mathcal{P}(n) : \|\mathcal{K}\| \leq M\}$, being a closed subset of $\mathcal{P}(n)$. According to (2), any solution of (7.1) in \mathcal{X} has to be positive definite. We have, for any $\mathcal{K} \in \mathcal{X}$,

$$\begin{aligned}
& \|\mathcal{G} + \sum_{j=1}^k \mathcal{D}_i^* \varrho(\mathcal{K}) \mathcal{D}_i\| \\
& \leq \|\mathcal{G}\| + \left\| \sum_{j=1}^k \mathcal{D}_i^* \varrho(\mathcal{K}) \mathcal{D}_i \right\| \\
& \leq \|\mathcal{G}\| + \sum_{j=1}^k \|\mathcal{D}_i^*\| \|\mathcal{D}_i\| \|\varrho(\mathcal{K})\| \\
& = \|\mathcal{G}\| + N \|\varrho(\mathcal{K})\|. \tag{7.5}
\end{aligned}$$

Since, for all singular values, $s(\varrho(\mathcal{K})) \leq N_1$ holds, it follows that $\|\varrho(\mathcal{K})\| \leq N_1 n$. Thus, (7.5) implies

$$\begin{aligned}
& \|\mathcal{G} + \sum_{j=1}^k \mathcal{D}_i^* \varrho(\mathcal{K}) \mathcal{D}_i\| \leq \|\mathcal{G}\| + N N_1 n \\
& \leq M - N N_1 n + N N_1 n = M.
\end{aligned}$$

Define now an operator $\mathcal{T} : \mathcal{X} \rightarrow \mathcal{X}$ by

$$\mathcal{T}(\mathcal{K}) = \mathcal{G} + \sum_{j=1}^k \mathcal{D}_i^* \varrho(\mathcal{K}) \mathcal{D}_i,$$

for $\mathcal{K} \in \mathcal{X}$. Then it is clear that finding positive definite solution(s) (PDS) of the equation (7.1) is equivalent to finding fixed point(s) of \mathcal{T} .

Now, for any $\mathcal{K}, \mathcal{L}, \mathcal{J} \in \mathcal{X}$, we have

$$\begin{aligned}
& \|\max\{\mathcal{T}(\mathcal{K}), \mathcal{T}(\mathcal{L})\} - \mathcal{T}(\mathcal{J})\| \\
& = \left\| \max\left\{ \mathcal{G} + \sum_{j=1}^k \mathcal{D}_i^* \varrho(\mathcal{K}) \mathcal{D}_i, \mathcal{G} + \sum_{j=1}^k \mathcal{D}_i^* \varrho(\mathcal{L}) \mathcal{D}_i \right\} - \left(\mathcal{G} + \sum_{j=1}^k \mathcal{D}_i^* \varrho(\mathcal{J}) \mathcal{D}_i \right) \right\| \\
& = \max \left\{ \left\| \sum_{j=1}^k \mathcal{D}_i^* \varrho(\mathcal{K}) \mathcal{D}_i - \sum_{j=1}^k \mathcal{D}_i^* \varrho(\mathcal{L}) \mathcal{D}_i \right\|, \left\| \sum_{j=1}^k \mathcal{D}_i^* \varrho(\mathcal{L}) \mathcal{D}_i - \sum_{j=1}^k \mathcal{D}_i^* \varrho(\mathcal{J}) \mathcal{D}_i \right\| \right\} \\
& \leq \sum_{j=1}^k \|\mathcal{D}_i^*\| \|\mathcal{D}_i\| \max\{\|\varrho(\mathcal{K}) - \varrho(\mathcal{L})\|, \|\varrho(\mathcal{L}) - \varrho(\mathcal{J})\|\} \\
& \leq N \max\{\|\varrho(\mathcal{K})\| + \|\varrho(\mathcal{L})\|, \|\varrho(\mathcal{L})\| + \|\varrho(\mathcal{J})\|\} \\
& \leq N(N_1 n + N_1 n) \\
& = 2N N_1 n.
\end{aligned}$$

Thus, for any $\mathcal{K}, \mathcal{L}, \mathcal{J} \in \mathcal{X}$, we have

$$\|\max\{\mathcal{T}(\mathcal{K}), \mathcal{T}(\mathcal{L})\} - \mathcal{T}(\mathcal{J})\| \leq 2N N_1 n. \tag{7.6}$$

For some fixed $\mathcal{K}, \mathcal{L}, \mathcal{J} \in \mathcal{X}$, from (7.2) and (7.3), we have

$$\begin{aligned} & (2NN_1n)^2 \max\{s^+(\mathcal{K}), s^+(\mathcal{L})\} + s^+(\mathcal{J}) + 3 \\ & \leq a \max \left\{ \begin{array}{l} \left(\max\{s^+(\mathcal{K}), |s^+(\mathcal{G} + \sum_{j=1}^k \mathcal{D}_i^* \varrho(\mathcal{K}) \mathcal{D}_i)|\} - |s^+(\mathcal{G} + \sum_{j=1}^k \mathcal{D}_i^* \varrho(\mathcal{L}) \mathcal{D}_i)| \right)^2, \\ \left(\max\{s^+(\mathcal{K}), s^+(\mathcal{L})\} - |s^+(\mathcal{G} + \sum_{j=1}^k \mathcal{D}_i^* \varrho(\mathcal{L}) \mathcal{D}_i)| \right)^2, \\ \left(\max\{s^+(\mathcal{L}), |s^+(\mathcal{G} + \sum_{j=1}^k \mathcal{D}_i^* \varrho(\mathcal{L}) \mathcal{D}_i)|\} - |s^+(\mathcal{G} + \sum_{j=1}^k \mathcal{D}_i^* \varrho(\mathcal{J}) \mathcal{D}_i)| \right)^2 \end{array} \right\} \\ & - b \left(\max\{s^+(\mathcal{K}), |s^+(\mathcal{G} + \sum_{j=1}^k \mathcal{D}_i^* \varrho(\mathcal{L}) \mathcal{D}_i)|\} - |s^+(\mathcal{G} + \sum_{j=1}^k \mathcal{D}_i^* \varrho(\mathcal{J}) \mathcal{D}_i)| \right)^2, \end{aligned}$$

that is,

$$\begin{aligned} 2NN_1n(\|\max\{\mathcal{K}, \mathcal{L}\}\| + \|\mathcal{J}\| + 3) & \leq a \max \left\{ \begin{array}{l} \|\max\{\mathcal{K}, \mathcal{T}(\mathcal{K})\} - \mathcal{T}(\mathcal{L})\|^2, \\ \|\max\{\mathcal{K}, \mathcal{L}\} - \mathcal{T}(\mathcal{L})\|^2, \\ \|\max\{\mathcal{L}, \mathcal{T}(\mathcal{L})\} - \mathcal{T}(\mathcal{J})\|^2 \end{array} \right\} \\ & - b \|\max\{\mathcal{K}, \mathcal{T}(\mathcal{L})\} - \mathcal{T}(\mathcal{J})\|^2. \end{aligned}$$

Therefore, from (7.6) we have

$$\begin{aligned} & (\|\max\{\mathcal{K}, \mathcal{L}\}\| + \|\mathcal{J}\| + 3) \|\max\{\mathcal{T}(\mathcal{K}), \mathcal{T}(\mathcal{L})\} - \mathcal{T}(\mathcal{J})\|^2 \\ & \leq a \max \left\{ \begin{array}{l} \|\max\{\mathcal{K}, \mathcal{T}(\mathcal{K})\} - \mathcal{T}(\mathcal{L})\|^2, \\ \|\max\{\mathcal{K}, \mathcal{L}\} - \mathcal{T}(\mathcal{L})\|^2, \\ \|\max\{\mathcal{L}, \mathcal{T}(\mathcal{L})\} - \mathcal{T}(\mathcal{J})\|^2 \end{array} \right\} \\ & - b \|\max\{\mathcal{K}, \mathcal{T}(\mathcal{L})\} - \mathcal{T}(\mathcal{J})\|^2. \end{aligned} \tag{7.7}$$

Let $g_b: \mathcal{P}(n) \times \mathcal{P}(n) \rightarrow \mathbb{R}_+$ be defined by

$$g_b(\mathcal{K}, \mathcal{L}, \mathcal{J}) = \|\max\{\mathcal{K}, \mathcal{L}\} - \mathcal{J}\|^2 \text{ for all } \mathcal{K}, \mathcal{L}, \mathcal{J} \in \mathcal{P}(n).$$

Then $(\mathcal{P}(n), g_b)$ is a complete EBgbDS with coefficient $\omega(\mathcal{K}, \mathcal{L}, \mathcal{J}) = \|\max\{\mathcal{K}, \mathcal{L}\}\| + \|\mathcal{J}\| + 3$. It follows from (7.7) that

$$\begin{aligned} & \omega(\mathcal{K}, \mathcal{L}, \mathcal{J}) g_b(\mathcal{T}(\mathcal{K}), \mathcal{T}(\mathcal{L}), \mathcal{T}(\mathcal{J})) \\ & \leq a \max \{g_b(\mathcal{K}, \mathcal{T}(\mathcal{K}), \mathcal{T}(\mathcal{L})), g_b(\mathcal{K}, \mathcal{L}, \mathcal{T}(\mathcal{L})), g_b(\mathcal{L}, \mathcal{T}(\mathcal{L}), \mathcal{T}(\mathcal{J}))\} \\ & \quad - b g_b(\mathcal{K}, \mathcal{T}(\mathcal{L}), \mathcal{T}(\mathcal{J})). \end{aligned} \tag{7.8}$$

Let $\mathcal{M}(\varrho_1, \varrho_2, \varrho_3, \varrho_4, \varrho_5) = \varrho_1 - a \max\{\varrho_2, \varrho_3, \varrho_4\} + b \varrho_5$, $0 < b$ and $0 \leq a < 1$ in (7.8), we have

$$\mathcal{M} \left(\begin{array}{c} \omega(\mathcal{K}, \mathcal{L}, \mathcal{J}) g_b(\mathcal{T}(\mathcal{K}), \mathcal{T}(\mathcal{L}), \mathcal{T}(\mathcal{J})), g_b(\mathcal{K}, \mathcal{T}(\mathcal{K}), \mathcal{T}(\mathcal{L})), \\ g_b(\mathcal{K}, \mathcal{L}, \mathcal{T}(\mathcal{L})), g_b(\mathcal{L}, \mathcal{T}(\mathcal{L}), \mathcal{T}(\mathcal{J})), g_b(\mathcal{K}, \mathcal{T}(\mathcal{L}), \mathcal{T}(\mathcal{J})) \end{array} \right) \leq 0.$$

Then the formulated results follow from Theorem 3.2. \square

8 Numerical experiments

The experiment is conducted on a macOS Mojave version 10.14.6 CPU@1.6 GHz intel core i5 8GB with the programming language MATLAB R2020b (Online). Error is the symbol for the trace norm of the residual ($\text{Res}(\mathcal{K}) = \|\mathcal{K}_{n+1} - \mathcal{K}_n\|_{tr}$) and CPU is the estimate time. In all investigations, tolerance, $tol = 10^{-10}$ has been allocated.

Example 7. Consider matrices with coefficients constructed at random by

$$\mathcal{D}_1 = (1/2^n) \times \text{rand}(n); \quad \mathcal{D}_2 = (1/2^n) \times \text{rand}(n);$$

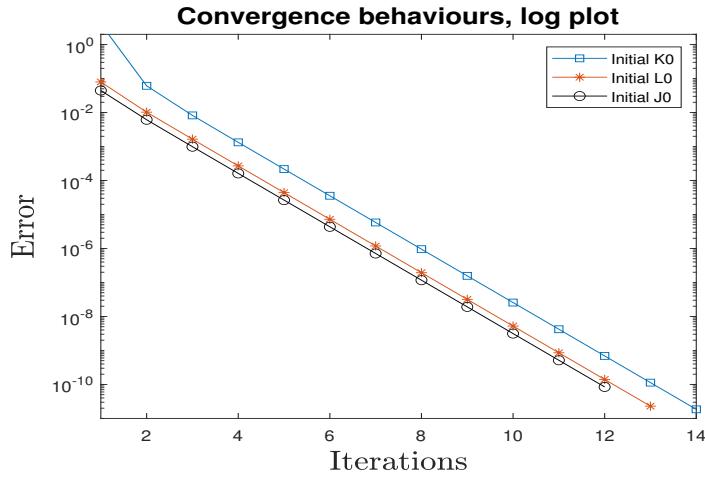


Figure 1. Iteration vs Error graph

where $\mathcal{D}_1, \mathcal{D}_2 \in \mathcal{C}^{n \times n}$. For $n = 4$, we obtain

$$\mathcal{D}_1 = \begin{bmatrix} 0.0445 & 0.0609 & 0.0193 & 0.0117 \\ 0.0545 & 0.0047 & 0.0165 & 0.0488 \\ 0.0205 & 0.0367 & 0.0474 & 0.0122 \\ 0.0406 & 0.0259 & 0.0622 & 0.0620 \end{bmatrix},$$

$$\mathcal{D}_2 = \begin{bmatrix} 0.0501 & 0.0506 & 0.0569 & 0.0024 \\ 0.0265 & 0.0223 & 0.0121 & 0.0591 \\ 0.0456 & 0.0046 & 0.0270 & 0.0477 \\ 0.0311 & 0.0369 & 0.0468 & 0.0349 \end{bmatrix},$$

$$\mathcal{G} = \begin{bmatrix} 1.0015 & 0.0025 & 0.0017 & 0.0025 \\ 0.0025 & 1.0073 & 0.0037 & 0.0055 \\ 0.0017 & 0.0037 & 1.0022 & 0.0032 \\ 0.0025 & 0.0055 & 0.0032 & 1.0053 \end{bmatrix}.$$

We use the initial values

$$\mathcal{K}_0 = \begin{bmatrix} 0.0006 & 0.0004 & 0.0004 & 0.0005 \\ 0.0004 & 0.0006 & 0.0003 & 0.0006 \\ 0.0004 & 0.0003 & 0.0004 & 0.0005 \\ 0.0005 & 0.0006 & 0.0005 & 0.0010 \end{bmatrix}, \quad \mathcal{L}_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\mathcal{J}_0 = \begin{bmatrix} 1.0031 & 0.0050 & 0.0034 & 0.0050 \\ 0.0050 & 1.0146 & 0.0076 & 0.0110 \\ 0.0034 & 0.0076 & 1.0044 & 0.0064 \\ 0.0050 & 0.0110 & 0.0064 & 1.0106 \end{bmatrix},$$

where $\mathcal{K}_0, \mathcal{L}_0, \mathcal{J}_0 \in \mathcal{P}(n)$. For $b = 0.1$, $a = 0.9$, the PDS is

$$\hat{\mathcal{K}} = \begin{bmatrix} 1.0173 & 0.0136 & 0.0151 & 0.0155 \\ 0.0136 & 1.0189 & 0.0151 & 0.0131 \\ 0.0151 & 0.0151 & 1.0174 & 0.0144 \\ 0.0155 & 0.0131 & 0.0144 & 1.0208 \end{bmatrix}$$

Figures 1 and 2 depict, respectively, the surface plot of $\hat{\mathcal{K}}$ and the graphical representation of convergence:

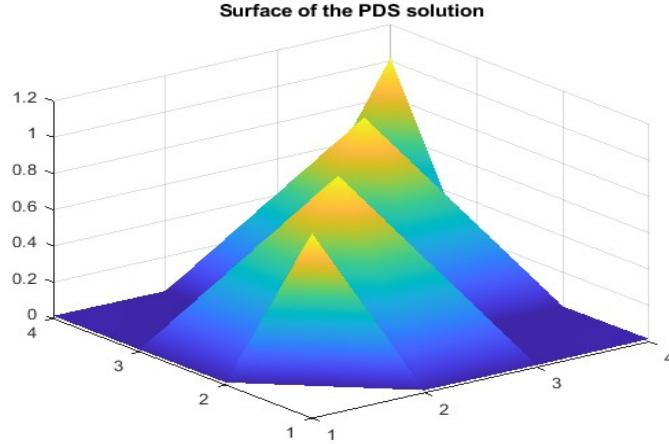


Figure 2. Solution's surface plot

Example 8. Consider the NME with full complex matrices coefficients $\mathcal{D}_1, \mathcal{D}_2, \mathcal{G} \in \mathcal{C}^{4 \times 4}$:

$$\begin{aligned} \mathcal{D}_1 &= \begin{bmatrix} 0.0048 + 0.0194i & 0.0044 - 0.0067i & 0.0092 - 0.0037i & 0.0116 + 0.0157i \\ 0.0168 - 0.0062i & 0.0045 + 0.0022i & 0.0128 + 0.0032i & 0.0087 - 0.0087i \\ 0.0171 - 0.0030i & 0.0107 + 0.0165i & 0.0183 + 0.0187i & 0.0177 + 0.0108i \\ 0.0193 - 0.0093i & 0.0152 + 0.0064i & 0.0032 - 0.0063i & 0.0079 + 0.0194i \end{bmatrix}, \\ \mathcal{D}_2 &= \begin{bmatrix} 0.0389 + 0.0040i & -0.0134 + 0.0038i & -0.0075 + 0.0026i & 0.0314 + 0.0174i \\ -0.0124 + 0.0071i & 0.0044 + 0.0090i & 0.0065 + 0.0161i & -0.0174 + 0.0137i \\ -0.0061 + 0.0044i & 0.0330 + 0.0136i & 0.0374 + 0.0015i & 0.0215 + 0.0102i \\ -0.0186 + 0.0001i & 0.0129 + 0.0094i & -0.0126 + 0.0025i & 0.0387 + 0.0123i \end{bmatrix}, \\ \mathcal{G} &= \begin{bmatrix} 1.0031 + 0.0000i & -0.0008 - 0.0008i & -0.0000 + 0.0003i & 0.0007 + 0.0004i \\ -0.0008 + 0.0008i & 1.0011 + 0.0000i & 0.0004 + 0.0013i & -0.0002 + 0.0005i \\ -0.0000 - 0.0003i & 0.0004 - 0.0013i & 1.0033 + 0.0000i & 0.0012 - 0.0002i \\ 0.0007 - 0.0004i & -0.0002 - 0.0005i & 0.0012 + 0.0002i & 1.0024 + 0.0000i \end{bmatrix}. \end{aligned}$$

We begin with the following initializations to demonstrate the convergence of the sequence $\{\mathcal{K}_n\}$ specified in (7.4):

$$\begin{aligned} \mathcal{K}_0 &= \begin{bmatrix} 0.0944 + 0.0000i & 0.0035 + 0.0476i & 0.0433 + 0.0116i & 0.0385 + 0.0235i \\ 0.0035 - 0.0476i & 0.0671 - 0.0000i & 0.0747 - 0.0532i & 0.0386 - 0.0104i \\ 0.0433 - 0.0116i & 0.0747 + 0.0532i & 0.1805 + 0.0000i & 0.0918 + 0.0201i \\ 0.0385 - 0.0235i & 0.0386 + 0.0104i & 0.0918 - 0.0201i & 0.1219 - 0.0000i \end{bmatrix}, \\ \mathcal{L}_0 &= \begin{bmatrix} 0.0000 + 2.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i \\ 0.0000 + 0.0000i & 0.0000 + 2.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i \\ 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 2.0000i & 0.0000 + 0.0000i \\ 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 2.0000i \end{bmatrix}, \\ \mathcal{J}_0 &= \begin{bmatrix} 1.0062 + 0.0000i & -0.0016 - 0.0016i & -0.0001 + 0.0006i & 0.0014 + 0.0007i \\ -0.0016 + 0.0016i & 1.0022 + 0.0000i & 0.0008 + 0.0026i & -0.0004 + 0.0009i \\ -0.0001 - 0.0006i & 0.0008 - 0.0026i & 1.0066 + 0.0000i & 0.0023 - 0.0004i \\ 0.0014 - 0.0007i & -0.0004 - 0.0009i & 0.0023 + 0.0004i & 1.0048 + 0.0000i \end{bmatrix}, \end{aligned}$$

where $\mathcal{K}_0, \mathcal{L}_0, \mathcal{J}_0 \in \mathcal{P}(n)$. Table 1 shows the details of iteration, the PDS is given by $\hat{\mathcal{K}}$ and convergence graph is shown in Figure 3. From the table, it is clear that the real part of eigenvalues are positive for three different initializations, hence the common solution is PDS.



Table 1

Int. Mat.	$\varrho(\mathcal{K})$	(a, b)	Dim.	Iter No.	CPU	Error	Min(Eig)
\mathcal{K}_0	\mathcal{K}^3	(0.98,0.05)	4	7	0.019362	0	0.0016 + 0.0003i
\mathcal{L}_0	\mathcal{L}^3	(0.98,0.05)	4	7	0.014226	0	0.0016 + 0.0003i
\mathcal{J}_0	\mathcal{J}^3	(0.98,0.05)	4	6	0.013140	0	0.0016 + 0.0003i

$$\hat{\mathcal{K}} = \begin{bmatrix} 1.0068 - 0.0000i & -0.0014 - 0.0006i & 0.0003 + 0.0002i & 0.0023 + 0.0010i \\ -0.0014 + 0.0006i & 1.0037 + 0.0000i & 0.0025 + 0.0008i & 0.0016 + 0.0003i \\ 0.0003 - 0.0002i & 0.0025 - 0.0008i & 1.0063 + 0.0000i & 0.0020 + 0.0001i \\ 0.0023 - 0.0010i & 0.0016 - 0.0003i & 0.0020 - 0.0001i & 1.0080 - 0.0000i \end{bmatrix}.$$

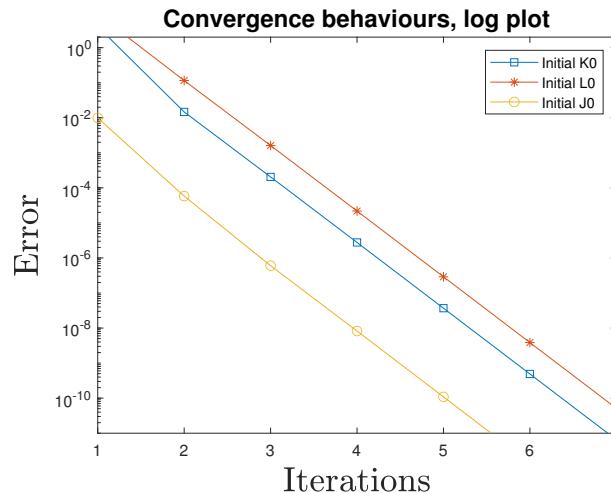


Figure 3. Iteration vs Error graph

Conclusion

We introduce a notion of extended Branciari generalized b -distance space and implicit relation, and derive new fixed point results based on a new implicit contractive condition. Following this, we demonstrate the weak well-posed property and generalized Ulam-Hyers stability in the underlying space. We use these findings to obtain the solution of fractional integral equations, which are an equivalent form of Riesz-Caputo fractional differential equations with antiperiodic boundary values. In addition, the solution of nonlinear matrix equations is discussed. All concepts, outcomes, and applications are illustrated by appropriate examples.

Competing interests

The authors declare that they have no competing interests.

Funding

Not Applicable

Availability of data and materials

Not Applicable

Acknowledgment

Authors are grateful to the learned referees for useful suggestions which helped us to improve the text of the first draft.

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