A SCHEMATIC APPROACH TO MURASUGI SUMS OF RELATIVE TRISECTIONS AND OF BOUNDED ACHIRAL LEFSCHETZ FIBRATIONS

ABHIJEET GHANWAT, SUHAS PANDIT, AND SELVAKUMAR A

Abstract. In this article, we discuss the notion of a Murasugi sum of relative trisections as well as a Murasugi sum of bounded achiral Lefschetz fibrations of compact oriented 4–manifolds with connected boundary. We give another proof of a result of Castro, et al., using a schematic approach, which states that the compact oriented 4−manifold associated to a Murasugi sum of relative trisections of two compact oriented 4–manifolds X and X' with connected boundary is the boundary connected sum of X and X'. We also prove the same result for a Murasugi sum of bounded achiral Lefschetz fibrations. As a corollary to the above results, we prove Gabai's theorem which states that the closed oriented 3−manifold associated to a Murasugi sum of open book of two closed oriented 3–manifolds M and M' is the connected sum of M and M'. Finally, we make some remarks on a Murasugi sum of bounded achiral Lefschetz fibrations and associated relative trisections.

1. Introduction

The notion of a trisection of a closed oriented smooth 4−manifold and the notion of a relative trisection of a compact 4−manifold with connected boundary is introduced by Gay and Kirby [3] as an analog of Heegaard splittings of 3−manifold. They also showed the existence of a trisection on a closed oriented smooth 4−manifold as well as a relative trisection on a compact oriented smooth 4−manifold with connected boundary.

In this article, we discuss the notion of a Murasugi sum of relative trisections. This notion is introduced in [2] by Castro, et al. They showed that given relative trisections $\mathcal{T}$ and $\mathcal{T}'$ of the compact oriented 4–manifolds X and X' with connected boundaries, respectively and given a Murasugi sum $\mathcal{OB}*\mathcal{OB}'$ of the open books $\mathcal{OB}$ and $\mathcal{OB}'$ on ∂X and $\partial X'$ induced by $\mathcal{T}$ and $\mathcal{T}'$, respectively, there exists a relative trisection $\mathcal{T}_{X\natural X'}$ of $X\natural X'$ such that the open book on $\partial(X\natural X')$ is the Murasugi sum $\mathcal{OB}*\mathcal{OB}'$, see [[2]; Theorem 3.22]. The relative trisection $\mathcal{T}_{X\natural X'}$ is obtained by plumbing $\mathcal{T}$ and $\mathcal{T}'$ by a Murasugi sum. Their proof uses Gabai's theorem [6] which states that the closed oriented 3−manifold associated to a Murasugi sum of open books of two closed oriented 3–manifolds M and M' is the connected sum of M and M'. Here, we give another proof of the above result of Castro, et al., using a different and a schematic approach. In the proof, we do not use Gabai's theorem for a Murasugi sum of open books of 3–manifolds. In fact, we prove Gabai's result for a Murasugi sum of open books of closed 3–manifolds as a corollary to this result.

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Now, we give an outline of the article. In Section 2, we discuss the notions of a relative trisection of a compact oriented 4–manifold with connected boundary, a relative trisection diagram and an open book of a closed oriented 3–manifold. In Section 3, we recall the construction of a compact oriented 4–manifold with connected boundary from a relative trisection diagram as it is required for the article. In Section 4, we explicitly define the notion of a Murasugi sum of two relative trisection diagrams along rectangles and give another proof of the above result of Castro, et al., [[2]; Theorem 3.22], using a different and a schematic approach, see Theorem 4.4. As a corollary to Theorem 4.4, we prove a special case of Gabai's result for a Murasugi sum along rectangles of open books of closed 3–manifolds, see Corollary 4.5. Theorem 4.4 and Corollary 4.5 can be extended to Murasugi sum along polygons, see Subsection 4.1. In order to avoid the complexity of notations in Murasugi sum along polygons and to increase the clarity of our arguments, we have given only the proofs of Theorem 4.4 and Corollary 4.5 for the Murasugi sum along rectangles in detail. In Section 5, we recall the notions of a bounded achiral Lefschetz fibration of a compact oriented 4–manifold, an abstract bounded achiral Lefschetz fibration and define a Murasugi sum of two abstract bounded achiral Lefschetz fibrations. We show that the compact oriented $4-$manifold associated to a Murasugi sum of bounded achiral Lefschetz fibrations of two compact oriented 4–manifolds X and X' with connected boundary is the boundary connected sum of X and X', see Theorem 5.5. This result is probably known to the experts in the field, but as it is not available in the literature to the best of our knowledge, we write the proof here. In [1], Castro et al. associated a relative trisection to a bounded achiral Lefschetz fibration. In Subsection 5.5, we discuss a few connections between a Murasugi sum of bounded achiral Lefschetz fibrations and associated relative trisections.

2. Preliminaries

In this section, we recall the necessary notions needed for this article.

2.1. Relative trisections of $4-$manifolds.

In this subsection, we recall the notion of a relative trisection of a compact 4–manifold with connected boundary. Let us begin with the following definition.

Definition 2.1. *A genus p cut system α on a compact oriented surface $\Sigma = \Sigma_{g,b}$ of genus g with b boundary components is a collection of $g-p$ disjoint non-separating simple closed curves on Σ which cut Σ into a surface P of genus p.*

In Figure 1, the collection of red curves is a genus p cut system on the compact oriented surface Σ of genus g with non-empty boundary. Similarly, the collection of blue curves shown in Figure 1 is also a genus p cut system on Σ.

Definition 2.2. *Let $\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_{g-p}\}$ be a genus p cut system on a compact oriented surface $\Sigma = \Sigma_{g,b}$. A relative compression body C_α over Σ is the compact oriented $3-$manifold obtained by attaching $(g-p)$ copies of $3-$dimensional $2-$handles $H_{\alpha_i} = (D^2 \times D^1)_i$ to $[1,2] \times \Sigma$ along curves in $\alpha \subset \{2\} \times \Sigma$, i.e.,*

$$C_\alpha = ([1,2] \times \Sigma) \bigcup_{\alpha_i \in \alpha} H_{\alpha_i}$$

In other words, a relative compression body C_α is a relative cobordism from Σ to Σ_α, where Σ_α is the surface obtained from Σ by performing surgery along α.

Note that

$$\partial C_\alpha = \partial_1 C_\alpha \cup ([1,2] \times \partial\Sigma) \cup \partial_2 C_\alpha,$$

where $\partial_1 C_\alpha = \{1\} \times \Sigma$ and $\partial_2 C_\alpha = (\{2\} \times \Sigma)_\alpha = \Sigma_\alpha$.

Next, we recall the notion of a relative trisection of a compact 4–manifold with connected boundary. In order to do that we briefly recall the following decomposition of $Z_k = \natural_k S^1 \times D^3$ and a decomposition of ∂Z_k from [1].

Let $D = \{re^{i\theta} \mid 0 \leq r \leq 1 \text{ and } -\frac{\pi}{3} \leq \theta \leq \frac{\pi}{3}\}$ be a one-third of the unit disc with $\partial D = \partial^- D \cup \partial^0 D \cup \partial^+ D$, where $\partial^- D = \{re^{\frac{-i\pi}{3}} \mid 0 \leq r \leq 1\}$, $\partial^0 D = \{e^{\frac{i\pi}{3}} \mid -\frac{\pi}{3} \leq \theta \leq \frac{\pi}{3}\}$ and $\partial^+ D = \{re^{\frac{i\pi}{3}} \mid 0 \leq r \leq 1\}$. Let $g, k, p \geq 0$, $b \geq 1$ be the integers such that $g+p+b-1 \geq k \geq 2p+b-1$ and $g \geq p$. Let P be a compact oriented surface of genus p with b boundary components. Let $U = D \times P \cong \natural_{2p+b-1} S^1 \times D^3$. Then $\partial U = \partial^- U \cup \partial^0 U \cup \partial^+ U$, where $\partial^\pm U = \partial^\pm D \times P$ and $\partial^0 U = (\partial^0 D \times P) \cup (D \times \partial P)$. For $n = k - (2p+b-1)$, let $\#_n S^1 \times S^2 = V^+ \cup V^-$ be the genus $g-p$ Heegaard splitting of $\#_n S^1 \times S^2$ which is obtained by $g-k+p+b-1$ stabilizations of the standard genus n Heegaard splitting of $\#_n S^1 \times S^2$.

Now, consider $Z_k = (\natural_n S^1 \times D^3) \natural (D \times P)$ which is diffeomorphic to $\natural_k S^1 \times D^3$, where the boundary connected sum of $\natural_n S^1 \times D^3$ and $D \times P$ is taken along a point on the Heegaard surface of the genus $g-p$ Heegaard splitting $V^+ \cup V^-$ of $\partial(\natural_n S^1 \times D^3) = \#_n S^1 \times S^2$ and a point in the interior of $\{0\} \times P \subset \partial(D \times P)$. From the above discussion, the boundary $Y_k = \partial Z_k$ of Z_k decomposes as follows:

$$\partial Z_k = Y_k = Y^+_{g,k,p,b} \cup Y^0_{g,k,p,b} \cup Y^-_{g,k,p,b},$$

where $Y^\pm_{g,k,p,b} = \partial^\pm U \natural V^\pm$ and $Y^0_{g,k,p,b} = \partial^0 U$.

Now, we are ready to define the notion of a relative trisection of a compact 4–manifold with connected boundary.

Definition 2.3. *Consider the non-negative integers g, k, p, b with $b > 0$ and $g + p + b - 1 \geq k \geq 2p + b - 1$. A $(g, k; p, b)-$relative trisection of a smooth compact connected oriented $4-$manifold W with connected boundary is a decomposition of W into three co-dimension 0 submanifolds W_1, W_2 and W_3 such that*

(1) For each $1 \leq i \leq 3$, there is a diffeomorphism $\psi_i : W_i \to Z_k = \natural_k S^1 \times D^3$,
(2) taking indices mod 3, $\psi_i(W_i \cap W_{i-1}) = Y^-_{g,k,p,b}$ and $\psi_i(W_i \cap W_{i+1}) = Y^+_{g,k,p,b}$.

Remark 2.4.

(1) A relative trisection on W induces an open book on the boundary ∂W of W.
(2) The manifold $M = Y^+_{g,k,p,b} \cup Y^-_{g,k,p,b}$ is a sutured manifold with $\partial M = (\{e^{\frac{-i\pi}{3}}\} \times P \cup \partial^- D \times \partial P) \bigcup (\{e^{\frac{i\pi}{3}}\} \times P \cup \partial^+ D \times \partial P)$ and the sutures $\{0\} \times \partial P$.
(3) The standard sutured Heegaard diagram associated to the above sutured Heegaard splitting of the sutured manifold M is given by the standard model (Σ, θ, η) as shown in Figure 1, where the collection of blue curves depicts the genus p cut system θ and the collection of red curves depicts the genus p cut system η on compact oriented surface Σ of genus g with b boundary components.

Recall that *an open book decomposition* structure of a manifold M is a pair (B, π), where B is a co-dimension 2 submanifold with a trivial normal bundle in M and $\pi : M \setminus B \to S^1$ is a locally trivial fibration such that the fibration in

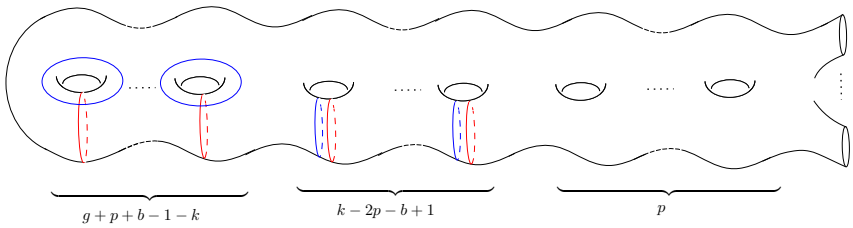

FIGURE 1. The standard model (Σ, θ, η)

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a neighborhood of B looks like the trivial fibration of $(D^2 \times B) \setminus \{0\} \times B \to S^1$ sending (r, θ, x) to θ, where $x \in B$ and (r, θ) are polar coordinates on D^2. For each $\theta \in S^1$, the closure of $\pi^{-1}(\theta)$ is called a page and the monodromy of the fibration is called the monodromy of the open book. We also denote the open book (B, π) by $\mathcal{OB}(P, \phi)$, where P is the page and ϕ is the monodromy of the open book (B, π).

As $\pi : M \setminus \mathcal{N}(B) \to S^1$ is a locally trivial fiber bundle over S^1, we can construct M up to diffeomorphism by filling the boundary of the fiber bundle using $D^2 \times B$. Hence, the manifold M is completely determined by the locally trivial fiber bundle π over S^1 in the complement of a tubular neighborhood $\mathcal{N}(B) = D^2 \times B$ of B in M. But, any locally trivial fiber bundle over S^1 with fiber P, a compact manifold with non-empty boundary, is canonically isomorphic to the fiber bundle

$$\frac{[0,1] \times P}{(1,x) \sim (0, \phi(x))} =: \mathcal{MT}(P, \phi),$$

for some diffeomorphism ϕ of P. The manifold $\mathcal{MT}(P, \phi)$ is called the mapping torus associated to P and ϕ. Hence, we have a natural way of constructing the manifold M using P and ϕ. This leads to the notion of an abstract open book.

An *abstract open book* associated with a manifold M is a pair (P, ϕ), where P is a compact surface with non-empty boundary and ϕ is a diffeomorphism of P which is the identity near the boundary such that M is diffeomorphic to

$$M_{(P,\phi)} = \mathcal{MT}(P, \phi) \cup_{id} D^2 \times \partial P,$$

where id denotes the identity map of $S^1 \times \partial P$.

The map ϕ in the above definition is called the monodromy of the abstract open book. For more details, refer [4].

Throughout this article, by an open book on the boundary of a relatively trisected 4–manifold, we mean the open book induced by the relative trisection.

2.2. **Relative trisection diagrams.** In this subsection, we recall the notion of a relative trisection diagram. We begin with the following definition.

Definition 2.5. *Let α_i, β_i, $i = 1, 2$ be genus p cut systems on a compact oriented surface Σ. We say that two triples $(\Sigma, \alpha_1, \beta_1)$ and $(\Sigma, \alpha_2, \beta_2)$ are diffeomorphism and handle slide equivalent if there exists a diffeomorphism $f : \Sigma \to \Sigma$ such that*

- *$f(\alpha_1)$ and α_2 are related by a sequence of handle slides.*
- *$f(\beta_1)$ and β_2 are related by a sequence of handle slides.*

Definition 2.6. *Consider the non-negative integers g, k, p, b with $b > 0$ and $g + p + b - 1 \geq k \geq 2p + b - 1$. A $(g, k; p, b)$–relative trisection diagram is a 4–tuple $(\Sigma, \alpha, \beta, \gamma)$, where*

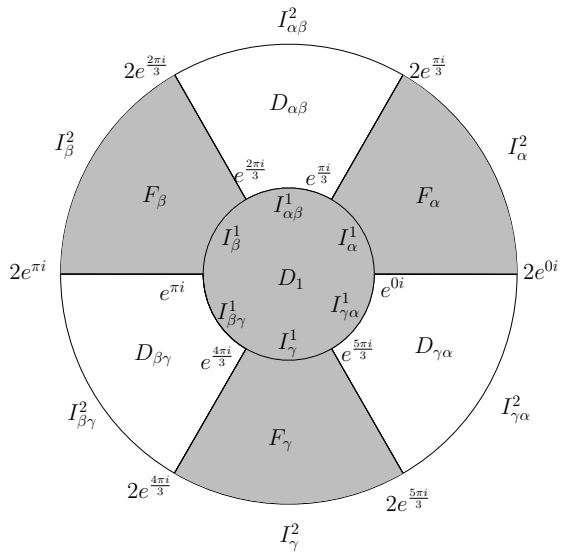


FIGURE 2. The above picture describes various parts of the disc D_2 of radius 2.

(1) Σ is a compact oriented surface of genus g with b boundary components,
(2) each of α, β and γ is a genus p cut system on the surface Σ,
(3) each triple (Σ, α, β), (Σ, β, γ) and (Σ, α, γ) is diffeomorphism and handle slide equivalent to the standard triple (Σ, θ, η) shown in Figure 1.

In [1], Castro, Gay and Pinzón-Caicedo showed that there is a one-to-one correspondence between the relative trisections on a smooth compact oriented $4-$manifold X with connected boundary up to diffeomorphism and the relative trisection diagrams associated to X up to diffeomorphism and handle slide equivalence of relative trisection diagrams. Moreover, an explicit algorithm is given to determine the page and the monodromy of the open book on the boundary from a relative trisection diagram.

3. Construction of a compact $4-$manifold from a relative trisection

In this section, we recall the construction of a compact $4-$manifold $Y_{\mathcal{D}}$ using a given $(g, k; p, b)$–relative trisection diagram $\mathcal{D} = (\Sigma, \alpha, \beta, \gamma)$ from [1]. Let $D_2 = \{re^{i\theta} : 0 \leq r \leq 2;\ \theta \in [0, 2\pi]\} \subset \mathbb{R}^2$ be the disc of radius 2. Let $I^1_\alpha, I^1_\beta, I^1_\gamma, I^2_\alpha, I^2_\beta, I^2_\gamma, I^1_{\alpha\beta}, I^1_{\beta\gamma}, I^1_{\gamma\alpha}, I^2_{\alpha\beta}, I^2_{\beta\gamma}, I^2_{\gamma\alpha}, F_\alpha, F_\beta, F_\gamma, D_{\alpha\beta}, D_{\beta\gamma}, D_{\gamma\alpha}$ and D_1 be as labelled in Figure 2. Now, we describe the steps of the construction.

Step 1.

Consider $M_1 = F \times \Sigma$, where $F = D_1 \cup F_\alpha \cup F_\beta \cup F_\gamma$, refer Figure 2.

Step 2.

For each i, attach $(I^2_\alpha \times (3\text{–dimensional 2–handle } H_{\alpha_i}))$ along $I^2_\alpha \times \alpha_i \subset I^2_\alpha \times \Sigma \subset \partial M_1$. Here, $\{t\} \times H_{\alpha_i}$ is attached along $\{t\} \times \alpha_i \subset \{t\} \times \Sigma$. Similarly, for each i, attach $(I^2_\beta \times (3\text{–dimensional 2–handle } H_{\beta_i}))$ and $(I^2_\gamma \times (3\text{–dimensional 2–handle } H_{\gamma_i}))$

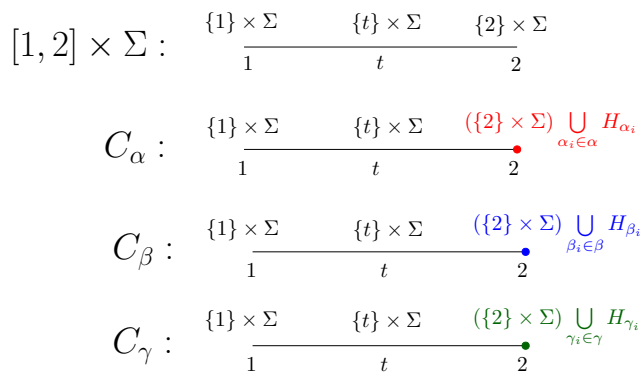


FIGURE 3. In the above figure, the horizontal line in black at the top depicts $\Sigma\times[1,2]$ and the horizontal black line with the red dot depicts the compression body C_α. Similarly, the horizontal lines with blue and green dots depict the compression bodies C_β and C_γ, respectively.

along $I^2_\beta\times\beta_i\subset I^2_\beta\times\Sigma\subset\partial M_1$ and $I^2_\gamma\times\gamma_i\subset I^2_\gamma\times\Sigma\subset\partial M_1$, respectively. We denote the resulting manifold by M_2, refer to Figure 4.

Recall that for $\eta\in\{\alpha,\beta,\gamma\}$, the relative compression body $C_\eta=([1,2]\times\Sigma)\bigcup_{\eta_i\in\eta}H_{\eta_i}$, is obtained by attaching the 2–handles H_{η_i}'s along η_i's in $\eta\subset\{2\}\times\Sigma$, with $\partial C_\eta=\partial_1C_\eta\cup([1,2]\times\partial\Sigma)\cup\partial_2C_\eta$, where $\partial_1C_\eta=\{1\}\times\Sigma=\Sigma$ and $\partial_2C_\eta=(\{2\}\times\Sigma)_\eta=\Sigma_\eta$.

A schematic diagram of the manifold M_2 is shown in Figure 4, in which

- the grey disc D_1 represents the manifold $D_1\times\Sigma$,
- for each $0\le\theta\le\frac{\pi}{3}$, the line segment $l_\theta=\{re^{i\theta}\ :\ 1\le r\le 2\}$ with the red vertex $\{2e^{i\theta}\}$ represents the relative compression body $C^\theta_\alpha\cong C_\alpha$. For the schematic diagram of C_α, refer Figure 3,
- for each $\frac{2\pi}{3}\le\theta\le\frac{3\pi}{3}$, the line segment $l_\theta=\{re^{i\theta}\ :\ 1\le r\le 2\}$ with the blue vertex $\{2e^{i\theta}\}$ represents the relative compression body $C^\theta_\beta\cong C_\beta$,
- for each $\frac{4\pi}{3}\le\theta\le\frac{5\pi}{3}$, the line segment $l_\theta=\{re^{i\theta}\ :\ 1\le r\le 2\}$ with the green vertex $\{2e^{i\theta}\}$ represents the relative compression body $C^\theta_\gamma\cong C_\gamma$.

Let $N_{\alpha\beta}=C^{\frac{\pi}{3}}_\alpha\cup(I^1_{\alpha\beta}\times\Sigma)\cup C^{\frac{2\pi}{3}}_\beta$, $N_{\beta\gamma}=C^{\frac{3\pi}{3}}_\beta\cup(I^1_{\beta\gamma}\times\Sigma)\cup C^{\frac{4\pi}{3}}_\gamma$ and $N_{\gamma\alpha}=C^{\frac{5\pi}{3}}_\gamma\cup(I^1_{\gamma\alpha}\times\Sigma)\cup C^{\frac{6\pi}{3}}_\alpha$.

Step 3.

Consider $M_3=M_2\cup(D_{\alpha\beta}\times\mathcal{N}(\partial\Sigma))\cup(D_{\beta\gamma}\times\mathcal{N}(\partial\Sigma))\cup(D_{\gamma\alpha}\times\mathcal{N}(\partial\Sigma))$, where $\mathcal{N}(\partial\Sigma)$ is a collar of $\partial\Sigma$ in Σ.

Step 4.

By [[1]; Lemma 13 and Corollary 14,], there exist unique diffeomorphisms $f_{\alpha\beta}:\Sigma_\alpha\to\Sigma_\beta$, $f_{\beta\gamma}:\Sigma_\beta\to\Sigma_\gamma$ and $f_{\gamma\alpha}:\Sigma_\gamma\to\Sigma_\alpha$ such that if we glue $I^2_{\alpha\beta}\times\Sigma_\alpha$ to M_3 by identifying $2e^{\frac{\pi i}{3}}\times\Sigma_\alpha$ to $\partial_2(C^{\frac{\pi}{3}}_\alpha)=2e^{\frac{\pi i}{3}}\times\Sigma_\alpha$ using the identity map, $I^2_{\alpha\beta}\times\mathcal{N}(\partial\Sigma_\alpha)$

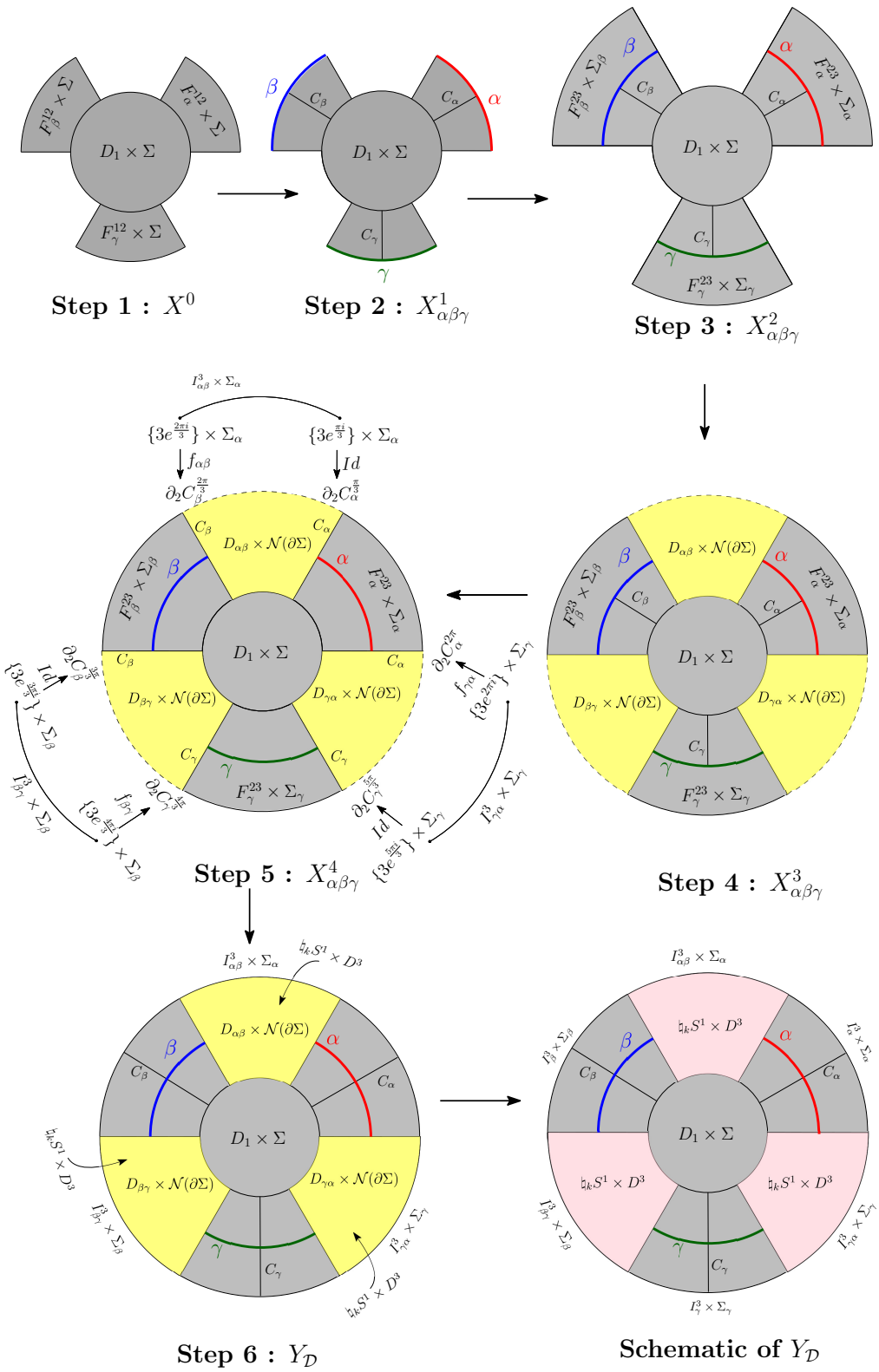


FIGURE 4. Schematic of step-by-step construction of a compact 4−manifold from a relative trisection diagram.

to $I^2_{\alpha\beta} \times \mathcal{N}(\partial\Sigma)$ using the identity map and $2e^{\frac{2\pi i}{3}} \times \Sigma_\alpha$ to $\partial_2(C^{\frac{\pi}{3}}_\beta) = 2e^{\frac{\pi i}{3}} \times \Sigma_\beta$ using the map $f_{\alpha\beta}$ and thicken, then we get a 4–manifold with the interior boundary component –up to diffeomorphism– $N_{\alpha\beta} \cup (D_{\alpha\beta} \times \partial\Sigma) \cup I^2_{\alpha\beta} \times \Sigma_\alpha$ diffeomorphic to $\#_k S^1 \times S^2$. By the same way, we can glue $I^2_{\beta\gamma} \times \Sigma_\beta$ and $I^2_{\gamma\alpha} \times \Sigma_\gamma$ to M_3 by using the maps $f_{\beta\gamma}$ and $f_{\gamma\alpha}$, respectively to get a compact 4–manifold M_4 with four boundary components among which three interior boundary components are diffeomorphic to $\#_k S^1 \times S^2$ and the exterior boundary component admits an open book with page Σ_α.

Step 5.

In [8], Laudenbach-Poénaru showed that any diffeomorphism of $\#_k S^1 \times S^2$ can be extended to a diffeomorphism of $\natural_k S^1 \times D^3$. Hence, by filling the interior boundary components of M_4 by $\natural_k S^1 \times D^3$'s, we get a unique –up to diffeomorphism– compact oriented $4-$manifold $Y_{\mathcal{D}}$ with boundary $\partial Y_{\mathcal{D}}$ such that $Y_{\mathcal{D}}$ admits a $(g, k; p, b)-$relative trisection induced by the $(g, k; p, b)-$relative trisection diagram $(\Sigma, \alpha, \beta, \gamma)$ and has the induced open book structure on ∂Y.

Now, we describe the open book decomposition $\mathcal{OB}$ on $\partial Y_{\mathcal{D}}$.

$$\partial Y_{\mathcal{D}} = \mathcal{MT}(\Sigma_\alpha, \phi) \bigsqcup_{Id} D_2 \times \partial\Sigma_\alpha,$$

where

$$\mathcal{MT}(\Sigma_\alpha, \phi) = \frac{(I^2_\alpha \cup I^2_{\alpha\beta}) \times \Sigma_\alpha \sqcup (I^2_\beta \cup I^2_{\beta\gamma}) \times \Sigma_\beta \sqcup (I^2_\gamma \cup I^2_{\gamma\alpha}) \times \Sigma_\gamma}{(f_{\alpha\beta}, f_{\beta\gamma}, f_{\gamma\alpha})},$$

with the maps

(1) $f_{\alpha\beta} : \{2e^{\frac{2\pi i}{3}}\} \times \Sigma_\alpha \to \{2e^{\frac{2\pi i}{3}}\} \times \Sigma_\beta$,
(2) $f_{\beta\gamma} : \{2e^{\frac{4\pi i}{3}}\} \times \Sigma_\beta \to \{2e^{\frac{4\pi i}{3}}\} \times \Sigma_\gamma$,
(3) $f_{\gamma\alpha} : \{2e^{2\pi i}\} \times \Sigma_\gamma \to \{2e^{2\pi i}\} \times \Sigma_\alpha$,

are as described in **Step 4** as well as in Figure 4. Therefore, one can easily see that the monodromy of the open book $\mathcal{OB}$ is $\phi = f_{\gamma\alpha} \circ f_{\beta\gamma} \circ f_{\alpha\beta}$.

4. Murasugi sum of relative trisections

In this section, we begin by recalling the notion of a Murasugi sum of two relative trisection diagrams along rectangles and related results. The Murasugi sum along polygons will be discussed later in Subsection 4.1

Definition 4.1. *Let Σ and Σ' be two compact surfaces with boundary. Suppose that c and c' are the properly embedded arcs in Σ and Σ'. Let $R = c \times [-1, 1]$ and $R' = c' \times [-1, 1]$ be the rectangular neighborhoods of c and c' in Σ and Σ', respectively. A Murasugi sum of Σ and Σ' along c and c' is a compact surface*

$$\Sigma * \Sigma' = \Sigma \bigcup_{R=R'} \Sigma',$$

where R and R' are identified by the map $g : R \to R'$ so that $g(c \times \{-1, 1\}) = \partial c' \times [-1, 1]$.

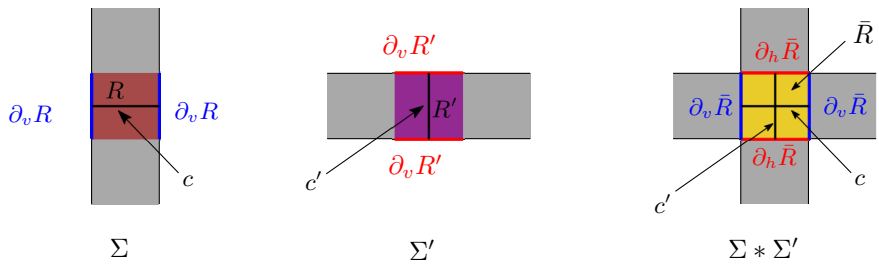


FIGURE 5. Local picture of the Murasugi sum of Σ and Σ' along $c \subset \Sigma$ and $c' \subset \Sigma'$, respectively.

Note that there is a natural embedding of each Σ and Σ' in $\Sigma * \Sigma'$.

Recall that a *Murasugi sum of two open books* (Σ, ϕ) and (Σ', ϕ') is an open book $(\Sigma * \Sigma', \phi * \phi')$, where $\Sigma * \Sigma'$ is a Murasugi sum of Σ and Σ' and the monodromy $\phi * \phi' : \Sigma * \Sigma' \to \Sigma * \Sigma'$ is given by $\phi * \phi' = \widetilde{\phi} \circ \widetilde{\phi'}$. Here, the map $\widetilde{\phi} : \Sigma * \Sigma' \to \Sigma * \Sigma'$ is the extension of $\phi : \Sigma \to \Sigma$ by the identity map in the complement of Σ in $\Sigma * \Sigma'$ and the map $\widetilde{\phi'} : \Sigma * \Sigma' \to \Sigma * \Sigma'$ is the extension of $\phi' : \Sigma' \to \Sigma'$ by the identity map in the complement of Σ' in $\Sigma * \Sigma'$.

Let $\mathcal{D} = (\Sigma, \alpha, \beta, \gamma)$ and $\mathcal{D}' = (\Sigma', \alpha', \beta', \gamma')$ be two relative trisection diagrams. Let $\alpha = \{\alpha_1, ..., \alpha_k\}$, $\beta = \{\beta_1, ..., \beta_k\}$ and $\gamma = \{\gamma_1, ..., \gamma_k\}$. Let $f_{\alpha\beta} : \Sigma_\alpha \to \Sigma_\beta$, $f_{\beta\gamma} : \Sigma_\beta \to \Sigma_\gamma$ and $f_{\gamma\alpha} : \Sigma_\gamma \to \Sigma_\alpha$ be the maps corresponding to the relative trisection diagram $\mathcal{D}$ as described in **Step 4** in Section 3 (see also [[1]; Lemma 13 and Corollary 14,]). Let $\mathcal{A}_\alpha = \{a_1, \ldots, a_l\}$, $\mathcal{A}_\beta = \{b_1 = f_{\alpha\beta}(a_1), \ldots, b_l = f_{\alpha\beta}(a_l)\}$ and $\mathcal{A}_\gamma = \{c_1 = f_{\beta\gamma}(b_1), \ldots, c_l = f_{\beta\gamma}(b_l)\}$ be the arc systems of Σ_α, Σ_β and Σ_γ, respectively. By an *arc system* $\mathcal{A}_\alpha = \{a_1, \ldots, a_l\}$, we mean a collection of disjoint properly embedded arcs $a_1, \ldots, a_l$ in Σ_α such that cutting Σ_α along a_i's yields a disc. Let $F_{\alpha\beta} : \Sigma \to \Sigma$ be a diffeomorphism such that $F_{\alpha\beta}(\alpha_i) = \beta_i$ and $F_{\alpha\beta}(a_i) = b_i$. Let $F_{\beta\gamma} : \Sigma \to \Sigma$ be a diffeomorphism such that $F_{\beta\gamma}(\beta_i) = \gamma_i$ and $F_{\beta\gamma}(b_i) = c_i$.

Let c and c' be properly embedded arcs in Σ and Σ' with rectangular neighborhoods $R = c \times [-1, 1]$ and $R' = c' \times [-1, 1]$ of c and c' in Σ and Σ' disjoint from α and α', respectively. Let $\Sigma * \Sigma'$ be the Murasugi sum of Σ and Σ' along c and c'. Let $\bar{F}_{\alpha\beta} : \Sigma * \Sigma' \to \Sigma * \Sigma'$ and $\bar{F}_{\beta\gamma} : \Sigma * \Sigma' \to \Sigma * \Sigma'$ be the extensions of the maps $F_{\beta\gamma}$ and $F_{\beta\gamma}$ by the identity map in the complement of Σ in $\Sigma * \Sigma'$.

Definition 4.2. *A Murasugi sum of the relative trisection diagrams $\mathcal{D} = (\Sigma, \alpha, \beta, \gamma)$ and $\mathcal{D}' = (\Sigma', \alpha', \beta', \gamma')$ along the properly embedded arcs $c \subset \Sigma$ and $c' \subset \Sigma'$ disjoint from α and α' respectively, is a relative trisection diagram*

$$\mathcal{D} * \mathcal{D}' = (\Sigma * \Sigma', \alpha * \alpha', \beta * \beta', \gamma * \gamma'),$$

where

(1) $\alpha * \alpha' = \alpha \cup \alpha'$,
(2) $\beta * \beta' = \beta \cup \beta^*$, *where* $\beta^* = \bar{F}_{\alpha\beta}(\beta')$,
(3) $\gamma * \gamma' = \gamma \cup \gamma^*$, *where* $\gamma^* = (\bar{F}_{\beta\gamma} \circ \bar{F}_{\alpha\beta})(\gamma')$.

Remark 4.3.

(1) *Our proof of Theorem 4.4 establishes the fact that the Murasugi sum diagram $\mathcal{D} * \mathcal{D}'$ is indeed a relative trisection diagram.*
(2) *In case $\alpha' = \beta' = \gamma' = \emptyset$, it is easy to see that the Murasugi sum diagram $\mathcal{D} * \mathcal{D}' = (\Sigma * \Sigma', \alpha, \beta, \gamma)$ is a relative trisection diagram of the relatively trisected manifold $X = Y_{\mathcal{D}} \cup (D_2 \times \Sigma' \setminus \mathring{R}')$ which is obtained by gluing the relatively trisected manifold $Y_{\mathcal{D}}$ associated to $\mathcal{D}$ and the trivially trisected manifold $D_2 \times (\Sigma' \setminus \mathring{R}')$ along $D_2 \times (\partial c \times [-1, 1]) \subset \partial Y_{\mathcal{D}}$ and $D_2 \times (c \times \{\pm 1\}) \subset \partial(D_2 \times (\Sigma' \setminus \mathring{R}'))$ induced by the map g as given in Definition 4.1, where the relative trisection of X is the natural relative trisection of X induced by the relative trisections of $Y_{\mathcal{D}}$ and $D_2 \times \Sigma'$.*

For a detailed example of Murasugi sum of relative trisection discussed by Castro, et al., we refer to [2].

Theorem 4.4. *Let $\mathcal{D} = (\Sigma, \alpha, \beta, \gamma)$ and $\mathcal{D}' = (\Sigma', \alpha', \beta', \gamma')$ be $(g, k; p, b)-$ and $(g', k'; p', b')$–relative trisection diagrams of the relative trisections $\mathcal{T}$ and $\mathcal{T}'$ of the 4–manifolds X and X', respectively. Let $\mathcal{OB}(\Sigma_\alpha * \Sigma'_{\alpha'}, \phi_\alpha * \phi_{\alpha'})$ be a Murasugi sum of $\mathcal{OB}(\Sigma_\alpha, \phi_\alpha)$ and $\mathcal{OB}(\Sigma'_{\alpha'}, \phi_{\alpha'})$ along the properly embedded arcs $c \subset \Sigma_\alpha$ and $c' \subset \Sigma'_{\alpha'}$, where $\mathcal{OB}(\Sigma_\alpha, \phi_\alpha)$ and $\mathcal{OB}(\Sigma'_{\alpha'}, \phi_{\alpha'})$ are the induced open books on ∂X and $\partial X'$, respectively. Then there exists a relative trisection $\mathcal{T}_{X \natural X'}$ of the compact 4–manifold $X \natural X'$ such that*

(1) *the relative trisection diagram of the relative trisection $\mathcal{T}_{X \natural X'}$ is the Murasugi sum $\mathcal{D} * \mathcal{D}'$ of the relative trisection diagrams $\mathcal{D}$ and $\mathcal{D}'$ along c and c',*
(2) *the induced open book on $\partial(X \natural X')$ is the Murasugi sum $\mathcal{OB}(\Sigma_\alpha * \Sigma'_{\alpha'}, \phi_\alpha * \phi_{\alpha'})$ of $\mathcal{OB}(\Sigma_\alpha, \phi_\alpha)$ and $\mathcal{OB}(\Sigma'_{\alpha'}, \phi_{\alpha'})$.*

Proof. In order to prove the theorem, it is enough to establish the following three steps:
Step a: There exists a compact 4–manifold W with a relative trisection $\mathcal{T}_W$ such that its relative trisection diagram is the Murasugi sum $\mathcal{D} * \mathcal{D}'$ of the relative trisection diagrams $\mathcal{D}$ and $\mathcal{D}'$ along c and c'.
Step b: The induced open book on ∂W is the Murasugi sum $\mathcal{OB}(\Sigma_\alpha * \Sigma'_{\alpha'}, \phi_\alpha * \phi_{\alpha'})$ of $\mathcal{OB}(\Sigma_\alpha, \phi_\alpha)$ and $\mathcal{OB}(\Sigma'_{\alpha'}, \phi_{\alpha'})$.
Step c: The 4–manifold W is diffeomorphic to the boundary connected sum $X \natural X'$ of X and X'.

Let us establish **Step a:** We construct the 4–manifold W by appropriate gluing of the 4–manifolds X_2 and X'_2, where the 4–manifold X_2 is obtained by performing sequence of operations on the 4–manifold from $X = X_{\mathcal{D}}$ and 4–manifold X'_2 is obtained by performing sequence of operations on the 4–manifold $X' = X_{\mathcal{D}'}$, refer Figure 6 and Figure 7. For the construction of the 4–manifolds $X = X_{\mathcal{D}}$ and $X' = X_{\mathcal{D}'}$ from the trisection diagrams $\mathcal{D}$ and $\mathcal{D}'$, respectively refer Section 3. For the pictorial description of the 4–manifolds $X = X_{\mathcal{D}}$ and $X' = X_{\mathcal{D}'}$, refer Figure 6 (A) and (A').

We can regard the arcs $c \subset \Sigma_\alpha$ and $c' \subset \Sigma'_{\alpha'}$ as the arcs $c \subset \Sigma$ and $c' \subset \Sigma'$ disjoint from $\alpha \subset \Sigma$ and $\alpha' \subset \Sigma'$, respectively. Let $\Sigma * \Sigma'$ be the Murasugi sums of Σ and Σ', along $c \subset \Sigma$ and $c' \subset \Sigma'$ and $\Sigma_\alpha * \Sigma'$ be the Murasugi sum of Σ_α and Σ' along $c \subset \Sigma_\alpha$ and $c' \subset \Sigma'$. Let X_1 and X'_1 be the 4–manifolds constructed from the relative trisection diagrams $\mathcal{D}_1 = (\Sigma * \Sigma', \alpha, \beta, \gamma)$ and $\mathcal{D}'_1 = (\Sigma_\alpha * \Sigma', \alpha', \beta', \gamma')$,

respectively, refer Figure 6 (A_1) and (A_1'). Here, $\mathcal{D}_1$ and $\mathcal{D}_1'$ are indeed relative trisection diagrams, see Remark 4.3(2). Now, we redraw the schematic diagrams of the relatively trisected 4–manifolds X_1 and X_1' as shown in Figure 6 $(\widetilde{A}_1)$ and $(\widetilde{A}_1')$.

We construct a new compact oriented 4–manifold X_2 from X_1 by attaching $(I^2_\alpha \times (\text{3–dimensional 2–handle } H_{\alpha_i'}))$ along $I^2_\alpha \times \alpha_i' \subset I^2_\alpha \times \Sigma_\alpha * \Sigma' \subset \partial X_1$. Here, $\{t\} \times H_{\alpha_i'}$ is attached along $\{t\} \times \alpha_i' \subset \{t\} \times \Sigma_\alpha * \Sigma'$, for $t \in I^2_\alpha$. For a schematic of X_2, refer Figure 7 (A_2).

By recalling **Step 2** of the construction of X_1' from the relative trisection diagram $\mathcal{D}_1' = (\Sigma_\alpha * \Sigma', \alpha', \beta', \gamma')$ as described in Section 3, we construct a 4–manifold X_2' from X_1' by removing $(\mathring{I}^2_{\alpha'} \times (\text{3–dimensional 2–handles } H_{\alpha_i'}\text{'s}))$, where $\mathring{I}^2_{\alpha'}$ is the interior of $I^2_{\alpha'}$, i.e.,

$$X_2' = X_1' \setminus \left(\bigcup_{\alpha_i' \in \alpha'} \mathring{I}^2_{\alpha'} \times (\text{3–dimensional 2–handle } H_{\alpha_i'}) \right).$$

For a schematic of X_2', refer Figure 7 (A_2'). Note that the schematic of both the 4–manifolds X_2 and X_2' as shown in Figure 7(A_2) and (A_2') do not represent a relative trisection.

Now, we construct the desired 4–manifold W by gluing X_2 and X_2' as follows: Let $\Sigma_\alpha * \Sigma', \Sigma_\beta * \Sigma'$ and $\Sigma_\gamma * \Sigma'$ are the Murasugi sums of Σ_α and Σ', Σ_β and Σ', Σ_γ and Σ' along c and c', $f_{\alpha\beta}(c)$ and c', $(f_{\beta\gamma} \circ f_{\alpha\beta})(c)$ and c', respectively. First, we describe the gluing regions $M \subset \partial X_2$ and $M' \subset \partial X_2'$ depicted in Figure 7. The gluing region $M \subset \partial X_2$ is

$$M = \left(M_{\alpha\beta} \bigsqcup_{f_{\alpha\beta} * Id_{\Sigma'}} M_\beta \bigsqcup_{Id_\beta} M_{\beta\gamma} \bigsqcup_{f_{\beta\gamma} * Id_{\Sigma'}} M_\gamma \bigsqcup_{Id_\gamma} M_{\gamma\alpha} \right) \bigcup_{\alpha_i' \in \alpha'} \left(H^0_{\alpha_i'} \cup H^{\frac{\pi}{3}}_{\alpha_i'} \right),$$

where

(1) $M_{\alpha\beta} = I^2_{\alpha\beta} \times (\Sigma_\alpha * \Sigma')$,
(2) $M_\beta = I^2_\beta \times (\Sigma_\beta * \Sigma')$ and $M_{\beta\gamma} = I^2_{\beta\gamma} \times (\Sigma_\beta * \Sigma')$,
(3) $M_\gamma = I^2_\gamma \times (\Sigma_\gamma * \Sigma')$ and $M_{\gamma\alpha} = I^2_{\gamma\alpha} \times (\Sigma_\gamma * \Sigma')$,
(4) $Id_\alpha : \Sigma_\alpha * \Sigma' \to \Sigma_\alpha * \Sigma'$, $Id_\beta : \Sigma_\beta * \Sigma' \to \Sigma_\beta * \Sigma'$ and $Id_\gamma : \Sigma_\gamma * \Sigma' \to \Sigma_\gamma * \Sigma'$ are the identity maps,
(5) $f_{\alpha\beta} * Id_{\Sigma'} : \Sigma_\alpha * \Sigma' \to \Sigma_\beta * \Sigma'$ and $f_{\beta\gamma} * Id_{\Sigma'} : \Sigma_\beta * \Sigma' \to \Sigma_\gamma * \Sigma'$,
(6) $H^0_{\alpha_i'}$ and $H^{\frac{\pi}{3}}_{\alpha_i'}$ are the 3−dimensional 2–handles attached along $\alpha_i' \subset e^{i0} \times \Sigma_\alpha * \Sigma'$ and $\alpha_i' \subset e^{\frac{i\pi}{3}} \times \Sigma_\alpha * \Sigma'$, respectively.

The gluing region $M' \subset \partial X_2'$ is

$$M' = I^2_{\alpha'} \times (\Sigma_\alpha * \Sigma') \bigcup_{\alpha_i' \in \alpha'} \left(H^0_{\alpha_i'} \cup H^{\frac{\pi}{3}}_{\alpha_i'} \right),$$

where $H^0_{\alpha_i'}$ and $H^{\frac{\pi}{3}}_{\alpha_i'}$ are the 3−dimensional 2–handles attached along $\alpha_i' \subset e^{i0} \times \Sigma_\alpha * \Sigma'$ and $\alpha_i' \subset e^{\frac{i\pi}{3}} \times \Sigma_\alpha * \Sigma'$, respectively.

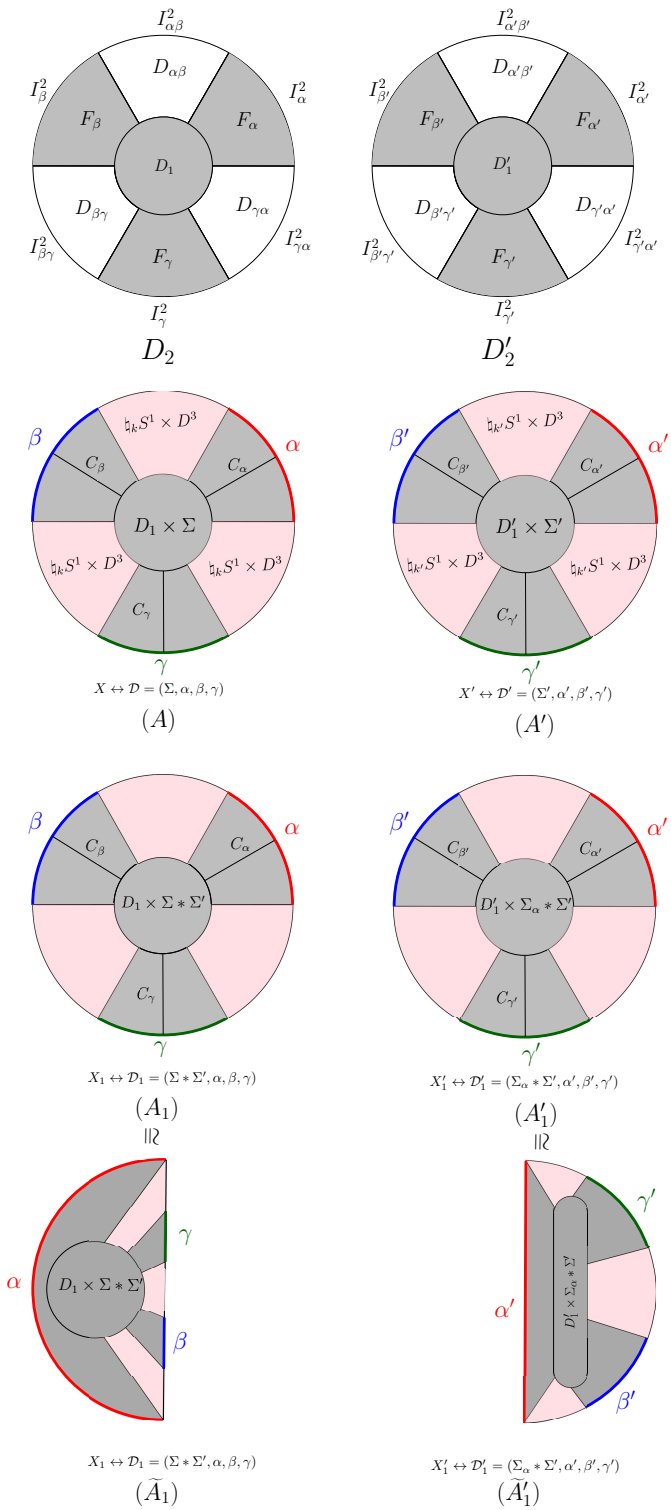


FIGURE 6. The above figure depicts a schematics of the construction of the 4–manifolds X, X_1, X' and X'_1.

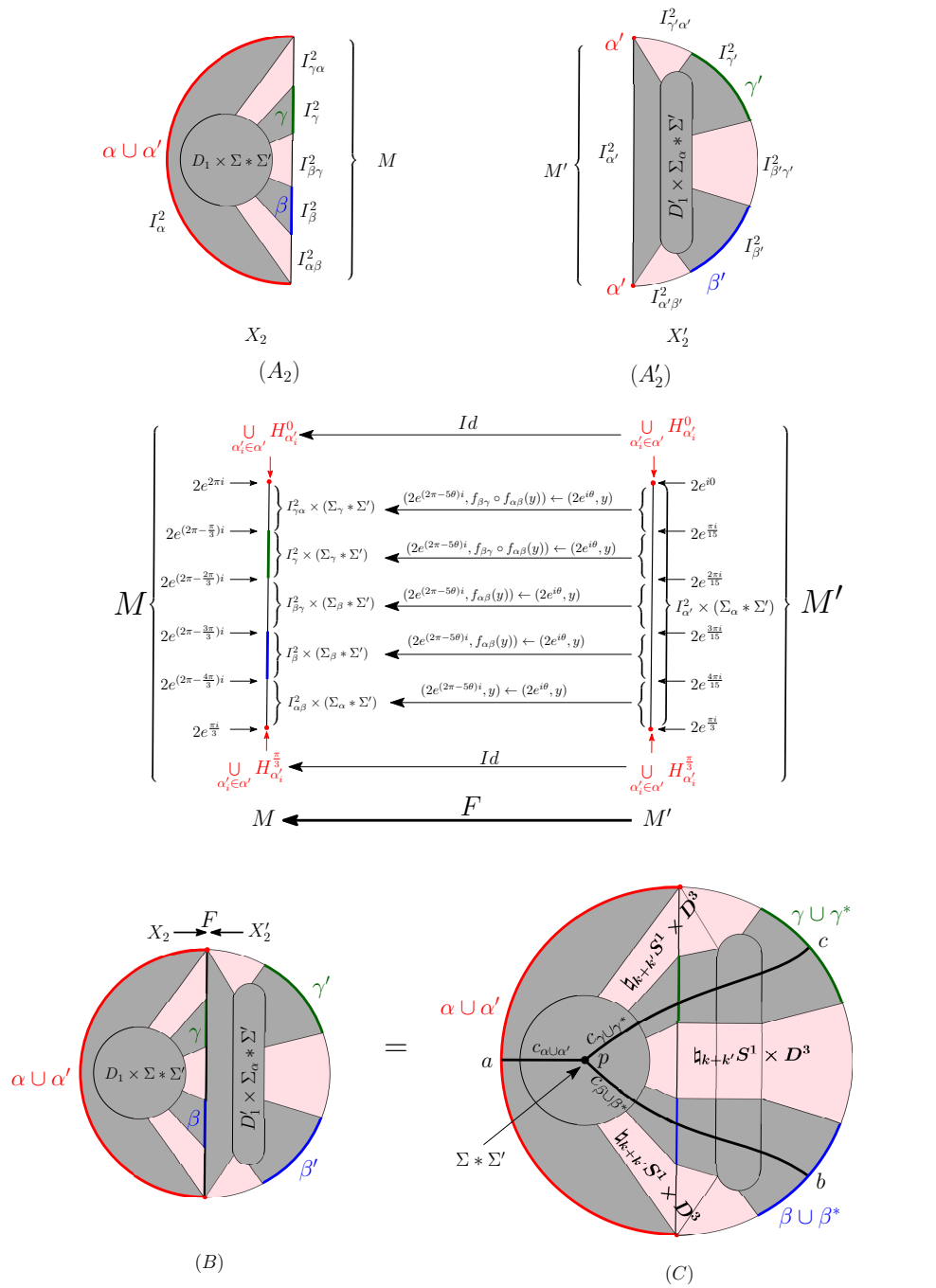


FIGURE 7. The above figure (A_2) and (A'_2) depict the schematics of the construction of the 4–manifolds X_2 and X'_2, respectively. The middle figure describes the gluing map F between the gluing regions $M \subset \partial X_2$ and $M' \subset \partial X'_2$. The figure (C) depicts a schematic of the desired trisected 4–manifold W with the relative trisection $\mathcal{T}_W$.

We define the gluing map $F : M' \to M$ as follows: Let $w \in M'$. If $w = (2e^{i\theta}, y) \in I^2_{\alpha'} \times (\Sigma_\alpha * \Sigma')$ then

$$F(w) = \begin{cases} (2e^{(2\pi-5\theta)i}, f_{\beta\gamma} \circ f_{\alpha\beta}(y)) \in M_{\gamma\alpha}, & \text{if } 0 \leq \theta \leq \frac{\pi}{15} \\ (2e^{(2\pi-5\theta)i}, f_{\beta\gamma} \circ f_{\alpha\beta}(y)) \in M_{\gamma}, & \text{if } \frac{\pi}{15} \leq \theta \leq \frac{2\pi}{15} \\ (2e^{(2\pi-5\theta)i}, f_{\alpha\beta}(y)) \in M_{\beta\gamma}, & \text{if } \frac{2\pi}{15} \leq \theta \leq \frac{3\pi}{15} \\ (2e^{(2\pi-5\theta)i}, f_{\alpha\beta}(y)) \in M_{\beta}, & \text{if } \frac{3\pi}{15} \leq \theta \leq \frac{4\pi}{15} \\ (2e^{(2\pi-5\theta)i}, y) \in M_{\alpha\beta}, & \text{if } \frac{4\pi}{15} \leq \theta \leq \frac{\pi}{3} \end{cases}$$

and if $w \in H^0_{\alpha'_i} \cup H^{\frac{\pi}{3}}_{\alpha'_i}$, $\alpha'_i \in \alpha$ then $F(w) = w$.

Now, by following the gluing map F and the schematic of the 4–manifold W as shown in Figure 7, one can observe that W admits the desired trisection $\mathcal{T}_{\mathcal{W}}$ in which

(1) the core of $\mathcal{T}_W$ is $\Sigma * \Sigma'$ which is represented by the point p in Figure 7(C),
(2) the three line segments joining the point p to the points a, b, c in Figure 7(C) represents the three compression bodies $C_{\alpha\cup\alpha'}, C_{\beta\cup\beta^*}, C_{\gamma\cup\gamma^*}$, respectively,
(3) the three pink regions in Figure 7(C) represent the three sectors which are diffeomorphic to $\natural_{k+k'} S^1 \times D^3 = (\natural_k S^1 \times D^3)\natural(\natural_{k'} S^1 \times D^3)$.

This completes the argument for **Step a**.

Step b: Let $f_{\alpha\beta} : \Sigma_\alpha \to \Sigma_\beta$, $f_{\beta\gamma} : \Sigma_\beta \to \Sigma_\gamma$ and $f_{\gamma\alpha} : \Sigma_\gamma \to \Sigma_\alpha$ be the maps described in **Step 4** of the construction of the 4–manifold X in Section 3. Note that the monodromy of the open book $\mathcal{OB}(\Sigma_\alpha, \phi_\alpha)$ on ∂X is $\phi_\alpha = f_{\gamma\alpha} \circ f_{\beta\gamma} \circ f_{\alpha\beta}$. Let $f_{\alpha'\beta'} : \Sigma'_{\alpha'} \to \Sigma'_{\beta'}$, $f_{\beta'\gamma'} : \Sigma'_{\beta'} \to \Sigma'_{\gamma'}$ and $f_{\gamma'\alpha'} : \Sigma'_{\gamma'} \to \Sigma'_{\alpha'}$ be the maps described in **Step 4** of the construction of the 4–manifold X' in Section 3. Note that the monodromy of the open book $\mathcal{OB}(\Sigma'_{\alpha'}, \phi_{\alpha'})$ on $\partial X'$ is $\phi_{\alpha'} = f_{\gamma'\alpha'} \circ f_{\beta'\gamma'} \circ f_{\alpha'\beta'}$.

Let $P_{\alpha\alpha'} = \Sigma_\alpha * \Sigma'_{\alpha'}$, $P_{\alpha\beta'} = \Sigma_\alpha * \Sigma'_{\beta'}$, $P_{\alpha\gamma'} = \Sigma_\alpha * \Sigma'_{\gamma'}$ be the Murasugi sums of Σ_α and $\Sigma'_{\alpha'}$, Σ_α and $\Sigma'_{\beta'}$, Σ_α and $\Sigma'_{\gamma'}$ along c and c', c and $f_{\alpha'\beta'}(c')$, c and $(f_{\beta'\gamma'} \circ f_{\alpha'\beta'})(c')$, respectively. Let $P_{\alpha\alpha'} = \Sigma_\alpha * \Sigma'_{\alpha'}$, $P_{\beta\alpha'} = \Sigma_\beta * \Sigma'_{\alpha'}$, $P_{\gamma\alpha'} = \Sigma_\gamma * \Sigma'_{\alpha'}$ be the Murasugi sums of Σ_α and $\Sigma'_{\alpha'}$, Σ_β and $\Sigma'_{\alpha'}$, Σ_γ and $\Sigma'_{\alpha'}$ along c and c', $f_{\alpha\beta}(c)$ and c', $(f_{\beta\gamma} \circ f_{\alpha\beta})(c)$ and c', respectively.

Now, we describe the induced open book decomposition $\mathcal{OB}$ on ∂W as follows:

$$\partial W = \mathcal{MT}(\Sigma_\alpha * \Sigma'_{\alpha'}, \phi) \bigsqcup_{Id} \widetilde{D} \times \partial(\Sigma_\alpha * \Sigma'_{\alpha'}),$$

where $\widetilde{D}$ is a 2 disc obtained by gluing the discs D_2 and D'_2 along the boundary arcs $(I^2_{\alpha\beta} \cup I^2_\beta \cup I^2_{\beta\gamma} \cup I^2_\gamma \cup I^2_{\gamma\alpha}) \subset \partial D_2$ and $I^2_{\alpha'} \subset \partial D'_2$. Here,

$$\mathcal{MT}(P_{\alpha\alpha'}, \phi) = \frac{(J_1 \times P_{\alpha\alpha'}) \sqcup (J_2 \times P_{\alpha\beta'}) \sqcup (J_3 \times P_{\alpha\gamma'}) \sqcup (\{2e^{2\pi i}\} \times P_{\alpha\alpha'}) \sqcup (\{2e^{2\pi i}\} \times P_{\gamma\alpha'})}{(G_1, G_2, G_3, G_4, G_5)},$$

where,

(1) $J_1 = I^2_\alpha \cup I^2_{\alpha'\beta'}, J_2 = I^2_{\beta'} \cup I^2_{\beta'\gamma'}$ and $J_3 = I^2_{\gamma'} \cup I^2_{\gamma'\alpha'}$,
(2) $G_1 = Id_{\Sigma_\alpha} * f_{\alpha'\beta'} : \{2e^{\frac{2\pi i}{3}}\} \times \Sigma_\alpha * \Sigma'_{\alpha'} \to \{2e^{\frac{2\pi i}{3}}\} \times \Sigma_\alpha * \Sigma'_{\beta'}$,
(3) $G_2 = Id_{\Sigma_\alpha} * f_{\beta'\gamma'} : \{2e^{\frac{4\pi i}{3}}\} \times \Sigma_\alpha * \Sigma'_{\beta'} \to \{2e^{\frac{4\pi i}{3}}\} \times \Sigma_\alpha * \Sigma'_{\gamma'}$,
(4) $G_3 = Id_{\Sigma_\alpha} * f_{\gamma'\alpha'} : \{2e^{2\pi i}\} \times \Sigma_\alpha * \Sigma'_{\gamma'} \to \{2e^{2\pi i}\} \times \Sigma_\alpha * \Sigma'_{\alpha'}$,
(5) $G_4 = (f_{\beta\gamma} \circ f_{\alpha\beta}) * Id_{\Sigma'_{\alpha'}} : \{2e^{2\pi i}\} \times \Sigma_\alpha * \Sigma'_{\alpha'} \to \{2e^{2\pi i}\} \times \Sigma_\gamma * \Sigma'_{\alpha'}$,
(6) $G_5 = f_{\gamma\alpha} * Id_{\Sigma'_{\alpha'}} : \{2e^{2\pi i}\} \times \Sigma_\gamma * \Sigma'_{\alpha'} \to \{2e^{2\pi i}\} \times \Sigma_\alpha * \Sigma'_{\alpha'}$.

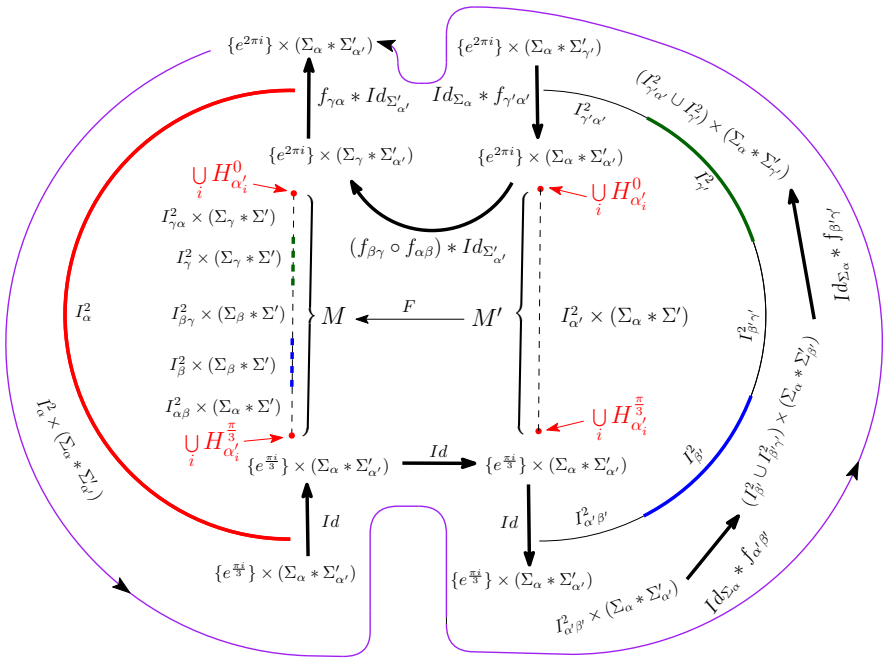


FIGURE 8. Tracing the monodromy of the open book $\mathcal{OB}$ on ∂W.

For the above decomposition of $\mathcal{MT}(P_{\alpha\alpha'}, \phi)$, refer Figure 8. Now, from the above description of the open book $\mathcal{OB}$ of ∂W, one can easily see that the monodromy of the open book $\mathcal{OB}$ is

$$\begin{aligned}
\phi &= G_5 \circ G_4 \circ G_3 \circ G_2 \circ G_1 \\
&= (f_{\gamma\alpha} * Id_{\Sigma'_{\alpha'}}) \circ ((f_{\beta\gamma} \circ f_{\alpha\beta}) * Id_{\Sigma'_{\alpha'}}) \circ (Id_{\Sigma_\alpha} * f_{\gamma'\alpha'}) \circ (Id_{\Sigma_\alpha} * f_{\beta'\gamma'}) \circ (Id_{\Sigma_\alpha} * f_{\alpha'\beta'}) \\
&= ((f_{\gamma\alpha} \circ f_{\beta\gamma} \circ f_{\alpha\beta}) * Id_{\Sigma'_{\alpha'}}) \circ (Id_{\Sigma_\alpha} * (f_{\gamma'\alpha'} \circ f_{\beta'\gamma'} \circ f_{\alpha'\beta'})) \\
&= \phi_\alpha * Id_{\Sigma'_{\alpha'}} \circ Id_{\Sigma_\alpha} * \phi_{\alpha'} \\
&= \phi_\alpha * \phi_{\alpha'}
\end{aligned}$$

Hence, the induced open book $\mathcal{OB}$ on ∂W is the Murasugi sum $\mathcal{OB}(\Sigma_\alpha * \Sigma'_{\alpha'}, \phi_\alpha * \phi_{\alpha'})$ of $\mathcal{OB}(\Sigma_\alpha, \phi_\alpha)$ and $\mathcal{OB}(\Sigma'_{\alpha'}, \phi_{\alpha'})$.

Step c: In order to establish **Step c**, we show that there is a properly embedded 3–disc $\mathcal{D}^3$ in W such that

$$W \setminus \mathcal{D}^3 = W_1 \sqcup W_2,$$

where

(1) $W_1 \cup \mathcal{D}^3$ is diffeomorphic to X,
(2) $W_2 \cup \mathcal{D}^3$ is diffeomorphic to X'.

In order to construct the desired disc $\mathcal{D}^3 \subset W$, we need to recall the following:

Let $R = c \times [-1, 1]$ and $R' = c' \times [-1, 1]$ be the rectangular neighborhoods of c and c' in Σ and Σ', respectively. Let $\bar{R}$ be the rectangle in $\Sigma * \Sigma'$ obtained by identifying $R \subset \Sigma$ and $R' \subset \Sigma'$ by the map g as in Definition 4.1. We regard $\bar{R}$ as

$[-1,1]\times[-1,1]$ with $c\times\{0\}=[-1,1]\times\{0\}$ and $c'\times\{0\}=\{0\}\times[-1,1]$. We denote the vertical boundary $\{\pm1\}\times[-1,1]$ of $\bar{R}$ by $\partial_v^{\pm}\bar{R}$ and the horizontal boundary $[-1,1]\times\{\pm1\}$ of $\bar{R}$ by $\partial_h^{\pm}\bar{R}$. Note that the rectangular regions R and R' can also be regarded as rectangular neighborhoods of c and c' in Σ_α and $\Sigma'_{\alpha'}$, respectively.

The disc $\mathcal{D}^3$ which we are going to construct is the union of five 3–dimensional discs $\mathcal{D}^3_+, \mathcal{D}^3_-, \mathbf{D}^3, \mathcal{D}'^3_+$ and $\mathcal{D}'^3_-$, i.e.,

$$\mathcal{D}^3=\mathcal{D}^3_+\cup\mathcal{D}^3_-\cup\mathbf{D}^3\cup\mathcal{D}'^3_+\cup\mathcal{D}'^3_-,$$

where

(1) the disc $\mathbf{D}^3=I^2_{\alpha'}\times\bar{R}\subset M'\subset\partial X'_2\subset W$,
(2) $\mathcal{D}^3_+$ and $\mathcal{D}^3_-$ are the discs in W which are naturally carried over W from the properly embedded discs $\mathcal{D}^3_{v_+}$ and $\mathcal{D}^3_{v_-}$ in X_1, respectively,
(3) $\mathcal{D}'^3_+$ and $\mathcal{D}'^3_-$ are the discs in W which are naturally carried over W from the properly embedded discs $\mathcal{D}'^3_{h_+}$ and $\mathcal{D}'^3_{h_-}$ in X'_1, respectively.

Now, we describe the properly embedded discs $\mathcal{D}^3_{v_+}$ and $\mathcal{D}^3_{v_-}$ in X_1 and the properly embedded discs $\mathcal{D}'^3_{h_+}$ and $\mathcal{D}'^3_{h_-}$ in X'_1 as follows: Recall that X_1 and X'_1 are the 4–manifolds constructed from the relative trisection diagrams $\mathcal{D}_1=(\Sigma*\Sigma',\alpha,\beta,\gamma)$ and $\mathcal{D}'_1=(\Sigma_\alpha*\Sigma',\alpha',\beta',\gamma')$, respectively, refer Figure 6 (A_1) and (A'_1). Now, one can easily see that X_1 and X'_1 can be decomposed as follows:

$$X_1=X\bigsqcup_{(Id\times g)}(D_2\times(\Sigma'\setminus\mathring{R}'))$$

$$X'_1=X'\bigsqcup_{(Id\times g^{-1})}(D'_2\times(\Sigma_\alpha\setminus\mathring{R})),$$

where

(1) $g:R\to R'$ is the map as in Definition 4.1 used to identify the rectangular neighborhoods R and R' to get $\bar{R}$,
(2) the map $(Id\times g):D_2\times(\partial c\times[-1,1])\subset\partial X\to D_2\times(c'\times\{\pm1\})$ is given by $(Id\times g)(x,y)=(x,g(y))$.
(3) the map $(Id\times g^{-1}):D'_2\times(\partial c'\times[-1,1])\subset\partial X'\to D'_2\times(c\times\{\pm1\})$ is given by $(Id\times g^{-1})(x,y)=(x,g^{-1}(y))$.

In the above decomposition of X_1 and X'_1, consider the properly embedded discs $\mathcal{D}^3_{v_+},\mathcal{D}^3_{v_-}\subset X_1$ and $\mathcal{D}'^3_{h_+},\mathcal{D}'^3_{h_-}\subset X'_1$, respectively as follows:

(1) $\mathcal{D}^3_{v_+}=D_2\times(\partial^+c\times[-1,1])=D_2\times\partial_v^+\bar{R}$,
(2) $\mathcal{D}^3_{v_-}=D_2\times(\partial^-c\times[-1,1])=D_2\times\partial_v^-\bar{R}$, where $\partial c=\partial^+c\cup\partial^-c$,
(3) $\mathcal{D}'^3_{h_+}=D'_2\times(\partial^+c'\times[-1,1])=D'_2\times\partial_h^+\bar{R}$,
(4) $\mathcal{D}'^3_{h_-}=D'_2\times(\partial^-c'\times[-1,1])=D'_2\times\partial_h^-\bar{R}$, where $\partial c'=\partial^+c'\cup\partial^-c'$.

Now, by following the step by step construction of W from X_1 and X'_1, one can easily see that the properly embedded discs $\mathcal{D}^3_{v_+},\mathcal{D}^3_{v_-}\subset X_1$ and $\mathcal{D}'^3_{h_+},\mathcal{D}'^3_{h_-}\subset X'_1$ canonically embeds into W and we denote these discs in W as $\mathcal{D}^3_+,\mathcal{D}^3_-,\mathcal{D}'^3_+$ and $\mathcal{D}'^3_-$, respectively. Note that the gluing map $F:M\to M'$ is the identity map when

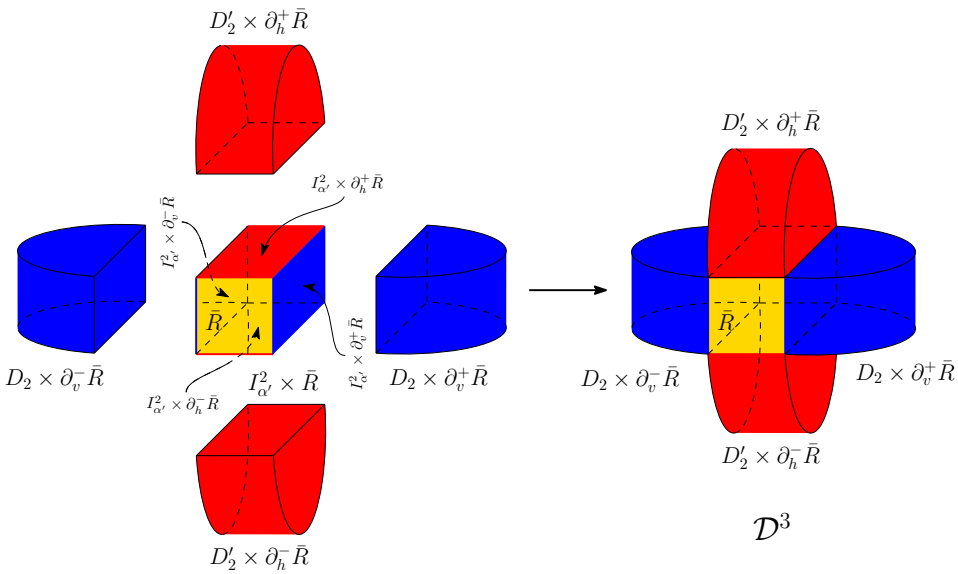


FIGURE 9. The desired disc $\mathcal{D}^3$.

restricted to $\{t\} \times \partial_v \bar{R}$ for each $t \in I^2_{\alpha'}$. Therefore,

$$\begin{aligned}\mathcal{D}^3 &= \mathcal{D}^3_+ \cup \mathcal{D}^3_- \cup \mathbf{D}^3 \cup \mathcal{D}'^3_+ \cup \mathcal{D}'^3_- \\ &= \left(D_2 \times \partial_v^+ \bar{R} \cup D_2 \times \partial_v^- \bar{R}\right) \bigcup_{Id_v} \left(I^2_{\alpha'} \times \bar{R}\right) \bigcup_{Id_h} \left(D_2' \times \partial_h^+ \bar{R} \cup D_2' \times \partial_h^- \bar{R}\right)\end{aligned}$$

is a properly embedded 3–dimensional disc in W, where $Id_v : I^2_{\alpha'} \times \partial_v^{\pm} \bar{R} \to M'$ is the restriction of the map F to $I^2_{\alpha'} \times \partial_v^{\pm} \bar{R}$ and $Id_h : I^2_{\alpha'} \times \partial_h^{\pm} \bar{R} \to I^2_{\alpha'} \times \partial_h^{\pm} \bar{R}$ is the identity map, see Figure 9. Now, by the construction of the disc $\mathcal{D}^3$, one can easily see that $\mathcal{D}^3$ divides W into W_1 and W_2, i.e.

$$W \setminus \mathcal{D}^3 = W_1 \sqcup W_2,$$

where

(1) $W_1 = X \bigcup (D_2' \times (\Sigma_\alpha \setminus R))$
(2) $W_2 = \left(D_2 \times (\Sigma' \setminus R') \bigcup (X' \setminus (\mathring{I}^2_{\alpha'} \times H_{\alpha'}))\right) \bigcup (I^2_\alpha \times H_{\alpha'})$, where $H_{\alpha'} = \bigcup_{\alpha'_i \in \alpha'} H_{\alpha'_i}$.

From the above description of W_1 and W_2 and the gluing maps involved in the construction of W, we can see that $W_1 \cup \mathcal{D}^3$ is diffeomorphic to X and $W_2 \cup \mathcal{D}^3$ is diffeomorphic to X'. This completes the proof of the theorem.

□

Now, we obtain the following Gabai's result from [6] as a corollary to Theorem 4.4.

Corollary 4.5. *Let (P, ϕ) and (P', ϕ') be two abstract open books. Then, $M_{(P,\phi)*(P',\phi')}$ is diffeomorphic to $M_{(P,\phi)} \# M_{(P',\phi')}$.*

Proof. Let $\mathcal{OB}(P,\phi)$ and $\mathcal{OB}(P',\phi')$ be the open books on $M_{(P,\phi)}$ and $M_{(P',\phi')}$, respectively. Let X and X' be the compact oriented 4–manifolds with $\partial X = M_{(P,\phi)}$ and $\partial X' = M_{(P',\phi')}$, respectively. By [3], there exist relative trisections $\mathcal{T}$ and $\mathcal{T}'$ on X and X' such that the open books $\mathcal{OB}$ and $\mathcal{OB}'$ on ∂X and $\partial X'$ induced by $\mathcal{T}$ and $\mathcal{T}'$ coincide with the open books $\mathcal{OB}(P,\phi)$ and $\mathcal{OB}(P',\phi')$, respectively. By Theorem 4.4, there exists a relative trisection $\mathcal{T}_{X\natural X'}$ on $X\natural X'$ such that the open book on $\partial(X\natural X')$ induced by $\mathcal{T}_{X\natural X'}$ is the Murasugi sum $\mathcal{OB}(P * P', \phi * \phi')$ of $\mathcal{OB}(P,\phi)$ and $\mathcal{OB}(P',\phi')$. Now, the proof of the Corollary follows as $\partial(X\natural X') = \partial X \# \partial X' = M_{(P,\phi)} \# M_{(P',\phi')}$. □

4.1. **Remarks on general Murasugi sum of relative trisections.** Recall that in Definition 4.1, we have defined the notion of a Murasugi sum of compact surfaces along embedded rectangles R and R' in the surfaces. We would like to point out that this notion is a particular case of the notion of a Murasugi sum of compact surfaces along embedded discs P and P' as $2n$–gons in the surfaces.

Definition 4.6. *Let Σ and Σ' be two compact surfaces with boundary. Suppose that P and P' are the embedded discs in Σ and Σ' as $2n$–gons with the consecutive edges $e_1, e_2, \ldots, e_{2n}$ and $e'_1, e'_2, \ldots, e'_{2n}$, respectively such that the alternating edges $e_2, e_4, e_6, \ldots, e_{2n}$ of P are contained in $\partial\Sigma$ and the alternating edges $e'_1, e'_3, e'_5, \ldots, e'_{2n-1}$ of P' are contained in $\partial\Sigma'$. A Murasugi sum of Σ and Σ' along $P \subset \Sigma$ and $P' \subset \Sigma'$ is a compact surface*

$$\Sigma *_{\mathbf{P}} \Sigma' = \Sigma \bigcup_{P=P'} \Sigma'$$

obtained by identifying Σ and Σ' along the polygonal discs P and P' by a diffeomorphism $g : P \to P'$ so that $g(e_i) = e'_i$.

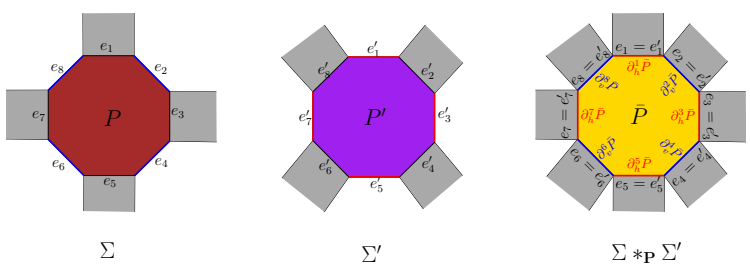


FIGURE 10. Local picture of the Murasugi sum of Σ and Σ' along the 8–gons $P \subset \Sigma$ and $P' \subset \Sigma'$.

Now, it is easy to see that the notion of a general Murasugi sum of two abstract open books (Σ,ϕ) and (Σ',ϕ') along embedded polygonal discs P and P' in Σ and Σ' with the alternating edges of P and P' in $\partial\Sigma$ and $\partial\Sigma'$ can be defined analogously as described at the beginning of this section.

We also observe that given two relative trisection diagrams $(\Sigma,\alpha,\beta,\gamma)$ and $(\Sigma',\alpha',\beta',\gamma')$ and given the embedded discs P and P' in Σ and Σ' as $2n$–gons with the alternating edges of P and P' in $\partial\Sigma$ and $\partial\Sigma'$, respectively such that P is away from α curves in Σ and P' is away from α' curves in Σ', we can define a Murasugi sum of the given relative trisection diagrams in the same way defined in Definition 4.2.

Theorem 4.4 and Corollary 4.5 can be generalized for the general Murasugi sum of trisections and the general Murasugi sum of open books. The proofs follow the same line of arguments in the proof of Theorem 4.4 and Corollary 4.5 by replacing

(1) R, R' and $\bar{R}$ by P, P' and $\bar{P}$, respectively,
(2) $\partial_h^{\pm}\bar{R}$ by $\partial_h^1\bar{P}, \partial_h^3\bar{P}, \partial_h^5\bar{P}, \ldots, \partial_h^{2n-1}\bar{P}$ and $\partial_v^{\pm}\bar{R}$ by $\partial_v^2\bar{P}, \partial_v^4\bar{P}, \partial_v^6\bar{P}, \ldots, \partial_v^{2n}\bar{P}$, and
(3) the disc $\mathcal{D}^3$ as shown in Figure 9 by the disc $\mathcal{D}^3$ obtained by appropriate gluing as shown in Figure 11.

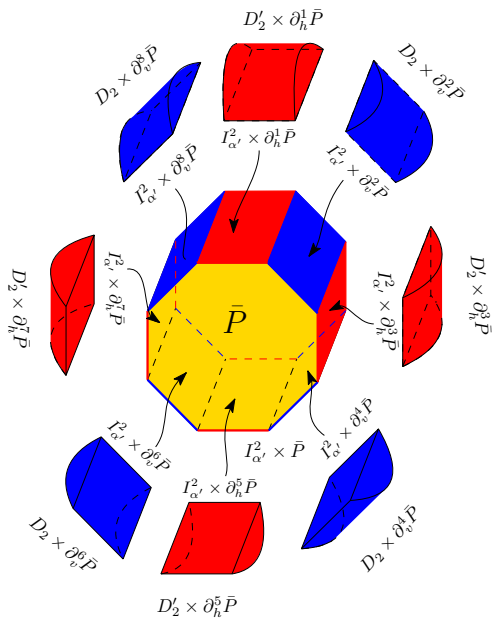


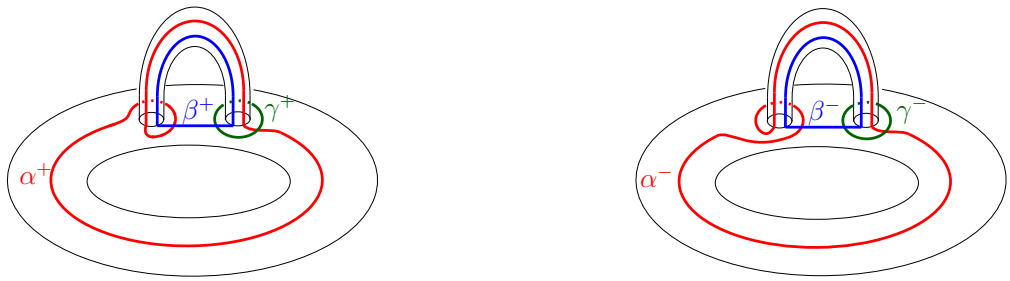
FIGURE 11. The desired disc $\mathcal{D}^3$ in case of Murasugi sum of relative trisections diagrams along 8-gons

4.2. **Stabilizations of relative trisections.** Now, we discuss the notion of the stabilization of a relative trisection on a compact 4−manifold with connected boundary.

Recall that the 4−dimensional disc D^4 admits the relative trisection diagrams $\mathcal{D}^+ = (\Sigma_{1,2},\ \alpha^+,\ \beta^+,\ \gamma^+)$ and $\mathcal{D}^- = (\Sigma_{1,2},\ \alpha^-,\ \beta^-,\ \gamma^-)$ as shown in Figure 12 with induced open books of $S^3 = \partial D^4$ with page an annulus and the monodromies the positive and negative Dehn twists, respectively. For more details, refer to Figure 15.

Definition 4.7. *A stabilization of a relative trisection diagram $\mathcal{D} = (\Sigma,\ \alpha,\ \beta,\ \gamma)$ is a relative trisection diagram $\mathcal{D}^{\pm}_{st} = \mathcal{D} * \mathcal{D}^{\pm}$, where $\mathcal{D}^{\pm}$ are the relative trisection diagrams of the 4–disc D^4 as described above.*

Definition 4.8. *A $\pm$ve stabilization of an abstract open book (Σ, ϕ) is an abstract open book $S_{\pm}(\Sigma, \phi) = (\Sigma_{st}, \phi \circ d_{\gamma}^{\pm})$ with $\Sigma_{st} = \Sigma \cup 1-$handle, where $d_{\gamma}^{\pm}$ the $\pm$ve Dehn twists along a simple closed curve γ in Σ_{st}, respectively such that γ intersects the co-core of the 1–handle exactly once.*



$\mathcal{D}^+ = (\Sigma_{1,2},\ \alpha^+,\ \beta^+,\ \gamma^+)$ $\mathcal{D}^- = (\Sigma_{1,2},\ \alpha^-,\ \beta^-,\ \gamma^-)$

FIGURE 12. Relative trisection diagrams $\mathcal{D}^+$ and $\mathcal{D}^-$ of the 4–disc D^4 with induced open books of S^3 with page an annulus and the monodromies the positive and negative Dehn twists, respectively.

One can easily see the following proposition:

Proposition 4.9. *Let $\mathcal{D}$ be a relative trisection diagram. Then,*

(1) $X_{\mathcal{D}^{\pm}_{st}} = X_{\mathcal{D}}$,
(2) the induced open book of $\partial X_{\mathcal{D}^{\pm}_{st}}$ is a $\pm$ve stabilization of the induced open book of $\partial X_{\mathcal{D}}$.

Proof. By Theorem 4.4, we can see that

$$X_{\mathcal{D}^{\pm}_{st}} = X_{\mathcal{D}} \natural X_{\mathcal{D}^{\pm}} = X_{\mathcal{D}} \natural D^4 = X_{\mathcal{D}}.$$

□

5. Murasugi sum of bounded achiral Lefschetz fibrations

In this section, we discuss the notion of a Murasugi sum of two abstract bounded achiral Lefschetz fibrations.

5.1. Bounded achiral Lefschetz fibration.

Let us discuss the notion of bounded achiral Lefschetz fibrations on 4–manifolds.

Definition 5.1. *Let X be a compact, connected, oriented smooth 4–manifold with non-empty boundary. A smooth surjective map $\pi : X \to D^2$ is said to be a bounded achiral Lefschetz fibration (BALF) if*

(1) the map π has finitely many critical (or singular) points q_i's, all lie in the interior of X with the critical (or singular) values $f(q_i) = p_i$ and $p_i \neq p_j$, for $i \neq j$,
(2) for each critical point q_i and its corresponding critical value p_i, there exist coordinate charts $U_{q_i} = \mathbb{C}^2$ around $q_i = (0,0)$ and $V_{p_i} = \mathbb{C}$ around $p_i = 0$ in X and D^2, which agree with the orientations of X and D^2, respectively such that the restriction of π on U_{q_i} is the map $\pi : U_{q_i} \to V_{p_i}$ defined as $\pi(z_1, z_2) = z_1^2 + z_2^2$ or $\pi(z_1, z_2) = z_1^2 + \bar{z}_2^2$,
(3) for a regular value p, the fiber is a compact, connected, oriented surface with non-empty boundary.

The fiber of π over a singular value is called a *singular fiber* and the fiber of π over a regular value is called a *regular fiber*. If a singularity is locally modeled by the complex map $\pi(z_1, z_2) = z_1^2 + z_2^2$, we call the singularity a *Lefschetz singularity* and if a singularity is locally modeled by the complex map $\pi(z_1, z_2) = z_1^2 + \bar{z}_2^2$, we call the singularity an *achiral Lefschetz singularity*.

5.2. **Topology of a bounded achiral Lefschetz fibration.** The following description is from [7] and [10]. Consider a BALF $\pi : X \to D^2$ on a compact oriented 4–manifold X with non–empty boundary ∂X. Let $q_1, q_2, ..., q_k$ be the critical points of π and let $\pi(q_i) = p_i$ be the critical value corresponding to the critical point q_i. Fix a regular value p in the interior of D^2 and an identification of the fiber $\pi^{-1}(p)$ with a compact oriented surface $\Sigma = \Sigma_{g,n}$ of genus g with n boundary components. For each i, choose an arc α_i in D^2 connecting the point p and the singular value p_i such that α_i and α_j intersect only at p, when $i \neq j$. The labels p_i's for the singular values and the arcs α_i's can be chosen in the anticlockwise sense with respect to a small circle around p.

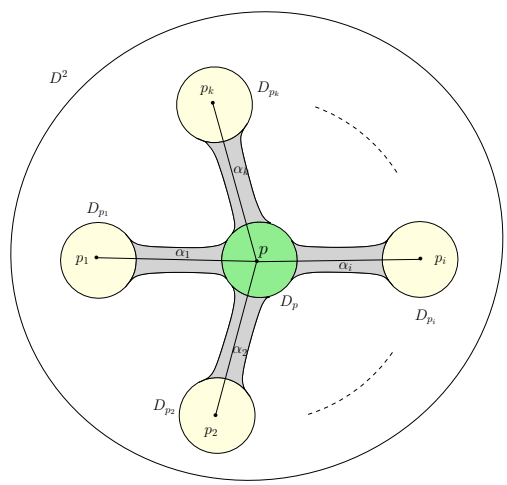


FIGURE 13. The critical values of the Lefschetz fibration π in D^2.

For each critical value p_i, choose a disc D_{p_i} centered at p_i and for the regular value p, choose a disc D_p centered at p such that the discs D_p and D_{p_i}'s are disjoint from each other in D^2, see Figure 13.

In the complement of the critical values, the map

$$\pi|_{X \setminus (\cup_{i=1}^k \pi^{-1}(p_i))} : X \setminus (\cup_{i=1}^k \pi^{-1}(p_i)) \to D^2 \setminus \{p_1, p_2, ..., p_k\}$$

is a locally trivial fiber bundle. As every fiber bundle over a disc is trivial, $\pi^{-1}(D_p) = D_p \times \Sigma$. Let $\nu(\alpha_i)$ denote a regular neighborhood of the arc α_i in D^2.

Note that $X_0 = \pi^{-1}(D_p \cup (\cup_{i=1}^k \nu(\alpha_i)) \cup (\cup_{i=1}^k D_{p_i}))$ is diffeomorphic to $D_p \times \Sigma$ with the 2–handles H_{p_i}'s attached along the simple closed curves γ_i's in the fibers $x_i \times \Sigma \subset \partial D^2 \times \Sigma$, where x_i is the intersection of ∂D_p and α_i. The 2–handle H_{p_i} corresponds to the singularity q_i and the framing of the handle attachment is -1 with respect to the surface framing if q_i is a Lefschetz singularity and is $+1$ with respect to the surface framing if q_i is an achiral Lefschetz singularity.

We call the curve γ_i the *vanishing cycle* corresponding to the singularity q_i. The monodromy of the fibration π around each singular value p_i is given by a positive (negative) handed Dehn twist about the simple closed curve γ_i in Σ. Recall that by a positive (negative) *Dehn twist* on an embedded circle c in a surface Σ, we mean a diffeomorphism obtained by cutting Σ along c and twisting $360°$ to the right (left) and re-gluing it back. The Dehn twist d_{γ_i} is positive if q_i is a Lefschetz singularity and negative if q_i is an achiral Lefschetz singularity. The vanishing cycle γ_i collapses to the critical point q_i on a singular fiber as one gets near the critical point q_i. The boundary monodromy of the Lefschetz fibration $\pi : X_0 = \pi^{-1}(D) \to D$ is the product $d_{\gamma_1} d_{\gamma_2} \dots d_{\gamma_k}$ (we will write compositions of Dehn twists in a monodromy as words from left-to-right) of Dehn twists d_{γ_i}'s along the vanishing cycles γ_i's, where

$$D = D_p \cup (\cup_{i=1}^{k} \nu(\alpha_i)) \cup (\cup_{i=1}^{k} D_{p_i}).$$

Since there are no critical values of the Lefschetz fibration π outside the disc D and the disc D^2 is isotopic to the disc D, we can assume $D^2 = D$ and $X^4 = X_0$. From the above discussion, we can associate a pair (Σ, Γ) to the BALF $\pi : X \to D^2$, where Σ is a compact oriented surface of genus g with non-empty boundary and $\Gamma = (d_{\gamma_1}, d_{\gamma_2}, \dots, d_{\gamma_k})$ is an ordered k–tuple of Dehn twists along the curves γ_i's in Σ.

Remark 5.2. *This association is not unique as it depends on an identification of a regular fiber with Σ and a choice of the arcs α_i's joining the regular value p and the singular values p_i's, once we fix a regular value p.*

Note that the BALF $\pi : X \to D^2$ induces the open book $\mathcal{OB}(\Sigma, \phi)$ on the boundary ∂X with page Σ and the monodromy the product $\phi = d_{\gamma_1} d_{\gamma_2} \dots d_{\gamma_k}$ of Dehn twists $d_{\gamma_1}, d_{\gamma_2}, \dots, d_{\gamma_k}$ along the curves $\gamma_1, \gamma_2, \dots, \gamma_k$ in Σ, respectively. The open book $\mathcal{OB}(\Sigma, \phi)$ on ∂X has the following decomposition:

$$\partial X = \pi^{-1}(\partial D^2) \cup D^2 \times \partial \Sigma.$$

Definition 5.3. *An abstract bounded achiral Lefschetz fibration (BALF) is a pair (Σ, Γ), where Σ is a compact oriented surface with non–empty boundary and $\Gamma = (d_{\gamma_1}, d_{\gamma_2}, ..., d_{\gamma_k})$ is an ordered collection of Dehn twists along the simple closed curves $\gamma_1, \gamma_2, ..., \gamma_k$ on Σ.*

Given an ordered collection of Dehn twists $\Gamma = (d_{\gamma_1}, d_{\gamma_2}, ..., d_{\gamma_k})$ along the simple closed curves $\gamma_1, \gamma_2, ..., \gamma_k$ on a compact oriented surface Σ with non-empty boundary, we can construct a BALF over D^2 as follows: First, consider the trivial bundle $\pi : D^2 \times \Sigma \to D^2$. Let $p_1, p_2, ..., p_k$ be distinct points in $\partial D^2 = S^1$ ordered in the anticlockwise sense. For each i, consider the curve γ_i in the fiber $p_i \times \Sigma \subset \partial D^2 \times \Sigma$. Attach a 4–dimensional 2–handle H_i along γ_i to $D^2 \times \Sigma$ with framing -1 (respectively, $+1$) with respect to the fiber framing if the Dehn twist d_{γ_i} is positive (respectively, negative). The resulting manifold $X = (D^2 \times \Sigma) \cup H_1 \cup H_2 \cup ... \cup H_k$ admits an achiral Lefschetz fibration $\pi : X \to D^2$ with the regular fiber Σ and the vanishing cycles $\gamma_1, \gamma_2, ..., \gamma_k$. We denote the manifold X associated to the abstract BALF by $X_{(\Sigma,\Gamma)}$. We say that the pair (Σ, Γ) is an abstract BALF of a compact 4–manifold X if X is diffeomorphic to the 4–manifold $X_{(\Sigma,\Gamma)}$.

One can see that there is a one-to-one correspondence between the equivalence classes of BALFs $\pi : X \to D^2$ having the regular fibers, a surface $\Sigma_{g,n}$ of genus g with n boundary components and the equivalence classes of abstract BALF (Σ, Γ),

where Γ is an ordered k–tuples of Dehn twists $(d_{\gamma_1}, d_{\gamma_2}, ..., d_{\gamma_k})$ on the surface $\Sigma_{g,n}$, up to cyclic permutations, the Hurwitz equivalence and the global conjugations by orientation preserving diffeomorphisms of Σ. For the notions of cyclic permutations, the Hurwitz equivalence and the global conjugations refer [5].

5.3. **Murasugi sum of bounded achiral Lefschetz fibrations.** In this subsection, we define the notion of a Murasugi sum of two abstract BALFs.

Definition 5.4. *Let (Σ, Γ) and (Σ', Γ') be two BALFs, where $\Gamma = (d_{\gamma_1}, d_{\gamma_2}, ..., d_{\gamma_k})$ and $\Gamma' = (d_{\gamma'_1}, d_{\gamma'_2}, ..., d_{\gamma'_{k'}})$ are the ordered collection of Dehn twists along the simple closed curves $\gamma_1, \gamma_2, ..., \gamma_k$ and $\gamma'_1, \gamma'_2, ..., \gamma'_{k'}$ on the compact oriented surfaces Σ and Σ', respectively. A Murasugi sum $(\Sigma, \Gamma) * (\Sigma', \Gamma')$ of (Σ, Γ) and (Σ', Γ') is a BALF $(\Sigma * \Sigma', \Gamma * \Gamma')$, where*

(1) $\Sigma * \Sigma'$ is a Murasugi sum of Σ and Σ',
(2) $\Gamma * \Gamma' = (d_{\gamma_1}, d_{\gamma_2}, ..., d_{\gamma_k}, d_{\gamma'_1}, d_{\gamma'_2}, ..., d_{\gamma'_{k'}})$.

Theorem 5.5. *Let (Σ, Γ) and (Σ', Γ') be two BALFs, where $\Gamma = (d_{\gamma_1}, d_{\gamma_2}, ..., d_{\gamma_k})$ and $\Gamma' = (d_{\gamma'_1}, d_{\gamma'_2}, ..., d_{\gamma'_{k'}})$ are the ordered collection of Dehn twists along the simple closed curves $\gamma_1, \gamma_2, ..., \gamma_k$ and $\gamma'_1, \gamma'_2, ..., \gamma'_{k'}$ on the compact oriented surfaces Σ and Σ', respectively. Let $(\Sigma * \Sigma', \Gamma * \Gamma')$ be a Murasugi sum of (Σ, Γ) and (Σ', Γ'). Then, the 4–manifold $X_{(\Sigma,\Gamma)*(\Sigma',\Gamma')}$ is diffeomorphic to the boundary connected sum $X_{(\Sigma,\Gamma)} \natural X_{(\Sigma',\Gamma')}$ of $X_{(\Sigma,\Gamma)}$ and $X_{(\Sigma',\Gamma')}$. Moreover, the open book on the boundary of $X_{(\Sigma,\Gamma)} \natural X_{(\Sigma',\Gamma')}$ is a Murasugi sum of the open books on the boundaries of $X_{(\Sigma,\Gamma)}$ and $X_{(\Sigma',\Gamma')}$.*

Proof. Let $\Sigma * \Sigma'$ be the Murasugi sum of Σ and Σ' along the properly embedded arcs $c \subset \Sigma$ and $c' \subset \Sigma'$. Let $\bar{R}$ be the rectangle in $\Sigma * \Sigma'$ obtained by identifying $R \subset \Sigma$ and $R' \subset \Sigma'$ as in Definition 4.1. We regard $\bar{R}$ as $[-1, 1] \times [-1, 1]$ with $c \times \{0\} = [-1, 1] \times \{0\}$ and $c' \times \{0\} = \{0\} \times [-1, 1]$. We denote the vertical boundary $\{\pm 1\} \times [-1, 1]$ of $\bar{R}$ by $\partial_v^{\pm} \bar{R}$ and the horizontal boundary $[-1, 1] \times \{\pm 1\}$ of $\bar{R}$ by $\partial_h^{\pm} \bar{R}$.

In order to prove the theorem, we show that there is a properly embedded 3–disc $\mathcal{D}^3$ in $X_{(\Sigma,\Gamma)*(\Sigma',\Gamma')}$ such that

$$X_{(\Sigma,\Gamma)*(\Sigma',\Gamma')} \setminus \mathcal{D}^3 = X \sqcup X',$$

where

(1) $X \cup \mathcal{D}^3$ is diffeomorphic to $X_{(\Sigma,\Gamma)}$,
(2) $X' \cup \mathcal{D}^3$ is diffeomorphic to $X_{(\Sigma',\Gamma')}$.

In order to construct the required disc $\mathcal{D}^3$, we recall the construction of 4–manifold $X_{(\Sigma,\Gamma)*(\Sigma',\Gamma')}$ as follows: Let D^2 be the 2–disc together with the points $p_1, p_2, \dots p_k$ appearing in the anticlockwise sense on the upper semicircle of ∂D^2 and the points $p'_1, p'_2, \dots, p'_{k'}$ appearing in the anticlockwise sense on the lower semicircle of ∂D^2 as shown in Figure 14. Now, the 4–manifold $X_{(\Sigma,\Gamma)*(\Sigma',\Gamma')}$ can be obtained from $D^2 \times (\Sigma * \Sigma')$ by attaching the 2–handles H_{γ_i}'s and $H_{\gamma'_j}$'s along $\gamma_i \subset \{p_i\} \times \Sigma * \Sigma'$ and $\gamma'_j \subset \{p'_j\} \times \Sigma * \Sigma'$ for each $i = 1, 2, \dots k$ and $j = 1, 2, \dots, k'$, respectively. The 2–handle H_{γ_i} is attached with framing -1 (respectively, $+1$) with respect to the fiber farming if the Dehn twist d_γ is positive (respectively, negative) and The 2–handle $H_{\gamma'_j}$ is attached with the framing -1 (respectively,

+1) with respect to the fiber farming if the Dehn twist $d_{\gamma'_j}$ is positive (respectively, negative).

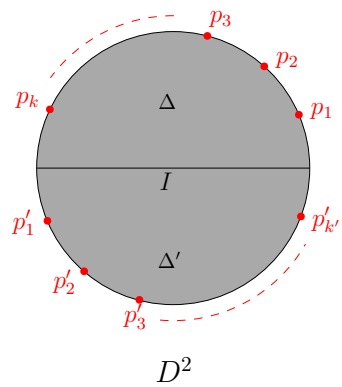


FIGURE 14. The various parts of the Disc $D^2 = \Delta \cup \Delta'$.

Now, we use the above construction of $X_{(\Sigma,\Gamma)*(\Sigma',\Gamma')}$ to describe the desired properly embedded disc $\mathcal{D}^3$ as follows: Let Δ and Δ' be the upper and the lower half 2–discs in D^2 as shown in Figure 14. Let $I = \Delta \cap \Delta'$. Let $\mathcal{D}^3$ be the subset of $X_{(\Sigma,\Gamma)*(\Sigma',\Gamma')}$ as given below:

$$\mathcal{D}^3 = \left(\Delta \times \partial_v^+ \bar{R} \cup \Delta \times \partial_v^- \bar{R}\right) \bigcup \left(I \times \bar{R}\right) \bigcup \left(\Delta' \times \partial_h^+ \bar{R} \cup \Delta' \times \partial_h^- \bar{R}\right).$$

One can easily see that $\mathcal{D}^3$ is a properly embedded disc in $X_{(\Sigma,\Gamma)*(\Sigma',\Gamma')}$. Observe that

$$X_{(\Sigma,\Gamma)*(\Sigma',\Gamma')} \setminus \mathcal{D}^3 = X \sqcup X',$$

where

(1) $X = \left(\bigcup_{i=1}^{k} H_{\gamma_i} \cup (\Delta \times \Sigma)\right) \cup (\Delta' \times (\Sigma \setminus R)$,

(2) $X' = (\Delta \times (\Sigma' \setminus R')) \cup \left((\Delta' \times \Sigma) \bigcup_{i=1}^{k'} H_{\gamma'_i}\right)$.

By recalling the construction of $X_{(\Sigma,\Gamma)}$ and $X_{(\Sigma',\Gamma')}$ from $D^2 \times \Sigma$ and $D^2 \times \Sigma'$, respectively, we observe that the 4–manifolds $X \cup \mathcal{D}^3 = X_{(\Sigma,\Gamma)} \setminus (\Delta' \overset{\circ}{\times} R)$ which is diffeomorphic to the 4–manifolds $X_{(\Sigma,\Gamma)}$ and $X' \cup \mathcal{D}^3 = X_{(\Sigma',\Gamma')} \setminus (\Delta \overset{\circ}{\times} R')$ which is diffeomorphic to the 4–manifold $X_{(\Sigma',\Gamma')}$. This completes the proof of the theorem. □

Another argument for Gabai's result from [6] (refer Corollary 4.5) is as follows: Let (Σ, ϕ) and (Σ', ϕ') be two abstract open books. By Lickorish [9], we can write –up to isotopy– $\phi = d_{\gamma_1} d_{\gamma_2} \dots d_{\gamma_k}$ and $\phi' = d_{\gamma'_1} d_{\gamma'_2} \dots d_{\gamma'_{k'}}$, where d_{γ_i}'s and $d_{\gamma'_j}$'s are the Dehn twists along the simple closed curves γ_i's and γ'_j's in Σ and Σ', respectively. Consider the compact oriented 4–manifolds $X_{(\Sigma,\Gamma)}$ and $X_{(\Sigma',\Gamma')}$ associated to the abstract BALFs (Σ, Γ) and (Σ', Γ'), respectively, where $\Gamma = (d_{\gamma_1}, d_{\gamma_2}, \dots, d_{\gamma_k})$ and $\Gamma' = (d_{\gamma'_1}, d_{\gamma'_2}, \dots, d_{\gamma'_{k'}})$. Note that $\partial X_{(\Sigma',\Gamma')} = M_{(\Sigma,\phi)}$ and $\partial X_{(\Sigma',\Gamma')} = M_{(\Sigma',\phi')}$. By Theorem 5.5, the open book on the boundary $\partial(X_{(\Sigma,\gamma)} \natural X_{(\Sigma',\Gamma')})$ is

the Murasugi sum $\mathcal{OB}(\Sigma * \Sigma', \phi * \phi')$. Now, the proof of the Corollary follows as $\partial(X_{(\Sigma,\Gamma)} \natural X_{(\Sigma',\Gamma')}) = \partial X_{(\Sigma,\Gamma)} \# \partial X_{(\Sigma',\Gamma')} = M_{(\Sigma,\phi)} \# M_{(\Sigma',\phi')}$.

5.4. **Stabilizations of Lefschetz fibrations.** Now, we discuss the notion of the stabilization of a BALF on a compact 4−manifold.

Note that the 4−disc D^4 admits a bounded Lefschetz fibration $(\mathcal{A}, \Gamma^+ = (d_c^+))$, where $\mathcal{A}$ is an annulus and d_c^+ is the positive Dehn twist along the central curve c of $\mathcal{A}$. The 4−disc D^4 also admits a bounded achiral Lefschetz fibration $(\mathcal{A}, \Gamma^- = (d_c^-))$, where d_c^- is the negative Dehn twist along the central curve c of the annulus $\mathcal{A}$.

Definition 5.6. *A stabilization of an abstract BALF (Σ, Γ) is an abstract BALF $S_\pm(\Sigma, \Gamma) = (\Sigma_{st}, \Gamma_{st}^\pm)$ with $\Sigma_{st} = \Sigma \cup 1-$handle and $\Gamma_{st}^\pm = (\Gamma, d_\gamma^\pm)$, where d_γ^+ and d_γ^- are the positive and the negative Dehn twists along a simple closed curve γ in Σ_{st}, respectively such that γ intersects the co-core of the 1–handle exactly once.*

One can easily see the following proposition:

Proposition 5.7. *Let (Σ, Γ) be an abstract BALF. Let $(\mathcal{A}, \Gamma^\pm = (d_c^\pm))$ be the BALFs of D^4. Then,*

$$S_\pm(\Sigma, \Gamma) = (\Sigma, \Gamma) * (\mathcal{A}, \Gamma^\pm).$$

Hence,

$$X_{S_\pm(\Sigma,\Gamma)} = X_{(\Sigma,\Gamma)*(\mathcal{A},\Gamma^\pm)}.$$

As an application of the above proposition, we have the following proposition:

Proposition 5.8. *Let (Σ, Γ) be an abstract BALF. Then,*

$$X_{S_\pm(\Sigma,\Gamma)} = X_{(\Sigma,\Gamma)}.$$

Proof. From Proposition 5.7, we have $X_{S_\pm(\Sigma,\Gamma)} = X_{(\Sigma,\Gamma)*(\mathcal{A},\Gamma^\pm)}$.

Hence by Theorem 5.5, we can see that $X_{S_\pm(\Sigma,\Gamma)} = X_{(\Sigma,\Gamma)} \natural D^4 = X_{(\Sigma,\Gamma)}$. □

5.5. **Remarks on a Murasugi sum of relative trisections and a Murasugi sum of BALFs.** In this subsection, we discuss some remarks on a Murasugi sum of relative trisections and a Murasugi sum of bounded achiral Lefschetz fibrations using results in the previous sections.

Let $\pi : X \to D^2$ be a BALF of a compact oriented 4–manifold X associated to an abstract BALF (Σ, Γ). One can use an algorithm given in [1] to get a relative trisection $\mathcal{T}$ of X obtained from the BALF π such that the open books on ∂X induced by $\mathcal{T}$ and the BALF π coincide, see [[1]; Corollary 18]. This algorithm also allows us to get a relative trisection diagram $\mathcal{D}$ of the relative trisected 4–manifold X from the abstract BALF (Σ, Γ). We denote the relative trisection diagram $\mathcal{D}$ associated to the BALF (Σ, Γ) by $\mathcal{D}_{(\Sigma,\Gamma)}$. More precisely, we state the result as follows:

Theorem 5.9 (Castro, Gay and Pinzón-Caicedo [1])**.** *Let $\pi : X \to D^2$ be a BALF with regular fiber a compact surface P of genus p with b boundary components, and with n vanishing cycles. Then, the manifold X admits a $(p+n, 2p+b-1; p, b)$–relative trisection $\mathcal{T}$ such that the induced open books on ∂X by the trisection $\mathcal{T}$ and the BALF π coincide.*

The proof of this result follows from the following lemma.

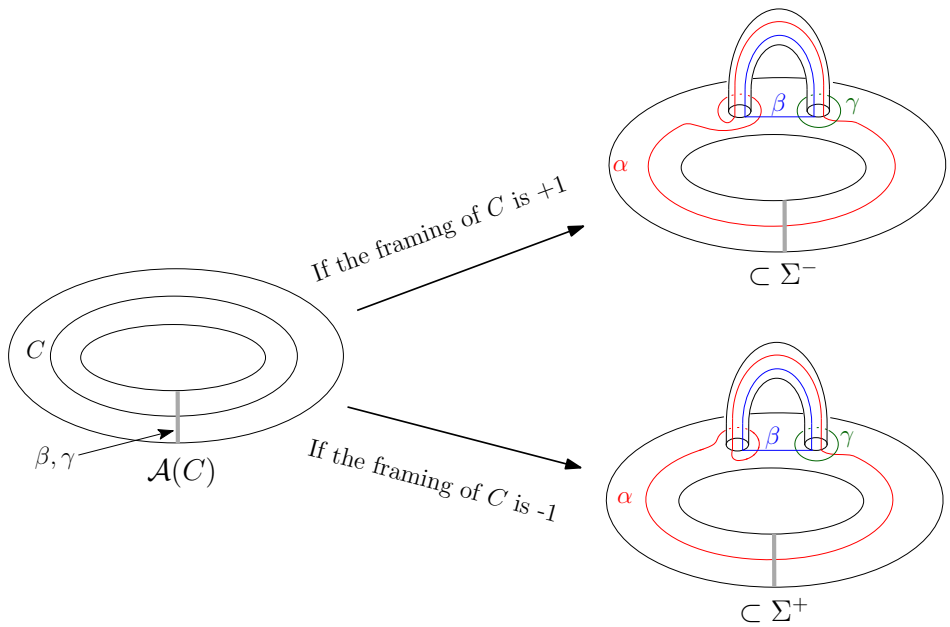


FIGURE 15. The above figure shows how to get $\Sigma^\pm$ from Σ by replacing an annular neighborhood $\mathcal{A}(C)$–depicted on the left– of the simple closed curve C in Σ by twice punctured torus–depicted on the right– with new α, β, γ curves according to the framing ∓ 1 of C.

Lemma 5.10 (Castro, Gay and Pinzón-Caicedo [1])**.** *Let $(\Sigma, \alpha, \beta, \gamma)$ be a relative trisection diagram of a compact oriented 4–manifold X. Consider a simple closed curve $C \subset \Sigma$ disjoint from α and transverse to β and γ. Let $(\Sigma^\pm, \alpha^\pm, \beta^\pm, \gamma^\pm)$ be the result of removing an annular neighborhood $\mathcal{A}(C)$ of C together with the β and γ arcs running across this neighborhood and replacing it with a twice-punctured torus as in Figure 15 with β and γ arcs as drawn, and with one new α, β and γ curve as drawn. Then $(\Sigma^\pm, \alpha^\pm, \beta^\pm, \gamma^\pm)$ is a relative trisection diagram for a trisected 4–manifold $X' = X'_1 \cup X'_2 \cup X'_3$, where X' is obtained by attaching a 2–handle to X along $C \subset P$ with framing ∓ 1 relative to P such that the open book on $\partial X'$ has page P with monodromy $d_C^{\pm 1} \circ \mu$. Here, d_C is a right-handed Dehn twist about C.*

Now, we have the following theorem.

Theorem 5.11. *Let $\mathcal{D}_{(\Sigma_1,\Gamma_1)}$ and $\mathcal{D}_{(\Sigma_2,\Gamma_2)}$ are the relative trisection diagrams associated to the abstract BALFs (Σ_1, Γ_1) and (Σ_2, Γ_2), respectively. Let $(\Sigma, \Gamma) = (\Sigma_1 * \Sigma_2, \Gamma_1 * \Gamma_2)$ be a Murasugi sum of the abstract BALFs (Σ_1, Γ_1) and (Σ_2, Γ_2). Then, the relative trisection diagram $\mathcal{D}_{(\Sigma,\Gamma)}$ associated to the abstract BALF (Σ, Γ) is (diffeomorphism and handle slide equivalent to) a Murasugi sum $\mathcal{D}_{(\Sigma_1,\Gamma_1)} * \mathcal{D}_{(\Sigma_2,\Gamma_2)}$ of $\mathcal{D}_{(\Sigma_1,\Gamma_1)}$ and $\mathcal{D}_{(\Sigma_2,\Gamma_2)}$.*

Proof. The proof of the theorem follows by the definitions of the Murasugi sum of relative trisections and the Murasugi sum of BALFs and by the algorithm provided in [1] to get a relative trisection diagram associated to a given abstract BALF. □

One can easily see the following corollary of the above theorem.

Corollary 5.12. *Let (Σ, Γ) be an abstract BALF. Then a relative trisection associated to a stabilisation $S_{\pm}(\Sigma, \Gamma)$ of the abstract BALF (Σ, Γ) is a stabilisation $\mathcal{D}_{st}^{\pm}$ of a relative trisection $\mathcal{D} = \mathcal{D}_{(\Sigma,\Gamma)}$ associated to the abstract BALF (Σ, Γ).*

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Tata Institute of Fundamental Research, 1st Homi Bhabha Road, Colaba, Mumbai – 400 005, India.

Email address: abhijeet@math.tifr.res.in

Indian Institute of Technology Madras, IIT PO. Chennai – 600 036, Tamil Nadu, India.

Email address: suhas@iitm.ac.in

The Institute of Mathematical Sciences, Chennai, IV Cross Road, CIT Campus, Taramani, Chennai – 600 113, Tamil Nadu, India.

Email address: aselvammaths@gmail.com

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