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QUALITATIVE BEHAVIOR OF A CHEMICAL REACTION SYSTEM WITH FRACTIONAL DERIVATIVES

MESSAOUD BERKAL AND JUAN F. NAVARRO

ABSTRACT. Chemical reactions can be successfully investigated using fractional differential equations. This work aims to explore new outcomes such as fixed points, local stability, types of bifurcations, limit cycles for a fractional-order chemical reaction system which arises in chemical reactions. The equilibrium point of this system is presented and method of linearization is successfully implemented to analyse the asymptotic behavior of solutions about the positive equilibrium point. Taking advantage of the bifurcation theory, we examine the existence of a period-doubling bifurcation and a Neimark-Sacker bifurcation. We find that this system undergoes period-doubling and Neimark-Sacker bifurcations under some specific values of bifurcation parameter. Furthermore, the maximum Lyapunov characteristic exponents are shown. We present some 2D diagrams for the phase portraits, local stability, closed invariant curves, types of bifurcations, and the maximum Lyapunov exponents to ensure the chaotic behavior of the considered model. The used techniques can be applied to deal with other high-order models.

1. Introduction

A chemical reaction is defined as a chemical process in which one or more chemical substances produce new chemical substances. Chemical reactions occur in biology, chemistry, nature, etc. Lighting a match, smelting iron, taking medications, burning fuels, brewing beer, making glass and pottery, and cooking are useful examples for chemical reactions that have been well known and utilized for hundreds of years. Moreover, one of the most common chemical reactions occur in plants is called photosynthesis which works on converting carbon dioxide and water into food and oxygen. The oxidation reaction can be clearly seen in iron when it is converted into rust. Electrochemical or redox reactions are used in batteries to produce electrical energy. Furthermore, enzymes in our bodies react with other substances to achieve a specific duty such as digestion. In particular, a certain type of enzyme called amylase starts by breaking down sugars we eat into simpler forms.

Mathematics science is widely used in investigating many chemical models. Several nonlinear phenomena such as stability, bifurcation, chaos, periodic oscillation, boundedness, etc., which emerge in chemical reaction systems have attracted many researchers in recent years. We may efficiently examine various chemical reaction models and expose the relationship between various chemical variables by using certain useful mathematical techniques. Then, we can better understand the intrinsic relationship between many chemical factors and serving humanity. In particular, fractional order differential equations are a vital tool to investigate the dynamical behavior of chemical phenomena. For instance, Xu et al. [29] investigated the existence and uniquencess of the solution of a new fractional

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order delayed Brusselator chemical reaction model. The stability and Hopf bifurcation of this system were also analysed. In [30], Xu et al. analysed the stability, the existence of limit cycle and the existence of Hopf bifurcation for the fractional-order delayed Oregonator system. They found that the stability is controlled by time delay. Din [12] obtained the solutions of a 3-dimensional chaotic system and discussed the global stability using Lyapunov function. The authors in [14] explored the discretization, stability, flip and Hopf bifurcation, and chaos control for Schnakenberg system. A model of three independent intermediate substances was discussed in [19] to investigate chaotic oscillations. Kol'tsov [19] used analytically and numerically methods to study the chaotic behavior of this reaction. Furthermore, Monwanou et al. [22] studied the fixed points of a nonlinear dynamical system of reactions between four molecules. The stability of the equilibrium points, bifurcations structures, Lyapunov exponent, and phase portraits were nicely presented. Bodale and Oancea [7] investigated the dynamics of the dynamics of Willamowski-Rössler model and obtained chaos control in chemical reactions. The steady state multiplicity, limit cycles, power spectra, time series and phase portraits, quasi-periodic and chaotic behaviors, the time-delay reconstruction diagrams, Hopf bifurcation and bifurcation figures of four problems emerged from chemical and biochemical engineering were successfully presented in [6]. Finally, Olabodé et al. [24] analysed chemical systems governing by a forced modified Van der Pol-Duffing oscillator.
The Caputo derivative fro a function f is defined as follows:

Definition 1.1. [25] *Let* $f:(0,\infty)\to \mathrm{R}$ *be a continuous function. Then, The Caputo fractional derivative of order* $0<\alpha\leq 1$ *is given by*

$$\mathscr{D}^{\alpha}F(t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha}F(\tau)d\tau. \tag{1}$$

Here, Γ represents Euler's Gamma function. This derivative has been applied by many researchers [1, 16, 32, 31].
We now discuss the originality of the considered system as shown in [8, 15]. Consider a two-species chemical reaction described by $\mathscr{X}$ and $\mathscr{Y}$ as follows:

$$\mathscr{Z}\xrightarrow{a_1}\mathscr{Y},\quad \mathscr{X}\xrightarrow{a_2}\mathscr{Z},\quad \mathscr{Y}\xrightarrow{a_3}\mathscr{X},\quad 2\mathscr{X}\xrightarrow{a_4}\mathscr{Y},\quad 2\mathscr{X}+\mathscr{Y}\xrightarrow{a_5}3\mathscr{X}, \tag{2}$$

where $\mathscr{X},\mathscr{Y}$ are chemical species and $\mathscr{Z}$ represents the environment. The parameters $a_1, a_2,\cdots, a_5$ are positive numbers representing reaction rate coefficients.
The motivation of this paper comes from various aspects shown as follows. First, the lack in the analyses of types of bifurcations in the literature review. Second, most chemical reactions can be extensively investigated using fractional differential equations. The main purpose of this work is to analyse the equilibrium point, local stability, the period-doubling bifurcation and the Neimark-Sacker bifurcation of the following system:

$$\begin{cases} \mathscr{D}^{\alpha}x(t)=-Ax(t)-2x(t)^2+By(t)+x(t)^2y(t),\\ \mathscr{D}^{\alpha}y(t)=C+x(t)^2-By(t)-x(t)^2y(t), \end{cases} \tag{3}$$

where x and y are concentrations of species $\mathscr{X}$ and $\mathscr{Y}$, respectively. Here, α represents the fractional-order parameter and D^{α} is the fractional derivative.
We use some stability theorems to examine the stability conditions. We also present some numerical

examples to verify the constructed theoretical outcomes. Some 2D figures for the period-doubling and the Neimark-Sacker bifurcations, phase portraits, limit cycles and the maximum Lyapunov exponent are successfully illustrated. As a results, the novelty of this work is to present new outcomes in terms of types of bifurcations, limit cycles, phase portraits, and Lyapunov exponents for the considered dynamical system.

This paper is outlined as follows. Section 2 is devoted to discretize the considered model. In Section 3, we study the local stability of the fixed point of system (6). Moreover, Section 4 presents the period-doubling bifurcation while Section 5 shows the Neimark-Sacker bifurcation. In Section 6, some numerical simulations are extensively explained. Finally, Section 7 concludes this work.

2. Discretization process

There are several studies showing that discrete-time systems exhibit much more interesting dynamical behaviors such as bifurcations and chaos, much better than its counterpart in the continuous-time system. In this work, we aim to discretize the fractional-order chemical reaction system (3) using the piecewise constant argument method [17, 20, 27, 3] as follows.

$$\tag{4} \begin{cases} D_h^\alpha x(t) = -Ax\left(\left[\frac{t}{h}\right]h\right) - 2x\left(\left[\frac{t}{h}\right]h\right)^2 + By\left(\left[\frac{t}{h}\right]h\right) + x\left(\left[\frac{t}{h}\right]h\right)^2 y\left(\left[\frac{t}{h}\right]h\right), \\ D_h^\alpha y(t) = C + x\left(\left[\frac{t}{h}\right]h\right)^2 - By\left(\left[\frac{t}{h}\right]h\right) - x\left(\left[\frac{t}{h}\right]h\right)^2 y\left(\left[\frac{t}{h}\right]h\right), \end{cases}$$

where $h > 0$ is the discretization parameter and $[t]$ denotes the integer part $t \in [nh, (n+1)h)$, for $n = 0, 1, 2, \cdots$. The n-th iterative solution of system (4) is given by

$$\tag{5} \begin{cases} x_{n+1}(t) = x_n(nh) + \dfrac{h^\alpha}{\Gamma(\alpha+1)}\left(-Ax_n(nh) - 2x_n^2(nh) + By_n(nh) + x_n^2(nh)y_n(nh)\right), \\ y_{n+1}(t) = y_n(nh) + \dfrac{h^\alpha}{\Gamma(\alpha+1)}\left(C + x_n^2(nh) - By_n(nh) - x_n^2(nh)y_n(nh)\right), \end{cases}$$

where $t \in [nh, (n+1)h)$. When $t \to (n+1)h$, system (5) is transformed to

$$\tag{6} \begin{cases} x_{n+1} = x_n + \dfrac{h^\alpha}{\Gamma(\alpha+1)}\left(-Ax_n - 2x_n^2 + By_n + x_n^2 y_n\right), \\ y_{n+1} = y_n + \dfrac{h^\alpha}{\Gamma(\alpha+1)}\left(C + x_n^2 - By_n - x_n^2 y_n\right), \end{cases}$$

which is the discretized version of system (3).

3. Local stability of the fixed point of system (6)

In this section, we discuss the existence of a positive fixed point of the discrete chemical reaction system (6). We also present a clear analysis for the local stability of the positive fixed point. In order to find the fixed points of the system (6), we solve the following algebraic system

$$\begin{cases} x = x + \dfrac{h^{\alpha}}{\Gamma(\alpha+1)}\left(-Ax - 2x^2 + By + x^2 y\right), \\ y = y + \dfrac{h^{\alpha}}{\Gamma(\alpha+1)}\left(C + x^2 - By - x^2 y\right). \end{cases}$$

Therefore,

$$\begin{cases} 0 = -Ax - 2x^2 + By + x^2 y, \\ 0 = C + x^2 - By - x^2 y. \end{cases}$$

Hence, the unique positive fixed point is $E_+ = (x^*, y^*)$, where

$$x^* = \left(\frac{-C + \sqrt{A^2 + 4C}}{2}\right), \quad y^* = \frac{C + x^2}{B + x^2} = \left(\frac{4C + A^2 - A\sqrt{A^2 + 4C}}{2(B+C) + A^2 - A\sqrt{A^2 + 4C}}\right).$$

Now, we will analyze the stability of the fixed point E_+, with the help of the following definition and Lemma [2, 3, 4, 20].

Definition 3.1. *The following situations are valid for the fixed point (x, y) of any system.*

(1) *If $|\mu_1| < 1$ and $|\mu_2| < 1$, it is a sink point and locally asymptotically stable,*
(2) *If $|\mu_1| > 1$ and $|\mu_2| > 1$, it is a source point and locally unstable,*
(3) *If $|\mu_1| < 1$ and $|\mu_2| > 1$ or $(|\mu_1| > 1$ and $|\mu_2| < 1)$, it is a saddle point,*
(4) *If $|\mu_1| = 1$ or $|\mu_2| = 1$, it is non-hyperbolic.*

Lemma 3.2. *Consider the polynomial $\rho(\mu) = \mu^2 - \mathscr{T}\mu + \mathscr{D}$, where $\rho(1) > 0$, and μ_1 and μ_2 are the two roots of $\rho(\mu) = 0$. Then,*

(1) *$|\mu_1| < 1$ and $|\mu_2| < 1$ if and only if $\rho(-1) > 0$ and $\rho(0) < 1$.*
(2) *$|\mu_1| > 1$ and $|\mu_2| > 1$ if and only if $\rho(-1) > 0$ and $\rho(0) > 1$.*
(3) *$|\mu_1| < 1$ and $|\mu_2| > 1$ (or $|\mu_1| > 1$ and $|\mu_2| < 1$) if and only if $\rho(-1) < 0$.*
(4) *$\mu_1 = -1$ and $\mu_2 \neq 1$ if and only if $\rho(-1) = 0$ and $\mathscr{T} \neq 0, 2$.*
(5) *μ_1 and μ_2 are complex numbers and $|\mu_1| = |\mu_2| = 1$ if and only if $|\mathscr{T}| < 2$ and $\rho(0) = 1$.*

The Jacobian matrix at the fixed E_+ of the linearization of system (6) is given as

$$\mathscr{J}(E_+) = \begin{pmatrix} 1 + \dfrac{h^{\alpha}(-A + 2x^*(y^* - 2))}{\Gamma(\alpha+1)} & \dfrac{h^{\alpha}(B + (x^*)^2)}{\Gamma(\alpha+1)} \\ \dfrac{2x^* h^{\alpha}(1 - y^*)}{\Gamma(\alpha+1)} & 1 - \dfrac{h^{\alpha}(B + (x^*)^2}{\Gamma(\alpha+1)} \end{pmatrix}, \tag{7}$$

where its characteristic polynomial is

$$\rho(\mu) = \mu^2 - \mathscr{T}\mu + \mathscr{D}, \tag{8}$$

where

$$\begin{aligned}\mathscr{T} &= 2+\frac{h^{\alpha}(2x^*y^*-(x^*)^2-4x^*-(A+B))}{\Gamma(\alpha+1)},\\ \mathscr{D} &= 1+\frac{h^{\alpha}(2x^*y^*-(x^*)^2-4x^*-(A+B))}{\Gamma(\alpha+1)}+\frac{h^{2\alpha}(AB+2Bx^*+A(x^*)^2+2(x^*)^3)}{(\Gamma(\alpha+1))^2}.\end{aligned}$$

Hence,

$$\begin{aligned}\rho(0) &= 1+\frac{h^{\alpha}(2x^*y^*-(x^*)^2-4x^*-(A+B))}{\Gamma(\alpha+1)}+\frac{h^{2\alpha}(AB+2Bx^*+A(x^*)^2+2(x^*)^3)}{(\Gamma(\alpha+1))^2},\\ \rho(-1) &= 4+\frac{2h^{\alpha}(2x^*y^*-(x^*)^2-4x^*-(A+B))}{\Gamma(\alpha+1)}+\frac{h^{2\alpha}(AB+2Bx^*+A(x^*)^2+2(x^*)^3)}{(\Gamma(\alpha+1))^2},\\ \rho(1) &= \frac{h^{2\alpha}(AB+2Bx^*+A(x^*)^2+2(x^*)^3)}{(\Gamma(\alpha+1))^2}>0.\end{aligned}$$

Since $\rho(1)>0$, we can apply Lemma 3.2 and Definition 3.1 to state the following result.

Lemma 3.3. *For the unique positive fixed point E_+ of system (6), let*

$$\Delta=((x^*)^2+4x^*+A+B-2x^*y^*)^2-4(AB+2Bx^*+A(x^*)^2+2(x^*)^3),$$

$$h_1=\left[\frac{\Gamma(\alpha+1)((x^*)^2+4x^*+A+B-2x^*y^*-\sqrt{\Delta})}{AB+2Bx^*+A(x^*)^2+2(x^*)^3}\right]^{\frac{1}{\alpha}},$$

$$h_2=\left[\frac{\Gamma(\alpha+1)((x^*)^2+4x^*+A+B-2x^*y^*+\sqrt{\Delta})}{AB+2Bx^*+A(x^*)^2+2(x^*)^3}\right]^{\frac{1}{\alpha}}, h^*=\left[\frac{\Gamma(\alpha+1)((x^*)^2+4x^*+A+B-2x^*y^*)}{AB+2Bx^*+A(x^*)^2+2(x^*)^3}\right]^{\frac{1}{\alpha}}.$$

Then, the following statements are true.

(1) *If one set of the following conditions is true, then E_+ is locally asymptotically stable (sink):*
   - i- $((x^*)^2+4x^*+A+B-2x^*y^*)^2\geq 4(AB+2Bx^*+A(x^*)^2+2(x^*)^3)$ *and* $0<h<h_1$.
   - ii- $((x^*)^2+4x^*+A+B-2x^*y^*)^2< 4(AB+2Bx^*+A(x^*)^2+2(x^*)^3)$ *and* $0<h<h^*$.

(2) *If one set of the following conditions is true, then P^+ is unstable (source):*
   - i- $((x^*)^2+4x^*+A+B-2x^*y^*)^2\geq 4(AB+2Bx^*+A(x^*)^2+2(x^*)^3)$ *and* $h>h_1$.
   - ii- $((x^*)^2+4x^*+A+B-2x^*y^*)^2< 4(AB+2Bx^*+A(x^*)^2+2(x^*)^3)$ *and* $h>h^*$.

(3) *The fixed point E_+ is unstable (saddle) if*

$$((x^*)^2+4x^*+A+B-2x^*y^*)^2\geq 4(AB+2Bx^*+A(x^*)^2+2(x^*)^3),\ \text{and } h_1<h<h_2.$$

(4) *P is non-hyperbolic and the roots of polynomial (8) are $\mu_1=-1$ and $|\mu_2|\neq 1$ if*

$$((x^*)^2+4x^*+A+B-2x^*y^*)^2\geq 4(AB+2Bx^*+A(x^*)^2+2(x^*)^3),\ h=h_{1,2},$$

$$\text{and } h\neq\left[\frac{2\Gamma(\alpha+1)}{(x^*)^2+4x^*+A+B-2x^*y^*}\right].$$

(5) *P is non-hyperbolic and the roots of polynomial (8) are complex numbers with modulus one if*

$$((x^*)^2+4x^*+A+B-2x^*y^*)^2 < 4(AB+2Bx^*+A(x^*)^2+2(x^*)^3),\ h=h^*.$$

Theorem 3.4. *The unique positive fixed point E_+ of system (6) loses its stability*

i- *via a period-doubling bifurcation if*

$$\Delta \geq 0 \quad and \quad h=h_{1,2}=\left[\frac{\Gamma(\alpha+1)((x^*)^2+4x^*+A+B-2x^*y^*\mp\sqrt{\Delta})}{AB+2Bx^*+A(x^*)^2+2(x^*)^3}\right]^{\frac{1}{\alpha}},$$

i- *via a Neimark-Sacker bifurcation if*

$$\Delta < 0 \quad and \quad h=h^*=\left[\frac{\Gamma(\alpha+1)((x^*)^2+4x^*+A+B-2x^*y^*)}{AB+2Bx^*+A(x^*)^2+2(x^*)^3}\right]^{\frac{1}{\alpha}},$$

with

$$\Delta=((x^*)^2+4x^*+A+B-2x^*y^*)^2-4(AB+2Bx^*+A(x^*)^2+2(x^*)^3).$$

Proof. From [3, 2, 4, 10, 13], a period-doubling bifurcation can occur when a single eigenvalue of the Jacobian matrix at fixed point is equal to -1. So, the fixed point E_+ loses its stability by a period-doubling bifurcation when $\Delta \geq 0$ and $h=h_{1,2}$. If the Jacobian matrix at fixed point has a pair of complex conjugate eigenvalues with modulus equal one, then a Neimark-Sacker bifurcation can occur. Hence, the fixed point E_+ undergoes a Neimark-Sacker bifurcation when $\Delta<0$ and $h=h^*$. □

4. Periodic-doubling bifurcation

We investigate period-doubling bifurcation of the fixed point $E_+=(x^*,y^*)$ by using the bifurcation theory [21, 23, 26]. The analysis of this part is done via the following Lemma.

Lemma 4.1. *Assume that $U_{k+1}=F_\mu(U_k)$ is a n-dimensional discrete dynamical system where $\mu \in R$ is a bifurcation parameter. Let U^* be an equilibrium point of F_μ and suppose that the characteristic polynomial of the Jacobian matrix $J(U^*)=(b_{ij})_{n\times n}$ of n-dimensional map $F_\mu(U_k)$ is given by*

$$P_\mu(\lambda)=\lambda^n+b_1\lambda^{n-1}+\cdots+b_{n-1}\lambda+b_n, \tag{9}$$

where $b_i=b_i(\mu,u)$, $i=1,2,3,\cdots,n$ and u is a control parameter or another parameter to be deduced. Let $\Delta_0^\pm(\mu,u)=1$, $\Delta_1^\pm(\mu,u),\cdots,\Delta_n^\pm(\mu,u)$ be a sequence of the determinants defined by

$$\Delta_i^\pm(\mu,u)=\det(M_1\pm M_2),\quad i=1,2,\cdots,n, \tag{10}$$

where

$$M_1=\begin{pmatrix} 1 & b_1 & b_2 & \cdots & b_{i-1}\\ 0 & 1 & b_1 & \cdots & b_{i-2}\\ 0 & 0 & 1 & \cdots & b_{i-3}\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & 0 & \cdots & 1\end{pmatrix},\ M_2=\begin{pmatrix} b_{n-i+1} & b_{n-i+2} & \cdots & b_{n-1} & b_n\\ b_{n-i+2} & b_{n-i+3} & \cdots & b_n & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ b_{n-1} & b_n & \cdots & 0 & 0\\ b_n & 0 & 0 & \cdots & 0\end{pmatrix}. \tag{11}$$

Moreover, suppose that the following conditions hold:

H1- *Eigenvalue criterion:* $P_{\mu_0}(-1)=0$, $\Delta^{\pm}_{n-1}(\mu_0,u)>0$, $P_{\mu_0}(1)>0$, $\Delta^{\pm}_{i}(\mu_0,u)>0$, $i=n-2,n-4,\cdots,1$ *(or* 1*), when* n *is even (or odd), respectively.*

H2- *Transversality criterion:* $\dfrac{\sum_{i=1}^{n}(-1)^{n-i}b_i'}{\sum_{i=1}^{n}(-1)^{n-i}(n-i+1)b_{i-1}}\neq 0$, *where* b_i' *denotes derivative of* $b(\mu)$ *at* $\mu=\mu_0$*. Then, a period-doubling bifurcation occurs at critical value* μ_0*.*

Theorem 4.2. *System (6) undergoes a period-doubling bifurcation at the unique positive equilibrium point* E*, if the following conditions hold:*

$$
\begin{aligned}
1+\mathscr{D}&>0,\\
1+\mathscr{T}+\mathscr{D}&=0,\\
1-\mathscr{T}+\mathscr{D}&>0.
\end{aligned}
$$

Thus, the period-doubling bifurcation occurs at h *if the parameters* (A,B,C,h,α) *vary in a neighborhood of the set*

$$
\mathscr{B}_1=\left\{\begin{array}{c}(A,B,C,h,\alpha)\in\mathrm{R}^5\ \middle|\ h=h_1=\left[\dfrac{\Gamma(\alpha+1)((x^*)^2+4x^*+A+B-2x^*y^*-\sqrt{\Delta})}{AB+2Bx^*+A(x^*)^2+2(x^*)^3}\right]^{\frac{1}{\alpha}}\\[2ex] \Delta=((x^*)^2+4x^*+A+B-2x^*y^*)^2-4(AB+2Bx^*+A(x^*)^2+2(x^*)^3)\geq 0,\ \ \alpha\in(0,1]\end{array}\right\}.
$$

Or,

$$
\mathscr{B}_2=\left\{\begin{array}{c}(A,B,C,h,\alpha)\in\mathrm{R}^5\ \middle|\ h=h_2=\left[\dfrac{\Gamma(\alpha+1)((x^*)^2+4x^*+A+B-2x^*y^*+\sqrt{\Delta})}{AB+2Bx^*+A(x^*)^2+2(x^*)^3}\right]^{\frac{1}{\alpha}}\\[2ex] \Delta=((x^*)^2+4x^*+A+B-2x^*y^*)^2-4(AB+2Bx^*+A(x^*)^2+2(x^*)^3)\geq 0,\ \ \alpha\in(0,1]\end{array}\right\},
$$

with

$$
x^*=\left(\frac{-C+\sqrt{A^2+4C}}{2}\right),\qquad y^*=\left(\frac{4C+A^2-A\sqrt{A^2+4C}}{2(B+C)+A^2-A\sqrt{A^2+4C}}\right).
$$

Proof. Using Lemmas 4.1, 3.3, and Theorem 3.4, and from the evaluation of Eq. (8) of system (6) at E_+, we have

$$
\begin{aligned}
&\Delta_0^{\mp}(h)=1>0,\\
&\Delta_1^{+}(h)=1+\mathscr{D}>0,\\
&(-1)^2P_h(-1)=1+\mathscr{T}+\mathscr{D}=0,\\
&P_h(1)=1-\mathscr{T}+\mathscr{D}>0,
\end{aligned}
$$

if and only if

$$
h=h_{1,2}=\left[\frac{\Gamma(\alpha+1)((x^*)^2+4x^*+A+B-2x^*y^*\mp\sqrt{\Delta})}{AB+2Bx^*+A(x^*)^2+2(x^*)^3}\right]^{\frac{1}{\alpha}},
$$

and

$$\Delta = ((x^*)^2 + 4x^* + A + B - 2x^*y^*)^2 - 4(AB + 2Bx^* + A(x^*)^2 + 2(x^*)^3) \geq 0.$$

In addition, the transversality condition is

$$\frac{\mathscr{T}' + \mathscr{D}'}{\mathscr{T} + 2} = \frac{2(AB + 2Bx^* + A(x^*)^2 + 2(x^*)^3)}{\Gamma(\alpha)(A + B + (x^*)^2 + 4x^* - 2x^*y^*)} \left[\frac{\Gamma(\alpha+1)((x^*)^2 + 4x^* + A + B - 2x^*y^* \mp \sqrt{\Delta})}{AB + 2Bx^* + A(x^*)^2 + 2(x^*)^3}\right]^{\frac{\alpha-1}{\alpha}},$$

with $\mathscr{T}' = \frac{d\mathscr{T}}{dh}\Big|_{h=h_{1,2}}$ and $\mathscr{D}' = \frac{d\mathscr{D}}{dh}\Big|_{h=h_{1,2}}$.

Then, the period-doubling bifurcation occurs at $h = h_1$ and $h = h_2$. Thus, the proof is done. □

5. Neimark-Sacker bifurcation

This section uses the bifurcation theory [2, 3, 4, 5, 9, 11, 10, 13, 18, 28] to investigate the Neimark-Sacker bifurcation of the fixed point $E_+ = (x^*, y^*)$ if $(A, B, C, h, \alpha) \in \mathscr{B}_3$ where

$$\mathscr{B}_3 = \left\{ \begin{array}{c} (A,B,C,h,\alpha) \in \mathrm{R}^5 \left| h = h^* = \left[\frac{\Gamma(\alpha+1)((x^*)^2 + 4x^* + A + B - 2x^*y^*)}{AB + 2Bx^* + A(x^*)^2 + 2(x^*)^3}\right]^{\frac{1}{\alpha}} \right. \\ ((x^*)^2 + 4x^* + A + B - 2x^*y^*)^2 < 4(AB + 2Bx^* + A(x^*)^2 + 2(x^*)^3),\ \alpha \in (0,1] \end{array} \right\},$$

with

$$x^* = \left(\frac{-C + \sqrt{A^2 + 4C}}{2}\right), \quad y^* = \left(\frac{4C + A^2 - A\sqrt{A^2 + 4C}}{2(B+C) + A^2 - A\sqrt{A^2 + 4C}}\right).$$

Since h is the bifurcation parameter, and if h in neighbourhood of h, that is, $h = h^* + \bar{h}$ where $\bar{h} \ll 1$, then system (6) takes the form:

$$\text{(12)} \quad \begin{cases} x_{n+1} = x_n + \dfrac{(h^* + \bar{h})^\alpha}{\Gamma(\alpha+1)}\left(-Ax_n - 2x_n^2 + By_n + x_n^2 y_n\right) = \mathscr{F}_1(x_n, y_n, \bar{h}), \\ y_{n+1} = y_n + \dfrac{(h^* + \bar{h})^\alpha}{\Gamma(\alpha+1)}\left(C + x_n^2 - By_n - x_n^2 y_n\right) = \mathscr{F}_2(x_n, y_n, \bar{h}), \end{cases}$$

where $(A, B, C, h^*, \alpha) \in \mathscr{B}_3$. Using the change of variable $u_n = x_n - x^*$ and $v_n = y_n - y^*$, we can translate the equilibrium point E_+ to origin. Furthermore, expanding $\mathscr{F}_1$ and $\mathscr{F}_2$ as a Taylor series at origin to the third order, system (12) becomes

$$\text{(13)} \quad \begin{cases} u_{n+1} &= a_{11}u_n + a_{12}v_n + a_{13}u_n^2 + a_{14}u_n v_n + a_{15}u_n^2 v_n, \\ v_{n+1} &= a_{21}u_n + a_{22}v_n + a_{23}u_n^2 + a_{24}u_n v_n + a_{25}u_n^2 v_n, \end{cases}$$

where

$$a_{11}=1+\frac{(h^*+\bar{h})^{\alpha}(-A+2x^*(y^*-2))}{\Gamma(\alpha+1)},\ a_{12}=\frac{(h^*+\bar{h})^{\alpha}(B+(x^*)^2)}{\Gamma(\alpha+1)},\ a_{13}=\frac{(h^*+\bar{h})^{\alpha}(y^*-2)}{\Gamma(\alpha+1)},$$

$$a_{14}=\frac{4x^*(h^*+\bar{h})^{\alpha}}{\Gamma(\alpha+1)},\ a_{15}=\frac{(h^*+\bar{h})^{\alpha}}{\Gamma(\alpha+1)},\ a_{21}=\frac{2x^*(h^*+\bar{h})^{\alpha}(1-y^*)}{\Gamma(\alpha+1)},\ a_{22}=1-\frac{(h^*+\bar{h})^{\alpha}(B+(x^*)^2}{\Gamma(\alpha+1)},$$

$$a_{23}=\frac{x^*(h^*+\bar{h})^{\alpha}(1-y^*)}{\Gamma(\alpha+1)},\ a_{24}=\frac{-4x^*(h^*+\bar{h})^{\alpha}}{\Gamma(\alpha+1)},\ a_{25}=\frac{-(h^*+\bar{h})^{\alpha}}{\Gamma(\alpha+1)}.$$

The characteristic equation of the Jacobian matrix of system (13) evaluated at the origin is given by

$$\mu^2-\mathscr{T}(\bar{h})\mu+\mathscr{D}(\bar{h}))=0, \tag{14}$$

where

$$\begin{aligned}\mathscr{T}(\bar{h}) &=2+\frac{(h^*+\bar{h})^{\alpha}(2x^*y^*-(x^*)^2-4x^*-(A+B))}{\Gamma(\alpha+1)},\\ \mathscr{D}(\bar{h}) &=1+\frac{(h^*+\bar{h})^{\alpha}(2x^*y^*-(x^*)^2-4x^*-(A+B))}{\Gamma(\alpha+1)}+\frac{(h^*+\bar{h})^{2\alpha}(AB+2Bx^*+A(x^*)^2+2(x^*)^3)}{(\Gamma(\alpha+1))^2}.\end{aligned}$$

Moreover, the roots of Eq. (14) are

$$\mu_{1,2}=\frac{\mathscr{T}\mp i\sqrt{4\mathscr{D}-\mathscr{T}^2}}{2}.$$

It follows that $|\mu_{1,2}|_{\bar{h}=0}=\sqrt{\mathscr{D}(0)}$, and

$$\left.\frac{d|\mu_{1,2}(\bar{h})|}{d\bar{h}}\right|_{\bar{h}=0}=\frac{(AB+2Bx^*+A(x^*)^2+2(x^*)^3)}{2\Gamma(\alpha)\sqrt{\mathscr{D}(0)}}\left(\left[\frac{\Gamma(\alpha+1)((x^*)^2+4x^*+A+B-2x^*y^*)}{AB+2Bx^*+A(x^*)^2+2(x^*)^3}\right]^{\frac{\alpha-1}{\alpha}}\right)>0.$$

Moreover, it is required that, when $\bar{h}$, $\mu_{1,2}^j\neq 1$ for $j=1,2,3,4$, which is equivalent to $\mathscr{T}(0)\neq -2,0,1,2$. We have $|\mathscr{T}|<2$. Therefore, $\mathscr{T}\neq\pm 2$. We only require that $\mathscr{T}\neq 0,1$, so

$$h\neq\left[\frac{2\Gamma(\alpha+1)}{(A+B)+4x^*+(x^*)^2-2x^*y^*}\right]^{\frac{1}{\alpha}},\quad\left[\frac{\Gamma(\alpha+1)}{(A+B)+4x^*+(x^*)^2-2x^*y^*}\right]^{\frac{1}{\alpha}}. \tag{15}$$

In order to write the normal form for system (13) at $\bar{h}=0$, we will use the following linear transformation

$$\begin{pmatrix}u_n\\ v_n\end{pmatrix}=\begin{pmatrix}a_{12} & 0\\ \varepsilon-a_{11} & -\rho\end{pmatrix}\begin{pmatrix}\bar{x}_n\\ \bar{y}_n\end{pmatrix},$$

with $\varepsilon=\Re(\mu_{1,2})=\frac{\mathscr{T}(0)}{2}$ and $\rho=\Im(\mu_{1,2})=\frac{1}{2}\sqrt{(4\mathscr{D}(0)-\mathscr{T}(0)^2)}$. Using this transformation on system (13), we obtain

$$\begin{cases}\bar{x}_{n+1}= & \varepsilon\bar{x}_n-\rho\bar{y}_n+\tilde{\mathscr{G}}_1(\bar{x},\bar{y},h^*),\\ \bar{y}_{n+1}= & \rho\bar{x}_n+\varepsilon\bar{y}_n+\tilde{\mathscr{G}}_2(\bar{x},\bar{y},h^*),\end{cases} \tag{16}$$

where

$$\left\{\begin{array}{ll}\tilde{\mathscr{G}}_1(\bar{x},\bar{y},h^*) & = b_{13}\bar{x}_n^2 + b_{14}\bar{x}_n\bar{y}_n + b_{15}\bar{x}_n^3 + b_{16}\bar{x}_n^2\bar{y}_n, \\ \tilde{\mathscr{G}}_2(\bar{x},\bar{y},h^*) & = b_{23}\bar{x}_n^2 + b_{24}\bar{x}_n\bar{y}_n + b_{25}\bar{x}_n^3 + b_{26}\bar{x}_n^2\bar{y}_n,\end{array}\right. \tag{17}$$

with

$$\begin{aligned} & b_{13} = a_{13}a_{12} + a_{14}(\varepsilon - a_{11}),\ b_{14} = -\rho a_{14},\ b_{15} = a_{15}(\varepsilon - a_{11}),\ b_{16} = -\rho a_{15}a_{12}, \\ & b_{23} = \frac{(a_{12}a_{13}(\varepsilon - a_{11}) - a_{12}^2a_{23} + a_{14}(\varepsilon - a_{11})^2 - a_{12}a_{24}(\varepsilon - a_{11}))}{\rho},\ b_{24} = a_{12}a_{24} - a_{14}(\varepsilon - a_{11}), \\ & b_{25} = \frac{a_{12}a_{15}(\varepsilon - a_{11})^2 - a_{12}a_{25}(\varepsilon - a_{11})}{\rho},\ b_{26} = a_{12}^2a_{25} - a_{12}a_{15}(\varepsilon - a_{11}). \end{aligned}$$

Furthermore, from (17) we have

$$\begin{aligned} & \left.\frac{\partial^2\tilde{\mathscr{G}}_1}{\partial\bar{x}_n^2}\right|_{(0,0)} = 2b_{13},\ \left.\frac{\partial^2\tilde{\mathscr{G}}_1}{\partial\bar{x}_n\bar{y}_n}\right|_{(0,0)} = b_{14},\ \left.\frac{\partial^2\tilde{\mathscr{G}}_1}{\partial\bar{y}_n^2}\right|_{(0,0)} = 0,\ \left.\frac{\partial^3\tilde{\mathscr{G}}_1}{\partial\bar{x}_n^3}\right|_{(0,0)} = 6b_{15},\ \left.\frac{\partial^3\tilde{\mathscr{G}}_1}{\partial\bar{x}_n^2\bar{y}_n}\right|_{(0,0)} = 2b_{16}, \\ & \left.\frac{\partial^3\tilde{\mathscr{G}}_1}{\partial\bar{x}_n\bar{y}_n^2}\right|_{(0,0)} = 0,\ \left.\frac{\partial^3\tilde{\mathscr{G}}_1}{\partial\bar{y}_n^3}\right|_{(0,0)} = 0,\ \left.\frac{\partial^2\tilde{\mathscr{G}}_2}{\partial\bar{x}_n^2}\right|_{(0,0)} = 2b_{23},\ \left.\frac{\partial^2\tilde{\mathscr{G}}_2}{\partial\bar{x}_n\bar{y}_n}\right|_{(0,0)} = b_{24},\ \left.\frac{\partial^2\tilde{\mathscr{G}}_2}{\partial\bar{y}_n^2}\right|_{(0,0)} = 0, \\ & \left.\frac{\partial^3\tilde{\mathscr{G}}_2}{\partial\bar{x}_n^3}\right|_{(0,0)} = 6b_{25},\ \left.\frac{\partial^3\tilde{\mathscr{G}}_2}{\partial\bar{x}_n^2\bar{y}_n}\right|_{(0,0)} = 2b_{26},\ \left.\frac{\partial^3\tilde{\mathscr{G}}_2}{\partial\bar{x}_n\bar{y}_n^2}\right|_{(0,0)} = 0,\ \left.\frac{\partial^3\tilde{\mathscr{G}}_2}{\partial\bar{y}_n^3}\right|_{(0,0)} = 0. \end{aligned}$$

In order to ensure the occurrence of Neimark-Sacker bifurcation at $(0,0,h^*)$ of system (16), the following discriminatory quantity must not be zero:

$$\mathscr{L} = \Re(\mu_2 \mathrm{m}_{21}) - \Re\left(\frac{(1-2\mu_1)\mu_2^2}{1-\mu_1}\mathrm{m}_{20}\mathrm{m}_{11}\right) - \frac{1}{2}|\mathrm{m}_{11}|^2 - |\mathrm{m}_{02}|^2, \tag{18}$$

where

$$\begin{aligned} \mathrm{m}_{20} &= \frac{1}{8}\left[\frac{\partial^2\tilde{\mathscr{G}}_1}{\partial\bar{x}^2} - \frac{\partial^2\tilde{\mathscr{G}}_1}{\partial\bar{y}^2} + 2\frac{\partial^2\tilde{\mathscr{G}}_2}{\partial\bar{x}\partial\bar{y}} + i\left(\frac{\partial^2\tilde{\mathscr{G}}_2}{\partial\bar{x}^2} - \frac{\partial^2\tilde{\mathscr{G}}_2}{\partial\bar{y}^2} - 2\frac{\partial^2\tilde{\mathscr{G}}_1}{\partial\bar{x}\partial\bar{y}}\right)\right]\Bigg|_{\bar{h}=0}, \\ \mathrm{m}_{11} &= \frac{1}{4}\left[\frac{\partial^2\tilde{\mathscr{G}}_1}{\partial\bar{x}^2} + \frac{\partial^2\tilde{\mathscr{G}}_1}{\partial\bar{y}^2} + i\left(\frac{\partial^2\tilde{\mathscr{G}}_2}{\partial\bar{x}^2} + \frac{\partial^2\tilde{\mathscr{G}}_2}{\partial\bar{y}^2}\right)\right]\Bigg|_{\bar{h}=0}, \\ \mathrm{m}_{02} &= \frac{1}{8}\left[\frac{\partial^2\tilde{\mathscr{G}}_1}{\partial\bar{x}^2} - \frac{\partial^2\tilde{\mathscr{G}}_1}{\partial\bar{y}^2} - 2\frac{\partial^2\tilde{\mathscr{G}}_2}{\partial\bar{x}\partial\bar{y}} + i\left(\frac{\partial^2\tilde{\mathscr{G}}_2}{\partial\bar{x}^2} - \frac{\partial^2\tilde{\mathscr{G}}_2}{\partial\bar{y}^2} + 2\frac{\partial^2\tilde{\mathscr{G}}_1}{\partial\bar{x}\partial\bar{y}}\right)\right]\Bigg|_{\bar{h}=0}, \\ \mathrm{m}_{21} &= \frac{1}{16}\left[\frac{\partial^3\tilde{\mathscr{G}}_1}{\partial\bar{x}^3} + \frac{\partial^3\tilde{\mathscr{G}}_1}{\partial\bar{x}\partial\bar{y}^2} + \frac{\partial^3\tilde{\mathscr{G}}_2}{\partial\bar{x}^2\partial\bar{y}} + \frac{\partial^3\tilde{\mathscr{G}}_2}{\partial\bar{y}^3} + i\left(\frac{\partial^3\tilde{\mathscr{G}}_2}{\partial\bar{x}^3} + \frac{\partial^3\tilde{\mathscr{G}}_2}{\partial\bar{x}\partial\bar{y}^2} - \frac{\partial^3\tilde{\mathscr{G}}_1}{\partial\bar{x}^2\partial\bar{y}} - \frac{\partial^3\tilde{\mathscr{G}}_1}{\partial\bar{y}^3}\right)\right]\Bigg|_{\bar{h}=0}. \end{aligned}$$

By using partial derivatives, we find

$$\begin{aligned}
\mathrm{m}_{20} &= \frac{1}{4}\left(b_{13}+b_{14}+i(b_{23}-b_{24})\right),\\
\mathrm{m}_{11} &= \frac{1}{2}\left(b_{13}+ib_{23}\right),\\
\mathrm{m}_{02} &= \frac{1}{4}\left(b_{13}-b_{14}+i(b_{23}+b_{24})\right),\\
\mathrm{m}_{21} &= \frac{1}{8}\left(3b_{15}+b_{16}+i(3b_{25}-b_{26})\right).
\end{aligned}$$

The above analysis leads to the following result.

Theorem 5.1. *Assume that condition (15) is satisfied, and let $(A,B,C,h,\alpha)\in\mathscr{B}_3$ with $\mathscr{L}\neq 0$. Then, system (6) undergoes a Neimark-Sacker bifurcation at the fixed point $E_+=(x^*,y^*)$ when the bifurcation parameter h varies in a small neighbourhood of*

$$h^*=\left[\frac{\Gamma(\alpha+1)((x^*)^2+4x^*+A+B-2x^*y^*)}{AB+2Bx^*+A(x^*)^2+2(x^*)^3}\right]^{\frac{1}{\alpha}}.$$

Moreover, if $\mathscr{L}<0$ ($\mathscr{L}>0$) then an attracting (respectively, repelling) invariant closed curve bifurcates from the fixed point E_+ for $h>h^*$ (respectively, $h<h^*$).

6. Numerical computations and discussion

Example 6.1. *In this example, we consider system (6) when $A=12$, $B=1.2$, $C=13$, $\alpha=0.75$ and $h\in[0,0.13]$ with the initial conditions $\mathrm{C}_1=(1,6.3636)$ and $\mathrm{C}_2=(0.98,6.47)$. When $h^*=0.0893$, system (6) undergoes the Neimark-Sacker bifurcation at the positive fixed point $E_+=(x^*,y^*)$. The Neimark-Sacker bifurcation diagrams x_n and y_n of system (6) are shown in Fig. 1a and 1b, respectively. Moreover, the maximum Lyapunov exponent is plotted in Fig. 1c. It is observed that the fixed point is sink (local asymptotically stable) for $0<h<h^*$. At $h=h^*=0.0893$, the fixed point E_+ loses its stability. As a result, a closed invariant curve appears around the fixed point E_+ due to Neimark-Sacker bifurcation (see 1d), and the diameter of the closed invariant curve increases with the increase in the value of h. For further confirmation, we notice that with parametric values $(A,B,C,\alpha,h^*)=(12,1.2,13,0.75,0.0893)$, the Jacobian matrix of system (6) is given by*

$$J(P^+)=\begin{pmatrix} 0.4185 & 0.3909\\ -1.9061 & 0.6091\end{pmatrix},$$

where its eigenvalues are $\mu_1=0.5138-0.8579i$, $\mu_2=0.5138+0.8579i$ with $|\mu_{1,2}|\neq 1$. In this case, the discriminatory quantity (first Lyapunov exponent) $\mathscr{L}=-0.1169$, which proves the correctness of Theorem 5.1. Figs. 2 and Figs. 3 present some phase portraits for system (6) and the evolution of x_n under the values of the bifurcation parameter $h\in[0,0.0893]$ and $h\in[0.0893,0.114]$, respectively. For $0<h<0.0893$, the fixed point E_+ is stable and all orbits tend to E_+ (see Figs. 2). If $0.0893\leq h<0.114$, we find an attracting closed invariant curve Λ_s encircling the fixed point. Here, the point E_+ loses its stability because all trajectories asymptotically approaches the closed invariant curve Λ_s

(see Figs. 3). Furthermore, when $h \in [0.114, 0.13]$*, it is easy to see that a 5-,10-,20-,40-period orbits, quasi periodic orbits and attracting chaotic sets (see 4a,4b,4c and 4d).*

Example 6.2. *In this example, we take* $A = 2.92$*,* $B = 0.68$*,* $C = 0.57$*,* $\alpha = 0.5$*,* $h \in [0, 0.55]$ *and the initial condition* $\mathrm{C}_0 = (0.059, 0.316)$*. Then, system (6) undergoes a period-doubling bifurcation as the bifurcation parameter h varies in a small neighbourhood of* $h_1 = 0.2784$*. To confirm the existence of a period double bifurcation, the Jacobean matrix evaluated at this point is expressed by*

$$\mathscr{J}(E_+) = \begin{pmatrix} -0.9909 & 0.4249 \\ 0.0337 & 0.5751 \end{pmatrix}.$$

Hence, the characteristic polynomial is given by

$$\rho(\lambda) = \lambda^2 + 0.4158\lambda - 0.5842,$$

whose roots are $\mu_1 = -1$*,* $\mu_2 = 0.5842$ *where* $|\mu_2| \neq 1$*. We also have*

$$\begin{aligned}
&\Delta_0^{\mp}(h) = 1 > 0, \\
&\Delta_1^{+}(h) = 1 + \mathscr{D} = 0.4158 > 0, \\
&(-1)^2 P_h(-1) = 1 + \mathscr{T} + \mathscr{D} = 0, \\
&P_h(1) = 1 - \mathscr{T} + \mathscr{D} = 0.8317 > 0,
\end{aligned}$$

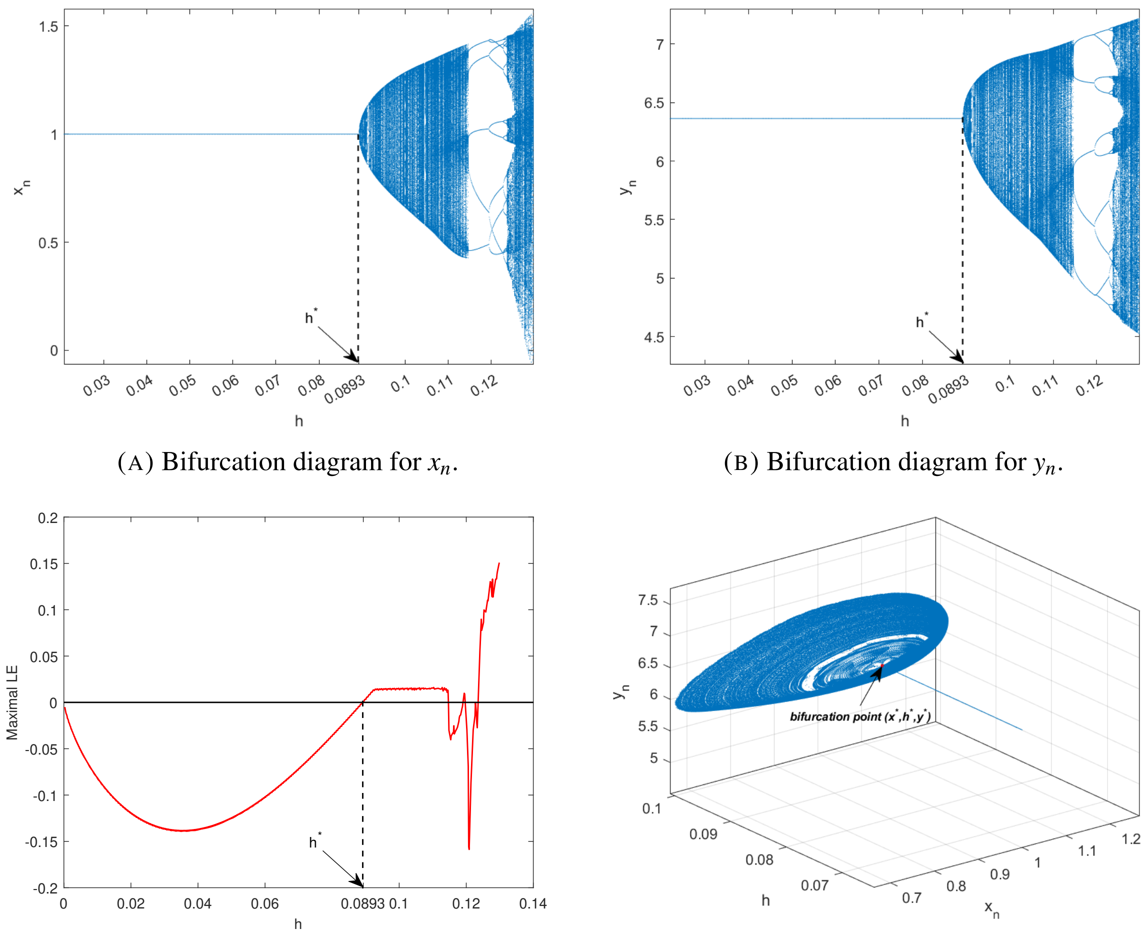
In addition, the transversality condition is written as

$$\frac{\mathscr{T}' + \mathscr{D}'}{\mathscr{T} + 2} = 3.5919 \neq 0.$$

Through these results, it is clear that the conditions of Theorem 4.2 are fulfilled. The fixed point E_+ *of system (6) is asymptotically stable when* $h < h_1$*, as shown in bifurcation diagrams of* x_n*,* y_n *and* $(x_n - h - y_n)-$*Space (see Figs 5a,5b and 5d, respectively). At* $h = h_1$*, the fixed point* E_+ *loses its stability due to a period-doubling bifurcation. Furthermore, when* $h > h_1$*, a period-doubling cascade in orbits of 2-,4-,8-period, quasi peridic and chaotic set. The maximum Lyapunov exponents are computed and the existence of chaotic regions in the parameter space is clearly depicted in Fig. 5c.*

7. Conclusion

This work has investigated the fixed point, local stability, types of bifurcations, closed invariant curves for a fractional-order chemical reaction system. We found that system (6) undergoes a period-doubling bifurcation at the unique positive fixed point E_+ if some conditions are satisfied, as shown in Theorem 4.2. Furthermore, if condition (15) is satisfied and $(A, B, C, h, \alpha) \in \mathscr{B}_3$ with $\mathscr{L} \neq 0$, then system (6) undergoes a Neimark-Sacker bifurcation at the fixed point $E_+ = (x^*, y^*)$. The Neimark-Sacker bifurcation diagrams x_n and y_n of system (6) are illustrated in Fig. 1a and 1b, respectively, when $h^* = 0.0893$. These diagrams are plotted under the parameter values $A = 12$, $B = 1.2$, $C = 13$, $\alpha = 0.75$ and $h \in [0, 0.13]$ with the initial conditions $\mathrm{C}_1 = (1, 6.3636)$ and $\mathrm{C}_2 = (0.98, 6.47)$. The maximum Lyapunov exponent is shown in Fig. 1c. It should be noted that the fixed point E_+ loses its stability at $h = h^* = 0.0893$. Hence, a closed invariant curve appears around the equilibrium point E_+ due to Neimark-Sacker bifurcation as shown in Fig. 1d. We have also considered that Now, system (6)



(A) Bifurcation diagram for x_n.

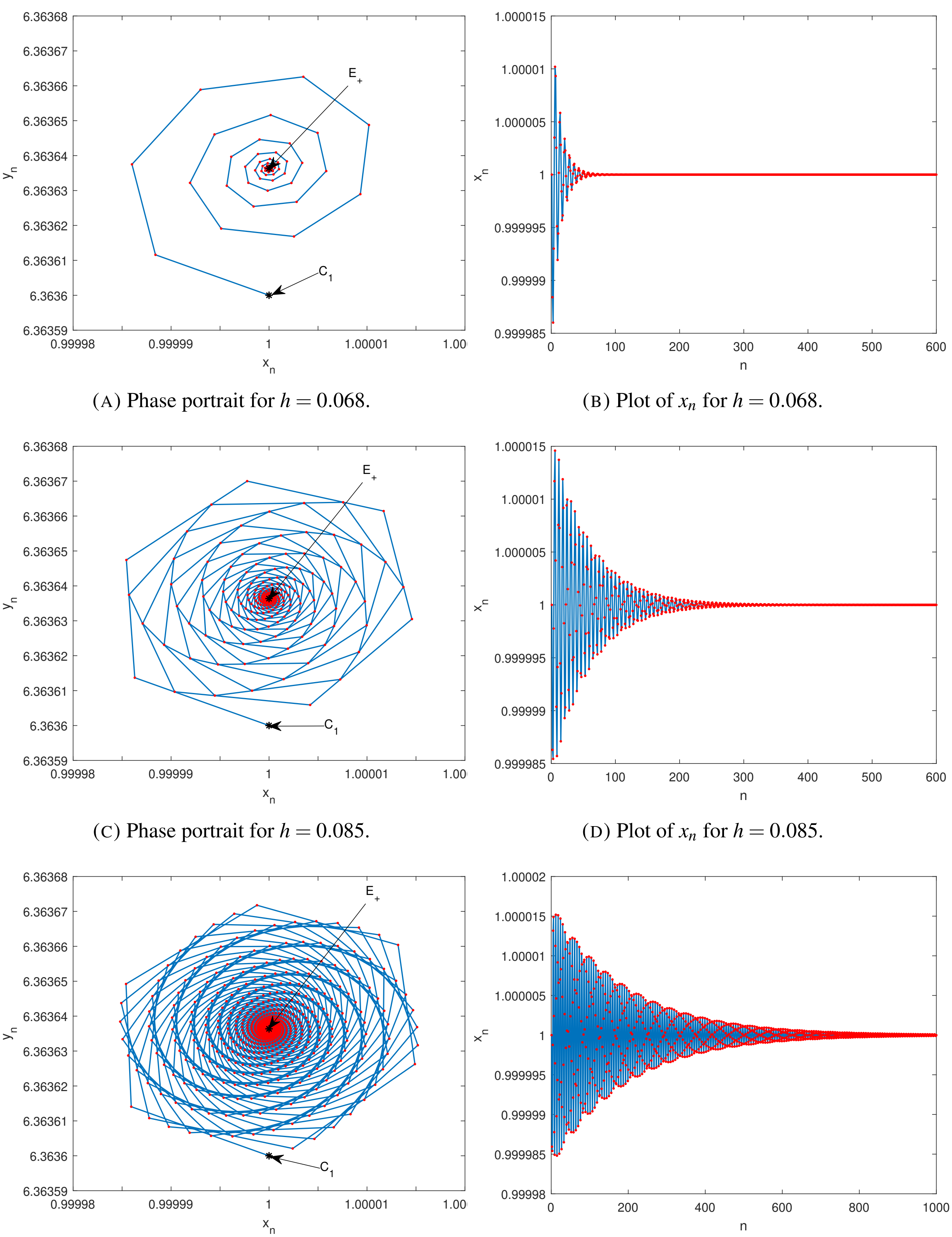
(B) Bifurcation diagram for y_n.

(C) Maximum Lyaponov exponents (MLE).

(D) Bifurcation diagram in $(x_n - h - y_n)$ Space.

FIGURE 1. Bifurcation diagrams x_n and y_n (Fig.1a and 1b resp.) with maximum Lyapunov exponent (Fig.1c) for $A = 12$, $B = 1.2$, $C = 13$, $\alpha = 0.75$ and $h \in [0, 0.13]$ and bifurcation diagram for $h \in [0, 0.1]$ in $(x_n - h - y_n)$ Space of system (6).

undergoes period-doubling bifurcation as bifurcation parameter h varies in a small neighbourhood of $h_1 = 0.2784$ where $A = 2.92$, $B = 0.68$, $C = 0.57$, $\alpha = 0.5$, $h \in [0, 0.55]$ and the initial condition $C_0 = (0.059, 0.316)$. The bifurcation diagram are shown in Figs. 5a,5b and 5d. Note that when $h = h_1$, the fixed point E_+ loses its stability due to a period-doubling bifurcation. If $h > h_1$, a period-doubling cascade in orbits of 2-,4-,8-period, quasi periodic and chaotic set. We have presented the maximum Lyapunov exponents in Fig. 5c. Finally, we can conclude that the used methods can be applied for dealing with other dynamical systems.



(A) Phase portrait for $h = 0.068$.

(B) Plot of x_n for $h = 0.068$.

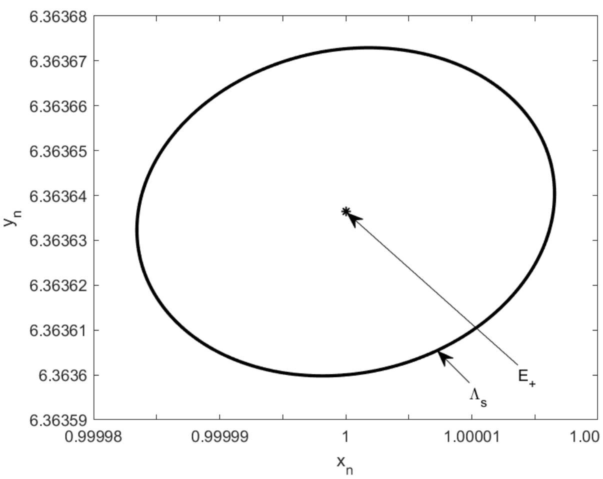
(C) Phase portrait for $h = 0.085$.

(D) Plot of x_n for $h = 0.085$.

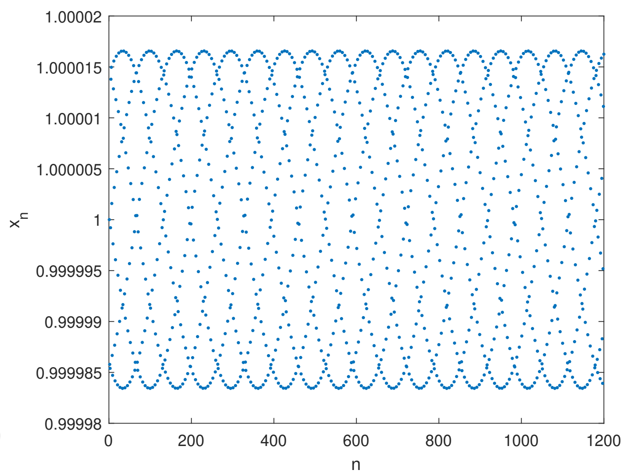
(E) Phase portrait for $h = 0.088$.

(F) Plot of x_n for $h = 0.088$.

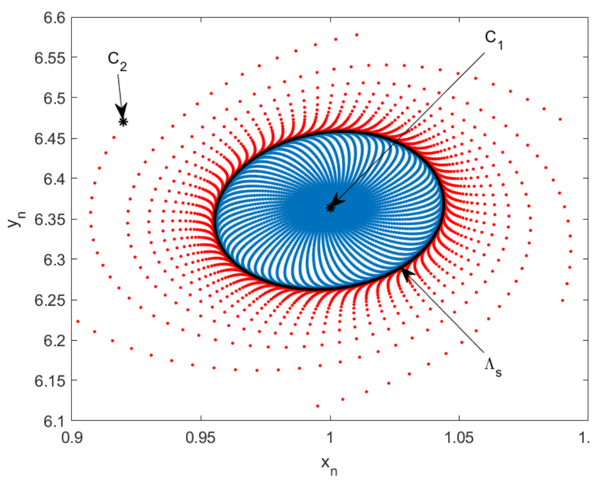
FIGURE 2. Phase portrait with plot of x_n of system (6) for different values of $h \in [0, 0.0893]$ and the initial condition C_1 and $A = 12$, $B = 1.2$, $C = 13$, $\alpha = 0.75$.



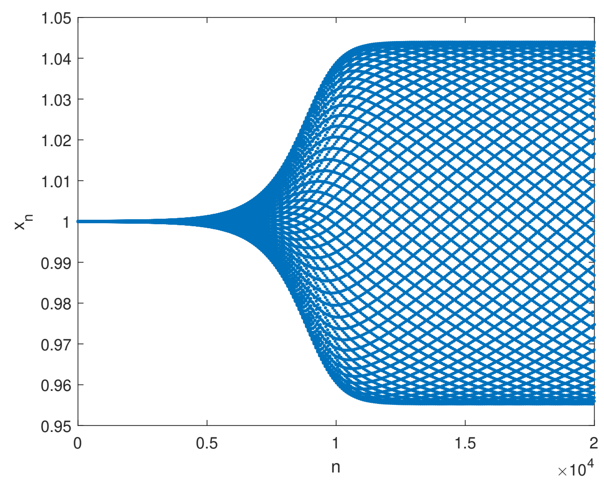
(A) Phase portrait for $h = h^* = 0.0893$.



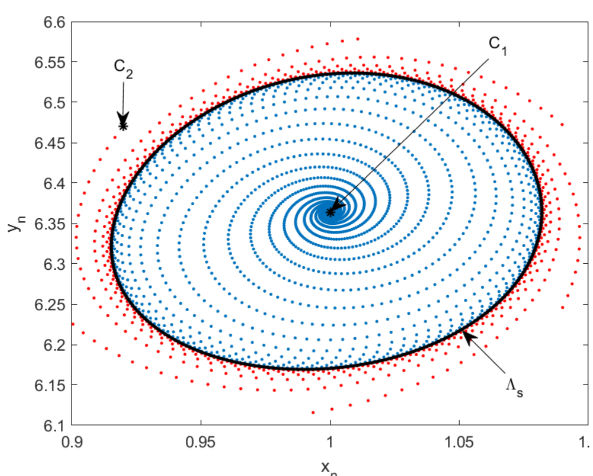
(B) Plot of x_n for $h = 0.0893$.



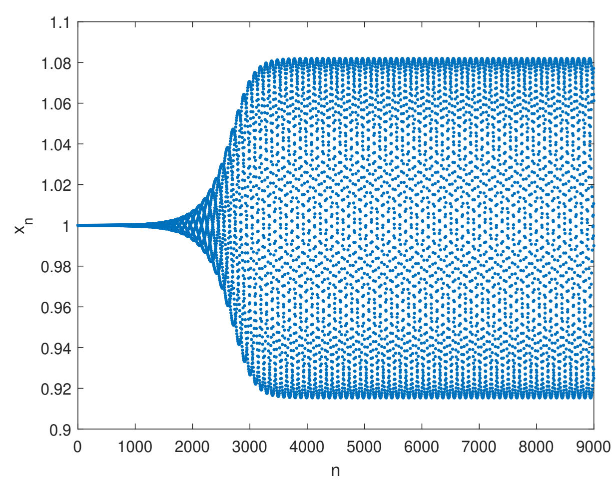
(C) Phase portrait for $h = 0.08947$.



(D) Plot of x_n for $h = 0.08947$.

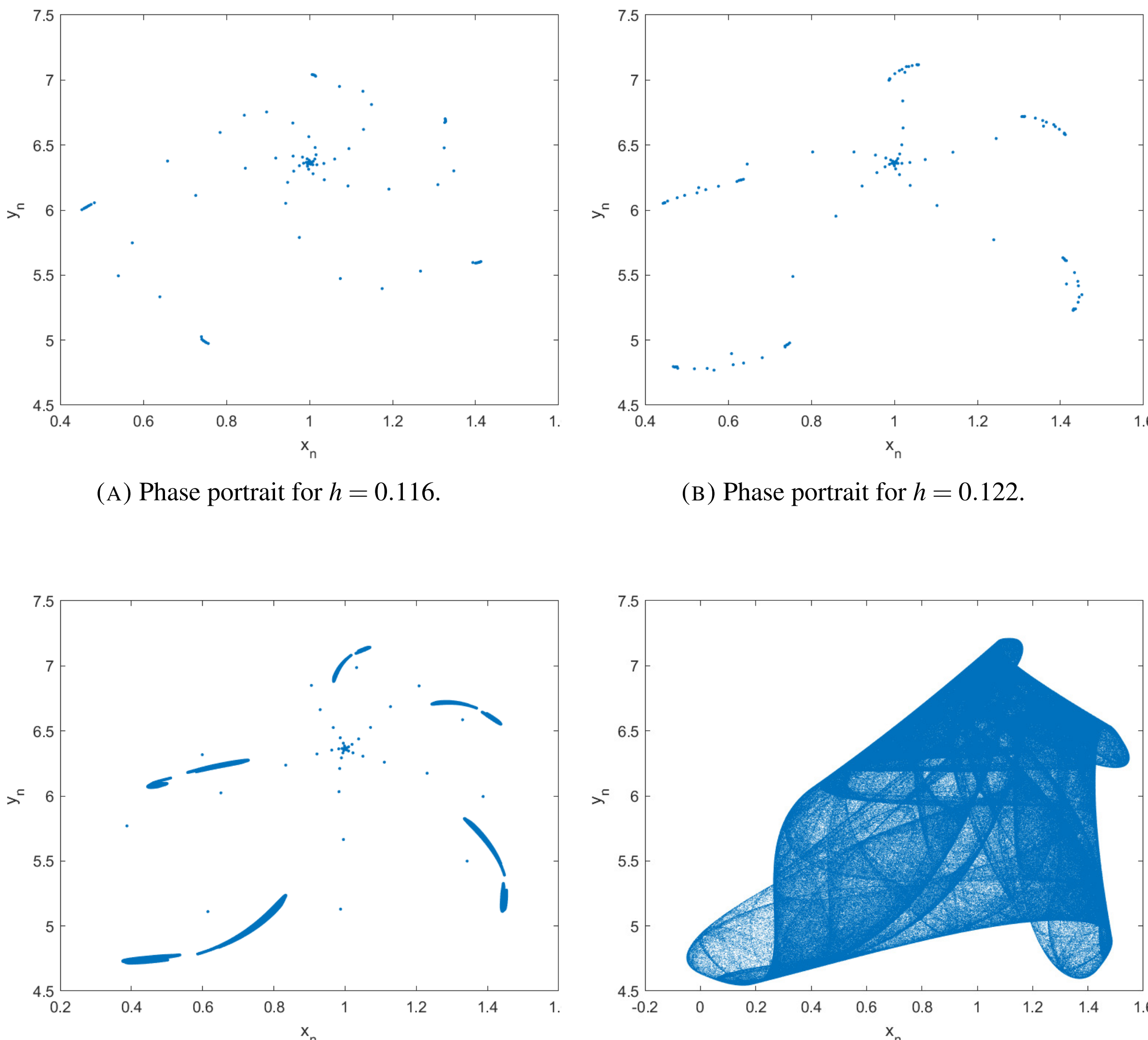


(E) Phase portrait for $h = 0.09$.



(F) Plot of x_n for $h = 0.088$.

FIGURE 3. Phase portrait with plot of x_n of system (6) for different values of $h \in [0.0893, 0.114]$ and the initial conditions C_1, C_2 and $A = 12$, $B = 1.2$, $C = 13$, $\alpha = 0.75$.



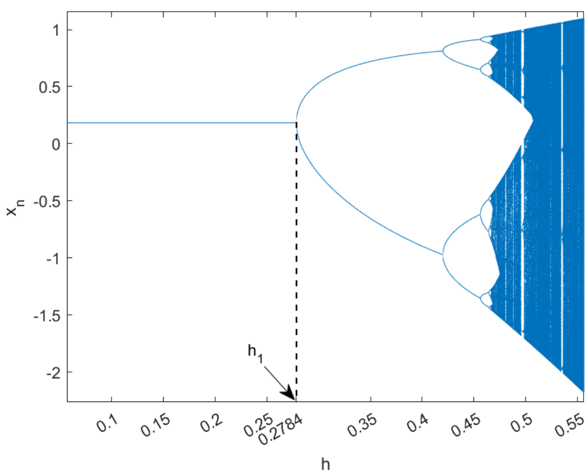
(A) Phase portrait for $h = 0.116$.

(B) Phase portrait for $h = 0.122$.

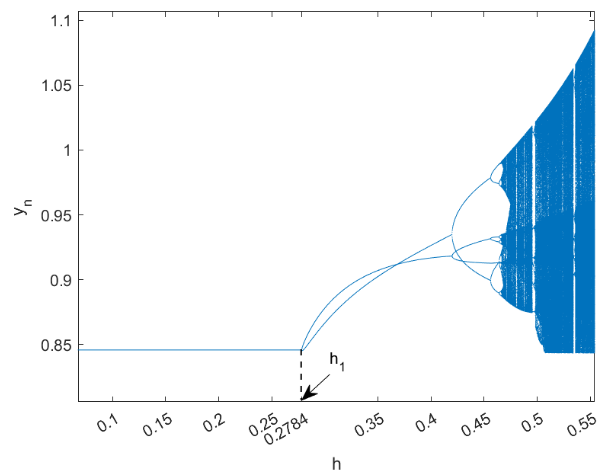
(C) Phase portrait for $h = 0.1236$.

(D) Phase portrait for $h = 0.129$.

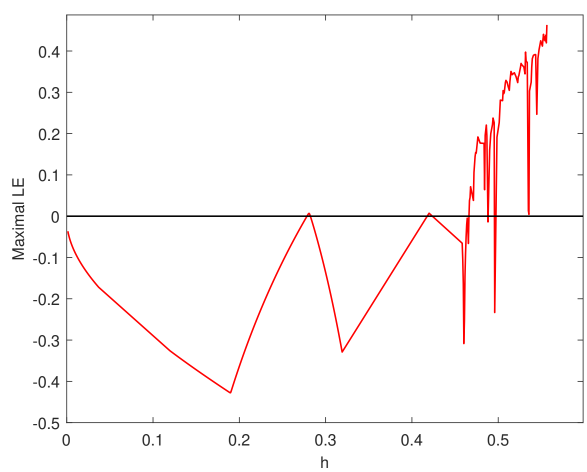
FIGURE 4. Phase portrait for system (6) for different values of $h \in [0.114, 0.13]$ and $A = 12$, $B = 1.2$, $C = 13$, $\alpha = 0.75$.



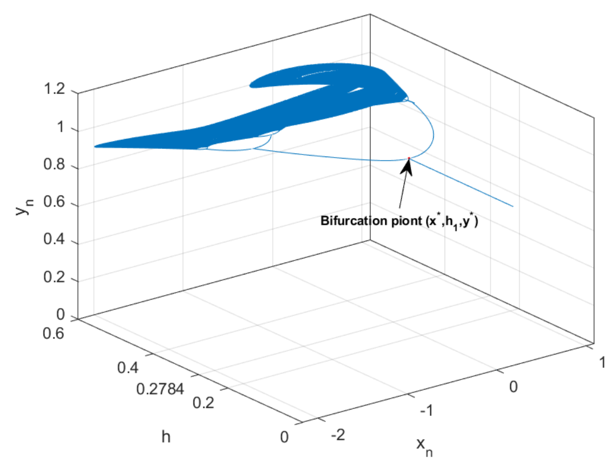
(A) Bifurcation diagrams x_n.



(B) Bifurcation diagrams y_n.



(C) Maximum Lyapunov exponent (MLE).



(D) Bifurcation diagrams in $(x_n - h - y_n)$ Space.

FIGURE 5. Bifurcation diagrams x_n and y_n (Fig.5a and 5b resp.) with maximum Lyapunov exponent (Fig.5c) for $A = 2.92$, $B = 0.68$, $C = 0.57$, $\alpha = 0.5$ and $h \in [0, 0.55]$ and bifurcation diagram for $h \in [0, 0.55]$ in $(x_n - h - y_n) - Space$ of system (6).

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Department of Applied Mathematics, University of Alicante, Alicante, San Vicente del Raspeig, 03690, Spain
Email address: `mb299@gcloud.ua.es`

Department of Applied Mathematics, University of Alicante, Alicante, San Vicente del Raspeig, 03690, Spain
Email address: `jf.navarro@ua.es`