ROCKY MOUNTAIN JOURNAL OF MATHEMATICS
Vol. , No. , YEAR
https://doi.org/rmj.YEAR..PAGE

NEW EXISTENCE AND GENERALIZED STABILITY RESULTS FOR COUPLED SYSTEMS OF FRACTIONAL DIFFERENTIAL EQUATIONS WITH ”MAXIMA”

KHADIDJA NISSE* AND LAMINE NISSE

ABSTRACT. In this work, we deal with initial value problems of coupled systems of nonlinear fractional differential equations with state deviating arguments and ”maxima” on the half line. First we prove a useful estimate for fractional integrals involving maxima. Then, adequate pseudo-metrics on the solutions space using Bielecki's idea are introduced and via some Perov's type fixed point theorem in gauge spaces, a global existence-uniqueness result is obtained under Lipschitz condition on the nonlinearity with merely continuous arguments. Our approach allowed us to get rid of strict conditions imposed in some recent results in the literature. Furthermore, some generalized concept of uniform stability of solutions is also presented and proved under the same conditions.

1. Introduction

Differential equations with ”maxima” form a special type of functional differential equations, which involve in addition of the current state of the unknown function, its maximum value over a certain past time interval. A typical example described by this type of equations and the most cited in the literature, is the system for regulating the voltage of a generator of constant current considered by E.P. Popov in 1966. Since then, differential equations with ”maxima” appeared in the modeling of many real world processes, especially those arising in the automatic control theory of various technical systems [4, 23]. For some recent studies on the existence results and stability properties on this subject, we refer to [8, 13, 14, 15, 21] and the references therein.

Fractional differential equations arise naturally when solving practical problems in many fields of science, for example in mechanics, physics, bio-chemistry, electrical engineering, medicine, etc. Therefore, an intensive development in the investigation of fractional differential systems has taken place in connection with the requirements of applied science [9, 16]. For some recent results on the topic, specially the question of existence and uniqueness of solutions under different forms of initial and boundary conditions, we refer to [1, 2, 3, 25, 30].

As it is noted in [20], when treating systems of equations, Perov and Kibenko [22] and also Precup [24], pointed out that better results can be obtained when endowing the involved space with vector-valued norms/metrics rather than norms/metrics of product spaces. In this direction, the authors in [20] presented Perov's type fixed point theorem for some contractive mappings in spaces endowed with a

*Corresponding author .

2020 *Mathematics Subject Classification.* 26A33, 93C23, 35E15, 47H10.

Key words and phrases. generalized spaces in Perov's sens, fixed point theorem, Bielecki's idea, existence and uniqueness of solution, stability.

family of vector-valued pseudo-metrics. As application, they proved also some existence results for systems of integral equations.

Recently, C.Guendouz et al. [12] studied the following coupled system of fractional differential equations with initial conditions

$$(1.1)\qquad \begin{cases} {}^{C}D^{\alpha}u(t) = f(t,u(t),v(t)), \\ {}^{C}D^{\beta}v(t) = g(t,u(t),v(t)), \qquad t \in J =]0,\infty[\\ u(0) = u_0, \\ v(0) = v_0, \end{cases}$$

where ${}^{C}D^{\alpha}$ and ${}^{C}D^{\beta}$ are the Caputo fractional derivatives, $\alpha,\beta \in]0,1[$, $f,g : J\times\mathbb{R}^2 \longrightarrow \mathbb{R}$ are given functions, $u_0, v_0 \in \mathbb{R}$. Using Perov's type fixed point theorem in generalized Banach spaces, the authors give sufficient conditions for the existence of a unique solution of (1.1) in C_b (the space of bounded continuous functions on J).

In 1956, A. Bielecki [6] initiated a new method of weighted norms for obtaining global existence results of ordinary differential equations. This method, known also as the equivalent norms/metrics method, which aims to optimize the use of Banach's type fixed point theorems, consists in choosing a suitable norm with respect to which the involved operator becomes a contraction. For a nice review of the obtained results and some extensions on the topic in the first 20 years, we refer to [7]. El-Raheem [11] employed this approach to one-term fractional differential equations on bounded intervals, satisfying Lipschitz condition with constant argument. Bălean and Mustafa [5] extended this idea, usually for one-term fractional differential equations but on unbounded intervals obeying to Lipschitz condition with variable argument. Up to now many papers make use of Bielecki's idea ([17, 26, 29] to cite only some of them), sometimes without mentioning this, for weakening the assumptions on the data's problem and (or) widening the class of the covered problems by their results.

Motivated by the above considerations about the advantage of the use of generalized gauge spaces in studying differential systems on one hand [20], and inspired by [5] on the other hand, we aim through this work to generalize and improve the existence-uniqueness result for (1.1) given in [12, Theorem 15]. More precisely, we consider the following nonlinear coupled system of fractional differential equations with state deviating arguments and "maxima":

$$(1.2)\qquad \begin{cases} {}^{C}D^{\alpha}u(t) = f\left(t, \max\limits_{\sigma\in[a(t),b(t)]} u(\sigma), \max\limits_{\sigma\in[a(t),b(t)]} v(\sigma), u(\tau_1(t)), v(\tau_2(t))\right), \quad t>0, \\ {}^{C}D^{\beta}v(t) = g\left(t, \max\limits_{\sigma\in[a(t),b(t)]} u(\sigma), \max\limits_{\sigma\in[a(t),b(t)]} v(\sigma), u(\tau_1(t)), v(\tau_2(t))\right), \quad t>0, \end{cases}$$

subject of the following initial conditions

$$(1.3)\qquad \begin{cases} u(t) = \varphi(t), \quad t\le 0, \\ v(t) = \psi(t), \quad t \le 0, \end{cases}$$

where ${}^{C}D^{\alpha}$ and ${}^{C}D^{\beta}$ denote the Caputo fractional derivative operators of order α and β in $]0,1[$ respectively, a, b and τ_i, $i=1,2$ are real continuous functions defined on $\mathbb{R}_+ = [0,+\infty[$ such that

$0 \leq a(t) \leq b(t) \leq t$, $f, g : \mathbb{R}_+ \times \mathbb{R}^4 \longrightarrow \mathbb{R}$ are nonlinear continuous functions, the initial conditions $\phi, \psi :]-\infty, 0] \longrightarrow \mathbb{R}$ are continuous functions.

Note that (1.2)-(1.3) is much more general than (1.1). In this study, we introduce useful vector-valued weighted pseudo-metrics on $X = \mathscr{C}(\mathbb{R}) \times \mathscr{C}(\mathbb{R})$, where $\mathscr{C}(\mathbb{R})$ denotes the set of all real continuous functions on $\mathbb{R}$. Then we reformulate (1.2)-(1.3) into a fixed point problem of generalized contractive mappings on X (in the sens of Definition 2.6), on which we apply some fixed point theorem of Perov's type, given in Theorem 2.7. Using this approach, we obtain a global existence-uniqueness result for (1.2)-(1.3) under more less restrictive conditions in comparison with those imposed in [12], allowing a large class of nonlinearities to be covered by our result (For further details, see Remark 3.6). In fact, the existence-uniqueness results obtained in this paper are improved generalizations and partial complement of many other recent results in the literature, such as those in [19, 18, 28, 15] (see Remark 3.4 and Remark 3.8). Furthermore, we give and we prove some generalized concept of uniform stability of the solutions.

The rest of the paper is organized as follows. In the next section we recall some useful definitions and properties from fractional calculus. We introduce also the fixed point theorem in generalized gauge spaces, on which our existence-uniqueness result is based, as well as some related concepts. The main result concerning the global existence-uniqueness result for (1.2)-(1.3) is established in section 3. Finally, in section 4, we present and discuss some generalized concept of uniform stability of solutions.

2. Preliminaries

Let us recall the notion of the fractional derivatives. For further details on some essential related properties, we refer to [9, 16].

Let n be a positive integer, α the positive real such that $n-1 < \alpha \leq n$ and d^n/dt^n the classical derivative operator of order n.

Definition 2.1. The Riemann-Liouville fractional integral, and the Riemann-Liouville fractional derivative, of a real function u defined on $\mathbb{R}_+$ of order α, are defined respectively by

$$I^{\alpha}_{0+} u(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)\, ds, \quad t > 0,$$
$$D^{\alpha}_{0+} u(t) := \frac{d^n}{dt^n} I^{n-\alpha}_{0+} u(t) := \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} u(s)\, ds, \quad t > 0,$$

where $\Gamma(.)$ is the Gamma function, provided that the right hand sides exist point wise.

Definition 2.2. The Caputo fractional derivative of a real function u defined on $\mathbb{R}_+$ of order α, denoted by ${}^C D^{\alpha}_{0+}$, is defined by

$${}^C D^{\alpha}_{0+} u(t) := \left(D^{\alpha}_{0+} \left[u - \sum_{k=0}^{n-1} \frac{u^{(k)}(0)}{k!} (.)^k \right] \right)(t), \quad t > 0,$$

provided that the right hand side exists point wise.

We denote by $\mathscr{M}_n(\mathbb{R}_+)$, the set of all square matrices of order n with positive real elements, I the identity matrix of order n and by O the zero matrix of order n.

Definition 2.3. [27] A square real matrix M of order n, is said to be convergent to zero, if $M^k \longrightarrow O$, as $k \longrightarrow \infty$.

Definition 2.4. [27]Let $M \in \mathscr{M}_n(\mathbb{R}_+)$ with eigenvalues λ_i, $1 \leq i \leq n$, that is $\lambda_i \in \mathbb{R}$ such that $det(M - \lambda_i I) = O$. Then

$$\rho(M) = \max_{1\leq i\leq n} |\lambda_i|$$

is called the spectral radius of M.

Theorem 2.5. [27, Theorem 3.15] *If $M \in \mathscr{M}_n(\mathbb{R}_+)$ with $\rho(M) < 1$, then $I - M$ is non singular, and*

$$(I-M)^{-1} = I + M + M^2 + ... + M^n + ...,$$

the series on the right is converging. Conversely, if the series on the right converges, then $\rho(M) < 1$.

We give now, the extension of Gheorghiu's theorem for generalized contractions on complete generalized gauge spaces introduced in [20].

Let E be a generalized gauge space endowed with a complete gauge structure $\mathfrak{D} = \{D_\nu\}_{\nu\in\mathscr{N}}$, where $\mathscr{N}$ is an index set. For further details on gauge spaces and generalized gauge spaces we refer to [10, 20].

Definition 2.6. [20] (Generalized contraction) Let $(E, \mathfrak{D})$ be a generalized gauge space with $\mathfrak{D} = \{D_\nu\}_{\nu\in\mathscr{N}}$. A map $T : D(T) \subset E \longrightarrow E$ is called a generalized contraction, if there exists a function $w : \mathscr{N} \longrightarrow \mathscr{N}$ and $M \in \mathscr{M}_n(\mathbb{R}_+)^{\mathscr{N}}$, $M = \{M_\nu\}_{\nu\in\mathscr{N}}$ such that

$$D_\nu(T(u), T(v)) \leq M_\nu D_{w(\nu)}(u, v), \quad \forall u, v \in D(T),\ \forall \nu \in \mathscr{N} \tag{2.1}$$

and

$$\sum_{n=1}^{\infty} M_\nu M_{w(\nu)} M_{w^2(\nu)} ... . M_{w^{n-1}(\nu)} D_{w^n(\nu)}(u, v) < \infty, \quad \forall u, v \in D(T),\ \forall \nu \in \mathscr{N}. \tag{2.2}$$

Theorem 2.7. [20, Theorem 2.1] *Let $(E, \mathfrak{D})$ be a complete generalized gauge space and let $T : E \longrightarrow E$ be a generalized contraction. Then, T has a unique fixed point in E, which can be obtained by successive approximations starting from any element of E.*

Recall that $\mathscr{C}(\mathbb{R})$ denotes the set of all real continuous functions on $\mathbb{R}$. In all what follows, $\mathscr{C}(\mathbb{R}) \times \mathscr{C}(\mathbb{R})$ is denoted by X.

Following the proof of [19, Lemma 1] with a slight adaptation, we get the system of integral equations equivalent to (1.2)-(1.3) given by the following lemma.

Lemma 2.8. *Let f, g, a, b and τ_i (with $i = 1, 2$) be continuous functions. Then, $(u, v) \in X$ is a solution of (1.2)-(1.3) if and only if (u, v) is a solution of the following system of integral equations*

$$(2.3)\quad \begin{cases} u(t) = \varphi_0 + \displaystyle\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\left(s, \max_{\sigma\in[a(s),b(s)]} u(\sigma), \max_{\sigma\in[a(s),b(s)]} v(\sigma), u(\tau_1(s)), v(\tau_2(s))\right) ds, & t > 0 \\ v(t) = \psi_0 + \displaystyle\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} g\left(s, \max_{\sigma\in[a(s),b(s)]} u(\sigma), \max_{\sigma\in[a(s),b(s)]} v(\sigma), u(\tau_1(s)), v(\tau_2(s))\right) ds, & t > 0 \\ u(t) = \varphi(t), & t \le 0 \\ v(t) = \psi(t), & t \le 0. \end{cases}$$

For $i = 1, 2$, let $T_i : X \to \mathscr{C}(\mathbb{R})$ be the operators defined for every $W := (u, v) \in X$ by

$$(2.4)\quad T_1(W)(t) = \begin{cases} \varphi_0 + \\ \displaystyle\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\left(s, \max_{\sigma\in[a(s),b(s)]} u(\sigma), \max_{\sigma\in[a(s),b(s)]} v(\sigma), u(\tau_1(s)), v(\tau_2(s))\right) ds, & t > 0 \\ \varphi(t), & t \le 0 \end{cases}$$

$$(2.5)\quad T_2(W)(t) = \begin{cases} \psi_0 + \\ \displaystyle\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} g\left(s, \max_{\sigma\in[a(s),b(s)]} u(\sigma), \max_{\sigma\in[a(s),b(s)]} v(\sigma), u(\tau_1(s)), v(\tau_2(s))\right) ds, & t > 0 \\ \psi(t), & t \le 0. \end{cases}$$

Remark 2.9. According to Lemma 2.8, (1.2)-(1.3) is equivalent to $(u, v) = T(u, v)$ where $T : X \to X$ is the operator defined by $T(u, v)(t) = (T_1(u, v)(t), T_2(u, v)(t))$ with T_1 and T_2 are given respectively by (2.4) and (2.5).

3. Existence-uniqueness results

The main purpose of this section, is to prove a global existence-uniqueness result for (1.2)-(1.3). To this end, we consider the following assumptions:

- (H_1) The functions $\tau_i : i = 1, 2$ are bounded respectively by h_1 and h_2.
- (H_2) There exist continuous positive real valued functions $L_i, M_i : i = 1, 2, 3, 4$ defined on $\mathbb{R}_+$, and satisfying

$$(i)\ |f(t, \xi_1, \eta_1, x_1, y_1) - f(t, \xi_2, \eta_2, x_2, y_2)| \le L_1(t)|\xi_1 - \xi_2| + L_2(t)|\eta_1 - \eta_2| + L_3(t)|x_1 - x_2| + L_4(t)|y_1 - y_2|$$

$$(ii)\ |g(t, \xi_1, \eta_1, x_1, y_1) - g(t, \xi_2, \eta_2, x_2, y_2)| \le M_1(t)|\xi_1 - \xi_2| + M_2(t)|\eta_1 - \eta_2| + M_3(t)|x_1 - x_2| + M_4(t)|y_1 - y_2|$$

whenever the left hand sides are defined.

Let introduce the positive continuous function:

$$A_\theta(t) = e^{t+\frac{\theta}{q}\int_0^t \left[\max\limits_{1\le i\le 4}\{L_i(\tau),M_i(\tau)\}\right]^q d\tau} \tag{3.1}$$

where $1 < p < \min\left\{\frac{1}{\alpha}, \frac{1}{1-\alpha}, \frac{1}{\beta}, \frac{1}{1-\beta}\right\}$, $\frac{1}{p}+\frac{1}{q} = 1$, $L_i, M_i : i = 1,2,3,4$ are the functions given by (H_2) and θ is a positive real number to be specified later.

Let $\mathfrak{D} = \{D_\nu\}_{\nu\in\mathscr{N}}$ be the complete generalized gauge structure on X defined for $W_1 = (u_1,v_1), W_2 = (u_2,v_2) \in X$ by:

$$D_\nu(W_1,W_2) = \begin{pmatrix} d_\nu(u_1,u_2) \\ \\ d_\nu(v_1,v_2) \end{pmatrix}, \tag{3.2}$$

where d_ν is the weighted pseudo-metric on $\mathscr{C}(\mathbb{R})$ given by

$$d_\nu(u,v) = \max_{t\in\nu}\left\{\frac{|u(t)-v(t)|}{A_\theta(t)}\right\}, \ \forall u,v\in\mathscr{C}(\mathbb{R}), \tag{3.3}$$

where $A_\theta(t)$ is defined in (3.1) and $\mathscr{N}$ denotes the set of all compact subsets of $\mathbb{R}$.

We start by an auxiliary lemma, which plays a key role in what follows.

Lemma 3.1. *Let F, r_1 and r_2 be continuous positive functions on $\mathbb{R}_+$ such that for every $t\ge 0$: $F(t)\le \max\limits_{1\le i\le 4}\{L_i(t),M_i(t)\}$ and $r_1(t)\le r_2(t)$.*

Then, for every $u_1,u_2\in\mathscr{C}(\mathbb{R})$, the following inequality holds true:

$$\begin{aligned}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}F(s)\max_{\sigma\in[r_1(s),r_2(s)]}|u_1(\sigma)-u_2(\sigma)|\,ds &\le C_{p,\alpha}(\theta)A_\theta(t)\times \\ &\max_{\sigma\in[r_1(t),r_2(t)]}\frac{|u_1(\sigma)-u_2(\sigma)|}{A_\theta(\sigma)}, \forall t>0\end{aligned} \tag{3.4}$$

where

$$C_{p,\alpha}(\theta) = \frac{p^{(1-\alpha)-\frac{1}{p}}\Gamma^{\frac{1}{p}}(p(\alpha-1)+1)}{\Gamma(\alpha)\,\theta^{1-\frac{1}{p}}}. \tag{3.5}$$

Proof. Let $t>0$, we have:

$$\begin{aligned}&\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}F(s)\max_{\sigma\in[r_1(s),r_2(s)]}|u_1(\sigma)-u_2(\sigma)|\,ds = \\ &\frac{1}{\Gamma(\alpha)}\int_0^t\left[(t-s)^{\alpha-1}e^s\right]\left[\frac{F(s)}{e^s}\max_{\sigma\in[r_1(s),r_2(s)]}|u_1(\sigma)-u_2(\sigma)|\right]ds.\end{aligned}$$

As a continuous function, any positive power of $\dfrac{F(.)}{e^{\cdot}} \max_{\sigma\in[r_1(.),r_1(.)]} |u_1(\sigma)-u_2(\sigma)|$ is locally integrable. Thus, with p and q given in (3.1), Hölder's inequality leads to

$$\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} F(s) \max_{\sigma\in[r_1(s),r_2(s)]} |u_1(\sigma)-u_2(\sigma)|\, ds \leq$$

$$\frac{1}{\Gamma(\alpha)} \left\{\int_0^t (t-s)^{p(\alpha-1)} e^{ps} ds\right\}^{\frac{1}{p}} \left\{\int_0^t \left[\frac{F(s)}{e^s} \max_{\sigma\in[r_1(s),r_2(s)]} |u_1(\sigma)-u_2(\sigma)|\right]^q ds\right\}^{\frac{1}{q}}$$

$$\leq \frac{1}{\Gamma(\alpha)} \left\{\int_0^t s^{p(\alpha-1)} e^{p(t-s)} ds\right\}^{\frac{1}{p}} \left\{\int_0^t \left[\frac{\max_{1\leq i\leq 4}\{L_i(s),M_i(s)\}}{e^s} \times \max_{\sigma\in[r_1(s),r_2(s)]} |u_1(\sigma)-u_2(\sigma)|\right]^q ds\right\}^{\frac{1}{q}}$$

$$\leq \frac{e^t}{\Gamma(\alpha)} \left\{\int_0^{pt} \left(\frac{X}{p}\right)^{p(\alpha-1)} e^{-X} \frac{dX}{p}\right\}^{\frac{1}{p}} \left\{\int_0^t \left[\frac{\max_{1\leq i\leq 4}\{L_i(s),M_i(s)\}}{e^s} \times \max_{\sigma\in[r_1(s),r_2(s)]} |u_1(\sigma)-u_2(\sigma)|\right]^q ds\right\}^{\frac{1}{q}}. \tag{3.6}$$

Note that

$$\left[\max_{\sigma\in[r_1(s),r_2(s)]} |u_1(\sigma)-u_2(\sigma)|\right]^q \leq \max_{\sigma\in[r_1(s),r_2(s)]} |u_1(\sigma)-u_2(\sigma)|^q \tag{3.7}$$

Indeed, since u_1 and u_2 are continuous, then let $\sigma_0 \in [r_1(s), r_2(s)]$ such that $\max_{\sigma\in[r_1(s),r_2(s)]} |u_1(\sigma)-u_2(\sigma)| = |u_1(\sigma_0)-u_2(\sigma_0)|$, hence:

$$\left[\max_{\sigma\in[r_1(s),r_2(s)]} |u_1(\sigma)-u_2(\sigma)|\right]^q = |u_1(\sigma_0)-u_2(\sigma_0)|^q$$
$$\leq \max_{\sigma\in[r_1(s),r_2(s)]} |u_1(\sigma)-u_2(\sigma)|^q.$$

On the other hand we have:

$$\left[\max_{1\leq i\leq 4}\{L_i(s),M_i(s)\}\right]^q = \frac{\frac{d}{ds}\left[\frac{1}{\theta} e^{\theta\int_0^s \left(\max_{1\leq i\leq 4}\{L_i(\eta),M_i(\eta)\}\right)^q d\eta}\right]}{e^{\theta\int_0^s \left(\max_{1\leq i\leq 4}\{L_i(\eta),M_i(\eta)\}\right)^q d\eta}}.$$

Consequently, in view of (3.1), one gets:

$$\left[\frac{\max\limits_{1\le i\le 4}\{L_i(s),M_i(s)\}}{e^s}\right]^q = \frac{\frac{d}{ds}\left[\frac{1}{\theta}e^{\theta\int_0^s\left(\max\limits_{1\le i\le 4}\{L_i(\eta),M_i(\eta)\}\right)^q d\eta}\right]}{[A_\theta(s)]^q}. \tag{3.8}$$

Returning now to (3.6), using (3.7) together with (3.8), we obtain

$$\begin{aligned}
&\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}F(s)\max_{\sigma\in[r_1(s),r_2(s)]}|u_1(\sigma)-u_2(\sigma)|\,ds \le \\
&\frac{e^t}{\Gamma(\alpha)}\left\{\int_0^{pt}\left(\frac{X}{p}\right)^{p(\alpha-1)}e^{-X}\frac{dX}{p}\right\}^{\frac{1}{p}}\times \\
&\left\{\int_0^t \frac{d}{ds}\left[\frac{1}{\theta}e^{\theta\int_0^s\left(\max\limits_{1\le i\le 4}\{L_i(\eta),M_i(\eta)\}\right)^q d\eta}\right]\frac{\max\limits_{\sigma\in[r_1(s),r_2(s)]}|u_1(\sigma)-u_2(\sigma)|^q}{[A_\theta(s)]^q}ds\right\}^{\frac{1}{q}} \\
&\le \frac{e^t}{\Gamma(\alpha)}\left\{p^{p(1-\alpha)-1}\Gamma(p(\alpha-1)+1)\right\}^{\frac{1}{p}}\left\{\frac{\theta^{-\frac{1}{q}}A_\theta(t)}{e^t}\max_{\sigma\in[r_1(t),r_2(t)]}\frac{|u_1(\sigma)-u_2(\sigma)|}{A_\theta(\sigma)}\right\}.
\end{aligned}$$

Which completes the proof. □

Note that, similar arguments as those used in the proof of Lemma 3.1 and under the same conditions, lead to the following inequality:

$$\begin{aligned}
&\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}F(s)\max_{\sigma\in[r_1(s),r_2(s)]}|u_1(\sigma)-u_2(\sigma)|\,ds \le C_{p,\beta}(\theta)A_\theta(t)\times \\
&\max_{\sigma\in[r_1(t),r_2(t)]}\frac{|u_1(\sigma)-u_2(\sigma)|}{A_\theta(\sigma)},\ \forall t>0.
\end{aligned} \tag{3.9}$$

Proposition 3.2. *Let $(H_1)-(H_2)$ be satisfied. Then, there exists a map $w:\mathcal{N}\longrightarrow\mathcal{N}$ such that for every $(u_1,v_1),(u_2,v_2)\in X$ and every $\nu\in\mathcal{N}$, the following inequality holds true:*

$$D_\nu(T(u_1,v_1),T(u_2,v_2)) \le M_{p,\alpha,\beta}(\theta)D_{w(\nu)}((u_1,v_1),(u_2,v_2)) \tag{3.10}$$

with

$$M_{p,\alpha,\beta}(\theta) := \begin{pmatrix} 2C_{p,\alpha}(\theta) & 2C_{p,\alpha}(\theta) \\ 2C_{p,\beta}(\theta) & 2C_{p,\beta}(\theta)\end{pmatrix} \tag{3.11}$$

Where $C_{p,\alpha}(\theta)$ is given by (3.5) ($C_{p,\beta}(\theta)$ is also given by (3.5) with β instead of α).

Proof. Let us define $w:\mathcal{N}\longrightarrow\mathcal{N}$ as follows:

$$w(\nu) = \begin{cases} \nu & :\text{if } \nu\subset\mathbb{R}_-=]-\infty,0], \\ [\min\{-h_1,-h_2\},\ \max\{\nu_m,h_1,h_2\}] & :\text{if no}, \end{cases} \tag{3.12}$$

where $v_m = \sup v$ and h_1, h_2 are the constants given by (H_1).

Note that according to (3.12), it follows that

(3.13) $$\text{For every } \ v \in \mathcal{N} : w^n(v) = w(v), \quad \forall n \geq 2.$$

Let $v \in \mathcal{N}$ such that $v \subset \mathbb{R}_-$. Thus, for every $t \in v$, we have:

$$|T_1(u_1,v_1)(t) - T_1(u_2,v_2)(t)| = |\varphi(t) - \varphi(t)| = 0.$$

Consequently, (3.10) holds true.

Let now $v \in \mathcal{N}$, where $v \cap \mathbb{R}_+ \neq \emptyset$ and let $t \in v$ such that $t > 0$. Using $(H_2(i))$, we get:

$$\begin{aligned}
&|T_1(u_1,v_1)(t) - T_1(u_2,v_2)(t)| \leq \\
&\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left\{ L_1(s) \left| \max_{\sigma\in[a(s),b(s)]} u_1(\sigma) - \max_{\sigma\in[a(s),b(s)]} u_2(\sigma) \right| \right. \\
&\left. \qquad + L_2(s) \left| \max_{\sigma\in[a(s),b(s)]} v_1(\sigma) - \max_{\sigma\in[a(s),b(s)]} v_2(\sigma) \right| \right\} ds \\
&+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \{ L_3(s) |u_1(\tau_1(s)) - u_2(\tau_1(s))| + L_4(s) |v_1(\tau_2(s)) - v_2(\tau_2(s))| \} ds \\
&\leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left\{ L_1(s) \max_{\sigma\in[a(s),b(s)]} |u_1(\sigma) - u_2(\sigma)| + L_2(s) \max_{\sigma\in[a(s),b(s)]} |v_1(\sigma) - v_2(\sigma)| \right\} ds \\
&+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left\{ L_3(s) \max_{\sigma\in\tau_1([0,s])} |u_1(\sigma) - u_2(\sigma)| + L_4(s) \max_{\sigma\in\tau_2([0,s])} |v_1(\sigma) - v_2(\sigma)| \right\} ds.
\end{aligned}$$

Taking into account (H_1), we obtain:

$$\begin{aligned}
&|T_1(u_1,v_1)(t) - T_1(u_2,v_2)(t)| \leq \\
&\leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left\{ L_1(s) \max_{\sigma\in[a(s),b(s)]} |u_1(\sigma) - u_2(\sigma)| + L_2(s) \max_{\sigma\in[a(s),b(s)]} |v_1(\sigma) - v_2(\sigma)| \right\} ds \\
&+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left\{ L_3(s) \max_{\sigma\in[-h_1,h_1]} |u_1(\sigma) - u_2(\sigma)| + L_4(s) \max_{\sigma\in[-h_2,h_2]} |v_1(\sigma) - v_2(\sigma)| \right\} ds.
\end{aligned}$$

Note first that according to (3.12), together with the fact that $b(t) \leq t$, the compacts $[a(s), b(s)], [-h_1, h_1], [-h_2, h_2]$ are included in $w(v)$. Now, using (3.4) to estimate the four integrals above, it follows that:

$$|T_1(u_1,v_1)(t) - T_1(u_2,v_2)(t)| \leq 2C_{p,\alpha}(\theta) A_\theta(t) \left[d_{w(v)}(u_1,u_2) + d_{w(v)}(v_1,v_2) \right].$$

Dividing the previous inequality by $A_\theta(t)$, then taking the supremum on v, we get:

(3.14) $$d_v(T_1(u_1,v_1), T_1(u_2,v_2)) \leq 2C_{p,\alpha}(\theta) \left[d_{w(v)}(u_1,u_2) + d_{w(v)}(v_1,v_2) \right].$$

Similarly, by means of (H_1) and $(H_2(ii))$ together with (3.9) we prove that the following inequality holds true for every $(u_1,v_1), (u_2,v_2) \in X$ and every $v \in \mathcal{N}$:

(3.15) $$d_v(T_2(u_1,v_1), T_2(u_2,v_2)) \leq 2C_{p,\beta}(\theta) \left[d_{w(v)}(u_1,u_2) + d_{w(v)}(v_1,v_2) \right].$$

Now, (3.10) yields immediately from (3.14) and (3.15). □

We are now ready to prove the following main result.

Theorem 3.3. *Under hypotheses* $(H_1)-(H_2)$*, the system* (1.2)-(1.3) *admits a unique global solution in* X.

Proof. Recall that, in view of Remark 2.9, the solutions of (1.2)-(1.3) are the fixed points of the operator T. Hence it is sufficient to show that T is a generalized contraction in the sens of Definition 2.6, then deduce the result from Theorem 2.7.

In view of Proposition 3.2, (2.1) holds true with $M_\nu = M_{p,\alpha,\beta}(\theta)$ which is independent of ν and consequently the series (2.2) becomes

$$\sum_{n=0}^{\infty} M_{p,\alpha,\beta}^{n+1}(\theta)\, D_{w^n(v)}(u,v). \tag{3.16}$$

Note that using the property of convergent matrices given in Theorem 2.5, we see that sufficient conditions, so that (3.16) is convergent are:

$$\rho\left(M_{p,\alpha,\beta}(\theta)\right) < 1 \tag{3.17}$$

and

$$sup\left\{D_{w^n(v)}(u,v) : n = 0,1,2,\dots\right\} < \infty. \tag{3.18}$$

Let us calculate the eigenvalues of $M_{p,\alpha,\beta}(\theta)$:

$$\det\left(M_{p,\alpha,\beta}(\theta) - \lambda I_2\right) = \lambda\left[\lambda - \left(C_{p,\alpha}(\theta) + C_{p,\beta}(\theta)\right)\right].$$

That is (3.17) holds true, if and only if

$$C_{p,\alpha}(\theta) + C_{p,\beta}(\theta) < 1. \tag{3.19}$$

So, let us choose θ in (3.1) sufficiently large such that (3.19) is fulfilled.

On the other hand, and according to (3.13), we have:

$$sup\left\{D_{w^n(v)}(u,v) : n = 0,1,2,\dots\right\} = sup\left\{D_v(u,v), D_{w(v)}(u,v)\right\},$$

which is clearly finite. This completes the proof. □

Remark 3.4. When f and g in (1.2) are independent of the first and the second arguments and furthermore $\tau_1(t) = \tau_2(t) = t$, then we have the following observations:

- (i) The assumptions of Theorem 3.3 are clearly much less restrictive than those of [12, Theorem 15]. Thus, as a special case of Theorem 3.3, Corollary 3.5 below, is an important improvement of [12, Theorem 15].
- (ii) The problem considered in [28] for $a_k = \tilde{a}_k = 0\,(k = 1,\dots,m)$ is extended in (1.2)-(1.3) to the half line. The assumptions of Theorem 3.3, even when the problem is considered on a bounded interval, are weaker than those of [28, Theorem 3.1] and [28, Theorem 3.2]. Thus, for some particular cases, Theorem 3.3 improves and complements the existence-uniqueness results of [28].

Corollary 3.5. *Suppose that the following hypothesis holds true*

(H) *There exist continuous positive real valued functions* $L_i : i = 1,2$, $M_i : i = 1,2$ *defined on* $\mathbb{R}_+$, *satisfying*
$|f(t,x_1,y_1) - f(t,x_2,y_2)| \leq L_1(t)|x_1 - x_2| + L_2(t)|y_1 - y_2|$
$|g(t,x_1,y_1) - g(t,x_2,y_2)| \leq M_1(t)|x_1 - x_2| + M_2(t)|y_1 - y_2|$
whenever the left hand sides are defined.

Then (1.1) *admits a unique solution in* X.

Remark 3.6. In order to compare our approach to that used in [12, Theorem 15], it should be noted that: firstly, choosing generalized gauge spaces setting (instead of generalized Banach spaces) allows us to get rid of hypotheses $(H_1(23)), (H_2)$ and replace $h_{i,\alpha}, h_{j,\beta} \in L^1(J,\mathbb{R}_+)$ by $h_{i,\alpha}, h_{j,\beta} \in \mathscr{C}(J,\mathbb{R}_+)$, which is most suitable for applications. Secondly, by means of the Bielecki's idea, we included $h_{i,\alpha}$, $h_{j,\beta}$ in the definition of the generalized gauge structure. This make the matrix M_ν in (2.1), and consequently its convergence to zero, independent of these data. That is, the use of suitable vector-valued weighted pseudo-metrics, allowed us also to get rid of the convergence to zero for the matrix (27) in [12, Theorem 15].

Applying the basic steps of the approach used in the study of the system (1.2)-(1.3) with appropriate adaptation, one acquires to a global existence-uniqueness result of the following initial value problem:

$$\begin{cases} {}^{C}D^{\alpha}u(t) = f(t, \max\limits_{\sigma\in[a(t),b(t)]} u(\sigma), u(g_1(t)), u(g_2(t)), \ldots, u(g_N(t))), & t > 0 \\ u(t) = \varphi(t), & t \leq 0. \end{cases} \tag{3.20}$$

Theorem 3.7. *Let the following conditions be satisfied:*

(C_1) $g_i : i = 1,\ldots,N$ *are continuous functions and bounded respectively by* h_1, ..., h_N .
(C_2) $f : \mathbb{R}_+ \times \mathbb{R}^{N+1} \longrightarrow \mathbb{R}$ *is a nonlinear continuous function, such that there exist continuous positive real valued functions* $L_i : i = 0,\ldots,N$, *defined on* $\mathbb{R}_+$, *satisfying*

$$|f(t,\xi,x_1,\ldots,x_N) - f(t,\eta,y_1,\ldots,y_N)| \leq L_0(t)|\xi - \eta| + \sum_{i=1}^{N} L_i(t)|x_i - y_i|.$$

Then, (3.20) *admits a unique global solution in* $\mathscr{C}(\mathbb{R})$.

Proof. Transform first (3.20) to the equivalent fixed point problem $T(u) = u$, where T is the operator defined on $\mathscr{C}(\mathbb{R})$ by:

$$T(u)(t) = \begin{cases} \varphi_0 + \displaystyle\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\left(s, \max_{\sigma\in[a(s),b(s)]} u(\sigma), u(g_1(s)), \ldots, u(g_N(s))\right) ds, & t > 0 \\ \varphi(t), & t \leq 0. \end{cases}$$

Let d_ν be the weighted pseudo-metric defined on $\mathscr{C}(\mathbb{R})$ by (3.3), where

$$A_\theta(t) = e^{t + \frac{\theta}{q}\int_0^t \left[\max\limits_{0\leq i\leq N}\{L_i(\tau)\}\right]^q d\tau}, \quad 1 < p < \min\left\{\frac{1}{\alpha}, \frac{1}{1-\alpha}\right\}, \quad \frac{1}{p} + \frac{1}{q} = 1$$

and $L_i : i = 1,\ldots,N$ are the functions given by (C_2).

Let $w : \mathscr{N} \longrightarrow \mathscr{N}$ be the mapping defined as follows:

$$w(\nu) := \begin{cases} \nu & \text{:if } \nu \subset \mathbb{R}_- =]-\infty, 0], \\ \\ \left[\min\limits_{0 \le i \le N} \{-h_i\},\ \max\{\nu_m, h_1, ..., h_N\}\right] & \text{:if no,} \end{cases}$$

where $\nu_m = \sup \nu$ and $h_i, i = 1, ..., N$ are the constants given by (C_2).

Now, the rest of the proof is similar to Theorem 3.3, so the details are omitted here. □

Remark 3.8.

(i) If $g_i(t) = t - \tau_i(t)\,(i = 1, ..., N)$, then the problem (3.20) is reduced to that considered in [19]. In this case, Theorem 3.7 provides a global existence-uniqueness result under conditions different from those established in [19, Theorem 1]. So Theorem 3.7 extends and complements the result of [19].

(ii) If f is independent of the second and the last $N - 1$ arguments, and furthermore $g_1(t) = t$, then the problem (3.20) is reduced to that considered in [18]. Theorem 3.7 includes and extends the global existence-uniqueness result of [18, Theorem 5.4.], without any need to obtain first a local existence result then applying the continuation method to acquire a global existence result, as it was done in [18].

4. Generalized stability

In some generalized sens, we introduce and discuss in this section, the uniform stability of solutions of (1.2)-(1.3). Before defining this concept, we give the following observation:
If (φ, ψ) and $(\tilde{\varphi}, \tilde{\psi})$ are two initial conditions of (1.2)-(1.3), then $D_{\tilde{\nu}}((\varphi, \psi), (\tilde{\varphi}, \tilde{\psi}))$ is defined only for $\tilde{\nu} \in \mathscr{N}$ such that $\tilde{\nu} \subset \mathbb{R}_-$.

Definition 4.1. (Generalized stability)

We say that the solution of (1.2)-(1.3) is uniformly stable if for every $\varepsilon = (\varepsilon_1, \varepsilon_2) > 0$, there exists $\delta = (\delta_1, \delta_2) > 0$ such that, for any two solutions $(u(t), v(t))$ and $(\tilde{u}(t), \tilde{v}(t))$ of (1.2) with the initial condition (1.3) and $(\tilde{u}(t), \tilde{v}(t)) = (\tilde{\varphi}(t), \tilde{\psi}(t))$ for $t \le 0$ respectively, the following holds true:

$$\forall \nu \in \mathscr{N}, \tilde{\nu} = \nu \cap \mathbb{R}_-, \text{ if } D_{\tilde{\nu}}((\varphi, \psi), (\tilde{\varphi}, \tilde{\psi})) < \delta, \text{ then, } D_{w(\nu)}((u, v), (\tilde{u}, \tilde{v})) < \varepsilon. \tag{4.1}$$

Remark 4.2. The notion given in Definition 4.1, is a generalization in the following sens: if $\nu \in \mathscr{N}$ is such that $\nu \subset \mathbb{R}_-$, then in view of (3.12), $w(\nu) = \nu$ and consequently (4.1) is reduced to the usual concept of uniform stability.

Theorem 4.3. *Let $(H_1) - (H_2)$ be satisfied. Then, the solution of (1.2)-(1.3) is uniformly stable.*

Proof. Let $(u(t), v(t))$ and $(\tilde{u}(t), \tilde{v}(t))$ be two solutions of (1.2) with the initial condition (1.3) and $(\tilde{u}(t), \tilde{v}(t)) = (\tilde{\varphi}(t), \tilde{\psi}(t))$ for $t \le 0$ respectively.

Let $\nu \in \mathscr{N}$ such that $w(\nu) \cap \mathbb{R}_+ = \emptyset$, that is $\nu = w(\nu) = \tilde{\nu}$. Then, by means of (2.3), we have:

$$D_{w(\nu)}((u, v), (\tilde{u}, \tilde{v})) = D_{\tilde{\nu}}((\varphi, \psi), (\tilde{\varphi}, \tilde{\psi})).$$

Consequently, (4.1) holds true with $\delta = \varepsilon$.

Let now $v \in \mathscr{N}$ with $w(v)\cap\mathbb{R}_+ \neq \emptyset$ and let $t \in w(v)$ such that $t>0$. From (2.3) together with (H_2), we have:

$$\begin{gathered}|u(t)-\tilde{u}(t)| \leq \\ |\varphi(0)-\tilde{\varphi}(0)| + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left\{L_1(s)\left|\max_{\sigma\in[a(s),b(s)]} u(\sigma) - \max_{\sigma\in[a(s),b(s)]} \tilde{u}(\sigma)\right|\right. \\ \left. +L_2(s)\left|\max_{\sigma\in[a(s),b(s)]} v(\sigma) - \max_{\sigma\in[a(s),b(s)]} \tilde{v}(\sigma)\right|\right\} ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \times \\ \{L_3(s)\,|u(\tau_1(s))-\tilde{u}(\tau_1(s))| + L_4(s)\,|v(\tau_2(s))-\tilde{v}(\tau_2(s))|\}\, ds.\end{gathered}$$

Using Lemma 3.1 and an argument similar to that used in the proof of Proposition 3.2, we prove that

$$|u(t)-\tilde{u}(t)| \leq |\varphi(0)-\tilde{\varphi}(0)| + 2C_{p,\alpha}(\theta)A_\theta(t)\left[d_{w^2(v)}(u,\tilde{u}) + d_{w^2(v)}(v,\tilde{v})\right].$$

Thus, (recall that $\tilde{v} := v\cap\mathbb{R}_-$)

$$\begin{aligned}|u(t)-\tilde{u}(t)| &\leq A_\theta(t)\max_{t\in\tilde{v}}\frac{|\varphi(t)-\tilde{\varphi}(t)|}{A_\theta(t)} + \\ &+2C_{p,\alpha}(\theta)A_\theta(t)\left[d_{w^2(v)}(u,\tilde{u}) + d_{w^2(v)}(v,\tilde{v})\right] \\ &= A_\theta(t)d_{\tilde{v}}(\varphi,\tilde{\varphi}) + 2C_{p,\alpha}(\theta)A_\theta(t)\left[d_{w^2(v)}(u,\tilde{u}) + d_{w^2(v)}(v,\tilde{v})\right].\end{aligned}$$

Dividing the above inequality by $A_\theta(t)$, then taking the supremum on $w(v)$ and noting that $w^2(v) = w(v)$, we get:

$$d_{w(v)}(u,\tilde{u}) \leq d_{\tilde{v}}(\varphi,\tilde{\varphi}) + 2C_{p,\alpha}(\theta)\left[d_{w(v)}(u,\tilde{u}) + d_{w(v)}(v,\tilde{v})\right]. \tag{4.2}$$

Similarly, we prove that the following inequality holds true for every $v \in \mathscr{N}$:

$$d_{w(v)}(v,\tilde{v}) \leq d_{\tilde{v}}(\psi,\tilde{\psi}) + 2C_{p,\beta}(\theta)\left[d_{w(v)}(u,\tilde{u}) + d_{w(v)}(v,\tilde{v})\right]. \tag{4.3}$$

Now, if we choose θ sufficiently large such that: $C_{p,\alpha}(\theta)+C_{p,\beta}(\theta) < \frac{1}{2}$, then from (4.2) and (4.3), it follows that:

$$\frac{1-2\left(C_{p,\alpha}(\theta)+C_{p,\beta}(\theta)\right)}{1-2C_{p,\beta}(\theta)} d_{w(v)}(u,\tilde{u}) \leq d_{\tilde{v}}(\varphi,\tilde{\varphi}) + \frac{1}{1-2C_{p,\beta}(\theta)} d_{\tilde{v}}(\psi,\tilde{\psi})$$

and

$$\frac{1-2\left(C_{p,\alpha}(\theta)+C_{p,\beta}(\theta)\right)}{1-2C_{p,\alpha}(\theta)} d_{w(v)}(v,\tilde{v}) \leq d_{\tilde{v}}(\psi,\tilde{\psi}) + \frac{1}{1-2C_{p,\alpha}(\theta)} d_{\tilde{v}}(\varphi,\tilde{\varphi}).$$

Therefore, for every $\varepsilon = (\varepsilon_1,\varepsilon_2) > 0$, there exists $\delta = (\delta_1,\delta_2) > 0$ given by

$$\delta_1 = \min\left\{\frac{1-2\left(C_{p,\alpha}(\theta)+C_{p,\beta}(\theta)\right)}{1-2C_{p,\beta}(\theta)}\varepsilon_1, \frac{(1-2C_{p,\alpha}(\theta))\left[1-2\left(C_{p,\alpha}(\theta)+C_{p,\beta}(\theta)\right)\right]}{1-2C_{p,\alpha}(\theta)}\varepsilon_2\right\}$$

and

$$\delta_2 = \min\left\{ \frac{(1-2C_{p,\beta}(\theta))\left[1-2\left(C_{p,\alpha}(\theta)+C_{p,\beta}(\theta)\right)\right]}{1-2C_{p,\beta}(\theta)}\varepsilon_1, \frac{1-2\left(C_{p,\alpha}(\theta)+C_{p,\beta}(\theta)\right)}{1-2C_{p,\alpha}(\theta)}\varepsilon_2 \right\},$$

such that (4.1) holds true. This completes the proof. □

Remark 4.4. The basic techniques of the approach used in this work for the study of systems of the form (1.2)-(1.3), can be easily adapted and applied to acquire similar results for coupled systems of n fractional functional differential equations with ”maxima”, for $n > 2$.

5. Applications

In this section, we provide two examples illustrating the significance of our main findings.

Example 5.1. *Let us consider the following system*

$$\begin{cases} {}^C D^{\frac{3}{2}} u(t) = e^{2t}(|\max_{[\frac{t}{2},t]} u(\sigma)| + |\max_{[\frac{t}{2},t]} v(\sigma)|) + \frac{2t+1}{1+|u(\sin t)|} + \frac{e^{2t}+t}{1+|v(\cos t)|}, & t>0, \\ {}^C D^{\frac{1}{2}} v(t) = e^{t+1}(|\max_{[\frac{t}{2},t]} u(\sigma)| + |\max_{[\frac{t}{2},t]} v(\sigma)|) + \frac{t+2}{1+|u(\sin t)|} + \frac{te^t+t^2}{1+|v(\cos t)|}, & t>0, \\ u(t) = 2t+1, & t\le 0, \\ v(t) = e^{2t}-t^2, & t \le 0. \end{cases} \tag{5.1}$$

(5.1) *is identified to* (1.2)-(1.3) *with*

$$\alpha = \tfrac{3}{2},\ f(t,\xi,\eta,x,y) = e^{2t}(|\xi|+|\eta|) + \frac{2t+1}{1+|x|} + \frac{e^{2t}+t}{1+|y|},\ \tau_1(t) = \sin t,\ a(t) = \tfrac{t}{2},\ \varphi(t) = 2t+1,$$

$$\beta = \tfrac{1}{2},\ g(t,\xi,\eta,x,y) = e^{t+1}(|\xi|+|\eta|) + \frac{t+2}{1+|x|} + \frac{te^t+t^2}{1+|y|},\ \tau_2(t) = \cos t,\ b(t) = t,\ \psi(t) = e^{2t}-t^2.$$

It is clear that (H_1) *is satisfied with* $h_1 = h_2 = 1$.

On the other hand, we have

$$\begin{aligned} |f(t,\xi_1,\eta_1,x_1,y_1) - f(t,\xi_2,\eta_2,x_2,y_2)| \le\ & e^{2t}(|\xi_1-\xi_2|+|\eta_1-\eta_2|) + (2t+1)|x_1-x_2| \\ & + (e^{2t}+t)|y_1-y_2| \end{aligned}$$

and

$$\begin{aligned} |g(t,\xi_1,\eta_1,x_1,y_1) - g(t,\xi_2,\eta_2,x_2,y_2)| \le\ & e^{t+1}(|\xi_1-\xi_2|+|\eta_1-\eta_2|) + (t+2)|x_1-x_2| \\ & + (te^t+t^2)|y_1-y_2|. \end{aligned}$$

Thus, (H_2) *is satisfied with*

$$L_1(t) = L_2(t) = e^{2t},\ L_3(t) = 2t+1,\ L_4(t) = e^{2t}+t \tag{5.2}$$

and

$$M_1(t) = M_2(t) = e^{t+1},\ M_3(t) = t+2,\ M_4(t) = te^t + t^2. \tag{5.3}$$

Then, all assumptions of Theorem 3.3 are satisfied and consequently (5.1) admits a unique global solution, which is furthermore uniformly stable according to Theorem 4.3.

We emphasise here that in our example, the functions given by (5.2)-(5.3) are far from satisfying the hypotheses of [12, Theorem 15].

Example 5.2. *Consider the following equation*

$$\begin{cases} {}^C D^{\frac{5}{7}} u(t) = \dfrac{\ln(t+1)}{1 + |\max\limits_{[0,\frac{t}{2}]} u(\sigma)| + e^{t^2}|u(e^{-2t})| + t|u(\frac{\cos t}{t+1})|}, & t > 0, \\ u(t) = t^2, & t \le 0. \end{cases} \tag{5.4}$$

(5.4) *is identified to* (3.20) *with* $\alpha = \frac{5}{7}$, $N = 2$, $a(t) = 0$, $b(t) = \frac{t}{2}$, $g_1(t) = e^{-2t}$, $g_2(t) = \dfrac{\cos t}{t+1}$, $\varphi(t) = t^2$ *and*

$$f(t,\xi,x,y) = \frac{\ln(t+1)}{1 + |\xi| + e^{t^2}|x| + t|y|}$$

It is not hard to see that (C_1) *and* (C_2) *are satisfied with* $h_1 = h_2 = 1$ *and*

$$L_0(t) = \ln(t+1), \quad L_1(t) = e^{t^2}\ln(t+1), \quad L_2(t) = t\ln(t+1). \tag{5.5}$$

Then, according to Theorem 3.7, (5.4) *admits a unique global solution.*

Note that since the Lipschitz constants given in (5.5) *(which depend on t) are clearly unbounded, many existing results in the literature fail to be applicable to* (5.4).

6. Conclusion

In this work, we investigate some systems of coupled fractional differential equations with deviating arguments and maxima on the half line, given by (1.2)-(1.3). We acquire a global existence-uniqueness result in the space $X = \mathscr{C}(\mathbb{R}) \times \mathscr{C}(\mathbb{R})$, under Lipschitz condition on the nonlinearity with merely continuous arguments and without any other restrictions. The approach used in this study relies mainly to some fixed point theorem of Perov's type in generalized gauge spaces. This suitable choice of the employed structure, allowed us to get rid of some strict conditions imposed in other recent works in the literature, such as [12]. In addition, the introduction of useful vector-valued pseudo-metrics on X, allowed us to get rid also of further conditions. By adapting the basic steps of the above approach to the initial value problem (3.20) of fractional differential equations with multi-deviating arguments and "maxima", we obtained a global existence-uniqueness result which generalizes and complements the results established in [15, 19, 18]. In some generalized sens, the uniform stability of solutions of

(1.2)-(1.3), is also introduced and discussed. We expect that this work, will open the way to improve and complement many other existence results for (integral, integro-differential) differential systems of integer or fractional order, especially on unbounded domains.

References

[1] B. Ahmed, A. Alsaedi , S. Aljoudi, S. K. Ntouyas, *On a coupled system of sequential fractional differential equations with variable coefficients and coupled integral boundary conditions*, Bull. Math. Soc. Sci. Math. Roumanie **60** (2017), 3-18.

[2] B. Ahmed, S. Hamdan, A. Elsaedi, S.K. Ntouyas, *A study of nonlinear coupled system of three fractional equations with nonlocal coupled boundary conditions*, Adv. Difference Equ. **178** (2021), DOI: 10.1186/s13662-021-03440-7

[3] S.S. Almuthaybiri, C.C. Tisdell, *Global existence theory for fractional differential equations: New advances via continuation methods for contractive maps*, Analysis **39** (2019), 117–128.

[4] D.D. Bainov, S.G. Hristova, *Differential Equations With Maxima*, Chapman and Hall CRC, Boca Raton, FL, USA, 2011.

[5] D. Băleanu, O.G. Mustafa, *On the global existence of solutions to a class of fractional differential equations*, Comput. Math. Appl. **59**, (2010), 1835–1841.

[6] A. Bielecki, *Une remarque sur la méthode de Banach-Cacciopoli-Tikhonov*, Bull. Acad. Polon. Sci. Sér. Sci. Math. **4** (1956), 216–264.

[7] C. Corduneanu, *Bielecki's method in the theory of integral equations*, Ann. Univ. Mariae Curie- Sklodowska Sect. A. **XXXVIII** (1984), 23—40.

[8] M. A. Darwish, K. Sadarangani, *On a quadratic integral equation with supremum involving Erdélki-Kober fractional order*, Mathematische Nachrichten **188** (2015), 566–576, DOI: 10.1002/mana.201400063

[9] K. Diethelm, *The Analysis of Fractional Differential Equations*, Springer, Berlin, 2004.

[10] J. Dugundji, *Topology*, Allyn and Bacon, Inc., 470 Atlantic Avenue, Boston, 1966.

[11] Z. F. Z. El-Raheem, *Modification of the application of a contraction mapping method on a class of fractional differential equation*, Appl. Math. Comput. **137** (2003), 371–374.

[12] C. Guendouz, J. E. Lazreg, J.J. Nieto, A. Ouahab, *Existence and compactness results for a system of fractional differential equations*, J. Funct. Spaces **2020** (2020), 1-12.

[13] Z. He, P. Wang, W. Ge, *Periodic boundary value problem for first order impulsive differential equations with supremum*, Indian J. Pure Appl. Math. **34** (2003), 133–143.

[14] V. Ilea, D. Otrocol, *Existence and uniqueness of the solution for an integral equation with supremum, via w-distances*, Symmetry **1554** (2020), DOI: 10.3390/sym12091554

[15] V. Ilea, D. Otrocol, *Functional differential equations with maxima, via step by step contraction principle*, Carpathian J. Math **37** (2021), 195–202.

[16] A. A. Kilbas, H. M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations* Elsevier, New York, 2006.

[17] M. Klimek, M. Blasik, *Existence and uniqueness of solution for a class of nonlinear sequential differential equations of fractional order*, Cent. Eur. J. Math. **10** (2012), DOI: 10.2478/s11533-012-0112-9

[18] C. Li, S. Sarwa, *Existence and continuation of solutions for Caputo type fractional differential equations*, Electron. J. Differential Equation **207** (2016).

[19] K. Nisse, L. Nisse, *An iterative method for solving a class of fractional functional differential equations with maxima*, Mathematics **6** (2018), DOI: 10.3390/math6010002

[20] A. Novac, R. Precup, *Perov type results in gauge spaces and their applications to integral systems on semi-axis*, Math. Slovaca **64** (2014), 961–972, DOI: 10.2478/s12175-014-0251-5

[21] A. Ouani, K. Nisse, L. Nisse, *Long-time solvability of functional differential equations with "supremum" involving increasing nonlinearity at infnite time*, Int. J. Nonlinear Anal. Appl. **14** (2023), 1437–1448.

[22] A. Perov, A. Kibenko, *On certain general method for investigation of boundary value problems*, Izv. Akad. Nauk. **30** (1966), 249–264.
[23] E. P.Popov, *Automatic Regulation and Control*, Nauka, Mosco, 1966.
[24] R. Precup, *The role of matrices that are convergent to zero in the study of semilinear operator systems*, Math. Comput. Modelling **49** (2009), 703–708, DOI: 10.1016/j.mcm.2008.04.006
[25] J. Tariboon, S.K. Ntouyas, S. Asawasamrit, C. Promsakon, *Positives solutions for Hadamard differential systems with fractional integral conditions on an unbounded domain*, Open Math. **15** (2017), 645–666, DOI: 10.1515/math-2017-0057
[26] C.C. Tisdell, *On the application of sequential and fixed-point theory method to fractional differential equations of arbitrary order*, J. Integral Equations Appl. **24** (2012), 283—319, DOI: 10.1216/JIE-2012-24-2-283
[27] R.S. Varga, *Matrix Iterative Analysis*, Institute of Computational Mathematics, Springer, Kent State University, Kent, OH 44242, USA, 1999.
[28] J. Wang, Y. Zhang, *Analysis of fractional order differential coupled systems*, Math. Methods Appl. Sci. **38** (2014), 3322-3338.
[29] P. Zhang, X. Hao, *Existence and uniqueness of solutions for nonlinear integro-differential equations on unbounded domains in Banach spaces*, Adv Difference Equ **247** (2018), DOI: 10.1186/s13662-018-1681-0
[30] P. Zhang, X. Hao, L. Liu, *Existence and uniqueness of the global solution for a class of nonlinear fractional integro-differential equations in a Banach space*, Adv Difference Equ **135** (2019), DOI: 10.1186/s13662-019-2076-6

DEPARTMENT OF MATHEMATICS, FACULTY OF EXACT SCIENCES, UNIVERSITY OF EL OUED, ALGERIA
Email address: nisse-khadidja@univ-eloued.dz

DEPARTMENT OF MATHEMATICS, FACULTY OF EXACT SCIENCES, UNIVERSITY OF EL OUED, ALGERIA
Email address: nisse-lamine@univ-eloued.dz