Expansions of real numbers with respect to two integer bases

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Abstract

We introduce and study expansions of real numbers with respect to two integer bases.

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1 Introduction

We construct an expansion of real numbers in $[0,1]$ by blowing up the first basic interval $[0, 1/b]$ in the classical b-adic expansion to $[0, 1/a]$ with $a < b$. The details of the construction are given in the next section. Due to the overlaps of basic intervals the properties of these new expansions of real numbers turn out to be similar to the properties of the meanwhile classical β-expansions, $\sum d_i \beta^{-i}$ for $\beta \in (1,2)$. The literature on β-expansion is modestly growing since the pioneering work of Alfred Renyi (1921-1970) in 1950th. We refer to [10] for an introduction to this topic. In section 3 we will prove Sidorov's theorem [9] in the context of a,b-expansions using an ergodic theoretical approach: Almost all $x \in [0,1]$ have a continuum of such expansions. Now it is natural to ask, if there are exceptional $x \in (0,1)$ having an unique a,b-expansions. We will show in section 4 that there are at least countable many such exceptions, if $b < a^2$ and that there is an uncountable set with positive Hausdorff dimension of such exceptions if $b < a^2 - 2 - (a\lceil b/a \rceil - b)$. The elementary number theoretical condition here is an analogue of the condition $\beta > \kappa(2)$, where $\kappa(2)$ is the transcendental Komornik-Loreti constant, which leads to a set of numbers with positive Hausdorff dimension, which have a unique β-expansion, see [8]. To characterize numbers which have a unique a,b-expansion, we again use ideas from dynamical systems. In the last section we consider sets of numbers with digits from a proper subset of $\{0, \ldots, b-1\}$ in their a,b-expansions. Hundred years ago Felix Hausdorff (1868-1942) [2] introduced his dimension to distinguish the size of such sets for b-adic expansions. We will determine the Hausdorff dimension of these sets with respect to a,b-expansions using a resent result of Mike Hochmann, see [3].

2 a, b-expansions

Fix integers a, b with $b > a > 1$. Consider the space of digits $\Sigma = \{0, \dots, b-1\}^{\mathbb{N}}$. For $d = (d_i) \in \Sigma$ and $n \in \mathbb{N}$ let $0_n(d)$ be the number of zeros in the sequence $(d_1, \dots, d_n)$. We consider the map $\pi : \Sigma \to [0,1]$ given by

$$\pi(d) = \sum_{i=1}^{\infty} d_i \left(\frac{1}{a}\right)^{0_i(d)} \left(\frac{1}{b}\right)^{i-0_i(d)}.$$

If $\pi(d) = x$ we call d an a, b-expansion of x. Several times we will use another description of the map π, which we now describe. For $j = 0, \dots, b-1$ let $T_j : [0,1] \to [0,1]$ be contractions given by

$$T_0(x) = \frac{x}{a} \quad \text{and} \quad T_j(x) = \frac{x}{b} + \frac{j}{b} \quad \text{for} \quad j = 1, \dots, b-1.$$

By induction we have

$$T_{d_1} \circ \cdots \circ T_{d_n}(x) = \left(\frac{1}{a}\right)^{0_n(d)} \left(\frac{1}{b}\right)^{n-0_n(d)} x + \sum_{i=1}^{n} d_i \left(\frac{1}{a}\right)^{0_n(d)} \left(\frac{1}{b}\right)^{n-0_n(d)}.$$

Hence for all $d \in \Sigma$ and all $x \in I$

$$\pi(d) = \lim_{n \to \infty} T_{d_1} \circ \cdots \circ T_{d_n}(x).$$

Obviously we have

$$[0,1] = \bigcup_{j=0}^{b-1} T_i([0,1]),$$

hence all $x \in [0,1]$ have at least one a, b-expansion. Such an expansion may be calculated using a greedy algorithm. Let $G : [0,1] \to [0,1]$ be given by

$$G(x) = \begin{cases} ax, & x \in [0, 1/b) \\ bx - j, & x \in [j/b, (j+1)/b) \\ 1, & x = 1 \end{cases} \quad \text{for} \quad j = 1, \dots, b-1 \quad .$$

For $x \in [0,1]$ we define the greedy expansion $g = g(x) = (g_i) \in \Sigma$ with respect to a and b by

$$g_i = \lfloor bG^{i-1}(x) \rfloor,$$

where $\lfloor x \rfloor$ is the greatest integer not greater than x. We have

Proposition 2.1 *For all $x \in [0,1]$ the greedy expansion $g(x) \in \Sigma$ with respect to a and b is an a, b-expansion of x, that is $\pi(g(x)) = x$.*

Proof. If $g_i = j$ we have $G^{i-1}(x) \in [j/b, (j+1)/b)$, which implies $G^{i-1}(x) \in T_{g_i}([0,1])$. Since G is defined by the inverse maps of T_i we obtain

$$x \in T_{g_1} \circ \cdots \circ T_{g_i}([0,1])$$

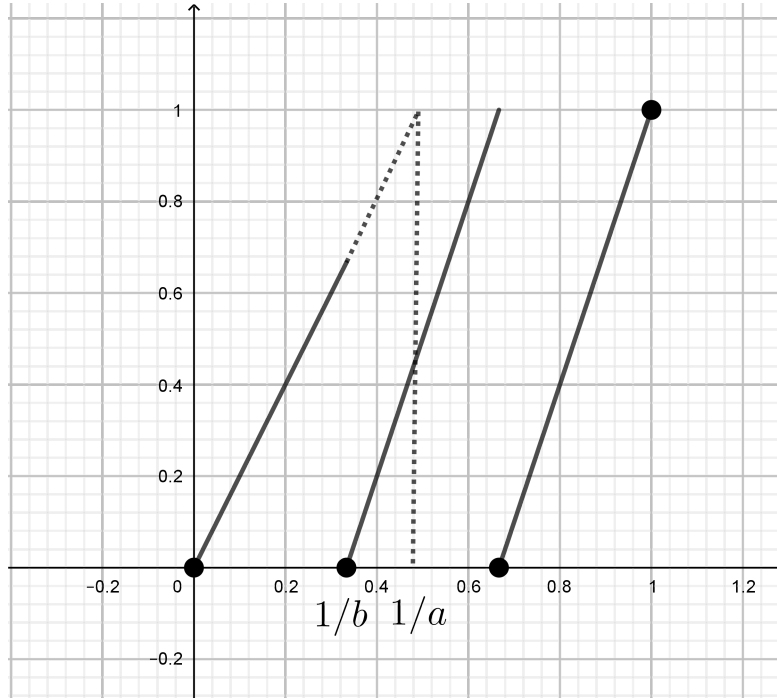for all $i \in \mathbb{N}$, but this implies $\pi(g(x)) = x$. $\qquad \square$

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Figure 1: The map G in the case $a = 2, b = 3$

3 A continuum of expansions

Due to the intersection of the open interval $(0, 1/a)$ and $(1/b, 2/b)$ in $(1/b, 1/a)$ for $b > a$ the a, b-expansion of $x \in (0, 1)$ will not be unique generically. We will first prove:

Theorem 3.1 *Almost all $x \in (0, 1)$ (in the sense of Lebesgue measure) have a continuum of a, b-expansions.*

Proof. Let $G : [0, 1] \to [0, 1]$ be the map defined in the last section. G is a piecewise linear expanding interval map and such maps are known to have an ergodic measure, which is equivalent to the Lebesgue measure, see [1] and [4]. By Poincare recurrence theorem for almost all $x \in [0, 1]$ there is a $k \geq 0$ such that $G^k(x) \in (1/b, 1/a)$. Note that each $x \in (1/b, 1/a)$ has two different a, b-expansions, one starting with 0 and other one starting with 1. Hence for almost all $x \in [0, 1]$ there is a $k \geq 0$ and a sequence $(d_1, \ldots, d_k) \in \{0, \ldots, b-1\}^k$ such that

$$x = T_{d_1} \circ \cdots \circ T_{d_k} \circ T_0(x_0) \quad \text{and} \quad x = T_{d_1} \circ \cdots \circ T_{d_k} \circ T_1(x_1),$$

where $x_0, x_1 \in (0, 1)$ and $x_0 \neq x_1$. For almost all x both numbers $x_1(x), x_2(x)$ have two different a, b-expansion hence almost all x have four different expansions. We use here that the intersection of two sets of full measure has full measure. Repeating this procedure $\aleph_0$ times we obtain $2^{\aleph_0}$ $a, b-$expansions for almost all $x \in [0, 1]$, using the fact a countable intersection of sets of full measure has full measure. $\qquad\square$

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It is natural to ask if there are $x \in (0, 1)$ that do not have a continuum of a, b-expansion. In the next section we will see that this is true.

4   Unique expansions

We consider the shift map $\sigma : \{0, \ldots, b-1\}^{\mathbb{N}} \to \{0, \ldots, b-1\}^{\mathbb{N}}$ given by $\sigma((d_k)) = (d_{k+1})$. Using this map we characterise numbers which have a unique a, b-expansion as follows:

Proposition 4.1 *The a, b-expansion (d_i) of $x \in [0, 1]$ is unique if and only if*

$$\pi(\sigma^k(d_i)) \in ([0, 1/b] \cup (1/a, 1]) \backslash \{j/b \mid j = 2, \ldots, b - 1\}$$

for all $k \geq 0$.

Proof. $\pi(d) = \pi(e)$ with $d \neq e$ holds if and only if there exists a $k \geq 0$ such that $d_k \neq e_k$ and $\pi(\sigma^k((d_i))) = \pi(\sigma^k((e_i)))$. But this is equivalent to

$$\pi(\sigma^k(d_i)) \in \bigcup_{l \neq m} T_l([0, 1]) \cap T_m([0, 1]) = [1/b, 1/a] \cup \{j/b \mid j = 2, \ldots, b - 1\}.$$

The proposition follows by contraposition. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Obviously 0 and 1 have a unique a, b-expansion. Using the last proposition we are able to prove:

Theorem 4.1 *If $a < b < a^2$ at least countable many $x \in (0, 1)$ have a unique a, b-expansion. If $a < b < a^2 - 2 - r$, where $r \in \{0, \ldots, b - 1\}$ is given by $b = la - r$, uncountable many $x \in (0, 1)$ have a unique a, b-expansion. Furthermore the Hausdorff dimension of the set of these numbers is positive.*

Proof. Let d be the periodic sequence in Σ given by $d = (d_i) = (0, a, 0, a, \ldots)$. We have

$$\pi(d) = \frac{1}{1 + a^{-1}b^{-1}} \cdot \frac{1}{b} < \frac{1}{b}.$$

On the other hand

$$\pi(\sigma(d)) = \frac{1}{1 + a^{-1}b^{-1}} \cdot \frac{a}{b} > \frac{1}{a}$$

by the assumption $a^2 > b > a$ for integers a, b. Furthermore $\pi(\sigma(d)) \neq j/b$ for $j = 2, \ldots, b - 1$. By proposition 4.1 $x = \pi(d) \in (0, 1)$ has an unique a, b-expansion. Considering sequences $d = (0, \ldots, 0, a, 0, a, \ldots)$ we see that there are at least countable many such $x \in (0, 1)$.

Let $b = al - r$ with $r \in \{0, 1 \ldots, b-1\}$ and consider the set $V = \{0l, l0\}^{\mathbb{N}}$ in Σ. The set is not invariant under the shift map σ but

$$U = \bigcup_{k=0}^{\infty} \sigma^k(V) = V \cup (\{0\} \times V) \cup (\{l\} \times V)$$

is a σ-invariant set. The sequence $d \in U$ with $d_1 = 0$, that has the largest projection under π, obviously is $d = (0, l, 0, 0, l, 0, l, 0, \ldots)$. We have

$$\pi(d) = \frac{l}{b}\left(\frac{1}{a} + \frac{1}{1+ab}\right) = \frac{b+r}{a^2 b} + \frac{b+r}{b(1+ab)}$$

$$= \frac{1}{a^2 b}(b + r + \frac{a(b+r)}{1+ab}) < \frac{1}{b}$$

by our assumption $b + r + 2 < a^2$ and by the inequality $a(b+r)/(1+ab) < 2$. The sequence $d \in U$ with $d_1 = l$, that has the smallest projection under π, obviously is $d = (l, 0, 0, l, 0, l, 0, \ldots)$. We have

$$\pi(d) = \frac{l}{b} + \frac{l}{ab + a^2 b^2} \geq \frac{1}{a} + \frac{1}{a^2 + a^3 b} > \frac{1}{a}$$

since $l \geq b/a$. We have thus shown that $\pi(\sigma^k(U)) = \pi(U) \subseteq [0, 1/b] \cup (1/a, 1]$ for all $k \geq 0$. We now prove that $\pi(U)$ is uncountable with positive Hausdorff dimension. The subset $\pi(V) \subseteq \pi(U)$ is the attractor of the iterated function system $(T_0 \circ T_l, T_l \circ T_0)$. This means $T_0 \circ T_l(\pi(V)) \cup T_l \circ T_0((\pi(V)) = \pi(V)$. The iterated functions system fulfills the open set condition, since $T_0 \circ T_l((0,1)) \cap T_l \circ T_0((0,1))) = \emptyset$. By [5] the set $\pi(V)$ (and hence $\pi(U)$ as well) has positive Hausdorff dimension and is thus uncountable. Note that $\pi(\sigma^k(d)) = j/b$ for some j and k hold for sequences d in a at most countable set $S \subseteq U$. Each $x \in \pi(U \backslash S)$ has unique a, b-expansion by proposition 4.1 and this set has positive Hausdorff dimension and is uncountable. $\qquad \square$

The upper bounds on b in our theorem are due to the ideas used in the proof. We do not know if these bound is sharp.

5 Restricted digits

Let $D \subseteq \{0, \ldots, b-1\}$ be subset of digits. We are interested in the set of numbers which have an a, b-expansion with digits in D, that is the set $\pi(D^{\mathbb{N}})$. We denote the cardinality of D by $|D|$. If $|D| = 1$ the set $\pi(D^{\mathbb{N}})$ obviously contains only one point and we already know that $\pi(D^{\mathbb{N}}) = [0, 1]$ if $|D| = b$. Moreover we obtain:

Theorem 5.1 *Let $D \subseteq \{0, \ldots, b-1\}$ with $1 < |D| < b$. If $0 \notin D$ we have*

$$\dim_H \pi(D^{\mathbb{N}}) = \frac{\log |D|}{\log b}.$$

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If $0 \in D$ and $\log(b)/\log(a) \notin \mathbb{Q}$ we have $\dim_H \pi(D^{\mathbb{N}}) = \min\{s, 1\}$ where s is given by

$$\left(\frac{1}{a}\right)^s + (|D| - 1)\left(\frac{1}{b}\right)^s = 1.$$

The same is true if $\min(D\backslash\{0\}) \geq b/a$ no matter where $\log(b)/\log(a) \in \mathbb{Q}$.

Proof. If $0 \notin D$ the set $\pi(D^{\mathbb{N}})$ contains the numbers in $[0, 1]$, which have a b-adic expansion with digits in D. The dimension formula goes back to Hausdorff [2] in this case. Now assume $0 \in D$ and $\log(b)/\log(a) \notin \mathbb{Q}$. The assumption means that $b^n \neq a^m$ for all $n, m \geq 1$. The identity of the maps

$$T_{d_1} \circ \cdots \circ T_{d_n} = T_{g_1} \circ \cdots \circ T_{g_m}$$

for some sequences $d = (d_1, \ldots, d_n)$ and $g = (g_1, \ldots, g_m)$ is equivalent to

$$\left(\frac{1}{a}\right)^{0_n(d)} \left(\frac{1}{b}\right)^{n-0_n(d)} x + \sum_{i=1}^{n} d_i \left(\frac{1}{a}\right)^{0_n(d)} \left(\frac{1}{b}\right)^{n-0_n(d)}$$

$$= \left(\frac{1}{a}\right)^{0_m(g)} \left(\frac{1}{b}\right)^{m-0_m(g)} x + \sum_{i=1}^{m} g_i \left(\frac{1}{a}\right)^{0_m(g)} \left(\frac{1}{b}\right)^{m-0_m(g)}$$

for all $x \in [0, 1]$. By our assumption this implies $n = m$ and $0_n(d) = 0_m(g)$. Furthermore we obtain $d_i = g_i$ for $i = 1, \ldots, n$. This means that there are no exact overlaps of intervals of the form $T_{d_1} \circ \cdots \circ T_{d_n}([0, 1])$. By Hochmann's result [3] this implies $\dim_H \pi(D^{\mathbb{N}}) = \min\{s, 1\}$, where s is the similarity dimension of the iterated function system $\{T_i | i \in D\}$. That means

$$\sum_{i \in D} |T_i'|^s = \left(\frac{1}{a}\right)^s + (|D| - 1)\left(\frac{1}{b}\right)^s = 1.$$

If $\min(D\backslash\{0\}) \geq b/a$, we have $T_i((0, 1)) \cap T_j((0, 1)) = \emptyset$ for $i \neq j$. In this case the result follows from the classical theory of iterated function systems fulfilling the open set condition, see [5], or again from [3]. □

If we have $\log(b)/\log(a) \in \mathbb{Q}$ and $\min(D\backslash\{0\}) < b/a$, there may occur exact overlaps of intervals $T_{d_1} \circ \cdots \circ T_{d_n}([0, 1])$. In this situation a combinatorial approach is necessary to calculate the Hausdorff dimension of $\pi(D^{\mathbb{N}})$. We refer here to [6].

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