ROCKY MOUNTAIN JOURNAL OF MATHEMATICS Vol., No., YEAR 2 3 4 5 6 7 8 https://doi.org/rmj.YEAR..PAGE THE BLASCHKE-LEBESGUE THEOREM REVISITED RYAN HYND 9 1. Introduction ¹⁰ A convex and compact subset of the Euclidean plane is a *constant width shape* provided the distance ¹¹ between two parallel supporting lines is the same in all directions. For convenience, we will always ¹² assume this distance is equal to one. In addition, we will say that the boundary of a constant width shape ¹³ is a *constant width curve* and refer to constant width shapes and their boundary curves interchangeably. ¹⁴ A simple example of a constant width curve is a circle of radius 1/2. Indeed, any two parallel supporting ¹⁵ lines touch the circle at the ends of a diametric chord. As we shall see below, there are a plethora of ¹⁶ curves of constant width. 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 FIGURE 1. A constant width curve with three pairs of parallel supporting line segments. 34 35

In what follows, an important example of a curve of constant width is a Reuleaux triangle. This shape is obtained as the intersection of three closed disks of radius one which are centered at the vertices of an equilateral triangle of side length one. In order to check that this shape has constant width, we only need to make the following observation. For any pair of parallel supporting lines, one

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1 touches a vertex of the associated equilateral triangle and the other touches a point on the circle of radius one centered at this vertex. As a result, the distance between these lines is necessarily equal to one. More generally, a Reuleaux polygon is a curve of constant width consisting of finitely many arcs 3 4 of circles of radius one. We note that the constant width condition necessitates that these shapes have an odd number of sides. 5



equilateral triangle.



24 A fundamental theorem in the study of constant width curves is due to independently Blaschke [3] and Lebesgue [12]. 25

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27 The Blaschke-Lebesgue theorem. Among all curves of constant width, Reuleaux triangles enclose the least area. 28

³⁰ The purpose of this note is to survey two proofs of the Blaschke–Lebesgue theorem. Our aim is to be as self-contained as possible without going too far astray from our goal of understanding why 32 this theorem holds.

33 This note is organized as follows. In the subsequent section, we will review the support function of ³⁴ a constant width shape; this is the principal tool employed throughout this paper. Then we will prove ³⁵ the Blaschke-Lebesgue theorem using variational methods following the approach of Harrell [10]. ³⁶ And in the final section, we will verify the Blaschke-Lebesgue theorem using fact that constant width ³⁷ curves can be closely approximated by Reuleaux polygons. This method was first used by Blaschke [3], ³⁸ although our argument is based on an exercise in the classic textbook of Yaglom and Boltjanskii [15]. ³⁹ We also acknowledge that there are several other proofs of the Blaschke-Lebesgue theorem including

40 those given in [7, 9, 5, 8, 13, 4].

We realize that many of the considerations in this note can be extended to shapes of constant width 41 $\overline{42}$ in the Euclidean space \mathbb{R}^3 . Good references for this three-dimensional shapes of constant width is



(A) A Reuleaux pentagon for which all circular arcs are the same length. We call this curve a regular Reuleaux pentagon.

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(B) A Reuleaux pentagon having boundary arcs of differing lengths. Such an example is an irregular Reuleaux pentagon.

the survey by Chakerian and Groemer [6] and the recent monograph by Martini, Montejano, and Oliveros [14]. However, despite several notable efforts such as [4, 1, 2], an analog of the Blaschke-Lebesgue theorem has not been established for constant width shapes in \mathbb{R}^3 . As a result, we will focus our attention on planar shapes. Our motivation is to give a clear and thorough account of the two-dimensional theory which may have a chance of being extended to the three-dimensional problem.

2. The support function

²⁸ In this section, we will study a basic concept used to analyze convex shapes. Suppose $K \subset \mathbb{R}^2$ is ²⁹ compact and convex. For a given $\theta \in \mathbb{R}$, set

$$h(\boldsymbol{\theta}) = \max_{x \in K} x \cdot u(\boldsymbol{\theta}),$$

where $u(\theta) = (\cos(\theta), \sin(\theta))$. This function $h : \mathbb{R} \to \mathbb{R}$ is known as the *support function* of *K* since the set of $x \in \mathbb{R}^2$ with $x \cdot u(\theta) \le h(\theta)$ is the supporting half-space of *K* which has outward normal $u(\theta)$.

As K is equal to the intersection of all half-spaces which include K, it follows that

$$\frac{\overline{37}}{38}_{39} (2.1) K = \bigcap_{\theta \in \mathbb{R}} \left\{ x \in \mathbb{R}^2 : x \cdot u(\theta) \le h(\theta) \right\}$$

That is, $x \in K$ if and only if $x \cdot u(\theta) \le h(\theta)$ for all $\theta \in \mathbb{R}$. It particular, a convex shape can be recovered from its support function. It is also not hard to verify that for any $y \in K$, $h(\theta) - y \cdot u(\theta)$ is the distance from y to the supporting line of K with outward normal $u(\theta)$.



FIGURE 4. This figure is the geometric interpretation of the value $h(\theta)$ as the distance from the origin to the line which supports *K* and has outward normal $u(\theta)$.

Note that *h* is continuous and 2π -periodic. In the sequel, we'll write $h \in C(\mathbb{S})$ for any function with these properties. It will also be useful for us to characterize which functions in $C(\mathbb{S})$ are support functions. The inequality in part (*ii*) of the following proposition was first noted by Kallay [11].

Proposition 2.1. Suppose $h \in C(\mathbb{S})$. The following are equivalent.

 $\overline{\mathbf{27}}$ (*i*) *h* is the support function of a convex and compact $K \subset \mathbb{R}^2$.

 $\overline{}_{\mathbf{28}}$ (ii) For each $\theta \in \mathbb{R}$ and $\phi \in [-\pi/2, \pi/2]$,

$$h(\theta + \phi) + h(\theta - \phi) \ge 2h(\theta)\cos(\phi).$$

 $\frac{30}{31}$ (iii) For each smooth $f : \mathbb{R} \to [0,\infty)$ with compact support,

Proof. $(i) \Rightarrow (ii)$ Using the angle sum-to-product formulae for sine and cosine, we find $u(\theta + \phi) + u(\theta - \phi) = 2u(\theta)\cos(\phi).$

 $\frac{37}{38}$ Therefore,

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$$2x \cdot u(\theta) \cos(\phi) = x \cdot u(\theta + \phi) + x \cdot u(\theta - \phi) \le h(\theta + \phi) + h(\theta - \phi)$$

$$40 \quad \text{for } x \in K. \text{ Since } \cos(\phi) \ge 0 \text{ for } \phi \in [-\pi/2, \pi/2], \text{ we also have}$$

$$41 \quad 2h(\theta) \cos(\phi) = \max_{x \in K} 2x \cdot u(\theta) \cos(\phi) \le h(\theta + \phi) + h(\theta - \phi).$$

(*ii*) \Rightarrow (*iii*) Suppose $f : \mathbb{R} \to [0,\infty)$ is smooth and has compact support. Then for $\phi \in [-\pi/2, \pi/2]$ $\phi \neq 0$, with $\phi \neq 0$,

$$\frac{\frac{3}{4}}{\frac{4}{5}} (2.4) \qquad 0 \leq \frac{1}{\phi^2} \int_{\mathbb{R}} f(\theta) \left(h(\theta + \phi) + h(\theta - \phi) - 2h(\theta) \cos(\phi) \right) d\theta$$

$$= \frac{1}{\phi^2} \int_{\mathbb{R}} h(\theta) \left(f(\theta + \phi) + f(\theta - \phi) - 2f(\theta) \cos(\phi) \right) d\theta$$

$$= \frac{1}{\phi^2} \int_{\mathbb{R}} h(\theta) \left[\frac{f(\theta + \phi) + f(\theta - \phi) - 2f(\theta)}{\phi^2} \right] + h(\theta) f(\theta) \left[2 \frac{(1 - \cos(\phi))}{\phi^2} \right] d\theta.$$
By our assumptions on f , we can send $\phi \to 0$ in the integral above to get
$$0 \leq \int_{\mathbb{R}} h(\theta) \left(f''(\theta) + f(\theta) \right) d\theta$$

$$0 \leq \int_{\mathbb{R}} h(\theta) (f''(\theta) + f(\theta)) d\theta.$$

13 $(iii) \Rightarrow (i)$ For $\varepsilon > 0$, we consider the mollification of h: set 14

$$h^{\varepsilon}(\theta) = \int_{\mathbb{R}} \eta^{\varepsilon}(\phi) h(\theta - \phi) d\phi$$

for $\theta \in \mathbb{R}$. Here $\eta^{\varepsilon}(t) = \varepsilon^{-1} \eta(t/\varepsilon)$, and $\eta : \mathbb{R} \to [0,\infty)$ is a smooth symmetric function which is 17 supported in [-1,1] and $\int_{\mathbb{R}} \eta(t) dt = 1$. As $h \in C(\mathbb{S})$, it is routine to show h^{ε} converges to h uniformly. 18 19 Moreover, h^{ε} is smooth and

(2.7)
$$(h^{\varepsilon})''(\theta) + h^{\varepsilon}(\theta) = \int_{\mathbb{R}} \left[(\eta^{\varepsilon})''(\phi) + \eta^{\varepsilon}(\phi) \right] h(\theta - \phi) d\phi$$

$$= \int_{\mathbb{R}} \left[(\eta^{\varepsilon})''(\theta - \phi) + \eta^{\varepsilon}(\theta - \phi) \right] h(\phi) d\phi$$

$$\geq 0$$

$$\geq 0$$

(2.9)24

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for each $\theta \in \mathbb{R}$. 25

For $u \neq 0$, we define 26

$$H^{\varepsilon}(u) = |u|h^{\varepsilon}(\theta), \quad u \in \mathbb{R}^2$$

where $\theta \in \mathbb{R}$ is chosen so that $u/|u| = u(\theta)$; and when u = 0, we set $H^{\varepsilon}(u) = 0$. Note that H^{ε} is 29 positively homogeneous and satisfies

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$$H^{\varepsilon}(u(\theta)) = h^{\varepsilon}(\theta)$$

31 for each $\theta \in \mathbb{R}$. In particular, H^{ε} is smooth away from the origin and direct computation yields 32 $DH^{\varepsilon}(u) \cdot u = H^{\varepsilon}(u)$ and $D^{2}H^{\varepsilon}(u)u = 0$ for $u \neq 0$. It follows that 33

$$D^{2}H^{\varepsilon}(u(\theta))(\alpha u(\theta) + \beta u'(\theta)) \cdot (\alpha u(\theta) + \beta u'(\theta)) = \beta^{2}((h^{\varepsilon})''(\theta) + h^{\varepsilon}(\theta)) \ge 0.$$

35 We conclude H^{ε} is convex. Sending $\varepsilon \to 0$, we find that H^{ε} converges locally uniformly to a positively 36 homogeneous and convex $H : \mathbb{R}^2 \to \mathbb{R}$ which fulfills 37

$$h(\boldsymbol{\theta}) = H(u(\boldsymbol{\theta}))$$

39 for all θ .

Define K as in (2.1) with the h we are currently studying, and let \tilde{h} be the support function of K. Fix 40 41 $\theta \in \mathbb{R}$. First observe that

$$\tilde{h}(\theta) = \max\{x \cdot u(\theta) : x \in K\} \le h(\theta),\$$

1	as $h(\theta) \ge x \cdot u(\theta)$ for all $x \in K$. We leave it as an exercise to show that				
2	$H(u(\theta)) = p \cdot u(\theta)$				
4	for any <i>p</i> belonging to the subdifferential of <i>H</i> at $u(\theta)$. In particular, for any $v \in \mathbb{R}^2$,				
5	$H(v) \ge H(u(\theta)) + p \cdot (v - u(\theta)) = p \cdot v.$				
7	Thus, $h(\theta) = p \cdot u(\theta)$ and $h(\phi) = H(u(\phi)) \ge p \cdot u(\phi)$ for all $\phi \in \mathbb{R}$. It follows that $p \in K$ and				
8	$h(\boldsymbol{ heta}) = p \cdot u(\boldsymbol{ heta}) \leq \tilde{h}(\boldsymbol{ heta}).$				
10	As a result, h is the support function of K .				
11 12	Corollary 2.2. The support function h is twice differentiable for almost every $\theta \in \mathbb{R}$ and				
13	$h''(oldsymbol{ heta})+h(oldsymbol{ heta})\geq 0$				
14 15	at any such θ .				
16	<i>Proof.</i> By the previous proposition,				
17 18	$h(\theta + \phi) - 2h(\theta) + h(\theta - \phi) \ge 2\left(\cos(\phi) - 1\right) L(\phi)$				
19	(2.10) $\frac{\psi^2}{\phi^2} \ge 2\left(\frac{\psi^2}{\phi^2}\right)h(\theta)$				
20 21	for all θ and each $\phi \in [-\pi/2, \pi/2]$ with $\phi \neq 0$. Since $1 \ge \cos(\phi) \ge 1 - \frac{1}{2}\phi^2$, we find				
22	$rac{h(heta+\phi)-2h(heta)+h(heta-\phi)}{2}\geq -\ h\ _{\infty}$				
23 24	ϕ^2 — IIII				
25 26 27	for all θ and $0 < \phi \le \pi/2$. It follows that there is a constant <i>b</i> for which $h(\theta) + \frac{b}{2}\theta^2$ is convex. By Alexandrov's theorem, <i>h</i> is twice differentiable for almost every $\theta \in \mathbb{R}$. If <i>h</i> is twice differentiable at θ , we can send ϕ to 0 in (2.10) to get $h''(\theta) \ge -h(\theta)$.				
28 29 30	<i>Remark</i> 2.3. For a convex and compact <i>K</i> with a smooth boundary, $h''(\theta) + h(\theta)$ is the radius of curvature of the boundary point with outward unit normal $u(\theta)$.				
31 32 33 34	2.1. Support function of a constant width curve. Let us now refine our consideration to constant width curves. We will use the fact that constant width shapes are strictly convex (Theorem 3.1.1 of [14]). In particular, if $K \subset \mathbb{R}^2$ has constant width, then for each $\theta \in \mathbb{R}$ there is a unique $\gamma(\theta) \in \partial K$ such that				
36	(2.11) $h(\theta) = \gamma(\theta) \cdot u(\theta).$				
37	³⁷ It turns out that this implies h is actually continuously differentiable.				
39	Lemma 2.4. The mapping $\gamma: \mathbb{R} \to \partial K$ is continuous and				
40	$h'(oldsymbol{ heta}) = \gamma(oldsymbol{ heta}) \cdot u'(oldsymbol{ heta}).$				
42	for all $ heta \in \mathbb{R}$.				

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¹ *Proof.* Suppose $\theta_k \to \theta$ as $k \to \infty$. As $\gamma(\theta_k) \in \partial K$, there is a subsequence $\gamma(\theta_{k_i})$ which converges to some $\xi \in \partial K$. Moreover, 2 3 4 5 6 7 8 $h(\theta) = \lim_{i \to \infty} h(\theta_{k_j}) = \lim_{i \to \infty} \gamma(\theta_{k_j}) \cdot u(\theta_{k_j}) = \xi \cdot u(\theta).$ By uniqueness, $\xi = \gamma(\theta)$. As this limit is independent of the subsequence, $\gamma(\theta_k) \to \gamma(\theta)$. We conclude that γ is continuous. Fix $\theta \in \mathbb{R}$, and note that as $\gamma(\theta) \in K$, $h(\theta + \tau) \geq \gamma(\theta) \cdot u(\theta + \tau)$ 9 10 for each $\tau > 0$. Therefore, 11 $\frac{h(\theta+\tau)-h(\theta)}{\tau} \geq \gamma(\theta) \cdot \frac{u(\theta+\tau)-u(\theta)}{\tau}.$ 12 13 14 Likewise, we find $\frac{h(\theta+\tau)-h(\theta)}{\tau} \leq \gamma(\theta+\tau) \cdot \frac{u(\theta+\tau)-u(\theta)}{\tau}.$ 15 16 As a result, $\lim_{\tau \to 0^+} \frac{h(\theta + \tau) - h(\theta)}{\tau} = \gamma(\theta) \cdot u(\theta).$ 17 18 Virtually the same considerations for $\tau < 0$ lead to 20 $\lim_{\tau \to 0^-} \frac{h(\theta + \tau) - h(\theta)}{\tau} = \gamma(\theta) \cdot u(\theta).$ 21 We conclude that $h'(\theta) = \gamma(\theta) \cdot u'(\theta)$ exists and that *h* is continuously differentiable. 23 In the proposition below, we will say $f \in C^{1,1}(\mathbb{S})$ provided $f : \mathbb{R} \to \mathbb{R}$ is 2π periodic, f is continuous 24 25 differentiable, and f' is Lipschitz continuous. 26 27 **Proposition 2.5.** Suppose K has constant width. Then 28 $h(\theta + \pi) + h(\theta) = 1$ 29 30 for all $\theta \in \mathbb{R}$. Moreover, $h \in C^{1,1}(\mathbb{S})$ and $0 < h''(\theta) + h(\theta) < 1$ 31 32 for almost all $\theta \in \mathbb{R}$. *Proof.* Fix $x \in K$. Recall that the distance from x to the supporting line with outward normal $u(\theta)$ is 34 $h(\theta) - x \cdot u(\theta)$. Likewise, the distance from x to the supporting line with outward normal $u(\theta + \pi)$ is 35 $h(\theta + \pi) - x \cdot u(\theta + \pi)$. Since $u(\theta + \pi) = -u(\theta)$, the distance between these supporting lines is equal 36 to 37 $h(\theta + \pi) - x \cdot u(\theta + \pi) + h(\theta) - x \cdot u(\theta) = h(\theta + \pi) + h(\theta).$ 38 As *K* has constant width, it must be that $h(\theta + \pi) + h(\theta) = 1$ for all θ . 39 Select a θ for which $h''(\theta)$ exists. As $h(\phi + \pi) = 1 - h(\phi)$ for all ϕ , h is twice differentiable at 40 $\theta + \pi$, as well. Moreover, 41 $h''(\theta + \pi) = -h''(\theta)$ 42

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$$h''(\theta) + h(\theta) = 1 - [h''(\theta + \pi) + h(\theta + \pi)] \le 1$$

We have also already concluded $h''(\theta) + h(\theta) > 0$ in the previous proposition. As

 $-h(\theta) < h''(\theta) < 1 - h(\theta)$

2 3 4 5 6 7 8 for almost every $\theta \in \mathbb{R}$, h'' is an essentially bounded function. As a result, h' is Lipschitz continuous. \square

Remark 2.6. The inequality $h''(\theta) + h(\theta) \le 1$ implies that the curvature of a smooth constant width curve is always greater than or equal to one. 11

12 The subsequent assertion is a converse to some of the facts derived above. It is also a useful tool in 13 generating shapes of constant width. 14

15 **Proposition 2.7.** Suppose satisfies $h : \mathbb{S} \to \mathbb{R}$ satisfies

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 $\begin{cases} h \in C^{1,1}(\mathbb{S}) \\ h(\theta + \pi) + h(\theta) = 1 \text{ for all } \theta \in \mathbb{R} \\ h''(\theta) + h(\theta) > 0 \text{ for almost every } \theta \in \mathbb{R}. \end{cases}$

Then h is the support function of the constant width shape K defined in (2.1). 21

22 *Proof.* Since $h \in C^{1,1}(\mathbb{S})$, h' is Lipschitz continuous. Thus, for smooth $f : \mathbb{R} \to [0,\infty)$ with compact 23 24 support we may integrate $h(\theta) f''(\theta)$ by parts twice to find

$$\int_{\mathbb{R}} h(\theta)(f''(\theta) + f(\theta))d\theta = \int_{\mathbb{R}} (h''(\theta) + h(\theta))f(\theta)d\theta \ge 0.$$

27 As explained in our proof of Proposition 2.7, h is the support function the convex and compact K 28 defined in (2.1). Since $h(\theta + \pi) + h(\theta) = 1$ for all θ , *K* has constant width. 29

30 Let us study a few examples.

31 **Example 2.8** (Disks). The support function of the circle of radius 1/2 centered at $a \in \mathbb{R}^2$ is given by 32 33

$$h(\theta) = \max_{|x-a| \le 1/2} x \cdot u(\theta) = \max_{|x-a| \le 1/2} (x-a) \cdot u(\theta) + a \cdot u(\theta) = \frac{1}{2} + a \cdot u(\theta)$$

Example 2.9 (Reuleaux triangle). Suppose K is a Reuleaux triangle with vertices 36

37 $\left(\frac{1}{2},\frac{1}{2\sqrt{3}}\right), \quad \left(-\frac{1}{2},\frac{1}{2\sqrt{3}}\right), \text{ and } \left(0,-\frac{1}{\sqrt{3}}\right).$ (2.12)38 39

⁴⁰ If $u(\theta)$ is an outward normal to ∂K at a vertex *a*, then $h(\theta) = u(\theta) \cdot a$. Furthermore, if $u(\theta)$ is an 41 outward normal to ∂K at a circular arc centered at b, then $h(\theta) = 1 + u(\theta) \cdot b$. Combining these 42 observations with some elementary case analysis leads to the following expression for the support

1	function of <i>K</i> . For $\theta \in [0, 2\pi]$,			
2 3 4		$\left(u(\theta)\cdot\left(\frac{1}{2},\frac{1}{2\sqrt{3}}\right),\right.$	$0 \le \theta \le \frac{\pi}{3}$	
5 6 7		$1+u(\boldsymbol{\theta})\cdot\left(0,-\frac{1}{\sqrt{3}}\right),$	$\frac{\pi}{3} \le \theta \le \frac{2\pi}{3}$	
8 9 10		$u(\theta)\cdot\left(-\frac{1}{2},\frac{1}{2\sqrt{3}}\right),$	$\frac{2\pi}{3} \le \theta \le \pi$	
11 12 13	$(2.13) h(\theta) = \langle$	$1+u(\theta)\cdot\left(\frac{1}{2},\frac{1}{2\sqrt{3}}\right),$	$\pi \le heta \le rac{4\pi}{3}$	
14 15 16		$u(\theta)\cdot\left(0,-\frac{1}{\sqrt{3}}\right),$	$\frac{4\pi}{3} \le \theta \le \frac{5\pi}{3}$	
17 18 19		$\left(1+u(\theta)\cdot\left(-\frac{1}{2},\frac{1}{2\sqrt{3}}\right),\right.$	$\frac{5\pi}{3} \leq \theta \leq 2\pi.$	
20	Example 2.10 (regular Reuleaux	x polygons). We can build on	our example above t	
21 22 23	function of a <i>regular</i> Reuleaux polygon. This is a Reuleaux polygon in which the larcs forming the boundary are all equal. Suppose $N \ge 3$ with N odd, and set			
24 25 26	$x_k = \left(\frac{\sin\left(\frac{1}{2}\right)}{2}\right)$	$\frac{\frac{k\pi}{N} - \sin\left(\frac{(k-1)\pi}{N}\right)}{2\sin\left(\frac{\pi}{N}\right)}, -\frac{\cos\left(\frac{k\pi}{N}\right)}{2\sin\left(\frac{\pi}{N}\right)}$	$\left(\frac{\frac{\pi}{k}}{2\sin\left(\frac{\pi}{N}\right)}\right)$	
27	for $k = 1, 2, \dots, 2N$. Next define	:	/	
28	(x_1, y_1)	for $\boldsymbol{\theta} \in \left[\frac{(k-1)\pi}{k\pi}\right]$	and $k = 1.3$ 2	

to express the support engths of the circular

$$h(\theta) = \begin{cases} x_k \cdot u(\theta), & \text{for } \theta \in \left[\frac{(k-1)\pi}{N}, \frac{k\pi}{N}\right] \text{ and } k = 1, 3, \dots, 2N-1 \\ 1 - x_k \cdot u(\theta), & \text{for } \theta \in \left[\frac{(k-1)\pi}{N}, \frac{k\pi}{N}\right] \text{ and } k = 2, 4, \dots, 2N. \end{cases}$$

33 It is straightforward exercise to verify that h is a support function of the N-sided regular Reuleaux ³⁴ polygon with vertices $x_1, x_3, \ldots, x_{2N-1}$. The Reuleaux triangle described in the example above is the ³⁵ case N = 3. We've also displayed the case N = 5 in Figure 3a.

³⁶/₃₇ Example 2.11 (Perturbation of a disk). Using Proposition 2.7, we can design a curve of constant width starting with any $g \in C^{1,1}(\mathbb{S})$ which satisfies 36 37

$$g(\theta + \pi) = -g(\theta)$$

$$g(\theta + \pi) = -g(\theta)$$
for all $\theta \in \mathbb{R}$. For $\delta > 0$, set
$$h(\theta) = \frac{1}{2} + \delta g(\theta)$$

1 for $\theta \in \mathbb{R}$. Note that $h \in C^{1,1}(\mathbb{S})$ and

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$$h(\theta) + h(\theta + \pi) = 1$$

 $\frac{2}{3}$ for all $\theta \in \mathbb{R}$. Also note for almost every $\theta \in \mathbb{R}$, $\frac{5}{6}$ $h''(\theta) + h(\theta) = \frac{1}{2} - \frac{1}{2}$

$$h''(\theta) + h(\theta) = \frac{1}{2} + \delta(g''(\theta) + g(\theta)) \ge 0$$

⁷ provided $\delta > 0$ is chosen sufficiently small. By Proposition 2.7, *h* is the support function of a constant ⁸ width curve.

Remark 2.12. We used this method to create the curve in Figure 1. For that specific example, we chose $\frac{10}{11} \delta = 1/160$ and $g(\theta) = \cos(3\theta) + \sin(7\theta)$.



FIGURE 5. On the left is a regular Reuleaux septagon K_1 , and on the right is a circle K_2 of radius 1/2; the middle curve is the convex combination $(4/5)K_1 + (1/5)K_2$.

Example 2.13 (convex combinations). Examples of constant width shapes can also be designed by forming convex combination of constant width shapes. In particular, it is routine to verify that if $K_1, K_2 \subset \mathbb{R}^2$ are constant width shapes and $\lambda \in [0, 1]$, the support function of

$$(1 - \lambda)K_1 + \lambda K_2 := \{(1 - \lambda)x_1 + \lambda x_2 \in \mathbb{R}^2 : x_1 \in K_1, x_2 \in K_2\}$$

³³/₃₄ is $(1 - \lambda)h_1 + \lambda h_2$. As $(1 - \lambda)h_1 + \lambda h_2$ satisfies the hypotheses of Proposition 2.7, $(1 - \lambda)K_1 + \lambda K_2$ is also a constant width shape. See Figure 5 for an example.

³⁶ **2.2.** *Parametrization of a constant width curve.* It is possible to parametrize the boundary curve of a ³⁷ constant width *K* using the support function of *K*. To do so, we will write an explicit formula for $\gamma(\theta)$ ³⁸ discussed in (2.11) and identify a few properties of this path.

³⁹/₄₀ **Proposition 2.14.** Suppose $K \subset \mathbb{R}^2$ has constant width and h is the support function of K and define $\gamma : \mathbb{R} \to \partial K$ via (2.11). (i) For $\theta \in \mathbb{R}$,

$$\gamma(\theta) = h(\theta)u(\theta) + h'(\theta)u'(\theta)$$

1 (*ii*) For $\theta \in \mathbb{R}$, $\gamma(\theta + \pi) = \gamma(\theta) - u(\theta)$ 3 4 5 6 (iii) γ is surjective. (iv) γ is injective on any interval $(\theta_0, \theta_1) \subset [0, 2\pi]$ for which $h''(\theta) + h(\theta) > 0$ for almost every $\theta \in (\theta_0, \theta_1)$. (2.15)7 8 9 (v) For $\theta_1, \theta_2 \in \mathbb{R}$, $|\gamma(\theta_1) - \gamma(\theta_2)| \le |\theta_1 - \theta_2|.$ (2.16)¹⁰ *Proof.* (*i*) As $\{u(\theta), u'(\theta)\}$ is an orthonormal basis of \mathbb{R}^2 , 11 $\gamma(\theta) = [\gamma(\theta) \cdot u(\theta)]u(\theta) + [\gamma(\theta) \cdot u'(\theta)]u'(\theta) = h(\theta)u(\theta) + h'(\theta)u'(\theta).$ 12 ¹³ (*ii*) This follows directly from the constant width condition $h(\theta + \pi) + h(\theta) = 1$. (*iii*) For each $x \in \partial K$, ¹⁴ there is at least one supporting plane for K which includes x. It follows that there is θ such that ¹⁵ $h(\theta) = x \cdot u(\theta)$. This in turn implies $\gamma(\theta) = x$. (*iv*) Suppose that $\gamma(\phi_0) = \gamma(\phi_1) =: x \in \partial K$ for $\phi_0, \phi_1 \in (\theta_0, \theta_1)$ with $\phi_0 < \phi_1$. Then $x \in \partial K$ has 16 ¹⁷ two distinct supporting lines $h(\phi_0) = y \cdot u(\phi_0)$ and $h(\phi_1) = y \cdot u(\phi_1)$. It follows that $h(\theta) = x \cdot u(\theta)$ for 18 $\theta \in (\phi_1, \phi_1)$, which contradicts (2.15). As a result, γ is injective on (θ_0, θ_1) . 19 (v) Direct computation gives 20 $\gamma'(\theta) = (h''(\theta) + h(\theta))u'(\theta)$ (2.17)21 for almost every $\theta \in \mathbb{R}$. Since $0 \le h''(\theta) + h(\theta) \le 1$, $|\gamma'(\theta)| \le 1$ for almost every θ . Therefore, 23 $|\gamma(\theta_1) - \gamma(\theta_2)| \leq |\theta_1 - \theta_2|$ for all $\theta_1, \theta_2 \in \mathbb{R}$. 24 A nice consequence of the above proposition is following. 25 26 **Theorem 2.15** (Barbier's theorem). The perimeter of a constant width curve is equal to π . 27 *Proof.* Let h be a support function of a constant width curve and γ the corresponding parametrization 28 discussed in the previous proposition. In view of (2.17), the perimeter of the curve is 29 30 $\int_0^{2\pi} |\gamma'(\theta)| d\theta = \int_0^{2\pi} (h''(\theta) + h(\theta)) d\theta = \int_0^{2\pi} h(\theta) d\theta = \frac{1}{2} \int_0^{2\pi} (h(\theta + \pi) + h(\theta)) d\theta = \pi.$ 31 32 33 **Corollary 2.16.** Among all curves of constant width, circles of radius 1/2 enclose the most area. 34 Moreover, circles are the only curves of constant width attaining the maximum possible area. 35 *Proof.* Suppose $K \subset \mathbb{R}^2$ is constant width shape and that K encloses area A. Barbier's theorem implies

37 that the perimeter of K is equal to π . According to the isoperimetric inequality, $4\pi A \le \pi^2$ and equality 38 holds if and only if K is a circle. We conclude as any circle of radius 1/2 has area $\pi/4$; and if $A = \pi/4$, 39 equality holds in the isoperimetric inequality. \square 40

It will also be useful to express the area of a constant width shape in terms of the support function. 41 ⁴² We will use A(K) to denote the area of a convex and compact $K \subset \mathbb{R}^2$.

Proposition 2.17. Suppose $K \subset \mathbb{R}^2$ has constant width and h is the support function of K. Then 2 3 4

$$A(K) = \frac{1}{2} \int_0^{2\pi} h\left(h'' + h\right) d\theta.$$

Proof. We will employ the parametrization $\gamma = (\gamma^1, \gamma^2)$ discussed above. As Since γ is Lipschitz continuous and parametrizes ∂K counterclockwise, Green's theorem gives 6

$$A(K) = \frac{1}{2} \int_0^{2\pi} \left(\gamma^1 (\gamma^2)' - \gamma^2 (\gamma^1)' \right) d\theta = \frac{1}{2} \int_0^{2\pi} h \left(h'' + h \right) d\theta.$$

¹¹ *Remark* 2.18. It is sometimes useful to integration by parts and express the area of K as 12

$$A(K) = \frac{1}{2} \int_0^{2\pi} (h^2 - h'^2) d\theta$$

Example 2.19. Suppose $N \ge 3$ is odd and K is the N-sided regular Reuleaux triangle with support function h given in example 2.10. The area of K is

$$\frac{17}{19} (2.19) = \frac{1}{2} \int_{0}^{2\pi} h(h''+h) d\theta = \frac{1}{2} \sum_{k=1}^{2N} \int_{(k-1)\pi/N}^{k\pi/N} h(h''+h) d\theta = \frac{1}{2} \sum_{k \text{ even}} \int_{(k-1)\pi/N}^{k\pi/N} (1-x_k \cdot u(\theta)) d\theta = \frac{1}{2} \sum_{k \text{ even}} \int_{(k-1)\pi/N}^{k\pi/N} (1-x_k \cdot u(\theta)) d\theta = \frac{1}{2} \sum_{k \text{ even}} \left\{ \frac{\pi}{N} - \frac{\left(\sin\left(\frac{k\pi}{N}\right) - \sin\left(\frac{(k-1)\pi}{N}\right)\right)^2 + \left(\cos\left(\frac{k\pi}{N}\right) - \cos\left(\frac{(k-1)\pi}{N}\right)\right)^2}{2\sin\left(\frac{\pi}{N}\right)} \right\} = \frac{1}{2} \sum_{k \text{ even}} \left\{ \frac{\pi}{N} - \frac{\left(\sin\left(\frac{k\pi}{N}\right) - \sin\left(\frac{(k-1)\pi}{N}\right)\right)^2}{2\sin\left(\frac{\pi}{N}\right)} \right\} = \frac{1}{2} \sum_{k \text{ even}} \left\{ \frac{\pi}{N} - \frac{1 - \cos\left(\frac{\pi}{N}\right)}{\sin\left(\frac{\pi}{N}\right)} \right\} = \frac{\pi}{2} \left(1 - \frac{1 - \cos\left(\frac{\pi}{N}\right)}{\frac{\pi}{N}\sin\left(\frac{\pi}{N}\right)} \right).$$

33 34 35 It is routine to check that this expression increases in N. Therefore, the Reuleaux triangle has the least area among all regular Reuleaux polygons.

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3. Variational methods

Let \mathscr{S} denote the space of support functions of constant width curves. That is, $h \in \mathscr{S}$ if and only if 38

$$\begin{cases} h \in C^{1,1}(\mathbb{S}) \\ h(\theta + \pi) + h(\theta) = 1 \text{ for all } \theta \in \mathbb{R} \end{cases}$$

 $h''(\theta) + h(\theta) \ge 0$ for almost every $\theta \in \mathbb{R}$. 42

Our goal is to characterize which $h \in \mathscr{S}$ minimize the area integral 2 3 4 5 $I(h) = \frac{1}{2} \int_{0}^{2\pi} h(h'' + h) d\theta.$ We will first argue that a minimizing h exists; so it will be important for us to identify a basic compactness property of the space \mathscr{S} . To this end, we will need a lemma. **Lemma 3.1.** Suppose $h \in \mathscr{S}$. There is $a \in \mathbb{R}^2$ for which $0 < h(\theta) - a \cdot u(\theta) < 1$ for all $\theta \in \mathbb{R}$ and $|h(\theta) - a \cdot u(\theta) - (h(\phi) - a \cdot u(\phi))| \le 1$ $\frac{1}{12} \text{ for all } \theta, \phi \in \mathbb{R}.$ *Proof.* Suppose K is the constant width shape associated with $h \in \mathscr{S}$ and $a \in K$. As $K \subset \mathbb{R}^2$ has diameter 1, K is a subset of the closed disk of radius 1 centered at a. This means 15 $a \cdot u(\theta) < h(\theta) < 1 + a \cdot u(\theta)$ 16 17 for $\theta \in \mathbb{R}$. 18 Next, fix $\theta, \phi \in \mathbb{R}$ and choose $x \in K$ such that $h(\theta) = x \cdot u(\theta)$. Since $h(\phi) \ge x \cdot u(\phi)$, 19 $h(\theta) - a \cdot u(\theta) - (h(\phi) - a \cdot u(\phi)) = (x - a) \cdot u(\theta) - (h(\phi) - a \cdot u(\phi))$ (3.1)20 $< (x-a) \cdot u(\theta) - (x-a) \cdot u(\phi)$ (3.2)21 $= (x-a) \cdot (u(\theta) - u(\phi))$ 22 (3.3)23 $\leq |x-a||u(\theta)-u(\phi)|$ (3.4)24 $< |\theta - \phi|.$ (3.5)25 26 Similarly, we find 27 $h(\theta) - a \cdot u(\theta) - (h(\phi) - a \cdot u(\phi)) \ge -|\theta - \phi|.$ 28 29 30 **Proposition 3.2.** Suppose $(h^k)_{k \in \mathbb{N}} \subset \mathscr{S}$. There is a sequence $(a^k)_{k \in \mathbb{N}} \subset \mathbb{R}^2$ for which $h^k(\theta) - a^k \cdot u(\theta)$ 31 32 has a subsequence which converges in $C^1(\mathbb{R})$ to some $h \in \mathscr{S}$. 33 $\overline{\mathbf{34}}$ Proof. For each $k \in N$, choose a^k as in the previous lemma. Then $h^k(\theta) - a^k \cdot u(\theta)$ is a uniformly bounded and equicontinuous family of 2π -periodic functions. By the Arzelá-Ascoli theorem, there is a subsequence $h^{k_j}(\theta) - a^{k_j} \cdot u(\theta)$ which converges uniformly to some continuous $h : \mathbb{R} \to \mathbb{R}$. Of 36 course, h is 2π -periodic and $h(\theta + \pi) + h(\theta) = 1$ for all $\theta \in \mathbb{R}$. In view of Proposition 2.7, we also 37 have 38 $h(\theta + \phi) + h(\theta - \phi) - 2h(\theta)\cos(\phi) = \lim_{i \to \infty} h^k(\theta + \phi) + h^k(\theta - \phi) - 2h^k(\theta)\cos(\phi) \ge 0$ 39 40 for all $\theta \in \mathbb{R}$ and $\phi \in [-\pi/2, \pi/2]$. Corollary 2.2, Lemma 2.4, and Proposition 2.5 also give that 41 $\overline{42}$ $h \in C^{1,1}(\mathbb{S})$ and $h''(\theta) + h(\theta) \ge 0$ for almost every $\theta \in \mathbb{R}$. Thus, $h \in \mathscr{S}$. Finally, as the sequence

of 2π -periodic functions $(h^k)'(\theta) - a^k \cdot u'(\theta)$ is uniformly Lipschitz continuous (by Proposition 2.5), $(h^{k_j})'(\theta) - a^{k_j} \cdot u'(\theta)$ also converges uniformly to $h'(\theta)$. \square We are ready establish the existence of an area minimizing constant width shape. A minor but useful observation we'll need along the way is that the area of such a shape does not changed if it is translated by a fixed vector $a \in \mathbb{R}^2$. In particular, $I(h) = I(\tilde{h})$ 7 8 for $h, h \in \mathscr{S}$ with $\tilde{h}(\theta) = h(\theta) + a \cdot u(\theta)$ for some $a \in \mathbb{R}^2$. **Corollary 3.3.** There is $h^* \in \mathscr{S}$ for which $I(h^*) \leq I(h)$ for all $h \in \mathscr{S}$. 11 *Proof.* Since any constant width shape K has diameter one and I(h) represents an area of such a shape, 12 $0 < I(h) < \pi$. Consequently, we may choose a minimizing sequence 13 $\inf \{I(h): h \in \mathscr{S}\} = \lim_{k \to \infty} I(h^k).$ 14 15 By the previous proposition, there is a sequence $(a^k)_{k \in \mathbb{N}} \subset \mathbb{R}^2$ and subsequence 16 $\tilde{h}^{k_j}(\boldsymbol{\theta}) = h^{k_j}(\boldsymbol{\theta}) - a^{k_j} \cdot \boldsymbol{u}(\boldsymbol{\theta})$ 17 18 which converges in $C^1(\mathbb{R})$ to some $h^* \in \mathscr{S}$. As a result, 19 20 $\inf \{I(h): h \in \mathscr{S}\} = \lim_{i \to \infty} I(h^{k_j})$ (3.6)21 22 23 24 $=\lim_{j\to\infty}I(\tilde{h}^{k_j})$ (3.7) $=\lim_{i\to\infty}\frac{1}{2}\int_0^{2\pi}((\tilde{h}^{k_j})^2-((\tilde{h}^{k_j})')^2)d\theta$ (3.8)25 26 $=\frac{1}{2}\int_{0}^{2\pi}((h^{*})^{2}-((h^{*})^{\prime})^{2})d\theta$ (3.9)27 28 $=I(h^{*}).$ (3.10)29 \square 30 We now discuss an important necessary condition for minimizers of I, which was derived by Harrell 31 [10]. A crucial element of the proof is that for a bounded, measurable, 2π -periodic $g: \mathbb{R} \to \mathbb{R}$ which 32 satisfies 33 $\int_{0}^{2\pi} g(\theta) \cos(\theta) d\theta = \int_{0}^{2\pi} g(\theta) \sin(\theta) d\theta = 0,$ 34 (3.11)35 36 there is $f \in C^{1,1}(\mathbb{S})$ which solves 37 $f''(\theta) + f(\theta) = g(\theta)$ 38 39 for almost every $\theta \in \mathbb{R}$. Indeed one way verify 40

 $\frac{40}{41} \qquad \qquad f(\theta) = \int_0^\theta \sin(\theta - \phi) g(\phi) d\phi$

⁴² is a solution. See Theorem 4 of [11] for a proof of this assertion.

Lemma 3.4 (Harrell's Lemma). Suppose $h \in \mathscr{S}$ minimizes $I : \mathscr{S} \to \mathbb{R}$ and 2 3 4 5 6 7 8 9 $\int_{0}^{2\pi} h(\theta) \cos(\theta) d\theta = \int_{0}^{2\pi} h(\theta) \sin(\theta) d\theta = 0.$ (3.12)Then $\{\theta \in \mathbb{R} : h(\theta) < 1/2 \text{ and } h''(\theta) + h(\theta) > 0\}$ and $\{\theta \in \mathbb{R} : h(\theta) > 1/2 \text{ and } h''(\theta) + h(\theta) < 1\}$ are null sets. Remark 3.5. The condition (3.12) is equivalent to the Steiner point of the associated constant with 11 shape being at the origin. 12 $\overline{13}$ *Proof.* We will show 14 $E = \{ \theta \in \mathbb{R} : h(\theta) < 1/2 \text{ and } h''(\theta) + h(\theta) > 0 \}$ 15 16 is a null set; the remaining assertion will follow from a similar proof. To this end, it suffices to prove 17 $E_n = \{ \theta \in \mathbb{R} : h(\theta) < 1/2 \text{ and } h''(\theta) + h(\theta) \ge 1/n \}$ 18 19 is a null set for each $n \in \mathbb{N}$ as $E = \bigcup_{n \in \mathbb{N}} E_n.$ 20 21 22 Fix $n \in \mathbb{N}$ and set $g(\theta) = -\chi_{E_n}(\theta) + \chi_{E_n + \pi}(\theta) - a \cdot u(\theta)$ 23 24 for $\theta \in \mathbb{R}$. Here $a \in \mathbb{R}^2$ is chosen so that g satisfies (3.11). It is also evident that that g is bounded, measurable, and 2π periodic. As mentioned above, there is $f \in C^{1,1}(\mathbb{S})$ which solves 25 26 $f''(\theta) + f(\theta) = g(\theta)$ 27 for almost every $\theta \in \mathbb{R}$. 28 Since $g(\theta + \pi) = -g(\theta)$ for all $\theta \in \mathbb{R}$, we have 29 $f''(\theta + \pi) + f(\theta + \pi) = g(\theta + \pi) = -g(\theta) = -(f''(\theta) + f(\theta))$ 30 31 for almost every θ . Thus, 32 $f(\theta + \pi) + f(\theta) = b \cdot u(\theta)$ 33 34 for some $b \in \mathbb{R}^2$. And since f is 2π periodic, $f(\theta) + f(\theta + \pi) = f((\theta + \pi) + \pi) + f(\theta + \pi) = b \cdot u(\theta + \pi) = -b \cdot u(\theta) = -(f(\theta + \pi) + f(\theta))$ 35 36 for all θ . As a result, 37 (3.13) $f(\theta + \pi) = -f(\theta)$ 38 for all θ , as well. 39 We claim that for all small enough $\delta > 0$, $h + \delta f \in \mathscr{S}$. It is clear that $h + \delta f \in C^{1,1}(\mathbb{S})$, and in view 40 41 of (3.13), $(h+\delta f)(\theta+\pi)+(h+\delta f)(\theta)=1$ 42

1 for all θ . Observe that if h and f are twice differentiable at θ , then 2 3 4 5 6 7 8 $(h+\delta f)''(\theta) + (h+\delta f)(\theta) = \begin{cases} h''(\theta) + h(\theta), & \theta \in E_n \cap (E_n + \pi) \\ h''(\theta) + h(\theta), & \theta \in E_n^c \cap (E_n + \pi)^c \\ h''(\theta) + h(\theta) - \delta, & \theta \notin E_n, \ \theta \notin (E_n + \pi) \\ h''(\theta) + h(\theta) + \delta, & \theta \notin E_n, \ \theta \in (E_n + \pi) \end{cases}$ is nonnegative provided $0 < \delta < 1/n$. This proves the claim. 9 10 In addition, since h is minimizes I among all functions in \mathscr{S} $I(h) \leq I(h+\delta f), \quad \delta \in (0,1/n].$ 11 12 13 14 15 16 17 However, if $E_n \cap [0, 2\pi]$ has positive measure, we find a contradiction as (3.14) $0 \ge \lim_{\delta \to 0^+} \frac{I(h) - I(h + \delta f)}{\delta}$ (3.15) $= \int_{0}^{2\pi} h(\theta) (f''(\theta) + f(\theta)) d\theta$ 18 19 $=\int_{0}^{2\pi}h(heta)(-\chi_{E_n}(heta)+\chi_{E_n+\pi}(heta))d heta$ (3.16) 20 $=\int_{0}^{2\pi}(h(\boldsymbol{\theta})-1/2)(-\boldsymbol{\chi}_{E_{n}}(\boldsymbol{\theta})+\boldsymbol{\chi}_{E_{n}+\pi}(\boldsymbol{\theta}))d\boldsymbol{\theta}$ 21 (3.17) 22 $=-\int_{0}^{2\pi}(h(\boldsymbol{\theta})-1/2)\boldsymbol{\chi}_{E_{n}}(\boldsymbol{\theta})d\boldsymbol{\theta}+\int_{0}^{2\pi}(h(\boldsymbol{\theta})-1/2)\boldsymbol{\chi}_{E_{n}+\pi}(\boldsymbol{\theta})d\boldsymbol{\theta}$ 23 (3.18)24 $=-\int_{0}^{2\pi}(h(\theta)-1/2)\chi_{E_n}(\theta)d\theta-\int_{0}^{2\pi}(h(\theta+\pi)-1/2)\chi_{E_n}(\theta+\pi)d\theta$ 25 (3.19) 26 $=-2\int_{E_n\cap[0,2\pi]}(h(\theta)-1/2)d\theta$ 27 (3.20)

30 Here we used that h satisfies (3.12), $h(\theta + \pi) + h(\theta) = 1$ for all θ , and the measure of $E_n \cap [0, 2\pi]$ 31 is the same as the measure of $(E_n + \pi) \cap [0, 2\pi]$. The latter fact follows as $E_n \cap [2\pi m, 2\pi (m+1)] =$ 32 $2\pi m + E_n \cap [0, 2\pi]$ for each $m \in \mathbb{Z}$. As a result, $E_n \cap [0, 2\pi]$ is a null set, and therefore, E_n is also a null 33 set. 34

35 We are just about ready to issue our first proof of the Blaschke-Lebesgue theorem. A final technical 36 assertion needed in our proof is as follows. 37

Lemma 3.6. Suppose $0 < b - a < \pi$. 38

> 0.

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(3.21)

(i)The unique $h_0: [a,b] \to \mathbb{R}$ which solves 39

40 $\begin{cases} h_0''(\theta) + h_0(\theta) = 0, \quad \theta \in [a,b] \\ h_0(a) = h_0(b) = 1/2 \end{cases}$ 41 42

1 *additionally satisfies*

$$h_0'(a+) = \tan\left(\frac{b-a}{2}\right)$$
 and $h_0'(b-) = -\tan\left(\frac{b-a}{2}\right)$.

 $\begin{array}{c}
\frac{2}{3} \\
\frac{3}{4} \\
\frac{5}{6} \\
\frac{6}{7} \\
8
\end{array}$ $\begin{array}{c}
h_0'(a+) = \tan\left(\frac{b-a}{2}\right) \\
\frac{1}{2} \\
\frac{1}{2}$

$$egin{cases} h_1''(m{ heta}) + h_1(m{ heta}) = 1, & m{ heta} \in [a,b] \ h_1(a) = h_1(b) = 1/2 \end{cases}$$

 $\frac{9}{10}$ also fulfills

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$$h_1'(a+) = -\tan\left(\frac{b-a}{2}\right)$$
 and $h_1'(b-) = \tan\left(\frac{b-a}{2}\right)$.

13 Remark 3.7. We could integrate the above equations explicitly. However, we will only need to know their endpoint derivatives below.

¹⁵ *Proof of the Blaschke–Lebesgue theorem.* Suppose $h \in \mathscr{S}$ minimizes $I : \mathscr{S} \to \mathbb{R}$ and that h satisfies ¹⁶ (3.12). Recall that $h \not\equiv 1/2$ or else h would maximize I. Assume there is $\phi \in \mathbb{R}$ for which $h(\phi) > 1/2$. ¹⁷ As h is continuous, there is some maximal interval $[\theta_0, \theta_1]$ including ϕ such that $h(\theta) > 1/2$ for ¹⁸ $\theta \in (\theta_0, \theta_1)$ and $h(\theta_0) = 1/2 = h(\theta_1)$. By Harrell's Lemma,

$$h''(\theta) + h(\theta) = 0$$
 for almost every θ with $h(\theta) > 1/2$.

In particular, $h''(\theta) + h(\theta) = 0$ for almost every $\theta \in [\theta_0, \theta_1]$; and since *h* is continuous, this equation actually holds at each $\theta \in (\theta_0, \theta_1)$. By Lemma 3.6,

$$h'(\theta_1-) = -\tan\left(\frac{\theta_1-\theta_0}{2}\right)$$

By an analogous argument, there is maximal interval $[\theta_1, \theta_2]$ for which h < 1/2 and $h''(\theta) + h(\theta) = 1$ for all $\theta \in (\theta_1, \theta_2)$. Lemma 3.6 gives that

$$h'(\theta_1+) = -\tan\left(\frac{\theta_2-\theta_1}{2}\right)$$

 $\frac{31}{32}$ Since *h* is continuously differentiable, it must be that

$$\tan\left(\frac{\theta_1-\theta_0}{2}\right)=\tan\left(\frac{\theta_2-\theta_1}{2}\right)$$

³⁵/₃₆ Since tan is increasing on $(0, \pi/2)$, $\theta_1 - \theta_0 = \theta_2 - \theta_1$. That is, both of the maximal intervals we discussed have the same length.

We can continue this argument to conclude that h'' + h is a function which alternates between 0 and 1 on intervals of the same length. As a result, *h* is the support function of a regular Reuleaux polygon. As we noted in Example 2.19, *h* must be the support function of a Reuleaux triangle.

 $\frac{41}{42}$ *Remark* 3.8. This proof shows that an area minimizing shape of constant width must be a Reuleaux triangle.

4. Approximation by Reuleaux polygons

In pursuing another strategy to prove the Blaschke-Lebesgue theorem, we will argue that each shape of constant width can be closely approximated by a Reuleaux polygon. Our proof is inspired by Theorem 6 of Kallay's paper [11]. The following assertion also implies that Reuleaux polygons are dense within the space of constant width shapes in the Hausdorff topology.

7 **Proposition 4.1.** Suppose K is a constant width shape with support function h and $\varepsilon > 0$. There is a Reuleaux polygon K_{ε} with support function h_{ε} such that

$$|h(\theta) - h_{\varepsilon}(\theta)| \leq \varepsilon$$
 and $|h'(\theta) - h'_{\varepsilon}(\theta)| \leq \varepsilon$

10 for each $\theta \in \mathbb{R}$. 11

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12 *Proof.* 1. By replacing h with

$$h_{oldsymbol{\delta}} = rac{h+oldsymbol{\delta}}{1+2oldsymbol{\delta}}$$

for $\delta > 0$ and small, we may suppose that the corresponding parametrization γ is injective. Indeed, it is routine to check that $h_{\delta} \in \mathscr{S}$ and 16

$$h_{\delta}''(\theta) + h_{\delta}(\theta) = \frac{h''(\theta) + h(\theta) + 2\delta}{1 + 2\delta} \ge \frac{2\delta}{1 + 2\delta} > 0$$

for almost every θ . Part (*iv*) of Proposition 2.14 gives that the parametrization associated with h_{δ} is 20 injective. Moreover, 21

$$|h(\theta) - h_{\delta}(\theta)| \le \frac{\delta}{1+2\delta}|2h(\theta) - 1|$$

23 and

$$|h'(heta)-h'_{\delta}(heta)|\leq rac{2\delta}{1+2\delta}|h'(heta)|$$

for all θ . Since h and h' are bounded functions, h_{δ} is a C^1 approximation of h with the desired 26 properties mentioned above. Consequently, we will suppose that γ is injective. 27

2. Suppose $n \in \mathbb{N}$ with

28 29 $\frac{2\pi}{n} \leq \varepsilon$ 30 and set 31 $\theta_i = \frac{i\pi}{n}$ 32 33 for i = 0, ..., n. For each i = 1, ..., n, we consider the 4-tuple of points 34 $\{\gamma(\theta_i), \gamma(\theta_{i-1}), \gamma(\theta_i + \pi), \gamma(\theta_{i-1} + \pi)\} \subset \partial K.$ 35 36 By our assumption that γ is injective, these are four distinct points. There are two solutions $z \in \mathbb{R}^2$ for which 37 38 $|z - \gamma(\theta_i)| = |z - \gamma(\theta_{i-1})| = 1.$

39 Let z_i be the solution z which additionally satisfies $|z - \gamma(\theta_i + \pi)| \le 1$ and $|z - \gamma(\theta_{i-1} + \pi)| \le 1$. See Figure 6. Also observe that 41

42 (4.1)
$$\gamma(\theta_i) - z_i = u(\psi_i)$$
 and $\gamma(\theta_{i-1}) - z_i = u(\phi_i)$

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FIGURE 6. This diagram illustrates how we can construct a Reuleaux polygon which approximates *K*. We simply partition the interval $0 = \theta_0 < \cdots < \theta_n = \pi$ and replace the part of ∂K between $\gamma(\theta_{i-1})$ and $\gamma(\theta_i)$ with an arc of a circle as shown in this diagram. Then we choose vertices on the other side of *K* to ensure the resulting curve has constant width.

 $\theta_{i-1} \leq \phi_i \leq \psi_i \leq \theta_i.$

²⁵ for angles ϕ_i and ψ_i with

3. Define

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$$h_{\varepsilon}(\theta) = \begin{cases} \gamma(\theta_{i-1}) \cdot u(\theta), & \theta \in [\theta_{i-1}, \phi_i] \\ 1 + z_i \cdot u(\theta), & \theta \in [\phi_i, \psi_i] \\ \gamma(\theta_i) \cdot u(\theta), & \theta \in [\psi_i, \theta_i] \end{cases}$$

³² for $\theta \in [0,\pi]$ and extend h_{ε} to $[\pi,2\pi]$ by setting

$$h_{\varepsilon}(\theta + \pi) = 1 - h_{\varepsilon}(\theta) \quad \theta \in [0, \pi].$$

It is straightforward to employ (4.1) and show h_{ε} extends to a 2π -periodic function which is continuously differentiable on \mathbb{R} . As $h''_{\varepsilon} + h_{\varepsilon}$ alternatives between 0 and 1 on successive intervals, h_{ε} is the support function of a Reuleaux triangle.

Suppose $\theta \in [0, \pi]$ and choose i = 1, ..., n such that $\theta \in [\theta_{i-1}, \theta_i]$. If $\theta \in [\theta_{i-1}, \phi_i]$, then

$$|h(\theta) - h_{\varepsilon}(\theta)| = |\gamma(\theta) \cdot u(\theta) - \gamma(\theta_{i-1}) \cdot u(\theta)$$

$$= |(\gamma(\theta) - \gamma(\theta_{i-1})) \cdot u(\theta)|$$

42 (4.4)
$$\leq |\gamma(\theta) - \gamma(\theta_{i-1})|$$

1 (4.5) $< \theta - \theta_{i-1}$ $\frac{2}{3} (4.6)$ $\frac{4}{4} (4.7)$ $\frac{5}{6} Here$ $\frac{6}{7} \theta \in [$ $\leq \frac{\pi}{n}$ $< \varepsilon$. Here we used the Lipschitz estimate (2.16). By virtually the same argument, $|h(\theta) - h_{\varepsilon}(\theta)| \leq \varepsilon$ when $\theta \in [\psi_i, \theta_i]$. Moreover, if $\theta \in [\phi_i, \psi_i]$, 8 $|h(\theta) - h_{\varepsilon}(\theta)| = |\gamma(\theta) \cdot u(\theta) - (1 + z_i \cdot u(\theta))|$ (4.8)9 (4.9) $= |\gamma(\theta) \cdot u(\theta) - (u(\theta) + z_i) \cdot u(\theta))|$ ¹⁰ (4.10) $\leq |\gamma(\theta) - (u(\theta) + z_i)|$ 11 12 $\leq |\gamma(\theta) - \gamma(\theta_i)| + |\gamma(\theta_i) - (u(\theta) + z_i)|$ (4.11) $= |\gamma(\theta) - \gamma(\theta_i)| + |u(\psi_i) - u(\theta)|$ **13** (4.12) 14 (4.13) $< \theta_i - \theta + \psi_i - \theta$ 15 $\leq \frac{2\pi}{n}$ (4.14)16 17 (4.15) $< \varepsilon$. 18 19 We conclude $|h(\theta) - h_{\varepsilon}(\theta)| \leq \varepsilon$ 20 21 for $\theta \in [0,\pi]$. Since $h(\theta + \pi) = 1 - h(\theta)$ and $h_{\varepsilon}(\theta + \pi) = 1 - h_{\varepsilon}(\theta)$ for $\theta \in [0,\pi]$, the estimate above 22 also holds for all $\theta \in [0, 2\pi]$. Finally, the bound 23 $|h'(\theta) - h'_{\epsilon}(\theta)| < \epsilon$ 24 for all $\theta \in [0, 2\pi]$ follows very similarly. We leave the details to the reader. 25 26 Employing the proposition above, we will argue that for any (possibly irregular) Reuleaux polygon ²⁷ the Reuleaux triangle has least area. This assertion is verified in the solution to problem 7.20 in 28 [15] and we shall follow this solution closely below. To this end, we will first need to establish a ²⁹ technical lemma. Let us denote C(x) and D(x) for the circle and open disk of radius one centered at x, 30 respectively. If $y, z \in C(x)$, we will write yz for the shorter segment within C(x) which joins y and z; 31 by abuse of notation, we will also write y_z for the length of this arc. In addition, $\Delta(abc)$ will denote a curvilinear triangle bounded by line segments or arcs of circles of radius one with vertices given by 33 *a*,*b* and *c*. 34 **Lemma 4.2.** Assume $p,q,r,s \in \mathbb{R}^2$ with |p-q| < 2, and that s and r are on the line segment between 35 p and q with 36 |s-p| = |r-q| = 1.37 Suppose $a \in C(p) \cap D(q)$ and $c \in C(q) \cap D(p)$ with 38 39 |a-c| = 1.40 41 (i) If $as \ge cr$, then $A(\Delta(asq)) \ge A(\Delta(cpr))$. $\overline{42}$ (ii) The area difference $A(\Delta(asq)) - A(\Delta(cpr))$ is nondecreasing in the length difference as - cr.





FIGURE 8. Here we take a curvilinear triangle $\Delta(c'p'r')$ which is congruent to $\Delta(cpr)$ and place it within $\Delta(asq)$. Note in particular, that c' is on the circle of radius one centered at p and that c' is between a and s. Also note r' = s, and p' = q. This implies $A(\Delta(asq)) \ge A(\Delta(c'p'r')) = A(\Delta(cpr)).$

²⁵/₋₋ As this would hold for any $\varepsilon > 0$, $A(T) \le A(K)$ and we would then conclude the Blaschke-Lebesgue theorem.

In order to prove the claim, we will argue that that for any *N*-sided Reuleaux polygon *P*, with *N* odd and N > 3, there is another Reuleaux polygon *P'* with N - 2 sides and having smaller area than *P*. In finitely many steps, we could then deduce that the Reuleaux triangle has area less than *P*. To this end, choose a pair $c, d \in P$ of neighboring vertices for which the distance between *c* and *d* is as small as any other pair of neighboring vertices. Let *q* the vertex opposite cd. There are a pair of arcs $bq \subset C(d)$ and $\frac{32}{33}$ $qa \subset C(c)$ in the boundary of *P*. There are also two solutions *z* of the equations

$$|b-z| = |a-z| = 1.$$

 $\frac{36}{37}$ We define p = z to be the solution closer to the arc cd.

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We will construct a new Reuleaux polygon P' from P by replacing the arc cd with the union of two arcs cp and pd and by replacing the two arcs bq and qa with the arc ba
ightharpoon C(p). See Figure 10. It is routine to check that P' has constant width. Moreover, p is a vertex of P' while c,d and q are no longer vertices. In particular, P' has N-2 vertices. Furthermore, we claim that the curvilinear triangle $\Delta(abq)$ has more area than $\Delta(cdp)$. Establishing this claim would complete our proof.





FIGURE 10. A Reuleaux polygon P as described in our proof of the Blaschke-Lebesgue theorem. We obtain an auxiliary Reuleaux polygon P' from P by replacing the arc \overrightarrow{cd} with the two arcs \overrightarrow{cp} and \overrightarrow{pd} and by replacing the two arcs \overrightarrow{bq} and \overrightarrow{qa} with \overrightarrow{ba} . The key here is that $A(P) \ge A(P')$ and P' has two fewer vertices than P'.

In view of (4.16), we have

$$as - cr \ge rd - sb \ge 0.$$

 $\frac{29}{30}$ By part (*ii*) of Lemma 4.2,

$$A(\Delta(sqa)) - A(\Delta(cpr)) \ge A(\Delta(dpr)) - A(\Delta(sqb)).$$

 $\frac{32}{33}$ Consequently,

 $\begin{array}{c|c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \\ 13 \\ 14 \\ 15 \\ 16 \\ 17 \\ 18 \\ 19 \\ 20 \end{array}$

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$$A(\Delta(aqb)) = A(\Delta(sqa)) + A(\Delta(sqb)) \ge A(\Delta(cpr)) + A(\Delta(dpr)) = A(\Delta(cpd)).$$

³⁵/₃₆ We conclude that $A(P) - A(P') = A(\Delta(aqb)) - A(\Delta(cpd)) \ge 0$, as claimed.

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