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PERIODIC SOLUTIONS FOR A MOSQUITO POPULATION SUPPRESSION MODEL BASED ON INCOMPLETE CYTOPLASMIC INCOMPATIBILITY

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ABSTRACT. In this paper, we discuss the periodic dynamics of a periodic switching mosquito population suppression model including the existence of periodic solutions and the stability analysis of the origin and each periodic solution. Under the assumption that the waiting period T between two consecutive releases of *Wolbachia*-infected males is equal to or less than the waiting release period threshold T^*, we give sufficient conditions for the model to have a unique or exactly two periodic solutions, respectively. Our results provide an important complement to the study of periodic solutions in the references. Some numerical examples are provided to illustrate the validity of our theoretical results.

1. Introduction

As a continued work of [9] and [10], we consider the following non-autonomous mosquito suppression model based on the incomplete cytoplasmic incompatibility of *Wolbachia*

$$\frac{dw(t)}{dt} = a\left(1 - s_h\frac{g(t)}{w(t)+g(t)}\right)(1-\xi(w(t)+g(t)))\,w(t) - \mu w(t), \tag{1.1}$$

where $w(t)$ represents the number of wild mosquitoes at time t, $g(t)$ represents the number of sexually active released *Wolbachia*-infected males at time t, $s_h \in (0,1)$ is the intensity of cytoplasmic incompatibility (CI for short), a is the number of eggs produced by per wild mosquito, per unit of time, μ is the constant death rate with $a > \mu$, and ξ is a small enough parameter used to characterize the survival probability $1-\xi(w(t)+g(t))$ of mosquitoes from eggs to adults. In this model, only sexually active released *Wolbachia*-infected males are included so that no independent dynamical equation for the *Wolbachia*-infected males is needed [11–14]. The relevant results for the situation when $s_h = 1$ have been stated in [4, 5, 17].

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Let T and $\overline{T}$ be the waiting period between two consecutive releases and the sexually active lifespan of *Wolbachia*-infected males, respectively. Assume that a constant number c of *Wolbachia*-infected males are released at discrete time points $T_i = iT$, $i = 0,1,2,\cdots$. Then $g(t)$ is a piecewise constant function. For the case when $T = \overline{T}$, the release function $g(t) = c$. The model (1.1) becomes

$$\frac{dw(t)}{dt} = a\left(1 - s_h\frac{c}{w(t)+c}\right)(1-\xi(w(t)+c))w(t) - \mu w(t). \tag{1.2}$$

By introducing the following two CI intensity thresholds

$$s_h^* = \frac{a-\sqrt{a\mu}}{a} \text{ and } s_h^{**} = \frac{a-\mu}{a},$$

and three release amount thresholds

$$c_0 = \frac{a(1-s_h)-\mu}{a\xi(1-s_h)},\ c_1 = \frac{a+\mu-2\sqrt{a\mu}}{a\xi s_h} \text{ and } c^* = \frac{-as_h+\sqrt{a^2s_h^2+4as_h(1-s_h)(a-\mu)}}{2a\xi(1-s_h)},$$

the authors in [9, 10] decomposed the parameter region $\Lambda = \{(s_h, c) : s_h \in (0,1), c \in (0,1/\xi)\}$ into the following three sub-regions

$$\Lambda_0 = \underbrace{\{s_h \in (0,s_h^*], c \in [c_0,1/\xi)\}}_{\Lambda_{01}} \cup \underbrace{\{s_h \in (s_h^*,1), c \in [c^*,1/\xi)\}}_{\Lambda_{02}^{(1)}} \cup \underbrace{\{s_h \in (s_h^*,1), c \in (c_1,c^*)\}}_{\Lambda_{02}^{(2)}},$$

$$\Lambda_1 = \underbrace{\{s_h \in (0,s_h^*], c \in (0,c_0)\}}_{\Lambda_{11}} \cup \underbrace{\{s_h \in (s_h^*,s_h^{**}), c \in (0,c_0)\}}_{\Lambda_{12}} \cup \underbrace{\{s_h \in (s_h^*,s_h^{**}), c = c_0\}}_{\Lambda_{13}}$$
$$\cup \underbrace{\{s_h \in (s_h^*,1), c = c_1\}}_{\Lambda_{14}},$$

$$\Lambda_2 = \underbrace{\{s_h \in (s_h^*.s_h^{**}), c \in (c_0,c_1)\}}_{\Lambda_{21}} \cup \underbrace{\{s_h \in [s_h^{**},1), c \in (0,c_1)\}}_{\Lambda_{22}}.$$

It is clear to see that model (1.2) has no positive equilibria for $(s_h,c) \in \Lambda_0$, a unique positive equilibrium for $(s_h,c) \in \Lambda_1$, and two positive equilibria for $(s_h,c) \in \Lambda_2$, respectively.

For the case when $T > \overline{T}$, which means a low release frequency of *Wolbachia*-infected males, $g(t)$ becomes the following piecewise constant function

$$g(t) = \begin{cases} c, & \text{for } t \in [iT, iT+\overline{T}), \\ 0, & \text{for } t \in [iT+\overline{T}, (i+1)T), \end{cases} \quad i = 0,1,2,\cdots,$$

and model (1.1) consists of two sub-equations as follows

$$\frac{dw(t)}{dt} = a\left(1 - s_h\frac{c}{w(t)+c}\right)(1-\xi(w(t)+c))w(t) - \mu w(t),\ t \in [iT, iT+\overline{T}), \tag{1.3}$$

and

$$\frac{dw(t)}{dt} = -a\xi w(t)(w(t)-A),\ t \in [iT+\overline{T},(i+1)T), \tag{1.4}$$

where $A = (a-\mu)/a\xi,\ i = 0,1,2,\cdots$.

A detailed study on the dynamics of model (1.3)-(1.4), such as the existence and stability of a unique or exactly two periodic solutions, has been given in recent works [9, 10] when $(s_h,c) \in \Lambda_{01} \cup \Lambda_{02}^{(1)} \cup \Lambda_{11} \cup \Lambda_{12} \cup \Lambda_{13}$. However, the complete stability analysis of the origin E_0 is still lacking in two regions $\Lambda_{02}^{(2)}$ and $\Lambda_{14} \cup \Lambda_2$. On the one hand, the available results in [9] show that when $(s_h,c) \in \Lambda_{02}^{(2)}$, the origin E_0 of model (1.3)-(1.4) is locally asymptotically stable if and only if $T < T^*(s_h,c)$, where $T^*(s_h,c)$ represents the waiting release period threshold, defined by

$$T^*(s_h,c) = \frac{a(1-(1-\xi c)(1-s_h))}{a-\mu}\overline{T}.$$

Naturally, we want to know whether there exists another waiting release period threshold $T^{**}(s_h,c) \in (\overline{T},T^*(s_h,c))$ such that it is possible for the origin E_0 to be globally asymptotically stable if $T \in (\overline{T},T^{**}(s_h,c))$. On the other hand, the main results in [10] suggest that when $(s_h,c) \in \Lambda_{14} \cup \Lambda_2$, the origin E_0 is unstable if $T > T^*(s_h,c)$, and the origin E_0 is locally asymptotically stable if $T < T^*(s_h,c)$, together with the existence of a locally asymptotically stable T-periodic solution with a larger initial value and an unstable T-periodic solution with a smaller initial value. However, how to determine the stability of the origin E_0 in the critical case $T = T^*(s_h,c)$ becomes the most difficult problem when $(s_h,c) \in \Lambda_{14} \cup \Lambda_2$. To deal with the stability of E_0 in two regions $\Lambda_{02}^{(2)}$ and $\Lambda_{14} \cup \Lambda_2$ to fill the gap when $T > \overline{T}$, the following two theorems give the answers.

Theorem 1.1. *Assume $(s_h,c) \in \Lambda_{02}^{(2)}$. Then there exists a unique waiting release period threshold $T^{**}(s_h,c) \in (\overline{T},T^*(s_h,c))$ such that the following statements are true.*

(1): *When $T \in (T^{**}(s_h,c),T^*(s_h,c))$, model (1.3)-(1.4) has exactly two T-periodic solutions, where the T-periodic solution with larger initial value is asymptotically stable, and the T-periodic solution with smaller initial value is unstable. The origin E_0 is asymptotically stable.*

(2): *When $T = T^{**}(s_h,c)$, model (1.3)-(1.4) has a unique T-periodic solution, which is semi-stable (stable on the right-side and unstable on the left-side). The origin E_0 is asymptotically stable.*

(3): *When $T \in (\overline{T},T^{**}(s_h,c))$, model (1.3)-(1.4) has no T-periodic solutions. The origin E_0 is globally asymptotically stable.*

Theorem 1.2. *Assume that $(s_h,c) \in \Lambda_{14} \cup \Lambda_2$ and $T = T^*(s_h,c)$. Then model (1.3)-(1.4) has a unique T-periodic solution, which is globally asymptotically stable. Meanwhile, the origin E_0 is unstable.*

In Section 2, we will give some preliminary work on proving theorems. In Sections 3 and 4, detailed proofs of Theorems 1.1 and 1.2 will be presented. Some numerical examples illustrate the two theoretical results. The discussion and analysis of all the conclusions regarding the release strategy $T > \overline{T}$ will be given in Section 5.

2. Preliminary

For any $t_0 \geq 0$ and $u > 0$, let $w(t;t_0,u)$ be the solution of model (1.3)-(1.4) with initial value $w(t_0) = u$. Then solution $w(t;t_0,u)$ is continuously differentiable with respect to u. Define functions $\bar{h}(u)$ and $h(u,T)$ as follows

$$\bar{h}(u) = w(\overline{T};0,u) \text{ and } h(u,T) = w(T;0,u).$$

Then $\bar{h}(u)$ and $h(u,T)$ are both continuously differentiable with respect to u and satisfy $h(u,T) = w(T;\overline{T},\bar{h}(u))$. Thus, $h(u,T)$ defines the Poincaré map of model (1.3)-(1.4). It is clear to see that the existence of T-periodic solutions of model (1.3)-(1.4) is equivalent to the existence of positive fixed points of map $h(u,T)$. To seek the T-periodic solutions of model (1.3)-(1.4), we only need to find a $u > 0$ such that $h(u,T) = u$. Now, we need to find solution $w(t;0,u)$ of model (1.3)-(1.4) with initial value $u > 0$.

For case $(s_h,c) \in \Lambda_{02}^{(2)}$, equation (1.3) can be rewritten as

$$\frac{dw}{dt} = -\frac{a\xi w((w-B)^2+D^2)}{w+c}, \tag{2.1}$$

which can be further decomposed into

$$\left(\frac{\alpha}{w} + \frac{\beta(w-B)}{(w-B)^2+D^2} + \frac{\gamma}{(w-B)^2+D^2}\right) dw = -a\xi dt, \tag{2.2}$$

where

$$B = \frac{A-(2-s_h)c}{2} \text{ and } D^2 = -\frac{a^2\xi^2 s_h^2 c^2 - 2a\xi s_h(a+\mu)c + (a-\mu)^2}{4a^2\xi^2}. \tag{2.3}$$

Integrating (2.2) from 0 to $\overline{T}$, we obtain

$$\bar{h}^{\alpha}(u)((\bar{h}(u)-B)^2+D^2)^{\frac{\beta}{2}} e^{\frac{\gamma}{D}\tan^{-1}(\frac{\bar{h}(u)-B}{D})} = u^{\alpha}((u-B)^2+D^2)^{\frac{\beta}{2}} e^{\frac{\gamma}{D}\tan^{-1}(\frac{u-B}{D})} e^{-a\xi\overline{T}}, \tag{2.4}$$

where

$$\alpha = -\beta = -\frac{a\xi}{a(1-s_h)(1-\xi c)+\mu} \text{ and } \gamma = 1 - \frac{a-\mu-a\xi c(2-s_h)}{2(a(1-s_h)(1-\xi c)-\mu)}.$$

For case $(s_h,c) \in \Lambda_{14}$, we define

$$0 < E^*(s_h,c) = \frac{a-\mu-a\xi c(2-s_h)}{2a\xi} < A.$$

From equation (1.3), we have for $t \in [0, \overline{T}]$

$$\frac{dw}{dt} = -\frac{a\xi w(w - E^*(s_h, c))^2}{w + c}, \ w(0) = u > 0, \tag{2.5}$$

which yields, by integrating from 0 to $\overline{T}$,

$$\bar{h}^{\alpha_1}(u)\ (\bar{h}(u) - E^*(s_h, c))^{\beta_1}\ e^{-\frac{\gamma_1}{\bar{h}(u) - E^*(s_h,c)}} = u^{\alpha_1}\ (u - E^*(s_h, c))^{\beta_1}\ e^{-\frac{\gamma_1}{u - E^*(s_h,c)}}\ e^{-a\xi \overline{T}}, \tag{2.6}$$

where

$$\alpha_1 = -\beta_1 = \frac{c}{E^{*2}(s_h, c)} \text{ and } \gamma_1 = 1 + \frac{c}{E^*(s_h, c)}.$$

For case $(s_h, c) \in \Lambda_2$, we define

$$0 < E_1(s_h, c) = \frac{a - \mu - a\xi c(2 - s_h) - \sqrt{\Delta}}{2a\xi} < E_2(s_h, c) = \frac{a - \mu - a\xi c(2 - s_h) + \sqrt{\Delta}}{2a\xi} < A,$$

where $\Delta = a^2\xi^2 s_h^2 c^2 - 2a\xi s_h(a + \mu)c + (a - \mu)^2 > 0$. From equation (1.3) again, we have for $t \in [0, \overline{T}]$

$$\frac{dw}{dt} = -\frac{a\xi w(w - E_1(s_h, c))(w - E_2(s_h, c))}{w + c}, \ w(0) = u > 0, \tag{2.7}$$

which gives, by integrating from 0 to $\overline{T}$,

$$\frac{\bar{h}(u)^{\alpha_2}}{u^{\alpha_2}}\ \frac{(\bar{h}(u) - E_1(s_h, c))^{\beta_2}}{(u - E_1(s_h, c))^{\beta_2}}\ \frac{(\bar{h}(u) - E_2(s_h, c))^{\gamma_2}}{(u - E_2(s_h, c))^{\gamma_2}} = e^{-a\xi \overline{T}}, \tag{2.8}$$

where

$$\alpha_2 = \frac{c}{E_1(s_h, c)E_2(s_h, c)},\ \beta_2 = -\frac{c + E_1(s_h, c)}{E_1(s_h, c)(E_2(s_h, c) - E_1(s_h, c))} \text{ and } \gamma_2 = \frac{c + E_2(s_h, c)}{E_2(s_h, c)(E_2(s_h, c) - E_1(s_h, c))}.$$

Integrating (1.4) from $\overline{T}$ to T yields

$$\frac{h(u, T)}{\bar{h}(u)} = \frac{A - h(u, T)}{m(A - \bar{h}(u))}, \tag{2.9}$$

where $m = e^{-Aa\xi(T - \overline{T})} = e^{-(a-\mu)(T - \overline{T})}$. Since $\bar{h}(u) \to 0$ and $h(u, T) \to 0$ as $u \to 0$, combining (2.4), (2.6) and (2.8), we have

$$h'_u(0, T) = \lim_{u \to 0} \frac{h(u, T)}{u} = \lim_{u \to 0} \left(\frac{h(u, T)}{\bar{h}(u)} \cdot \frac{\bar{h}(u)}{u}\right) = e^{(a-\mu)(T - T^*)}. \tag{2.10}$$

To estimate the stability of the origin E_0, we need to determine the existence of T-periodic solutions. Therefore, we first study the monotonicity of sequence $\{w(t + nT; 0, u)\}_{n=0}^{\infty}$ for $t \in [0, +\infty)$ and obtain the following lemma.

Lemma 2.1. *For solution $w(t; 0, u)$ of model (1.3)-(1.4) with $u > 0$, the following statements are true.*

(1): *If $h(u,T) > u$, then we have*

$$w(t+T;0,u) > w(t;0,u),\ t \in [0,+\infty),$$

and sequence $\{w(t+nT;0,u)\}_{n=0}^{\infty}$ is strictly increasing.

(2): *If $h(u,T) = u$, then we have*

$$w(t+T;0,u) = w(t;0,u),\ t \in [0,+\infty),$$

which is a T-periodic solution of model (1.3)-(1.4).

(3): *If $h(u,T) < u$, then we have*

$$w(t+T;0,u) < w(t;0,u),\ t \in [0,+\infty),$$

and sequence $\{w(t+nT;0,u)\}_{n=0}^{\infty}$ is strictly decreasing.

Proof. We just give the proof of **(1)**, and the proofs of **(2)** and **(3)** are similar to that of **(1)** and are omitted here. If **(1)** is not true, then there exists a $t' \in [t_0,+\infty)$ such that $w(t'+T;0,u) \leq w(t';0,u)$, where $t_0 \in (0,+\infty)$. Let

$$\bar{w}(t) = w(t+T;0,u) = w(t;0,w(T)) = w(t;0,h(u,T)).$$

By comparing $w(t)$ and $\bar{w}(t)$, there is a $\tilde{t} \in [0,t_0)$ such that

$$w(\tilde{t};0,u) < w(\tilde{t}+T;0,u) = w(\tilde{t};0,h(u,T)) = \bar{w}(\tilde{t}),$$

and

$$w(t';0,u) \geq w(t'+T;0,u) = w(t';0,h(u,T)) = \bar{w}(t'),$$

which is impossible due to the existence and uniqueness of the initial value problems. Meanwhile, the above proof also implies that sequence $\{w(t+nT;0,u)\}_{n=0}^{\infty}$ is strictly increasing. The proof is complete. □

By equations (1.4), (2.1), (2.5), and (2.7), the following three lemmas compare the sizes of $h(u,T)$ and u in $\Lambda_{02}^{(2)}$, Λ_{14} and Λ_2, respectively.

Lemma 2.2. *If $(s_h,c) \in \Lambda_{02}^{(2)}$, then the following statements are true.*

(1): *$u > \bar{h}(u)$ and $\bar{h}(u) < h(u,T)$, for $u \in (0,A]$.*

(2): *There exists a $\hat{u} > A$ such that*

$$u > A > h(u,T),\ u \in (A,\hat{u}),\ \textit{and}\ u > \bar{h}(u) > h(u,T),\ u \in (\hat{u},+\infty).$$

Lemma 2.3. *If $(s_h,c) \in \Lambda_{14}$, then the following statements are true.*

(1): *$u > \bar{h}(u)$ and $\bar{h}(u) < h(u,T)$, for $u \in (0,E^*(s_h,c))$.*

(2): *$u = \bar{h}(u)$ and $\bar{h}(u) < h(u,T)$, for $u = E^*(s_h,c)$.*

(3): *$u > \bar{h}(u)$ and $\bar{h}(u) < h(u,T)$, for $u \in (E^*(s_h,c), A]$.*

(4): *There exists a $\hat{u} > A$ such that*

$$u > A > h(u,T),\ u \in (A,\hat{u}),\ \text{and}\ u > \bar{h}(u) > h(u,T),\ u \in (\hat{u},+\infty).$$

Lemma 2.4. *If $(s_h,c) \in \Lambda_2$, then the following statements are true.*

(1): *$u > \bar{h}(u)$, and $\bar{h}(u) < h(u,T)$, for $u \in (0, E_1(s_h,c))$.*

(2): *$u \leq \bar{h}(u)$, and $\bar{h}(u) < h(u,T)$, for $u \in [E_1(s_h,c), E_2(s_h,c)]$.*

(3): *$u > \bar{h}(u)$, and $\bar{h}(u) < h(u,T)$, for $u \in (E_2(s_h,c), A]$.*

(4): *There exists a $\hat{u} > A$ such that*

$$u > A > h(u,T),\ u \in (A,\hat{u}),\ \text{and}\ u > \bar{h}(u) > h(u,T),\ u \in (\hat{u},+\infty).$$

According to Lemmas 2.1, 2.2, 2.3 and 2.4, we know that if the T-periodic solutions of model (1.3)-(1.4) exist, then the initial values of T-periodic solutions can only initiate from $(0,A)$. Next, we further discuss the existence and stability of T-periodic solutions for model (1.3)-(1.4) when $(s_h,c) \in \Lambda_{02}^{(2)}$ or $(s_h,c) \in \Lambda_{14} \cup \Lambda_2$, respectively.

3. Proof of Theorem 1.1

The following lemma shows the existence of T-periodic solutions when $(s_h,c) \in \Lambda_{02}^{(2)}$.

Lemma 3.1. *If $(s_h,c) \in \Lambda_{02}^{(2)}$ and $T \in (\overline{T}, T^*(s_h,c))$, then model (1.3)-(1.4) has at most two T-periodic solutions initiated from $(0,A)$.*

Proof. Since $T \in (\overline{T}, T^*(s_h,c))$, from (2.10), we have $h'_u(0,T) < 1$. Assume that model (1.3)-(1.4) has three T-periodic solutions initiated from u_1, u_2 and u_3, respectively, satisfying

$$0 < u_1 < u_2 < u_3 < A.$$

Then we have $h(u_i,T) = u_i$, $i = 1,2,3$, and hence one of the following four cases holds:

(i): $h'_u(u_1,T) = 1,\ h'_u(u_2,T) \geq 1,\ h'_u(u_3,T) \leq 1,$

(ii): $h'_u(u_1,T) \geq 1,\ h'_u(u_2,T) \leq 1,\ h'_u(u_3,T) = 1,$

(iii): $h'_u(u_1,T) = 1,\ h'_u(u_2,T) = 1,\ h'_u(u_3,T) = 1,$

(iv): $h'_u(u_1,T) \geq 1,\ h'_u(u_2,T) = 1,\ h'_u(u_3,T) \leq 1.$

From (26) in [9], we have

(3.1)
$$h'_u(u,T) = \frac{h(u,T)((Amh(u,T) - B(A-(1-m)h(u,T)))^2 + D^2(A-(1-m)h(u,T))^2)(u+c)}{uA((u-B)^2+D^2)(cA+(mA-(1-m)c)h(u,T))},$$

where B and D are defined in (2.3). Thus, case **(i)** corresponds to

$$Eu_1^2+Fu_1+G=0,\ Eu_2^2+Fu_2+G\geq 0 \text{ and } Eu_3^2+Fu_3+G\leq 0, \tag{3.2}$$

where

$$\begin{aligned}
E&=(B^2+D^2)(1-m)+2mAB+cA-mA^2,\\
F&=(1-m)(c(B^2+D^2)-2cAB)-cA^2(1+m)-2A(B^2+D^2),\\
G&=A(B^2+D^2)(A-c)+2cA^2B.
\end{aligned}$$

Set

$$P(u)=Eu^2+Fu+G. \tag{3.3}$$

Then, (3.2) is equivalent to

$$P(u_1)=0,\ P(u_2)\geq 0 \text{ and } P(u_3)\leq 0.$$

Since

$$P(A)=-mA(A(B^2+D^2)+A(A+c)(A-2B))<0,$$

the function (3.3) can only have one root in $(0,A)$ due to $G>0$ when $(s_h,c)\in\Lambda_{02}^{(2)}$. Thus, case **(i)** is impossible. Similarly, case **(ii)** is equivalent to

$$P(u_1)\geq 0,\ P(u_2)\leq 0 \text{ and } P(u_3)=0,$$

which also can be excluded. The contradiction in case **(iii)** is obvious because quadratic function (3.3) cannot have three roots.

For case **(iv)**, taking the perturbation from $h(u,T)-u$ to $h(u,T)-ku$ satisfying $k>1$, $mk<1$ and $(1+m)k<2$, we have four initial values v_1, v_2, v_3 and v_4, satisfying

$$0<u_1<v_1<v_2<u_2<v_3<v_4<u_3<A,$$

such that $h(v_i,T)=kv_i$ for $i=1,2,3,4$, and

$$h_u'(v_1,T)\geq k,\ h_u'(v_2,T)\leq k,\ h_u'(v_3,T)\geq k, \text{ and } h_u'(v_4,T)\leq k. \tag{3.4}$$

Following (3.1), we have

$$h_u'(v_i,T)=\frac{k((Amkv_i-B(A-(1-m)kv_i))^2+D^2(A-(1-m)kv_i)^2)(v_i+c)}{A((v_i-B)^2+D^2)(cA+(mA-(1-m)c)kv_i)},\ i=1,2,3,4,$$

which leads to (3.4) being equivalent to

$$\begin{aligned}
&E_kv_1^2+F_kv_1+G_k\geq 0,\ E_kv_2^2+F_kv_2+G_k\leq 0,\\
&E_kv_3^2+F_kv_3+G_k\geq 0,\ E_kv_4^2+F_kv_4+G_k\leq 0,
\end{aligned} \tag{3.5}$$

where

$$E_k = -(1-mk)mkA^2 + (1-m)(2mk^2AB + ckA + (1-m)k^2(B^2+D^2)),$$
$$F_k = -(1-m)k(B^2+D^2)(2A - ck(1-m)) - cA(1-mk)(A(1+mk)+2(1-m)kB),$$
$$G_k = ((1-mk)A^2 - (1-m)ckA)(B^2+D^2) + (1-mk)2cA^2B.$$

Set

$$P_k(v) = E_k v^2 + F_k v + G_k.$$

Then, from (3.5), we have

$$P_k(v_1) \geq 0,\ P_k(v_2) \leq 0,\ P_k(v_3) \geq 0, \text{ and } P_k(v_4) \leq 0,$$

which implies that $P_k(v)$ has three roots, a contradiction. In a similar way, we can exclude the possibility of four or more T-periodic solutions for model (1.3)-(1.4). The proof is complete. □

Next, we give detailed proof of Theorem 1.1.

Proof of Theorem 1.1. We begin with the existence and uniqueness of $T^{**}(s_h,c)$. Since equations (1.3) and (1.4) determine the solution $w(t;0,u)$ on $[iT, iT+\overline{T})$ and $[iT+\overline{T},(i+1)T)$, $i=0,1,2,\cdots$, respectively, when $(s_h,c)\in\Lambda_{02}^{(2)}$ and $u\in(0,A)$, we have

$$\frac{dw}{dt} = a\left(1 - s_h\frac{c}{w+c}\right)(1-\xi(w+c))w - \mu w < 0,\ t\in[iT, iT+\overline{T}), \tag{3.6}$$

and

$$\frac{dw}{dt} = a(1-\xi w)w - \mu w > 0,\ t\in[iT+\overline{T},(i+1)T),$$

which implies that $h(u,T)$ is continuously strictly increasing with respect to T. When $T=\overline{T}$, following (3.6), we have

$$h(u,\overline{T}) = w(\overline{T};0,u) = \bar{h}(u) < u,\ u\in(0,A). \tag{3.7}$$

When $T = T^*(s_h,c)$, from Theorem 3.1 **(2)** in [9], we know that model (1.3)-(1.4) has a unique globally asymptotically stable T-periodic solution, denoted as $w(t;0,u^*)$ for $u^*\in(0,A)$, which means that

$$h(u,T^*(s_h,c)) > u,\ u\in(0,u^*),\ \text{and } h(u,T^*(s_h,c)) < u,\ u\in(u^*,+\infty). \tag{3.8}$$

When $T < T^*(s_h,c)$, by (2.10), we get $h'(0,T) < 1$, which suggests that there exists a sufficiently small $\delta_0 > 0$ such that

$$h(u,T) < u,\ u\in(0,\delta_0). \tag{3.9}$$

According to (3.8) and (3.9), there must exists at least one initial value $\tilde{u}\in(\delta_0,u^*)$ such that

$$h(\tilde{u},T) > \tilde{u},\ \overline{T} < T < T^*(s_h,c). \tag{3.10}$$

Thus, from (3.7), (3.8), (3.9) and (3.10), there exists a unique $T^{**}(s_h,c) \in (\overline{T}, T^*(s_h,c))$ guaranteed by the continuity and monotonicity of $h(u,T)$ with respect to T such that model (1.3)-(1.4) has a unique T-periodic solution when $T = T^{**}(s_h,c)$. Furthermore, from Lemma 3.1, model (1.3)-(1.4) has exactly two T-periodic solutions if $T \in (T^{**}(s_h,c), T^*(s_h,c))$, and has no T-periodic solutions if $T \in (\overline{T}, T^{**}(s_h,c))$.

Next, we show the stability of T-periodic solutions under three releasing period strategies

$$\textbf{(1)}\ T \in (T^{**}(s_h,c), T^*(s_h,c)),\ \textbf{(2)}\ T = T^{**}(s_h,c) \text{ and } \textbf{(3)}\ T \in (\overline{T}, T^{**}(s_h,c)).$$

For case **(1)**, we assume that the two T-periodic solutions initiated from u_1 and u_2 satisfying $0 < u_1 < u_2 < A$. From Lemma 2.2, we have

$$\begin{aligned} &(u-u_1)(h(u,T)-u) > 0,\ u \in (0,u_1) \cup (u_1,u_2), \\ &(u-u_2)(h(u,T)-u) < 0,\ u \in (u_1,u_2) \cup (u_2,+\infty). \end{aligned} \tag{3.11}$$

We first show that T-periodic solution $w(t;0,u_1)$ is unstable. For any $\delta \in (0, \min\{u_1, u_2-u_1\})$, by Lemma 2.1 and (3.11), we get

$$h(u_1+\delta,T) > u_1+\delta \text{ and } h(u_1-\delta,T) < u_1-\delta,$$

which leads to

$$|w(T;0,u)-u_1| = |w(T;0,u)-w(T;0,u_1)| > \delta,\ u = u_1 \pm \delta.$$

Therefore, T-periodic solution $w(t;0,u_1)$ is unstable. We then show that T-periodic solution $w(t;0,u_2)$ is asymptotically stable.

We first prove the stability for case $u \in (u_1,u_2)$. Select an $\eta \in (0, u_2-u_1)$. For $u \in (u_1,u_2)$, if

$$\max_{t\in[0,T]} \{w(t;0,u_2) - w(t;0,u_2-\eta)\} \le \eta,$$

then taking $\varepsilon = \eta = \delta$, we have

$$w(t;0,u_2) - w(t;0,u) < \varepsilon.$$

If

$$\max_{t\in[0,T]} \{w(t;0,u_2) - w(t;0,u_2-\eta)\} > \eta,$$

by taking $\varepsilon = \eta$, then there is an $\eta_0 \in (0,\eta)$ such that

$$\max_{t\in[0,T]} \{w(t;0,u_2) - w(t;0,u_2-\eta_0)\} \le \eta.$$

Choosing $\delta = \eta_0$, we have

$$w(t;0,u_2) - w(t;0,u) < \max_{t\in[0,T]} \{w(t;0,u_2) - w(t;0,u_2-\eta_0)\} \le \eta = \varepsilon.$$

For the case $u \in (u_2, +\infty)$, the proof is similar and omitted here. Thus, T-periodic solution $w(t;0,u_2)$ is stable.

We next show the attractivity of T-periodic solution $w(t;0,u_2)$. From (3.11), we have

$$h(u,T) > u,\ u \in (u_1,u_2),\ \text{and}\ h(u,T) < u,\ u \in (u_2,+\infty),$$

which, by Lemma 2.1, means that sequence $\{w(t+nT;0,u)\}_{n=0}^{\infty}$ is strictly increasing with $u \in (u_1,u_2)$ and strictly decreasing with $u \in (u_2,+\infty)$. We assume by contradiction that, for $t_0 \in [0,+\infty)$,

$$\lim_{n\to+\infty} w(t_0+nT;0,u) = w(t_0;0,\hat{u}),\ \hat{u} \in (u_1,+\infty).$$

Then, we have

$$w(t_0+nT;0,\hat{u}) = \lim_{n\to+\infty} w(t_0+(n+1)T;0,\hat{u}) = \lim_{n\to+\infty} w(t_0+nT;0,\hat{u}) = w(t_0;0,\hat{u}).$$

Thus, solution $w(t;0,\hat{u})$ is another T-periodic solution, which is a contradiction. In conclusion, T-periodic solution $w(t;0,u_2)$ is asymptotically stable. Similarly, by $h(u,T) < u$ for $u \in (0,u_1)$, we can prove that the origin E_0 is asymptotically stable.

For case **(2)**, we denote the unique T-periodic solution by $w(t,0,u_3)$. By Lemma 2.2, we have

$$h(u,T) < u,\ u \in (0,u_3),\ \text{and}\ h(u,T) < u,\ u \in (u_3,+\infty).$$

Similar to the proof in case **(1)**, we can show that T-periodic solution $w(t;0,u_3)$ is semi-stable and the origin E_0 is asymptotically stable.

For case **(3)**, from Lemma 2.2, we have

$$h(u,T) < u,\ u \in (0,+\infty),$$

which yields the origin E_0 is globally asymptotically stable. The proof is complete. □

We provide a numerical example to confirm the results of Theorem 1.1 below.

Example 3.1. *Given*

$$a = 2,\ \mu = 0.05,\ \xi = 0.001\ \textit{and}\ \overline{T} = 14,$$

we have $s_h^* \approx 0.8419$. *If we choose* $s_h = 0.93 \in (s_h^*, 1)$, *then* $c_1 \approx 762.1207$ *and* $c^* \approx 912.3478$. *Taking* $c = 770 \in (c_1, c^*)$, *we have the waiting release period threshold* $T^*(0.93,770) \approx 14.1579$. *Thus, we can find a unique* $T^{**}(0.93,770) \approx 14.0212$ *(see Figure 1) by calculation* 1000 *points on* $h(u,T) - u$ *for* $u \in (0,200)$ *with respect to the initial value u such that* $h(u,T) - u$ *has two roots if* $T = 14.0600 \in (T^{**}(0.93,770), T^*(0.93,770))$, $h(u,T) - u$ *has a unique root if* $T = T^{**}(0.93,770)$, *and* $h(u,T) - u$ *has no roots if* $T = 14.0001 \in (\overline{T}, T^{**}(0.93,770))$. *This means that two T-periodic solutions occur for* $T \in (T^{**}, T^*)$, *a unique T-periodic solution appears for* $T = T^{**}$, *and no T-periodic solutions exist for* $T \in (\overline{T}, T^{**})$, *which confirms the results of Theorem 1.1. Figure 2 (a), (b) and (c)* *show the corresponding solution curves of model* (1.3)-(1.4) *when* $T = 14.0600$, $T = 14.0212$ *and* $T = 14.0001$, *respectively.*

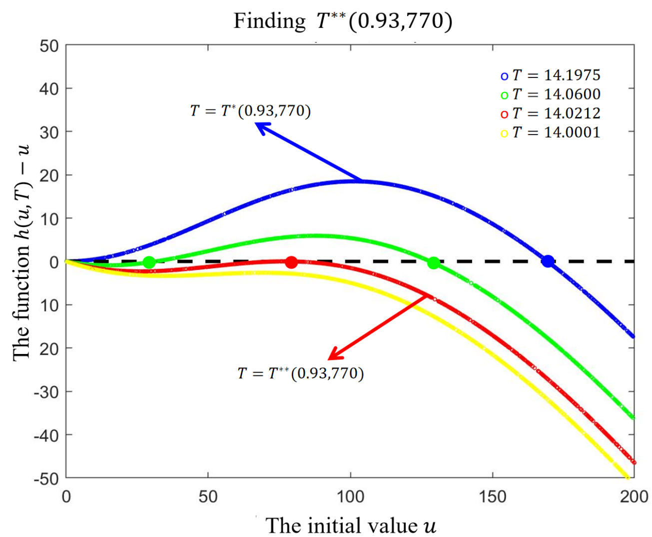

FIGURE 1. The waiting release period threshold $T^{**}(0.93,770)$ is based on the number of roots of $h(u,T)-u$, which is equivalent to the number of T-periodic solutions.

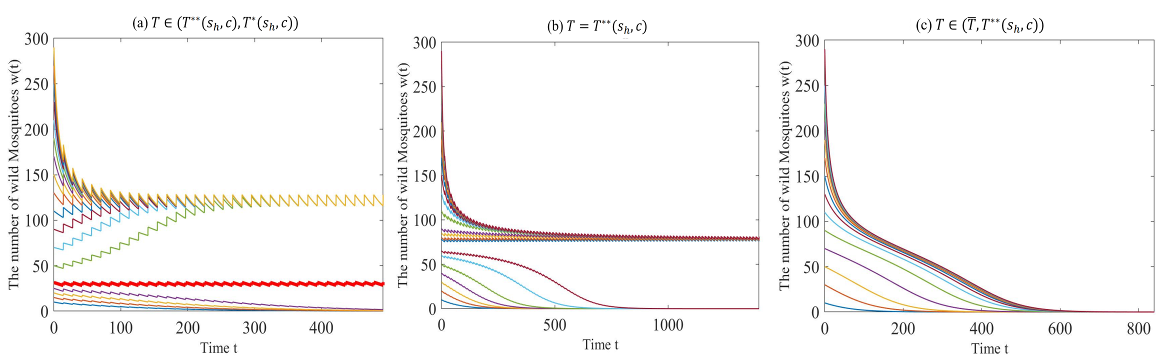

FIGURE 2. When $(s_h, c) \in \Lambda_{02}^{(2)}$, we plot the solution curves of model (1.3)-(1.4) with $T = 14.0600 \in (T^{**}(0.93,770), T^{*}(0.93,770))$ in panel (a), $T = 14.0212 \approx T^{**}(0.93,770)$ in panel (b) and $T = 14.0001 \in (\overline{T}, T^{**}(0.93,770))$ in panel (c).

4. Proof of Theorem 1.2

In this section, we elaborate in detail the existence and stability of T-periodic solutions and the origin E_0 when $(s_h, c) \in \Lambda_{14} \cup \Lambda_2$ and $T = T^*(s_h, c)$. From Lemmas 2.1, 2.2 and Theorem 1.3 in [10], the next lemma shows that model (1.3)-(1.4) has at least one T-periodic solution.

Lemma 4.1. *The following two conclusions hold.*

(1): *If $(s_h, c) \in \Lambda_{14}$, then there exists at least one $\bar{u} \in (E^*(s_h, c), A)$ such that $w(t; 0, \bar{u})$ is the T-periodic solution of model (1.3)-(1.4).*

(2): *If $(s_h, c) \in \Lambda_2$, then there exists at least one $\bar{v} \in (E_2(s_h, c), A)$ such that $w(t; 0, \bar{v})$ is the T-periodic solution of model (1.3)-(1.4).*

To estimate the stability of the origin E_0, we need to determine the specific number and stability of T-periodic solutions.

Proof of Theorem 1.2. We consider three cases: $(s_h, c) \in \Lambda_{14}$, $(s_h, c) \in \Lambda_{21}$ and $(s_h, c) \in \Lambda_{22}$, respectively, but only give the proof in detail for case $(s_h, c) \in \Lambda_{21}$. The proof for case $(s_h, c) \in \Lambda_{22}$ or $(s_h, c) \in \Lambda_{14}$ is similar and is omitted. By Lemma 4.1, we just need to show the uniqueness and global asymptotic stability of T-periodic solutions for model (1.3)-(1.4).

Firstly, we prove the uniqueness. From Lemma 2.4, we know that the Poincaré map $h(u, T)$ has no fixed points for $u \in [E_1(s_h, c), E_2(s_h, c)] \cup [A, \infty)$. Thus, we show that $h(u, T)$ has no fixed points for $(0, E_1(s_h, c))$ as well. In fact, since $T = T^*(s_h, c)$, from (2.8) and (2.9), we have

$$\frac{h(u,T)}{u} = \frac{A - h(u,T)}{A - u} \cdot \frac{(A-u)(E_1(s_h,c)-u)^{\frac{\beta_2}{\alpha_2}}(E_2(s_h,c)-u)^{\frac{\gamma_2}{\alpha_2}}}{(A-\bar{h}(u))(E_1(s_h,c)-\bar{h}(u))^{\frac{\beta_2}{\alpha_2}}(E_2(s_h,c)-\bar{h}(u))^{\frac{\gamma_2}{\alpha_2}}}.$$

Let

$$F(u) = (A-u)^{\alpha_2}(E_1(s_h,c)-u)^{\beta_2}(E_2(s_h,c)-u)^{\gamma_2},\ u \in (0, E_1(s_h,c)).$$

Then we have

$$F'(u) = F(u) \cdot \frac{-(\alpha_1 A + 1)(u - Q(s_h,c))}{(A-u)(u-E_1(s_h,c))(u-E_2(s_h,c))},$$

where

$$Q(s_h,c) = \frac{s_h A - c s_h - \xi c^2(1-s_h)}{1-(1-s_h)(1-\xi c)}.$$

If $u \in (0, \min\{E_1(s_h,c), Q(s_h,c)\})$, then $F(u)$ is strictly increasing with respect to u, which leads to

$$\frac{h(u,T)}{u} = \frac{A-h(u,T)}{A-u} \cdot \frac{F^{\frac{1}{\alpha_2}}(u)}{F^{\frac{1}{\alpha_2}}(\bar{h}(u))} > \frac{A-h(u,T)}{A-u},$$

that is,

$$h(u,T) > u.$$

Therefore, we only need to show that $Q(s_h,c) > E_1(s_h,c)$. Taking the derivative of $Q(s_h,c)$ with c yields

$$\frac{\partial Q(s_h,c)}{\partial c} = -1 - \frac{\xi s_h A(1-s_h)}{(s_h+\xi c(1-s_h))^2} < 0,$$

which means that $Q(s_h,c)$ is strictly decreasing with respect to c. By taking the derivative of $E_1(s_h,c)$, we have

$$\frac{\partial E_1(s_h,c)}{\partial c} = \frac{J_1(s_h,c)}{2\sqrt{\Delta}\left(s_h\left(a\left(1-\xi c s_h\right)+\mu\right)+(2-s_h)\sqrt{\Delta}\right)},$$

where

$$J_1(s_h,c) = 4a\mu s_h^2 - 4a^2\xi^2 s_h^2 c^2(1-s_h) + 8a\xi s_h c(a+\mu)(1-s_h) - 4(a-\mu)^2(1-s_h). \tag{4.1}$$

Similarly, taking the derivative of $J_1(s_h,c)$ with respect to c, we obtain

$$\frac{\partial J_1(s_h,c)}{\partial c} = 8a\xi s_h(1-s_h)(a(1-\xi c s_h)+\mu) > 0.$$

Substituting c_0 into (4.1), we have

$$J_1(s_h,c_0) := \frac{4J_2(s_h)}{1-s_h},$$

where

$$J_2(s_h) = 3a\mu s_h^2 - a\mu s_h^3 - 6a^2 s_h^2 + 4a^2 s_h^3 - a^2 s_h^4 + 4a^2 s_h - 4a\mu s_h - a^2 + 2a\mu - \mu^2. \tag{4.2}$$

Taking the first and second derivatives of $J_2(s_h)$ with respect to s_h, respectively, we get

$$\frac{dJ_2(s_h)}{ds_h} = 6a\mu s_h - 3a\mu s_h^2 - 12a^2 s_h + 12a^2 s_h^2 - 4a^2 s_h^3 + 4a^2 - 4a\mu,$$

and

$$\frac{d^2 J_2(s_h)}{ds_h^2} = 6a(1-s_h)(\mu - 2a(1-s_h)) < 0,$$

which means that $J_2(s_h)$ is a concave function. Substituting s_h^* and s_h^{**} to (4.2), respectively, we have

$$J_2(s_h^*) = \frac{a\mu\sqrt{a\mu}(\sqrt{a}-\sqrt{\mu})^2}{a^2} > 0 \text{ and } J_2(s_h^{**}) = 0.$$

Therefore, $E_1(s_h,c)$ is strictly increasing with respect to c. Thus, for $c \in (c_0,c_1)$, we have

$$Q(s_h,c) - E_1(s_h,c) > Q(s_h,c_1) - E_1(s_h,c_1) = \frac{R(s_h)}{2a^2\xi s_h(1-(1-s_h)(1-\xi c_1))},$$

where

$$R(s_h) = -2a\mu s_h^2 - 2a^2 + 6a\sqrt{a\mu} + 2a^2 s_h - 6a s_h\sqrt{a\mu} - 2\mu s_h\sqrt{a\mu} - 6a\mu + 2\mu\sqrt{a\mu} + 2a s_h^2\sqrt{a\mu} + 6a\mu s_h.$$

To determine the sign of $R(s_h)$, we take the first and second derivatives of $R(s_h)$ with respect to s_h, respectively,

$$\frac{dR(s_h)}{ds_h} = -4a\mu s_h + 2a^2 - 6a\sqrt{a\mu} - 2\mu\sqrt{a\mu} + 4as_h\sqrt{a\mu} + 6a\mu, \tag{4.3}$$

and

$$\frac{d^2R(s_h)}{ds_h^2} = -4a\mu + 4a\sqrt{a\mu} = 4a(\sqrt{a\mu} - \mu) > 0,$$

which implies that $dR(s_h)/ds_h$ is strictly increasing. Substituting s_h^* into (4.3), we have

$$\frac{dR(s_h)}{ds_h}\,|_{s_h=s_h^*} = 2(a-\sqrt{a\mu})(a-\mu) > 0,$$

which shows that $dR(s_h)/ds_h > 0$ for $s_h \in [s_h^*, s_h^{**}]$. Thus, $Q(s_h)$ is increasing with respect to $s_h \in [s_h^*, s_h^{**}]$. Substituting s_h^* into $R(s_h)$, we have $R(s_h^*) = 0$, which shows that $R(s_h) > 0$ for $s_h \in [s_h^*, s_h^{**}]$. Furthermore, we have

$$Q(s_h, c) - E_1(s_h, c) > 0.$$

Hence, $h(u,T)$ has no fixed points for $u \in (0, E_1(s_h, c))$ and so there are no T-periodic solutions in $(0, E_1(s_h, c))$ for model (1.3)-(1.4). The proof for the uniqueness of fixed points of $h(u,T)$ in $(E_2(s_h, c), A)$ is similar to Lemma 3.1 in [10]. And thus, the uniqueness proof is complete. In conclusion, from Lemmas 2.4 and 4.1, we obtain

$$(u-\bar{v})(h(u,T)-u) < 0, \; u \in (0,\bar{v}) \cup (\bar{v}, +\infty).$$

Similar to the proof of stability and attraction of Theorem 1.1, we can show that T-periodic solution $w(t;0,\bar{v})$ is globally asymptotically stable, where $\bar{v} \in (E_2(s_h, c), A)$. The proof is finished. □

A numerical example listed here can be used to confirm the results in Theorem 1.2.

Example 4.1. *Given*

$$a = 2, \; \mu = 0.05, \; \xi = 0.001 \textit{ and } \overline{T} = 14,$$

we have $s_h^* \approx 0.8419$, $s_h^{**} \approx 0.9750$, $c_0 = 1000 - 25/(1-s_h)$ *and* $c_1 = (1025 - 100\sqrt{10})/s_h$. *Thus, we have*

$$\Lambda_{14} = \{(s_h, c) : s_h \in (0.8419, 1), c = c_1\},$$
$$\Lambda_{21} = \{(s_h, c) : s_h \in (0.8419, 0.9750), c \in (c_0, c_1)\},$$
$$\Lambda_{22} = \{(s_h, c) : s_h \in [0.9750, 1), c \in (0, c_1)\}.$$

We plot the solution curves of model (1.3)-(1.4) *when* $(s_h, c) \in \Lambda_{14}$, $(s_h, c) \in \Lambda_{21}$ *or* $(s_h, c) \in \Lambda_{22}$ *by selecting same* 18 *initial values, respectively. In Figure 3 (a), (b), and (c), it shows that there exists a unique* T*-periodic solution when* $(s_h, c) \in \Lambda_{14} \cup \Lambda_{21} \cup \Lambda_{22}$ *and* $T = T^*(s_h, c)$. *It is globally asymptotically stable from Theorem 1.2.*

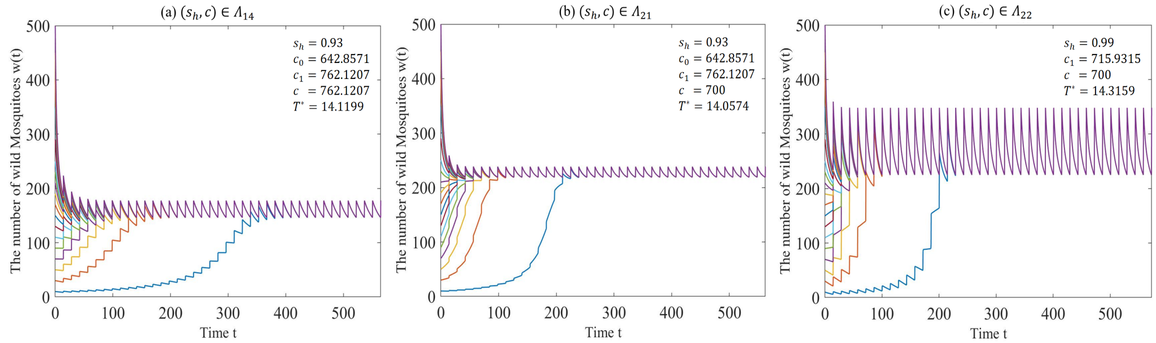

FIGURE 3. When $T = T^*(s_h, c)$, we plot the solution curves of model (1.3)-(1.4) with $(s_h, c) \in \Lambda_{14}$ in panel (a), $(s_h, c) \in \Lambda_{21}$ in panel (b) and $(s_h, c) \in \Lambda_{22}$ in panel (c).

5. Discussion

In this paper, we investigate an ordinary differential equation model with two sub-equations continuously switching between each other for wild mosquitoes population suppression by releasing *Wolbachia*-infected mosquitoes. A three-year field experiment [15] on two islands in Guangzhou confirms that releasing *Wolbachia*-infected mosquitoes is effective in eliminating more than 90% of the wild *Aedes albopictus* population. The suppression effect is mainly determined by three aspects, the CI intensity s_h, the release amount c and the waiting release period T between two consecutive releases. Different from the models studied in [2, 3, 6, 16] where complete CI is assumed. We focus on the effect of incomplete CI in current study. This consideration is necessary because numerous biological references show that incomplete CI is more common in practice, such as the *w*Ri in *Aedes albopictus* [8], the *cifB* in *Anopheles gambiae* [1] and the *w*Pip in *Culex pipiens* [7].

In the present study, we focus on the situation when the *Wolbachia*-infected mosquitoes are sexually active so that their death is negligible. When $T = \overline{T}$, according to the number of equilibria, two CI intensity thresholds s_h^*, s_h^{**} and three release amount thresholds c_0, c_1 and c^* divide the $s_h - c$ plane into three regions Λ_0, Λ_1 and Λ_2. Since releasing *Wolbachia*-infected mosquitoes requires a lot of manpower resources, we mainly consider the case $T > \overline{T}$ to control the manpower cost. According to the conclusions of Theorems 1.1 and 1.2 in this paper, Theorems 3.1 and 3.2 in [9] and Theorems 1.2 and 1.3 in [10], we know that the dynamical analysis of model (1.3)-(1.4) is quite complex. We give Table 1 to more intuitively reflect the existence and stability of the periodic solutions and the origin in each parameter region $\{(s_h, c, T) : \ s_h \in (0, 1), \ c \in (0, 1/\xi), \ T > \overline{T}\}$.

In a biological sense, the best we can hope for is that the origin E_0 is globally asymptotically stable, such as the parameter space (s_h, c, T) satisfying $\{(s_h, c) \in \Lambda_{01}, \ T \in (\overline{T}, T^*]\}$, $\{(s_h, c) \in \Lambda_{02}^{(1)}, \ T \in$

TABLE 1. The dynamic behaviors of wild mosquitoes under different release strategies of *Wolbachia*-infected mosquitoes.

(s_h, c)	$T \in (\overline{T}, +\infty)$	Num.T-ps	Stability	
			T-periodic solutions	The origin E_0
Λ_{01}	$T \in (\overline{T}, T^*]$	0	—	GAS
	$T \in (T^*, +\infty)$	1	GAS	US
$\Lambda_{02}^{(1)}$	$T \in (\overline{T}, T^*]$	0	—	GAS
	$T \in (T^*, +\infty)$	1	GAS	US
$\Lambda_{02}^{(2)}$	$T \in (\overline{T}, T^{**})$	0	—	GAS
	$T = T^{**}$	1	Semi-S	LAS
	$T \in (T^{**}, T^*)$	2	LAS_{larger}, $\text{US}_{smaller}$	LAS
	$T \in [T^*, +\infty)$	1	GAS	US
Λ_{11}	$T \in (\overline{T}, +\infty)$	1	GAS	US
Λ_{12}	$T \in (\overline{T}, +\infty)$	1	GAS	US
Λ_{13}	$T \in (\overline{T}, +\infty)$	1	GAS	US
Λ_{14}	$T \in (\overline{T}, T^*)$	2	LAS_{larger}, $\text{US}_{smaller}$	LAS
	$T \in [T^*, +\infty)$	1	GAS	US
Λ_{21}	$T \in (\overline{T}, T^*)$	2	LAS_{larger}, $\text{US}_{smaller}$	LAS
	$T \in [T^*, +\infty)$	1	GAS	US
Λ_{22}	$T \in (\overline{T}, T^*)$	2	LAS_{larger}, $\text{US}_{smaller}$	LAS
	$T \in [T^*, +\infty)$	1	GAS	US

*The following abbreviations are used: Num.T-ps: the number of T-periodic solutions, GAS: globally asymptotically stable, LAS: locally asymptotically stable, LAS_{larger}: the local asymptotically stable T-periodic solution with larger initial value, US: unstable, $\text{US}_{smaller}$: the unstable T-periodic solution with smaller initial value, Semi-S: Semi-stable.

$(\overline{T}, T^*]\}$ or $\{(s_h, c) \in \Lambda_{02}^{(2)},\ T \in (\overline{T}, T^{**}]\}$, which means that wild mosquitoes are total eliminated from the environment. Without exception, these three cases satisfy at least two of the following three conditions: stronger CI intensity, more release amount and more frequent releases. However, from the conclusions listed in Table 1, we can find that about 82.35% of the probability that the origin is unstable or locally asymptotically stable, together with the appearance of periodic solutions, which means that wild mosquitoes will most likely not be completely eliminated, but will remain at low levels. In fact, this situation is more common in nature because it is almost impossible to completely eliminate wild mosquitoes even with human intervention. Realistically, we can reduce the spread of mosquito-borne diseases by controlling mosquito populations to keep them below the epidemic risk threshold, which means that the periodic solutions need to lie below a certain threshold. Thus, to achieve such suppression, we can proceed in the following three aspects. First, the mosquito factory can screen *Wolbachia*-infected strains when producing *Wolbachia*-infected mosquitoes to improve the CI intensity. Second, to improve the probability of wild mosquitoes mating with *Wolbachia*-infected mosquitoes, we can evenly release as many *Wolbachia*-infected mosquitoes into the wild every time as the mosquito factory capacity allows. Finally, we can shorten the waiting period between two consecutive release of *Wolbachia*-infected mosquitoes to reduce offspring produced by wild females mating with wild males.

This paper, together with [9] and [10], gives a complete dynamical analysis of model (1.1) under the release strategy $T > \overline{T}$. We hope that our research can provide theoretical support for the real release of *Wolbachia*-infected mosquitoes into the wild to suppress wild mosquitoes.

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