Delta-shocks for the pressureless Euler equations with friction and time-dependent damping

Shiwei Li* Hui Wang

College of science, Henan University of Engineering, Zhengzhou, 451191, P. R. China

Abstract. This article is focused on the Riemann problem for the pressureless Euler equations with Coulomb-like friction and time-gradually-degenerate damping. With the introduction of a new variable, the considered equations are converted into the homogeneous form, whose Riemann problem is first solved with the characteristic method. Then by utilising relationship between variables, two types of solutions are obtained: vacuum and delta-shock solution. The generalized Rankine-Hugoniot relation and overcompressed entropy condition for the delta-shock are clarified. It is discovered that the presence of two external force terms causes the Riemann solutions to be non-self-similar. It is further proved that the non-self-similar Riemann solutions converge to the corresponding ones of pressureless Euler equations with friction as the time-dependent damping term vanishes, and the Riemann solutions converge to the corresponding self-similar solutions of pressureless Euler equations as the friction and damping terms vanish simultaneously. In the end, we apply the vanishing viscosity method to establish the stability of the non-self-similar solutions involving delta-shocks by introducing a time-dependent viscous system.

Keywords. Pressureless Euler equations; External force; Delta-shock; Vanishing viscosity method.

1. Introduction

Consider the pressureless Euler equations with friction and time-dependent damping

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2)_x = \alpha\rho - \dfrac{\mu}{1+t}\rho u, \end{cases} \tag{1.1}$$

where ρ is the gas density, u is the gas velocity, $\alpha\rho$ is the Coulomb-like friction term, α is a constant, $-\frac{\mu}{1+t}\rho u$ represents the time-gradually-degenerate damping, μ is a positive number to describe the scale of the damping. The equations (1.1) can be reduced from the balanced Euler equations [3, 15].

For $\alpha = \mu = 0$, the (1.1) is simplified as the homogeneous pressureless Euler system, which is capable of describing the motion of free particles sticking under collision [1, 36] and the formation of large-scale structures in the universe [25]. As the delta-shock solution appears in some cases, the homogeneous pressureless Euler system has attracted extensive attention ([2, 4, 16, 26]). The delta-shocks are an

* Corresponding author. E-mail address: lishiwei199102@163.com

8 Mar 2024 18:19:09 PST
231227-LiShiwei Version 2 - Submitted to Rocky Mountain J. Math.

important class of nonclassical waves for systems of conservation laws. The characteristic of the delta-shocks is that the delta functions appear in the state variables. Physically, they are capable of describing the concentration phenomenon. More research on the delta-shocks, please consult the papers [6, 7, 14, 17, 22–24, 27–30, 34, 35, 37, 38].

For the pressureless Euler equations with only one external force term, Shen [31] constructed two categories of non-self-similar Riemann solutions for the pressureless Euler equations with friction, i.e., (1.1) with $\mu = 0$. Zhang and He [40] obtained the exact Riemann solutions for the generalized pressureless Euler equations with dissipation. Zhang and Zhang [41] constructed two types of non-self-similar Riemann solutions for the equations of constant pressure fluid dynamics with nonlinear damping. For the pressureless Euler system with two source terms, the Riemann problem for the generalized pressureless Euler equations with a composite source term was investigated by Zhang, He and Ba [42]. Zhang and Zhang [43] discussed Riemann problem for Eulerian droplet model in consideration of the buoyancy and gravity forces.

Nevertheless, it is worth noting that many damping effects are usually time-dependent. For instance, a good example is $-\frac{\mu}{1+t}\rho u$ appearing in the governing equations [5, 12], which denotes the time-gradually-vanishing damping effect with physical parameter $\mu > 0$. These motivate us to discuss the Riemann problem, which is a fundamental problem associated with the systems of conservation laws, for the equations (1.1) with initial data

$$(\rho, u)(x, t = 0) = \begin{cases} (\rho_-, u_-), & x < 0, \\ (\rho_+, u_+), & x > 0. \end{cases} \tag{1.2}$$

The basic idea is to change the equations (1.1) into homogeneous equations through a suitable transformation. Then we solve the Riemann problem for the modified equations with the same Riemann initial data. Finally, the solutions for the original equations (1.1) with Riemann initial data (1.2) are constructed in terms of relationship between variables.

Performing a transformation $u(x,t) = \frac{\alpha}{1+\mu}(1+t) + \frac{1}{(1+t)^\mu}(v(x,t) - \frac{\alpha}{1+\mu})$, the (1.1) is rewritten as a homogeneous conservative form (2.1). The (2.1) is fully linearly degenerate such that the classical waves consist exclusively of contact discontinuities. With the aid of the characteristic method, the solution of (2.1)-(2.2) is constructed using two contact discontinuities and a vacuum when $u_- < u_+$. When $u_- > u_+$, we show that ρ and v_x will blow up simultaneously at a finite time, even when the initial data are smooth. Thereby, a solution involving delta-shock is constructed. For the delta-shock, we clarify the generalized Rankine-Hugoniot relation and over-compressive entropy condition, and exactly give the position, strength and propagation speed for the delta-shock. Then, by utilising the transformation of state variables $(\rho(x,t), u(x,t)) = (\rho(x,t), \frac{\alpha}{1+\mu}(1+t) + \frac{1}{(1+t)^\mu}(v(x,t) - \frac{\alpha}{1+\mu}))$, we construct two types of non-self-similar solutions including delta-shock and vacuum for (1.1)-(1.2).

To study the stability of the non-self-similar delta-shocks solutions for (1.1)-(1.2), we introduce the time-dependent viscous system

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2)_x = \alpha\rho - \dfrac{\mu}{1+t}\rho u + \varepsilon\Big(\dfrac{1}{(1+t)^\mu}\displaystyle\int_0^t \dfrac{1}{(1+s)^\mu}ds\Big)u_{xx}, \end{cases} \tag{1.3}$$

where $\int_0^t \frac{1}{(1+s)^\mu}ds = \ln(1+t)$ for $\mu = 1$, while $\int_0^t \frac{1}{(1+s)^\mu}ds = \frac{1}{1-\mu}((1+t)^{1-\mu} - 1)$ for $\mu \neq 1$. The idea for the (1.3) comes from the scalar conservation law with time-dependent viscosity

$$V_t + (F(V))_x = G(t)V_{xx}, \quad G(t) > 0.$$

Dafermos [8] proposed the hyperbolic systems of conservation laws with time-dependent viscosity $G(t) = \varepsilon t$, and investigated the Riemann solution of a class of hyperbolic systems of conservation laws using the viscosity method. Adopting this method, Tan, Zhang and Zheng [34] initially investigated the delta-shock for a triangular system of conservation laws. And since then, lots of researchers explored the stability of the solutions containing delta-shocks for various systems by making use of the method, see the papers [13, 18, 19, 32, 33, 39]. For nonlinear function $G(t)$, De la cruz [9] initially solved the Riemann problem for a 2×2 hyperbolic system with linear damping by the viscosity method. With the vanishing viscosity method, De la cruz and Juajibioy [10, 11] established the existence of Riemann solutions for a particular 2×2 system of conservation laws with linear damping and a generalized zero-pressure gas dynamics system with linear damping by introducing two time-dependent viscous systems. Following the method, Li [20, 21] studied the stability of the Riemann solutions involving the delta-shocks for two nonhomogeneous triangular systems of conservation laws.

According to the discussion in Section 2, we could observe that if the initial data belong to a bounded total variation space, the solutions to (2.1)-(2.2) have a similar structure. We, naturally, hope that these solutions are the limits of corresponding similar solutions of a viscous system as ε drops to zero. Under the transformation $u(x,t) = \frac{\alpha}{1+\mu}(1+t) + \frac{1}{(1+t)^\mu}(v(x,t) - \frac{\alpha}{1+\mu})$, the (1.3) is transformed into the following viscous system

$$\begin{cases} \rho_t + \Big(\rho\Big(\dfrac{\alpha}{1+\mu}(1+t) + \dfrac{1}{(1+t)^\mu}\Big(v - \dfrac{\alpha}{1+\mu}\Big)\Big)\Big)_x = 0, \\ (\rho v)_t + \Big(\rho v\Big(\dfrac{\alpha}{1+\mu}(1+t) + \dfrac{1}{(1+t)^\mu}\Big(v - \dfrac{\alpha}{1+\mu}\Big)\Big)\Big)_x = \varepsilon\Big(\dfrac{1}{(1+t)^\mu}\displaystyle\int_0^t \dfrac{1}{(1+s)^\mu}ds\Big)v_{xx}, \end{cases} \tag{1.4}$$

which can be considered as a viscous approximation of the modified homogeneous system (2.1). By seeking the solutions depending on the variable $\xi = \frac{x - \frac{\alpha((1+t)^2-1)}{2(1+\mu)} + \frac{\alpha}{1+\mu}\int_0^t \frac{1}{(1+s)^\mu}ds}{\int_0^t \frac{1}{(1+s)^\mu}ds}$ for the modified viscous system (1.4) with (2.2), we arrive at a two-point boundary value problem of high-order ordinary differential equations with the boundary value in the infinity, which has a weak solution $(\rho, v) \in L^1(-\infty,+\infty) \times C^2(-\infty,+\infty)$. It is proved that when $u_- > u_+$, in the limit $\varepsilon \to 0$, the solution of (1.4) with (2.2) converges weakly star to the solution containing delta-shock of (2.1)-(2.2), also see [11, 26] for more detail. As a consequence, the solution containing delta-shock to the (2.1)-(2.2) has stability under viscous perturbation. Additionally, it is intuitive to find that $(\rho(x,t), u(x,t)) = (\rho(x,t), \frac{\alpha}{1+\mu}(1+t) + \frac{1}{(1+t)^\mu}(v(x,t) - \frac{\alpha}{1+\mu}))$ is the solution of (1.3) with (1.2), provided that $(\rho(x,t), v(x,t))$ is the solution of (1.4) with (2.2). Accordingly, the solution involving delta-shock to (1.1)-(1.2) possesses also stability under viscous perturbation because ε and t are independent of each other.

The organization of the content is as follows. In Section 2, we convert the equations (1.1) to homogeneous form and solve the Riemann problem. In Section 3, we construct the solutions for (1.1)-(1.2). In Section 4, we analyze the limiting behavior of solutions for the viscous system (1.4) with (2.2) as $\varepsilon \to 0$, and establish the stability of the non-self-similar delta-shock solution to (1.1)-(1.2). In Section 5, we give the conclusion.

2. Riemann solutions to the homogeneous system (2.1)

This section solves the Riemann problem for the modified system (2.1). The equations (1.1) and the initial data (1.2) are converted to

$$\begin{cases} \rho_t + \Big(\rho\Big(\frac{\alpha}{1+\mu}(1+t) + \frac{1}{(1+t)^\mu}\Big(v - \frac{\alpha}{1+\mu}\Big)\Big)\Big)_x = 0, \\ (\rho v)_t + \Big(\rho v\Big(\frac{\alpha}{1+\mu}(1+t) + \frac{1}{(1+t)^\mu}\Big(v - \frac{\alpha}{1+\mu}\Big)\Big)\Big)_x = 0 \end{cases} \tag{2.1}$$

and

$$(\rho, v)(x, t=0) = \begin{cases} (\rho_-, u_-), & x < 0, \\ (\rho_+, u_+), & x > 0, \end{cases} \tag{2.2}$$

by introducing the transformation $u(x,t) = \frac{\alpha}{1+\mu}(1+t) + \frac{1}{(1+t)^\mu}(v(x,t) - \frac{\alpha}{1+\mu})$. A direct calculation shows that the (2.1) possesses a double characteristic root $\lambda = \frac{\alpha}{1+\mu}(1+t) + \frac{1}{(1+t)^\mu}(v - \frac{\alpha}{1+\mu})$. The only one right characteristic vector is found to be $\overrightarrow{r} = (1,0)^T$ satisfying $\nabla\lambda \cdot \overrightarrow{r} = 0$. As a consequence, the (2.1) is non-strictly hyperbolic and λ is linearly degenerate.

Suppose that $x = x(t)$ is a bounded discontinuity of solution of (2.1) and (ρ_l, v_l) and (ρ_r, v_r) are the left and right-side limits on the discontinuity of the solution $(\rho, v)(x,t)$. Then the Rankine-Hugoniot relation

$$\begin{cases} -x'(t)[\rho] + \Big[\rho\Big(\frac{\alpha}{1+\mu}(1+t) + \frac{1}{(1+t)^\mu}\Big(v - \frac{\alpha}{1+\mu}\Big)\Big)\Big] = 0, \\ -x'(t)[\rho v] + \Big[\rho v\Big(\frac{\alpha}{1+\mu}(1+t) + \frac{1}{(1+t)^\mu}\Big(v - \frac{\alpha}{1+\mu}\Big)\Big)\Big] = 0, \end{cases} \tag{2.3}$$

holds, where $[\rho] = \rho_l - \rho_r$ with $\rho_l = \rho(x(t) - 0, t)$ and $\rho_r = \rho(x(t) + 0, t)$.

Solving (2.3) yields the contact discontinuity

$$J: \quad x'(t) = \frac{\alpha}{1+\mu}(1+t) + \frac{1}{(1+t)^\mu}\Big(v_l - \frac{\alpha}{1+\mu}\Big) = \frac{\alpha}{1+\mu}(1+t) + \frac{1}{(1+t)^\mu}\Big(v_r - \frac{\alpha}{1+\mu}\Big). \tag{2.4}$$

Clearly, when $v_l = v_r$, we can connect (ρ_l, v_l) and (ρ_r, v_r) with the use of a contact discontinuity J.

We now construct the solutions for (2.1)-(2.2). The cases are divided into two types: $u_- < u_+$ and $u_- > u_+$. When $u_- < u_+$, the Riemann solution consisting of two contact discontinuities and a vacuum state (denoted by Vac)

$$(\rho, v)(x,t) = \begin{cases} (\rho_-, u_-), & x < x_1(t), \\ \text{Vac}, & x_1(t) < x < x_2(t), \\ (\rho_+, u_+), & x > x_2(t), \end{cases} \tag{2.5}$$

is constructed, in which

$$x_1(t) = \begin{cases} \frac{\alpha}{4}(1+t)^2 + \big(u_- - \frac{\alpha}{2}\big)\ln(1+t) - \frac{\alpha}{4}, & \text{for} \quad \mu = 1, \\ \frac{\alpha}{2(1+\mu)}((1+t)^2 - 1) + \Big(\frac{u_-}{1-\mu} - \frac{\alpha}{1-\mu^2}\Big)((1+t)^{1-\mu} - 1), & \text{for} \quad \mu \neq 1, \end{cases} \tag{2.6}$$

and

$$x_2(t) = \begin{cases} \frac{\alpha}{4}(1+t)^2 + \big(u_+ - \frac{\alpha}{2}\big)\ln(1+t) - \frac{\alpha}{4}, & \text{for} \quad \mu = 1, \\ \frac{\alpha}{2(1+\mu)}((1+t)^2 - 1) + \Big(\frac{u_+}{1-\mu} - \frac{\alpha}{1-\mu^2}\Big)((1+t)^{1-\mu} - 1), & \text{for} \quad \mu \neq 1. \end{cases} \tag{2.7}$$

8 Mar 2024 18:19:09 PST
231227-LiShiwei Version 2 - Submitted to Rocky Mountain J. Math.

Whereas, we cannot construct the solution by using the contact discontinuities for the case $u_- > u_+$. At this point, the singularity of solutions must happen in domain $\Theta = \{(x,t)|x_2(t) \leq x(t) \leq x_1(t), t \in R^+\}$, by virtue of the fact that the characteristic lines from the initial data may cause overlap in the Θ. This means that there is no solution in bounded variation space. Next we show that ρ and v_x will blow up simultaneously at a finite time, even when the initial data are smooth.

Discuss (2.1) subject to the smooth initial value $(\rho(x,0), v(x,0)) = (\rho_0(x), v_0(x))$ satisfying $v_0'(x) < 0$. The characteristic equations associated to the system (2.1) are

$$\frac{dx}{dt} = \frac{\alpha}{1+\mu}(1+t) + \frac{1}{(1+t)^\mu}\Big(v - \frac{\alpha}{1+\mu}\Big), \qquad \frac{dv}{dt} = 0, \qquad \frac{d\rho}{dt} = -\frac{\rho}{(1+t)^\mu} v_x. \tag{2.8}$$

And so the characteristic curve, which passes through any given point $(0,a)$ on the x-axis, can be given as

$$x = \begin{cases} a + \dfrac{\alpha}{4}((1+t)^2 - 1) + \Big(v_0(a) - \dfrac{\alpha}{2}\Big)\ln(1+t), & \mu = 1, \\[2ex] a + \dfrac{\alpha}{2(1+\mu)}((1+t)^2 - 1) + \Big(\dfrac{v_0(a)}{1-\mu} - \dfrac{\alpha}{1-\mu^2}\Big)((1+t)^{1-\mu} - 1), & \mu \neq 1, \end{cases} \tag{2.9}$$

on which v takes the constant value $v_0(a)$. Together with (2.1) and (2.8)-(2.9), we arrive at

$$v_x = \begin{cases} \dfrac{v_0'(a)}{1 + \ln(1+t)v_0'(a)}, & \mu = 1, \\[2ex] \dfrac{v_0'(a)}{1 + \frac{v_0'(a)}{1-\mu}((1+t)^{1-\mu} - 1)}, & \mu \neq 1, \end{cases} \qquad \rho = \begin{cases} \dfrac{\rho_0(a)}{1 + \ln(1+t)v_0'(a)}, & \mu = 1, \\[2ex] \dfrac{\rho_0(a)}{1 + \frac{v_0'(a)}{1-\mu}((1+t)^{1-\mu} - 1)}, & \mu \neq 1. \end{cases} \tag{2.10}$$

Due to $v_0'(a) < 0$ (if $\mu > 1$, we further assume $v_0'(x) + (\mu - 1) < 0$), we have

$$\lim_{t \to \left(\exp^{-\frac{1}{v_0'(a)}} - 1\right)} (\rho, v_x) = (\infty, \infty) \text{ for } \mu = 1, \qquad \lim_{t \to \left(\left(\frac{\mu - 1 + v_0'(a)}{v_0'(a)}\right)^{\frac{1}{1-\mu}} - 1\right)} (\rho, v_x) = (\infty, \infty) \text{ for } \mu \neq 1. \tag{2.11}$$

The formula (2.11) shows that ρ and v_x must blow up simultaneously at a finite time.

In this situation, motivated by [26, 34], the delta-shock solution should be kept in mind. To achieve this, the definitions of two-dimensional weighted delta function and delta-shock solution are introduced.

Definition 2.1. A two-dimensional weighted delta function $\omega(s)\delta_S$ supported on a smooth curve $S = \{(x(s), t(s)) : c \leq s \leq d\}$ is defined by

$$\big\langle \omega(s)\delta_S, \phi(x,t) \big\rangle = \int_c^d \omega(s)\phi(x(s), t(s))ds \tag{2.12}$$

for all test functions $\phi \in C_0^\infty(R \times R^+)$.

Definition 2.2. A pair (ρ, v) is recognized as a delta-shock type solution to the (2.1) in the sense of distributions if there exist a smooth curve S and a weight $\omega \in C^1(S)$ such that ρ and v are represented in the form:

$$\rho(x,t) = \rho_0(x,t) + \omega(t)\delta_S, \quad v(x,t) = v_0(x,t), \quad v(x,t)|_S = v_\delta(t), \tag{2.13}$$

where $\rho_0(x,t) = \rho_l(x,t) - (\rho_l(x,t) - \rho_r(x,t))H(x - x(t))$, $v_0(x,t) = v_l(x,t) - (v_l(x,t) - v_r(x,t))H(x - x(t))$, in which $(\rho_l, v_l)(x,t)$ and $(\rho_r, v_r)(x,t)$ are piecewise smooth solutions to the (2.1), $H(x)$ is the Heaviside function whose value is zero for negative argument and one for positive argument, and the formula

$$\begin{cases} \langle \rho, \phi_t \rangle + \Big\langle \rho\Big(\dfrac{\alpha}{1+\mu}(1+t) + \dfrac{1}{(1+t)^\mu}\Big(v - \dfrac{\alpha}{1+\mu}\Big)\Big), \phi_x \Big\rangle = 0, \\[2ex] \langle \rho v, \phi_t \rangle + \Big\langle \rho v\Big(\dfrac{\alpha}{1+\mu}(1+t) + \dfrac{1}{(1+t)^\mu}\Big(v - \dfrac{\alpha}{1+\mu}\Big)\Big), \phi_x \Big\rangle = 0 \end{cases} \tag{2.14}$$

8 Mar 2024 18:19:09 PST
231227-LiShiwei Version 2 - Submitted to Rocky Mountain J. Math.

holds for every test function $\phi \in C_0^\infty(R \times R^+)$, in which $\langle \rho v, \phi \rangle = \int\int_{R\times R^+} \rho_0 v_0 \phi dxdt + \langle \omega(t)v_\delta(t)\delta_S, \phi \rangle$.

The delta-shock solution to the system (2.1) is, by definitions, written in the following form

$$(\rho, v)(x,t) = \begin{cases} (\rho_l, v_l)(x,t), & x < x(t), \\ (\omega(t)\delta(x - x(t)), v_\delta(t)), & x = x(t), \\ (\rho_r, v_r)(x,t), & x > x(t). \end{cases} \tag{2.15}$$

If a pair (ρ, v) of the form (2.15) is a solution to the system (2.1) in the sense of distributions, then the generalized Rankine-Hugoniot relation

$$\begin{cases} \dfrac{dx(t)}{dt} = \sigma(t) = \dfrac{\alpha}{1+\mu}(1+t) + \dfrac{1}{(1+t)^\mu}\Big(v_\delta(t) - \dfrac{\alpha}{1+\mu}\Big), \\ \dfrac{d\omega(t)}{dt} = -[\rho]\sigma(t) + \Big[\rho\Big(\dfrac{\alpha}{1+\mu}(1+t) + \dfrac{1}{(1+t)^\mu}\Big(v - \dfrac{\alpha}{1+\mu}\Big)\Big)\Big], \\ \dfrac{d(\omega(t)v_\delta(t))}{dt} = -[\rho v]\sigma(t) + \Big[\rho v\Big(\dfrac{\alpha}{1+\mu}(1+t) + \dfrac{1}{(1+t)^\mu}\Big(v - \dfrac{\alpha}{1+\mu}\Big)\Big)\Big] \end{cases} \tag{2.16}$$

holds. As the process of proof is analogous to that in [26, 31], we omit it.

The exact relationship among the location, weight and propagation speed of the discontinuity are reflected by the generalized Rankine-Hugoniot relation (2.16). Apart from that, the entropy condition

$$\lambda(\rho_r, v_r) < \frac{dx(t)}{dt} < \lambda(\rho_l, v_l) \tag{2.17}$$

should be added to guarantee the uniqueness of such a discontinuity. The inequation (2.17) means that all characteristics on two sides of the discontinuity are incoming. A discontinuity is known as a delta-shock (symbolized by δ), provided that it satisfies generalized Rankine-Hugoniot relation (2.16) and entropy condition (2.17).

The section to follow is to solve the Riemann problem (2.1)-(2.2) for the case $u_- > u_+$. In such a case, the Riemann solution is a delta-shock of the form, besides two constant states,

$$(\rho, v)(x,t) = \begin{cases} (\rho_-, u_-), & x < x(t), \\ (\omega(t)\delta(x - x(t)), v_\delta(t)), & x = x(t), \\ (\rho_+, u_+), & x > x(t). \end{cases} \tag{2.18}$$

The next thing to be done is to calculate $x(t)$, $\omega(t)$ and $v_\delta(t)$ by solving the generalized Rankine-Hugoniot relation (2.16) with initial condition

$$t = 0: \ x(0) = 0, \ \omega(0) = 0 \tag{2.19}$$

under the entropy condition

$$u_+ < v_\delta(t) < u_-. \tag{2.20}$$

For simplicity sake, denote $[\bar\rho] = \rho_- - \rho_+$, $[\bar\rho\bar u] = \rho_- u_- - \rho_+ u_+$, $[\bar\rho\bar u^2] = \rho_- u_-^2 - \rho_+ u_+^2$. Integrating (2.16) yields

$$\begin{cases} \omega(t) = -[\bar\rho]\Big(x(t) - \displaystyle\int_0^t \Big(\frac{\alpha(1+s)}{1+\mu} - \frac{\alpha}{(1+\mu)(1+s)^\mu}\Big)ds\Big) + [\bar\rho\bar u]\int_0^t \frac{1}{(1+s)^\mu}ds, \\ \omega(t)v_\delta(t) = -[\bar\rho\bar u]\Big(x(t) - \displaystyle\int_0^t \Big(\frac{\alpha(1+s)}{1+\mu} - \frac{\alpha}{(1+\mu)(1+s)^\mu}\Big)ds\Big) + [\bar\rho\bar u^2]\int_0^t \frac{1}{(1+s)^\mu}ds. \end{cases} \tag{2.21}$$

8 Mar 2024 18:19:09 PST
231227-LiShiwei Version 2 - Submitted to Rocky Mountain J. Math.

Set

$$Q(t)=x(t)-\int_0^t\Big(\frac{\alpha(1+s)}{1+\mu}-\frac{\alpha}{(1+\mu)(1+s)^\mu}\Big)ds.$$

Using (2.21), we get

$$[\bar\rho]Q(t)v_\delta(t)-[\bar\rho\bar u]\Big(Q(t)+v_\delta(t)\int_0^t\frac{1}{(1+s)^\mu}ds\Big)+[\bar\rho\bar u^2]\int_0^t\frac{1}{(1+s)^\mu}ds=0, \tag{2.22}$$

which means that

$$[\bar\rho]\Big(\frac{Q(t)}{\int_0^t\frac{1}{(1+s)^\mu}ds}\Big)^2-2[\bar\rho\bar u]\frac{Q(t)}{\int_0^t\frac{1}{(1+s)^\mu}ds}+[\bar\rho\bar u^2]=0. \tag{2.23}$$

The equation (2.23) implies that the $v_\delta(t)$ is a constant. If $\rho_-\neq\rho_+$, by virtue of the discriminant $\Delta=4\rho_-\rho_+(u_--u_+)^2>0$, we calculate

$$\frac{Q(t)}{\int_0^t\frac{1}{(1+s)^\mu}ds}=v_\delta=\frac{\sqrt{\rho_-}u_-\pm\sqrt{\rho_+}u_+}{\sqrt{\rho_-}\pm\sqrt{\rho_+}}. \tag{2.24}$$

In consideration of the entropy condition (2.20),

$$v_\delta=\frac{\sqrt{\rho_-}u_-+\sqrt{\rho_+}u_+}{\sqrt{\rho_-}+\sqrt{\rho_+}} \tag{2.25}$$

is picked out. As a result, it follows from (2.16) that

$$x(t)=\begin{cases}\frac{\alpha}{4}((1+t)^2-1)+\Big(\frac{\sqrt{\rho_-}u_-+\sqrt{\rho_+}u_+}{\sqrt{\rho_-}+\sqrt{\rho_+}}-\frac{\alpha}{2}\Big)\ln(1+t), & \mu=1,\\ \frac{\alpha((1+t)^2-1)}{2(1+\mu)}+\Big(\frac{\sqrt{\rho_-}u_-+\sqrt{\rho_+}u_+}{\sqrt{\rho_-}+\sqrt{\rho_+}}-\frac{\alpha}{1+\mu}\Big)\frac{(1+t)^{1-\mu}-1}{1-\mu}, & \mu\neq1\end{cases} \tag{2.26}$$

and

$$\omega(t)=\begin{cases}\sqrt{\rho_-\rho_+}(u_--u_+)\ln(1+t), & \mu=1,\\ \sqrt{\rho_-\rho_+}(u_--u_+)\frac{(1+t)^{1-\mu}-1}{1-\mu}, & \mu\neq1.\end{cases} \tag{2.27}$$

If $\rho_-=\rho_+$, we solve (2.23) to obtain

$$\frac{Q(t)}{\int_0^t\frac{1}{(1+s)^\mu}ds}=v_\delta=\frac{u_-+u_+}{2}. \tag{2.28}$$

Then it follows from (2.16) that

$$\begin{cases}x(t)=\begin{cases}\frac{\alpha}{4}((1+t)^2-1)+\Big(\frac{u_-+u_+}{2}-\frac{\alpha}{2}\Big)\ln(1+t), & \mu=1,\\ \frac{\alpha((1+t)^2-1)}{2(1+\mu)}+\Big(\frac{u_-+u_+}{2}-\frac{\alpha}{1+\mu}\Big)\frac{(1+t)^{1-\mu}-1}{1-\mu}, & \mu\neq1,\end{cases}\\ \omega(t)=\begin{cases}\rho_-(u_--u_+)\ln(1+t), & \mu=1,\\ \rho_-(u_--u_+)\frac{(1+t)^{1-\mu}-1}{1-\mu}, & \mu\neq1.\end{cases}\end{cases} \tag{2.29}$$

The result can now be summarised as follows:

Theorem 2.3. If $u_->u_+$, then Riemann problem (2.1)-(2.2) admits a solution including delta-shock, which can be expressed as (2.18), where $v_\delta(t)=v_\delta$, $x(t)$ and $\omega(t)$ are shown in (2.25)-(2.27).

8 Mar 2024 18:19:09 PST
231227-LiShiwei Version 2 - Submitted to Rocky Mountain J. Math.

3. Construction of solutions to (1.1) with (1.2)

In this section, on the basis of the transformation of state variables

$$(\rho(x,t),u(x,t))=\Big(\rho(x,t),\frac{\alpha}{1+\mu}(1+t)+\frac{1}{(1+t)^{\mu}}\Big(v(x,t)-\frac{\alpha}{1+\mu}\Big)\Big),$$

we construct the solutions for the original equations (1.1) with initial data (1.2).

When $u_-<u_+$, the Riemann solution of (1.1)-(1.2) involves vacuum, which can be written as

$$(\rho,u)(x,t)=\begin{cases}\Big(\rho_-,\dfrac{\alpha}{1+\mu}(1+t)+\dfrac{1}{(1+t)^{\mu}}\Big(u_--\dfrac{\alpha}{1+\mu}\Big)\Big), & x<x_1(t),\\ \text{Vac}, & x_1(t)<x<x_2(t),\\ \Big(\rho_+,\dfrac{\alpha}{1+\mu}(1+t)+\dfrac{1}{(1+t)^{\mu}}\Big(u_+-\dfrac{\alpha}{1+\mu}\Big)\Big), & x>x_2(t),\end{cases} \tag{3.1}$$

where $x_1(t)$ and $x_2(t)$ are given by (2.6) and (2.7), respectively.

Definition 3.1. A pair (ρ,u) is referred to as a delta-shock solution to the equations (1.1) in the sense of distributions if there exist a smooth curve S and a weight $\omega\in C^1(S)$ such that ρ and u are represented in the form:

$$\rho(x,t)=\rho_0(x,t)+\omega(t)\delta_S,\quad u(x,t)=u_0(x,t),\quad u(x,t)|_S=u_\delta(t), \tag{3.2}$$

where $\rho_0(x,t)=\rho_l(x,t)-(\rho_l(x,t)-\rho_r(x,t))H(x-x(t))$, $u_0(x,t)=u_l(x,t)-(u_l(x,t)-u_r(x,t))H(x-x(t))$, the Heaviside function $H(x)$ is given by Definition 2.2, in which $(\rho_l,u_l)(x,t)$ and $(\rho_r,u_r)(x,t)$ are piecewise smooth solutions to the equations (1.1), and it satisfies

$$\langle\rho,\phi_t\rangle+\langle\rho u,\phi_x\rangle=0,\qquad \langle\rho u,\phi_t\rangle+\langle\rho u^2,\phi_x\rangle=-\Big\langle\alpha\rho-\frac{\mu}{1+t}\rho u,\phi\Big\rangle \tag{3.3}$$

for every test function $\phi\in C_0^\infty(R\times R^+)$, where $\langle\rho u,\phi\rangle=\int\int_{R\times R^+}\rho_0u_0\phi dxdt+\langle\omega(t)u_\delta(t)\delta_S,\phi\rangle$.

According to the above definition, if $u_->u_+$, we construct the following delta-shock solution for (1.1)-(1.2)

$$(\rho,u)(x,t)=\begin{cases}\Big(\rho_-,\dfrac{\alpha}{1+\mu}(1+t)+\dfrac{1}{(1+t)^{\mu}}\Big(u_--\dfrac{\alpha}{1+\mu}\Big)\Big), & x<x(t),\\ (\omega(t)\delta(x-x(t)),u_\delta(t)), & x=x(t),\\ \Big(\rho_+,\dfrac{\alpha}{1+\mu}(1+t)+\dfrac{1}{(1+t)^{\mu}}\Big(u_+-\dfrac{\alpha}{1+\mu}\Big)\Big), & x>x(t),\end{cases} \tag{3.4}$$

satisfying the generalized Rankine-Hugoniot relation

$$\begin{cases}\dfrac{dx(t)}{dt}=u_\delta(t),\\ \dfrac{d\omega(t)}{dt}=-[\rho]u_\delta(t)+[\rho u],\\ \dfrac{d(\omega(t)u_\delta(t))}{dt}=-[\rho u]u_\delta(t)+[\rho u^2]+\alpha\omega(t)-\dfrac{\mu}{1+t}\omega(t)u_\delta(t),\end{cases} \tag{3.5}$$

where the jump across the discontinuity is

$$[\rho u]=\rho_-\Big(\frac{\alpha}{1+\mu}(1+t)+\frac{1}{(1+t)^{\mu}}\Big(u_--\frac{\alpha}{1+\mu}\Big)\Big)-\rho_+\Big(\frac{\alpha}{1+\mu}(1+t)+\frac{1}{(1+t)^{\mu}}\Big(u_+-\frac{\alpha}{1+\mu}\Big)\Big).$$

Beyond that, the over-compressive entropy condition for the delta-shock

$$\frac{\alpha}{1+\mu}(1+t)+\frac{1}{(1+t)^{\mu}}\Big(u_{+}-\frac{\alpha}{1+\mu}\Big)<u_{\delta}(t)<\frac{\alpha}{1+\mu}(1+t)+\frac{1}{(1+t)^{\mu}}\Big(u_{-}-\frac{\alpha}{1+\mu}\Big) \tag{3.6}$$

should be imposed so as to ensure the uniqueness.

Theorem 3.2. When $u_{-}>u_{+}$, the Riemann problem (1.1)-(1.2) possesses a delta-shock solution which could be formulated as the formula (3.4), where

$$\begin{cases} u_{\delta}(t)=\dfrac{\alpha}{1+\mu}(1+t)+\dfrac{1}{(1+t)^{\mu}}\Big(\dfrac{\sqrt{\rho_{-}}u_{-}+\sqrt{\rho_{+}}u_{+}}{\sqrt{\rho_{-}}+\sqrt{\rho_{+}}}-\dfrac{\alpha}{1+\mu}\Big),\\ x(t)=\begin{cases}\dfrac{\alpha}{4}((1+t)^{2}-1)+\Big(\dfrac{\sqrt{\rho_{-}}u_{-}+\sqrt{\rho_{+}}u_{+}}{\sqrt{\rho_{-}}+\sqrt{\rho_{+}}}-\dfrac{\alpha}{2}\Big)\ln(1+t), & \mu=1,\\ \dfrac{\alpha((1+t)^{2}-1)}{2(1+\mu)}+\Big(\dfrac{\sqrt{\rho_{-}}u_{-}+\sqrt{\rho_{+}}u_{+}}{\sqrt{\rho_{-}}+\sqrt{\rho_{+}}}-\dfrac{\alpha}{1+\mu}\Big)\dfrac{(1+t)^{1-\mu}-1}{1-\mu}, & \mu\neq 1,\end{cases}\\ \omega(t)=\begin{cases}\sqrt{\rho_{-}\rho_{+}}(u_{-}-u_{+})\ln(1+t), & \mu=1,\\ \sqrt{\rho_{-}\rho_{+}}(u_{-}-u_{+})\dfrac{(1+t)^{1-\mu}-1}{1-\mu}, & \mu\neq 1.\end{cases}\end{cases} \tag{3.7}$$

Proof. The second equation in (3.5) can be simplified as

$$\frac{d\omega(t)}{dt}=-\Big(u_{\delta}(t)-\Big(\frac{\alpha(1+t)}{1+\mu}-\frac{\alpha}{(1+\mu)(1+t)^{\mu}}\Big)\Big)(\rho_{-}-\rho_{+})+(\rho_{-}u_{-}-\rho_{+}u_{+})\frac{1}{(1+t)^{\mu}}. \tag{3.8}$$

The third equality of (3.5) is reduced to

$$\begin{aligned}\frac{d\omega(t)}{dt}u_{\delta}(t)=&-u_{\delta}(t)\Big(\Big(\frac{\alpha(1+t)}{1+\mu}-\frac{\alpha}{(1+\mu)(1+t)^{\mu}}\Big)(\rho_{-}-\rho_{+})+(\rho_{-}u_{-}-\rho_{+}u_{+})\frac{1}{(1+t)^{\mu}}\Big)\\ &+2(\rho_{-}u_{-}-\rho_{+}u_{+})\Big(\frac{\alpha(1+t)}{1+\mu}-\frac{\alpha}{(1+\mu)(1+t)^{\mu}}\Big)\frac{1}{(1+t)^{\mu}}+(\rho_{-}u_{-}^{2}-\rho_{+}u_{+}^{2})\frac{1}{(1+t)^{2\mu}}\\ &+(\rho_{-}-\rho_{+})\Big(\frac{\alpha(1+t)}{1+\mu}-\frac{\alpha}{(1+\mu)(1+t)^{\mu}}\Big)^{2},\end{aligned} \tag{3.9}$$

where we have used the fact $(u_{\delta}(t)-(\frac{\alpha(1+t)}{1+\mu}-\frac{\alpha}{(1+\mu)(1+t)^{\mu}}))(1+t)^{\mu}=$ constant. We plug the equation (3.8) into the equation (3.9) to obtain

$$\begin{aligned}&(\rho_{-}-\rho_{+})\Big(\Big(u_{\delta}(t)-\Big(\frac{\alpha(1+t)}{1+\mu}-\frac{\alpha}{(1+\mu)(1+t)^{\mu}}\Big)\Big)(1+t)^{\mu}\Big)^{2}\\ &-2(\rho_{-}u_{-}-\rho_{+}u_{+})\Big(u_{\delta}(t)-\Big(\frac{\alpha(1+t)}{1+\mu}-\frac{\alpha}{(1+\mu)(1+t)^{\mu}}\Big)\Big)(1+t)^{\mu}+(\rho_{-}u_{-}^{2}-\rho_{+}u_{+}^{2})=0.\end{aligned} \tag{3.10}$$

Accordingly, $u_{\delta}(t)=\frac{\alpha}{1+\mu}(1+t)+\frac{1}{(1+t)^{\mu}}(v_{\delta}-\frac{\alpha}{1+\mu})$ is obtained by virtue of the entropy condition (3.6). Then it follows from (3.5) that (3.7). This completes the proof.

And eventually what we need to do is to show the delta-shock solution of (1.1)-(1.2) satisfying (3.3). Let P be any point on the delta-shock curve $S:(x,t)\mid x=x(t)$ and let Ω be a small ball centered at the point P. Then we assume that the intersection points of Ω and S are $P_1=(x(t_1),t_1)$ and $P_2=(x(t_2),t_2)$, where $t_1<t_2$, Ω_{-} and Ω_{+} are the left-hand part and right-hand part of Ω cut by S respectively. With

the use of Green's formula, we calculate

$$
\begin{aligned}
&\int\int_{\Omega}(\rho u\phi_t+\rho u^2\phi_x)dxdt\\
&=\int\int_{\Omega_-}\Big(\rho_-\Big(\frac{\alpha(1+t)}{1+\mu}+\frac{1}{(1+t)^{\mu}}\Big(u_--\frac{\alpha}{1+\mu}\Big)\Big)\phi_t+\rho_-\Big(\frac{\alpha(1+t)}{1+\mu}+\frac{1}{(1+t)^{\mu}}\Big(u_--\frac{\alpha}{1+\mu}\Big)\Big)^2\phi_x\Big)dxdt\\
&+\int\int_{\Omega_+}\Big(\rho_+\Big(\frac{\alpha(1+t)}{1+\mu}+\frac{1}{(1+t)^{\mu}}\Big(u_+-\frac{\alpha}{1+\mu}\Big)\Big)\phi_t+\rho_+\Big(\frac{\alpha(1+t)}{1+\mu}+\frac{1}{(1+t)^{\mu}}\Big(u_+-\frac{\alpha}{1+\mu}\Big)\Big)^2\phi_x\Big)dxdt\\
&+\int_{t_1}^{t_2}\omega(t)u_\delta(t)(\phi_t+u_\delta(t)\phi_x)dt\\
&=\int\int_{\Omega_-}\Big(\rho_-\Big(\frac{\alpha(1+t)}{1+\mu}+\frac{1}{(1+t)^{\mu}}\Big(u_--\frac{\alpha}{1+\mu}\Big)\Big)\phi\Big)_t dxdt\\
&+\int\int_{\Omega_+}\Big(\rho_+\Big(\frac{\alpha(1+t)}{1+\mu}+\frac{1}{(1+t)^{\mu}}\Big(u_+-\frac{\alpha}{1+\mu}\Big)\Big)\phi\Big)_t dxdt+\int_{t_1}^{t_2}\omega(t)u_\delta(t)d\phi\\
&+\int\int_{\Omega_-}\Big(\rho_-\Big(\frac{\alpha(1+t)}{1+\mu}+\frac{1}{(1+t)^{\mu}}\Big(u_--\frac{\alpha}{1+\mu}\Big)\Big)^2\phi\Big)_x dxdt\\
&+\int\int_{\Omega_+}\Big(\rho_+\Big(\frac{\alpha(1+t)}{1+\mu}+\frac{1}{(1+t)^{\mu}}\Big(u_+-\frac{\alpha}{1+\mu}\Big)\Big)^2\phi\Big)_x dxdt\\
&-\int\int_{\Omega_-}\Big(\alpha\rho_--\frac{\mu}{1+t}\rho_-\Big(\frac{\alpha(1+t)}{1+\mu}+\frac{1}{(1+t)^{\mu}}\Big(u_--\frac{\alpha}{1+\mu}\Big)\Big)\Big)\phi dxdt\\
&-\int\int_{\Omega_+}\Big(\alpha\rho_+-\frac{\mu}{1+t}\rho_+\Big(\frac{\alpha(1+t)}{1+\mu}+\frac{1}{(1+t)^{\mu}}\Big(u_+-\frac{\alpha}{1+\mu}\Big)\Big)\Big)\phi dxdt\\
&=\oint-\rho_-\Big(\frac{\alpha(1+t)}{1+\mu}+\frac{1}{(1+t)^{\mu}}\Big(u_--\frac{\alpha}{1+\mu}\Big)\Big)\phi dx+\rho_-\Big(\frac{\alpha(1+t)}{1+\mu}+\frac{1}{(1+t)^{\mu}}\Big(u_--\frac{\alpha}{1+\mu}\Big)\Big)^2\phi dt\\
&+\oint-\rho_+\Big(\frac{\alpha(1+t)}{1+\mu}+\frac{1}{(1+t)^{\mu}}\Big(u_+-\frac{\alpha}{1+\mu}\Big)\Big)\phi dx+\rho_+\Big(\frac{\alpha(1+t)}{1+\mu}+\frac{1}{(1+t)^{\mu}}\Big(u_+-\frac{\alpha}{1+\mu}\Big)\Big)^2\phi dt\\
&-\int_{t_1}^{t_2}\frac{d(\omega(t)u_\delta(t))}{dt}\phi dt-\int\int_{\Omega_-}\Big(\alpha\rho_--\frac{\mu}{1+t}\rho_-\Big(\frac{\alpha(1+t)}{1+\mu}+\frac{1}{(1+t)^{\mu}}\Big(u_--\frac{\alpha}{1+\mu}\Big)\Big)\Big)\phi dxdt\\
&-\int\int_{\Omega_+}\Big(\alpha\rho_+-\frac{\mu}{1+t}\rho_+\Big(\frac{\alpha(1+t)}{1+\mu}+\frac{1}{(1+t)^{\mu}}\Big(u_+-\frac{\alpha}{1+\mu}\Big)\Big)\Big)\phi dxdt\\
&=\int_{t_1}^{t_2}\Big(-[\rho u]u_\delta(t)+[\rho u^2]-\frac{d(\omega(t)u_\delta(t))}{dt}\Big)\phi dt\\
&-\int\int_{\Omega_-}\Big(\alpha\rho_--\frac{\mu}{1+t}\rho_-\Big(\frac{\alpha(1+t)}{1+\mu}+\frac{1}{(1+t)^{\mu}}\Big(u_--\frac{\alpha}{1+\mu}\Big)\Big)\Big)\phi dxdt\\
&-\int\int_{\Omega_+}\Big(\alpha\rho_+-\frac{\mu}{1+t}\rho_+\Big(\frac{\alpha(1+t)}{1+\mu}+\frac{1}{(1+t)^{\mu}}\Big(u_+-\frac{\alpha}{1+\mu}\Big)\Big)\Big)\phi dxdt\\
&=-\int_{t_1}^{t_2}\Big(\alpha\omega(t)-\frac{\mu}{1+t}\omega(t)u_\delta(t)\Big)\phi dt-\int\int_{\Omega_-}\Big(\alpha\rho_--\frac{\mu}{1+t}\rho_-\Big(\frac{\alpha(1+t)}{1+\mu}+\frac{1}{(1+t)^{\mu}}\Big(u_--\frac{\alpha}{1+\mu}\Big)\Big)\Big)\phi dxdt\\
&-\int\int_{\Omega_+}\Big(\alpha\rho_+-\frac{\mu}{1+t}\rho_+\Big(\frac{\alpha(1+t)}{1+\mu}+\frac{1}{(1+t)^{\mu}}\Big(u_+-\frac{\alpha}{1+\mu}\Big)\Big)\Big)\phi dxdt\\
&=-\int\int_{\Omega}\Big(\alpha\rho-\frac{\mu}{1+t}\rho u\Big)\phi dxdt
\end{aligned}
$$

for all $\phi \in C_0^\infty(R \times R^+)$. Analogously, we also can derive $\int\int_\Omega (\rho\phi_t + \rho u\phi_x)dxdt = 0$. The desired result is obtained.

Remark 3.3. When $u_- < u_+$, in the limit $\mu \to 0$, (3.1) gives

$$\lim_{\mu\to 0}(\rho,u)(x,t) = \begin{cases} (\rho_-, u_- + \alpha t), & x < u_-t + \frac{1}{2}\alpha t^2, \\ \text{Vac}, & u_-t + \frac{1}{2}\alpha t^2 < x < u_+t + \frac{1}{2}\alpha t^2, \\ (\rho_+, u_+ + \alpha t), & x > u_+t + \frac{1}{2}\alpha t^2, \end{cases} \tag{3.11}$$

which is identical with the vacuum Riemann solution of the pressureless Euler equations with friction.

When $u_- > u_+$, we send $\mu \to 0$ in (3.4) and (3.7) to get

$$\lim_{\mu\to 0}(\rho,u)(x,t) = \begin{cases} (\rho_-, u_- + \alpha t), & x < \frac{\sqrt{\rho_-}u_- + \sqrt{\rho_+}u_+}{\sqrt{\rho_-}+\sqrt{\rho_+}}t + \frac{1}{2}\alpha t^2, \\ \left(\varpi(t)\delta(x - x(t)), \frac{\sqrt{\rho_-}u_- + \sqrt{\rho_+}u_+}{\sqrt{\rho_-}+\sqrt{\rho_+}} + \alpha t\right), & x = \frac{\sqrt{\rho_-}u_- + \sqrt{\rho_+}u_+}{\sqrt{\rho_-}+\sqrt{\rho_+}}t + \frac{1}{2}\alpha t^2, \\ (\rho_+, u_+ + \alpha t), & x > \frac{\sqrt{\rho_-}u_- + \sqrt{\rho_+}u_+}{\sqrt{\rho_-}+\sqrt{\rho_+}}t + \frac{1}{2}\alpha t^2, \end{cases} \tag{3.12}$$

where

$$\varpi(t) = \lim_{\mu\to 0}\omega(t) = \sqrt{\rho_-\rho_+}(u_- - u_+)\lim_{\mu\to 0}\frac{(1+t)^{1-\mu} - 1}{1-\mu} = \sqrt{\rho_-\rho_+}(u_- - u_+)t.$$

The solution (3.12) is just the delta-shock solution of the pressureless Euler equations with friction. In consequence, the Riemann solutions of the (1.1) with (1.2) converge to the corresponding solutions of the pressureless Euler equations with friction when $\mu \to 0$.

Remark 3.4. When $u_- < u_+$, in the limit $\alpha, \mu \to 0$, it can be concluded from (3.1) that

$$\lim_{\alpha,\mu\to 0}(\rho,u)(x,t) = \begin{cases} (\rho_-, u_-), & x < u_-t, \\ \text{Vac}, & u_-t < x < u_+t, \\ (\rho_+, u_+), & x > u_+t. \end{cases} \tag{3.13}$$

The (3.13) is just the solution containing vacuum of the homogeneous pressureless Euler equations.

When $u_- > u_+$, letting $\mu, \alpha \to 0$ in (3.4) and (3.7), we obtain

$$\lim_{\alpha,\mu\to 0}(\rho,u)(x,t) = \begin{cases} (\rho_-, u_-), & x < \frac{\sqrt{\rho_-}u_- + \sqrt{\rho_+}u_+}{\sqrt{\rho_-}+\sqrt{\rho_+}}t, \\ \left(\sqrt{\rho_-\rho_+}(u_- - u_+)t\delta(x - x(t)), \frac{\sqrt{\rho_-}u_- + \sqrt{\rho_+}u_+}{\sqrt{\rho_-}+\sqrt{\rho_+}}\right), & x = \frac{\sqrt{\rho_-}u_- + \sqrt{\rho_+}u_+}{\sqrt{\rho_-}+\sqrt{\rho_+}}t, \\ (\rho_+, u_+), & x > \frac{\sqrt{\rho_-}u_- + \sqrt{\rho_+}u_+}{\sqrt{\rho_-}+\sqrt{\rho_+}}t, \end{cases}$$

which corresponds to the delta-shock solution to the homogeneous pressureless Euler equations. As a result, the limits of the non-self-similar solutions of (1.1)-(1.2) are identical with the self-similar solutions of the homogeneous pressureless Euler equations as $\alpha, \mu \to 0$.

8 Mar 2024 18:19:09 PST
231227-LiShiwei Version 2 - Submitted to Rocky Mountain J. Math.

4. Limit solutions of (1.4) with (2.2) as ε drops to zero

In this section, we analyse the limiting behavior of solutions for the modified viscous system (1.4) with initial value (2.2) as $\varepsilon \to 0$. These results suggest that the solutions to the homogeneous system (2.1) with (2.2) have stability. For convenience sake, we only discuss the case $u_- > u_+$. The case $u_- < u_+$ can be addressed by using the same process.

We seek the solutions, which depend on the variable $\xi = \frac{x - \frac{\alpha((1+t)^2-1)}{2(1+\mu)} + \frac{\alpha}{1+\mu}\int_0^t \frac{1}{(1+s)^\mu} ds}{\int_0^t \frac{1}{(1+s)^\mu} ds}$ for (1.4) with (2.2), to obtain the following two-point boundary value problem of high-order ordinary differential equations with the boundary value in the infinity:

$$\begin{cases} -\xi\rho_\xi + (\rho v)_\xi = 0, \\ -\xi(\rho v)_\xi + (\rho v^2)_\xi = \varepsilon v_{\xi\xi}, \end{cases} \tag{4.1}$$

and

$$(\rho, v)(\pm\infty) = (\rho_\pm, u_\pm). \tag{4.2}$$

By adopting the method as in [26], we can assert that the boundary value problem (4.1)-(4.2) possesses a weak solution $(\rho, v) \in L^1(-\infty, +\infty) \times C^2(-\infty, +\infty)$. Moreover, by combining with the works in [11, 26], we have the lemmas as follows:

Lemma 4.1. Let $(\rho^\varepsilon(\xi), v^\varepsilon(\xi))$ be the solution of (4.1)-(4.2). Suppose that ξ_σ^ε is the unique point satisfying $\xi_\sigma^\varepsilon = v^\varepsilon(\xi_\sigma^\varepsilon)$, $\xi_\sigma = \lim_{\varepsilon\to 0} \xi_\sigma^\varepsilon$ (pass to a subsequence if necessary). Then for each $\gamma > 0$,

$$\lim_{\varepsilon\to 0} (v^\varepsilon)'(\xi) = 0, \quad \text{for} \quad |\xi - \xi_\sigma| \geq \gamma,$$

$$\lim_{\varepsilon\to 0} v^\varepsilon(\xi) = \begin{cases} u_-, & \text{for} \quad \xi \leq \xi_\sigma - \gamma, \\ u_+, & \text{for} \quad \xi \geq \xi_\sigma + \gamma \end{cases}$$

uniformly in the above intervals.

Lemma 4.2. For any $\gamma > 0$, we have $\lim_{\varepsilon\to 0} \rho^\varepsilon(\xi) = \rho_-$ for $\xi < \xi_\sigma - \gamma$, and $\lim_{\varepsilon\to 0} \rho^\varepsilon(\xi) = \rho_+$ for $\xi > \xi_\sigma + \gamma$.

The next thing to do is to study the limiting behavior of ρ^ε in the neighborhood of $\xi = \sigma$ as $\varepsilon \to 0$. Set

$$\sigma = \xi_\sigma = \lim_{\varepsilon\to 0} \xi_\sigma^\varepsilon = \lim_{\varepsilon\to 0} v^\varepsilon(\xi_\sigma^\varepsilon) = v_\delta, \tag{4.3}$$

then we have

$$u_+ < \sigma < u_-. \tag{4.4}$$

We choose $\varphi \in C_0^\infty[\xi_4, \xi_5]$ which satisfies $\varphi(\xi) \equiv \varphi(\sigma)$ for ξ in a neighborhood Ω of $\xi = \sigma$, where $\xi_4 < \sigma < \xi_5$. When $\xi_\sigma^\varepsilon \in \Omega \subset (\xi_4, \xi_5)$, it can be concluded from (4.1) that

$$\begin{cases} -\int_{\xi_4}^{\xi_5} \rho^\varepsilon(v^\varepsilon - \xi)\varphi' d\xi + \int_{\xi_4}^{\xi_5} \rho^\varepsilon \varphi d\xi = 0, \\ -\int_{\xi_4}^{\xi_5} \rho^\varepsilon v^\varepsilon(v^\varepsilon - \xi)\varphi' d\xi + \int_{\xi_4}^{\xi_5} \rho^\varepsilon v^\varepsilon \varphi d\xi = \varepsilon \int_{\xi_4}^{\xi_5} v^\varepsilon \varphi'' d\xi. \end{cases} \tag{4.5}$$

From (4.5), it follows that

$$\begin{cases} \lim_{\varepsilon\to 0} \int_{\xi_4}^{\xi_5} (\rho^\varepsilon - H_\rho(\xi - \sigma))\varphi(\xi) d\xi = (-\sigma[\bar\rho] + [\bar\rho\bar u])\varphi(\sigma), \\ \lim_{\varepsilon\to 0} \int_{\xi_4}^{\xi_5} (\rho^\varepsilon v^\varepsilon - \bar H(\xi - \sigma))\varphi(\xi) d\xi = (-\sigma[\bar\rho\bar u] + [\bar\rho\bar u^2])\varphi(\sigma), \end{cases} \tag{4.6}$$

for all sloping test functions $\varphi \in C_0^\infty[\xi_4, \xi_5]$, where $H_\rho(x)$ is a step function whose value is ρ_- for $x < 0$ and ρ_+ for $x > 0$, $\bar{H}(x)$ is a step function whose value is $\rho_- u_-$ for $x < 0$ and $\rho_+ u_+$ for $x > 0$. By making use of the approximation process, it can be proved that (4.6) holds for all $\varphi \in C_0^\infty[\xi_4, \xi_5]$. As a result, ρ^ε and $\rho^\varepsilon v^\varepsilon$ converge to a sum of a step function and a weighted Dirac delta function with the weights $(-\sigma[\bar{\rho}] + [\bar{\rho}\bar{u}])$ and $(-\sigma[\bar{\rho}\bar{u}] + [\bar{\rho}\bar{u}^2])$ in the weak star topology of $C_0^\infty(R)$, respectively.

Additionally, we can deduce the following formula

$$\lim_{\varepsilon \to 0} \int_{\xi_4}^{\xi_5} (\rho^\varepsilon - H_\rho(\xi - \sigma))\varphi(\xi) d\xi \cdot \sigma = (-\sigma[\bar{\rho}\bar{u}] + [\bar{\rho}\bar{u}^2])\varphi(\sigma), \tag{4.7}$$

provided that we take the test function as $\varphi/(\bar{v}^\varepsilon + \mu)$ in (4.5) and let $\mu \to 0$, where $\bar{v}^\varepsilon$ is a modified function satisfying $v^\varepsilon(\sigma)$ inside Ω and v^ε outside Ω. Making use of (4.6) and (4.7), we calculate $\sigma = \frac{\sqrt{\rho_-}u_- + \sqrt{\rho_+}u_+}{\sqrt{\rho_-} + \sqrt{\rho_+}}$, which shows that $\omega_0 = -\sigma[\bar{\rho}] + [\bar{\rho}\bar{u}] = \sqrt{\rho_-\rho_+}(u_- - u_+)$.

So we draw the following conclusion.

Theorem 4.3. Let $u_- > u_+$. Assume that $(\rho^\varepsilon, v^\varepsilon)(x,t)$ is the solution depending on the variable $\xi = \frac{x - \frac{\alpha((1+t)^2-1)}{2(1+\mu)} + \frac{\alpha}{1+\mu}\int_0^t \frac{1}{(1+s)^\mu} ds}{\int_0^t \frac{1}{(1+s)^\mu} ds}$ of the modified viscous system (1.4) with initial data (2.2). Then the limit function $(\rho, v)(x,t)$ of $(\rho^\varepsilon, v^\varepsilon)(x,t)$ exists in the sense of distributions, and $(\rho, v)(x,t)$ solves (2.1)-(2.2). The solution $(\rho, v)(x,t)$ can be represented as

$$(\rho, v)(x,t) = \begin{cases} (\rho_-, u_-), & \dfrac{x - \frac{\alpha((1+t)^2-1)}{2(1+\mu)} + \frac{\alpha}{1+\mu}\int_0^t \frac{1}{(1+s)^\mu} ds}{\int_0^t \frac{1}{(1+s)^\mu} ds} < v_\delta, \\ \left(\omega_0 \displaystyle\int_0^t \frac{1}{(1+s)^\mu} ds \cdot \delta(x - x(t)), v_\delta\right), & \dfrac{x - \frac{\alpha((1+t)^2-1)}{2(1+\mu)} + \frac{\alpha}{1+\mu}\int_0^t \frac{1}{(1+s)^\mu} ds}{\int_0^t \frac{1}{(1+s)^\mu} ds} = v_\delta, \\ (\rho_+, u_+), & \dfrac{x - \frac{\alpha((1+t)^2-1)}{2(1+\mu)} + \frac{\alpha}{1+\mu}\int_0^t \frac{1}{(1+s)^\mu} ds}{\int_0^t \frac{1}{(1+s)^\mu} ds} > v_\delta, \end{cases} \tag{4.8}$$

where

$$v_\delta = \frac{\sqrt{\rho_-}u_- + \sqrt{\rho_+}u_+}{\sqrt{\rho_-} + \sqrt{\rho_+}}, \qquad \omega_0 = \sqrt{\rho_-\rho_+}(u_- - u_+). \tag{4.9}$$

This is a reminder that the strength ω_0 is different from the weight $\omega(t)$ in (2.27). The reason may be the introduction of the similarity variable.

Remark 4.4. As can be seen from Theorem 4.3, the delta-shock solution of (2.1)-(2.2) has stability under viscous perturbation. By reasoning and analysis, one can easily conclude that $(\rho^\varepsilon, u^\varepsilon)(x,t) = (\rho^\varepsilon, \frac{\alpha}{1+\mu}(1+t) + \frac{1}{(1+t)^\mu}(v^\varepsilon - \frac{\alpha}{1+\mu}))(x,t)$ solves the problem (1.3) and (1.2), provided that $(\rho^\varepsilon, v^\varepsilon)(x,t)$ solves the problem (1.4) and (2.2). Noticing that ε and t are independent of each other, then we have $\lim\limits_{\varepsilon\to 0}(\rho^\varepsilon, u^\varepsilon)(x,t) = (\rho, u)(x,t) = \left(\rho, \frac{\alpha}{1+\mu}(1+t) + \frac{1}{(1+t)^\mu}\left(v - \frac{\alpha}{1+\mu}\right)\right)(x,t)$ when $u_- > u_+$. The $(\rho, u)(x,t)$, which is just the solution of (1.1) with (1.2), can be given explicitly as

$$(\rho, u)(x,t) = \begin{cases} \left(\rho_-, \dfrac{\alpha}{1+\mu}(1+t) + \dfrac{1}{(1+t)^\mu}\left(u_- - \dfrac{\alpha}{1+\mu}\right)\right), & x < x(t), \\ \left(\omega_0 \displaystyle\int_0^t \frac{1}{(1+s)^\mu} ds \cdot \delta(x - x(t)), \dfrac{\alpha}{1+\mu}(1+t) + \dfrac{1}{(1+t)^\mu}\left(v_\delta - \dfrac{\alpha}{1+\mu}\right)\right), & x = x(t), \\ \left(\rho_+, \dfrac{\alpha}{1+\mu}(1+t) + \dfrac{1}{(1+t)^\mu}\left(u_+ - \dfrac{\alpha}{1+\mu}\right)\right), & x > x(t), \end{cases}$$

8 Mar 2024 18:19:09 PST
231227-LiShiwei Version 2 - Submitted to Rocky Mountain J. Math.

in which ω_0 and v_δ are given by (4.9), $x(t) = (v_\delta - \frac{\alpha}{1+\mu}) \int_0^t \frac{1}{(1+s)^\mu} ds + \frac{\alpha((1+t)^2-1)}{2(1+\mu)}$. Thereby, the non-self-similar delta-shock solution of (1.1) with (1.2) possesses also stability under reasonable viscous perturbation.

5. Conclusion

Considering the fact that lots of damping effects are usually time-dependent, this article discusses the Riemann problem for the pressureless Euler equations with Coulomb-like friction and time-gradually-degenerate damping, which can be reduced from the balanced Euler equations. By applying the variable substitution method, two kinds of solutions involving vacuum and delta-shock are obtained. The generalized Rankine-Hugoniot relation and entropy condition for the delta-shock are clarified. The presence of two external force terms causes the Riemann solutions to be non-self-similar. Moreover, we show that the non-self-similar solutions converge to the corresponding solutions to pressureless Euler equations with friction as the time-dependent damping term vanishes; the non-self-similar solutions converge to the corresponding self-similar solutions of pressureless Euler equations as the friction and damping terms vanish simultaneously. Finally, the stability of the non-self-similar delta-shocks solutions is proved by adopting vanishing viscosity method.

Declarations

Data Availability

No data were used to support this study.

Conflict of interest

The authors declare that they have no conflict of interest.

Acknowledgements

This paper is supported by the Science and Technology Research Program of Education Department of Henan Province (24A110002).

References

[1] Y. Brenier, E. Grenier, Sticky particles and scalar conservation laws, SIAM J. Numer. Anal. 35 (1998), 2317-2328.

[2] F. Bouchut, On zero pressure gas dynamics, in: B. Perthame (Ed.), in: Advances in Kinetic Theory and Computing: Selected Papers, Series on Advances in Mathematics for Applied Sciences: Volume 22, World Scientific, River Edge, NJ, (1994), 171-190.

[3] J. Clarke, C. Lowe, A class of analytical solutions to the Euler equations with source terms: Part II, Math. Comput. Modell. 38 (2003), 1101-1117.

[4] G. Chen, H. Liu, Formation of delta-shocks and vacuum states in the vanishing pressure limit of solutions to the isentropic Euler equations, SIAM J. Math. Anal. 34 (2003), 925-938.

[5] K. Luen. Cheung, S. Wong, Finite-time singularity formation for C^1 solutions to the compressible Euler equations with time-dependent damping, Appl. Anal. 100 (2021), 1774-1785.

[6] X. Ding, Z. Wang, Existence and uniqueness of discontinuous solutions defined by Lebesgue-Stieltjes integral, Sci. China Ser. A 39 (1996), 807-819.

[7] V. G. Danilov, V. M. Shelkovich, Dynamics of propagation and interaction of delta-shock waves in conservation laws systems, J. Differ. Equ. 211 (2005), 333-381.

[8] C. M. Dafermos, Solution of the Riemann problem for a class of hyperbolic systems of conservation laws by the viscosity method, Arch. Ration. Mech. Anal. 52(1) (1973), 1-9.

[9] R. De la cruz, Riemann problem for a 2×2 hyperbolic system with linear damping, Acta Appl. Math. 170(1) (2020), 631-647.

[10] R. De la cruz, J. Juajibioy, Vanishing viscosity limit for Riemann solutions to a 2×2 hyperbolic system with linear damping, Asymptotic Anal. 3 (2022), 275-296.

[11] R. De la cruz, J. Juajibioy, Delta shock solution for a generalized zero-pressure gas dynamics system with linear damping, Acta Appl. Math. 177 (2022), 1-25.

[12] S. Geng, Y. Lin, M. Mei, Asymptotic behavior of solutions to Euler equations with time-dependent damping in critical case, SIAM J. Math. Anal. 52 (2020), 1463-1488.

[13] J. Hu, One-dimensional Riemann problem for equations of constant pressure fluid dynamics with measure solutions by the viscosity method, Acta Appl. Math. 55 (1999), 209-229.

[14] D. Kong, C. Wei, Q. Zhang, Formation of singularities in one-dimensional Chaplygin gas, J. Hyperbolic Differ. Equ. 11 (2014), 521-561.

[15] C. Lowe, J. Clarke, A class of exact solutions for the Euler equations with sources: Part I, Math. Comput. Modell. 36 (2002), 275-291.

[16] J. Li, T. Zhang, S. Yang, The two-dimensional Riemann problem in gas dynamics, first ed., Longman Harlow, (1998).

[17] S. Li, Delta-shock for a class of systems of conservation laws of the Keyfitz-Kranzer type, Math. Nachr. (2023), 1-20. https://doi.org/10.1002/mana.202300053

[18] S. Li, Riemann problem for a class of non-strictly hyperbolic systems of conservation laws, Acta Appl. Math. 177(1) (2022), 1-19.

[19] S. Li, Delta shock wave as limits of vanishing viscosity for zero-pressure gas dynamics with energy conservation law, ZAMM-Z. Angew. Math. Me. (2022), e202100377.

[20] S. Li, Delta-shocks as limits of vanishing viscosity for a nonhomogeneous hyperbolic system, Z. Anal. Anwend. (2023).

[21] S. Li, Riemann problem for a 2×2 hyperbolic system with time-gradually-degenerate damping, Bound. Value Probl. 109 (2023). https://doi.org/10.1186/s13661-023-01798-z

[22] E. Yu. Panov, V. M. Shelkovich, δ'-shock waves as a new type of solutions to system of conservation laws, J. Differ. Equ. 228 (2006), 49-86.

[23] Y. Pang, L. Shao, Y. Wen, J. Ge, The δ' wave solution to a totally degenerate system of conservation laws, Chaos Soliton. Fract. 161 (2022), 112302.

[24] A. Qu, Z. Wang, Stability of the Riemann solutions for a Chaplygin gas, J. Math. Anal. Appl. 409 (2014), 347-361.

[25] S. F. Shandarin, Ya. B. Zeldovich, The large-scale structure of the universe: Turbulence, intermittency, structures in a self-gravitating medium, Rev. Modern Phys. 61 (1989), 185-220.

[26] W. Sheng, T. Zhang, The Riemann problem for the transportation equations in gas dynamics, Mem. Amer. Math. Soc. 137 (1999), 654.

[27] A. Sen, T. Raja. Sekhar, The limiting behavior of the Riemann solution to the isentropic Euler system for logarithmic equation of state with a source term, Math. Method. Appl. Sci. 44 (2021), 7207-7227.

[28] C. Shen, M. Sun, Exact Riemann solutions for the drift-flux equations of two-phase flow under gravity, J. Differ. Equ. 314 (2022), 1-55.

[29] S. Sheng, Z. Shao, The vanishing adiabatic exponent limits of Riemann solutions to the isentropic Euler equations for power law with a Coulomb-like friction term, J. Math. Phys. 60(10), (2019).

[30] C. Shen, The Riemann problem for the Chaplygin gas equations with a source term, Z. Angew. Math. Mech. 96 (2016), 681-695.

[31] C. Shen, The Riemann problem for the pressureless Euler system with the Coulomb-like friction term, IMA J. Appl. Math. 81 (2016), 76-99.

[32] A. Sen, T. Raja. Sekhar, Delta shock wave as self-similar viscosity limit for a strictly hyperbolic system of conservation laws, J. Math. Phys. 60(5) (2019), 051510.

[33] M. Sun, Delta shock waves for the chromatography equations as self-similar viscosity limits, Q. Appl. Math. 69 (2011), 425-443.

[34] D. Tan, T. Zhang, Y. Zheng, Delta shock waves as limits of vanishing viscosity for hyperbolic systems of conservation laws, J. Differ. Equ. 112 (1994), 1-32.

[35] D. Tan, T. Zhang, Two-dimensional Riemann problem for a hyperbolic system of nonlinear conservation laws I. Four-J cases, II. Initial data involving some rarefaction waves, J. Differ. Equ. 111 (1994), 203-282.

[36] E. Weinan, Yu. G. Rykov, Ya. G. Sinai, Generalized variational principles, global weak solutions and behavior with randon initial data for systems of conservation laws arising in adhesion particle dynamics, Comm. Phys. Math. 177 (1996), 349-380.

[37] Z. Wang, Q. Zhang, The Riemann problem with delta initial data for the one-dimensional Chaplygin gas equations, Acta Math. Sci. 32B (2012), 825-841.

[38] G. Yin, W. Sheng, Delta shocks and vacuum states in vanishing pressure limits of solutions to the relativistic Euler equations for polytropic gases, J. Math. Anal. Appl. 355 (2009), 594-605.

[39] H. Yang, Riemann problems for a class of coupled hyperbolic systems of conservation laws, J. Differ. Equ. 159 (1999), 447-484.

[40] Q. Zhang, F. He, The exact Riemann solutions to the generalized pressureless Euler equations with dissipation, B. Malays. Math. Sci. So. 43 (2020), 4361-4374.

[41] Y. Zhang, R. Zhang, The Riemann problem for the equations of constant pressure fluid dynamics with nonlinear damping, Int. J. Non-Lin. Mech. 133 (2021), 103712.

[42] Q. Zhang, F. He, Y. Ba, Delta-shock waves and Riemann solutions to the generalized pressureless Euler equations with a composite source term, Appl. Anal. 102 (2023), 576-589.

[43] Y. Zhang, Y. Zhang, The Riemann problem for the Eulerian droplet model with buoyancy and gravity forces, Eur. Phys. J. Plus 135 (2020), 1-16.