ON THE TIME-DELAYED FRACTIONAL MOBILE–IMMOBILE EQUATIONS: REGULARITY AND LARGE–TIME BEHAVIOR

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ABSTRACT. We consider the Cauchy problem driven by nonlinear fractional mobile–immobile equations (Fr-MIMEs) involving time–varying delays in Hilbert spaces. Based on fixed point arguments, smoothness of the resolvent operators, and local estimates, we show global solvability and regularity results for both linear and semilinear problems in which the nonlinearity terms are supposed to satisfy sublinear or superlinear growth conditions. Several qualitative aspects concerning the large–time behavior of solutions, such as dissipativity and stability, are investigated by establishing a new Halanay–type inequality. Moreover, thanks to the technique of measure of noncompactness, the existence of decay solutions with polynomial rates is also proved.

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1. INTRODUCTION

In the last decades, delay differential equations (DDEs) involving the fractional–order derivative operators in both finite and infinite dimensional spaces have been widely used to model and analyze various evolutionary processes arising in many fields in real life, such as physics, biology, bioengineering, control theory, demography, and medicine [BDST12, BNRV16, Ca99, CC08, El03, Er09, FWP17, Ju06, Ma06, OV20, Ni86, Wu96, ZWZ17]. In such models, the state of processes usually depends nonlinearly not only on the current state but also on the history state, and the fractional–order derivative operators normally

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describe the anomalous diffusion in materials with memory. Some interesting mathematical questions related to DDEs, including the global existence and large–time behavior of solutions generated by these equations, have attracted considerable attentions of many researchers; see [AK15, AKQ16, AY22, CTT20, DVVW95, KT20, KT23, KWS99, LP22, TKD23, TS20, TT20, TTG21] for example. On the other hand, another important topic in qualitative investigations of DDEs is about the regularity of solutions. This topic naturally emerges from both theoretical and applied issues. Some well–known situations mentioned here include creating the algorithms to find the approximation solutions and analyzing the numerical stability (see, e.g., [BZ13, LS23, LW22]); studying solvability and stability for some kinds of inverse problems [DL07, KTT22, LV14, Tu23a, Tu23b] and optimal problems [BK13, Wa72].

Inspired by the facts listed above and the recent works [ADT22, DTT23], in this paper we are primarily interested in analyzing the global existence, regulariy, stability and polynomial decay of solutions to nonlinear time–delayed Fr-MIMEs in Hilbert spaces, which states as follows

$$\nu_1 u'(t) + \nu_2 D_0^\alpha u(t) + Au(t) = f(t, u_\rho), t > 0, \tag{1.1}$$

$$u(s) = \xi(s), s \in [-q, 0], \tag{1.2}$$

where $\nu_1, \nu_2 > 0$, $\big(H, (\cdot,\cdot)\big)$ be a separable Hilbert space, the state function $u(\cdot)$ takes values in H, A is an unbounded linear operator on H, $D_0^\alpha, \alpha \in (0,1)$, is the Caputo fractional derivative of order α defined by

$$D_0^\alpha u(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} u'(s) ds, t > 0.$$

In Eq. (1.1), $u_\rho(t) = u(t - \rho(t))$ represents a time–varying delay term where ρ is a continuous function on $\mathbf{R}^+$ such that $-q \le t - \rho(t) \le t$. The nonlinear function $f : \mathbf{R}^+ \times H \to H$ is a given function and the initial datum ξ belongs to $C([-q, 0], H)$.

Before proceeding further, let us review some related works concerning Fr-MIMEs. It is worth mentioning that Fr-MIME is introduced for the first time in the seminal work by R. Schumer and his coauthors [SBMB03]. As remarked in [SBMB03], Fr-MIME is employed to describe the anomalous diffusion of solute in porous media. Subsequently, Fr-MIMEs have received a lot of attention, and, nowadays, there is a fairly long list of publications investigating the existence of numerical solutions for linear as well as nonlinear Fr-MIMEs [Ba21, JXQZ20, QXCG20, YLL20, ZLL19, ZW20]. However, the theoretical studies of solutions for Fr-MIMEs are not yet fully known. There have been some recent efforts to address qualitative questions for Fr-MIMEs [ADT22, DTT23]. In [DTT23], the authors have successfully established results on the existence, regularity in time, and stability in the Lyapunov sense of solutions to the Cauchy problem governed by nonlinear Fr-MIMEs. In addition, the global existence of decay solutions for nonlinear Fr-MIMEs with impulsive effects has been obtained in [ADT22]. We would like to make more contributions to the literature by considering the Fr-MIMEs with delays since the fact that there is no attempt at solving this problem.

Regarding the initial value problem (1.1)–(1.2), the analysis of the regularity and large–time behavior of solutions demands a few technical difficulties than the case without delays [DTT23]. The first one comes from the appearance of delay and nonlinearity terms, and the fact that two resolvent families $\mathcal{S}_\alpha(\cdot), \mathcal{R}_\alpha(\cdot)$ that are defined by (2.14), (2.15) in the next section no longer possess the semi-group property. The second one is the lack of appropriate functional inequalities. In order to overcome these difficulties, we first take the smoothness in time–space of $\mathcal{S}_\alpha(\cdot), \mathcal{R}_\alpha(\cdot)$ (see Lemma 2.1 below) into account and propose suitable assumptions on the regularity of both the initial data and the nonlinear perturbations. And then, with these assumptions in hand, the global existence and C^1–regularity of solutions for the problem (1.1)–(1.2) are proved by blending fixed point arguments and local estimates. In addition, regarding the large–time behavior of solutions, a new Halanay–type inequality of integral form is shown for analyzing dissipativity and stability. Finally, the boundedness of ρ and the measure of noncompactness are employed for the purpose of showing the existence of decay solution with polynomial rates. It should be noticeable that the last result is better in comparison with the counter part in [ADT22] due to the explicit rate of decay.

The present work is organized as follows. In the next section, we recall some basic facts related to the inhomogeneous initial value problem driven by the linear part of equation (1.1) and show important results on resolvent operators. A result on the C^1–regularity of the solution to this problem will be shown in Section 2. Section 3 is devoted to revealing the global existence and regularity of solutions to the corresponding nonlinear problem. In the last Section, we analyze the large–time behavior of solutions by showing the dissipativity and stability as well as the existence of decay solutions with polynomial rates.

2. Notations and preliminaries

In this section, our goal is to find a representation of the solution for the inhomogeneous initial value problem associated with the problem (1.1)–(1.2) and prove the regularity of its solution. For the sake of simplicity, we first list here some notations and conventions which will be used throughout this work. Let $(\cdot,\cdot)$, $\|\cdot\|$ be the inner product and the standard norm in H. For $a<b$, we denote by $C\big([a,b];H\big)$ the space of all continuous functions on $[a,b]$, taking values in H. This space is a Banach space when it is equipped with the norm

$$\|v\|_{C\left([a,b];H\right)} := \sup_{t\in[a,b]} \|v(t)\|.$$

In particular, the norms in $C\big([-q,0];H\big)$, $C\big([0,T];H\big)$ will be denoted by $\|\cdot\|_0, \|\cdot\|_\infty$, respectively. For $\gamma\in(0,1)$, we define the Hölder space $C^\gamma\big([a,b];H\big)$ consisting of all continuous functions $v:[a,b]\to H$ such that

$$\sup_{a\le t<t+h\le b} \frac{\|v(t+h)-v(t)\|}{h^\gamma} < \infty.$$

2.1. Resolvent operators.

In this subsection, we find a formula for mild solutions to the inhomogeneous initial value problem

$$\nu_1 u'(t) + \nu_2 D_0^\alpha u(t) + Au(t) = F(t), 0<t\le T, \tag{2.1}$$

$$u(s) = \xi(s), s\in[-q,0], \tag{2.2}$$

where $F\in L^1_{loc}\big(\mathbf{R}^+;H\big)$ and F is an exponentially bounded function. For this goal, we need the following assumption

(Ha) $A: D(A)\to H$ is densely defined, self-adjoint and positively definite operator with a compact resolvent.

It follows from the assumption (Ha) that there exists a sequence $\{\lambda_n\}_{n=1}^\infty$ satisfying

$$0<\lambda_1\le\lambda_2\le\cdots\le\lambda_n\le\cdots,$$

$\lim\limits_{n\to\infty}\lambda_n=\infty$ and a system of vectors $\{e_n\}_{n=1}^\infty\subset D(A)$, which forms an orthonormal basis of H such that $Ae_n=\lambda_n e_n$, for all $n\in\mathbb{N}^*$.

For $\gamma\in\mathbf{R}$, one can define the fractional power operator A^γ of A as follows

$$A^\gamma z := \sum_{n=1}^\infty \lambda_n^\gamma (z,e_n)\, e_n, z\in\mathbf{V}_\gamma := D(A^\gamma) = \left\{ z\in H: \sum_{n=1}^\infty \lambda_n^{2\gamma} |(z,e_n)|^2 < \infty \right\}.$$

It should be noted that $\mathbf{V}_\gamma$ is a Hilbert space with the norm

$$\|z\|_{\mathbf{V}_\gamma} = \left(\sum_{n=1}^\infty \lambda_n^{2\gamma} |(z,e_n)|^2 \right)^{\frac{1}{2}}, \ z\in D(A^\gamma).$$

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We can identify $\mathbf{V}_{-\gamma} = D(A^{-\gamma})$ with $\mathbf{V}_\gamma^*$, the dual space of $\mathbf{V}_\gamma$. Then $\mathbf{V}_{-\gamma}$ is a Hilbert space with the norm

$$\|h\|_{V_{-\gamma}} = \left(\sum_{n=1}^{\infty} \lambda_n^{-2\gamma} |\langle h, e_n\rangle|^2\right)^{\frac{1}{2}},$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $\mathbf{V}_{-\gamma}$ and $\mathbf{V}_\gamma$. Identifying H with its dual H^*, the following relations hold for all $\gamma \geq 0$:

$$\mathbf{V}_\gamma \subset H \simeq H^* \subset \mathbf{V}_{-\gamma}.$$

It is worth noting that $\langle f, z\rangle = (f, z)$ for all $f \in H, z \in \mathbf{V}_\gamma$.

Let

$$u(t) = \sum_{n=1}^{\infty} u_n(t) e_n, F(t) = \sum_{n=1}^{\infty} F_n(t) e_n, t \in (0, T]\ ,$$
$$\xi(s) = \sum_{n=1}^{\infty} \xi_n(s) e_n,\ s \in [-q, 0].$$

Hence, for all $n = 1, 2, \ldots$, one has

$$\nu_1 u_n'(t) + \nu_2 g_{1-\alpha} * u_n'(t) + \lambda_n u_n(t) = F_n(t), 0 < t \leq T, \tag{2.3}$$
$$u_n(s) = \xi_n(s),\ s \in [-q, 0], \tag{2.4}$$

where the notation '$*$' stands for the Laplace convolution with respect to the time t, i.e.,

$$(m * v)(t) = \int_0^t m(t-s) v(s) ds,$$

and $g_{1-\alpha}(t) = t^{-\alpha}/\Gamma(1-\alpha), t > 0$.

To find u_n satisfying Eqs. (2.3)–(2.4), we consider the following scalar integral equations

$$s(t) + \lambda(\ell * s)(t) = 1,\ t \geq 0, \tag{2.5}$$
$$r(t) + \lambda(\ell * r)(t) = \ell(t),\ t \geq 0, \tag{2.6}$$

where $\lambda > 0$ and ℓ is the unique solution of the following integral equation

$$\nu_1 \ell + \nu_2 g_{1-\alpha} * \ell = 1 \text{ on } [0, \infty). \tag{2.7}$$

It is well known (see, e.g. [**GLS90**, Theorem 2.3.1]), that Eqs. (2.5) and (2.6) are uniquely solved. In particular, see [**DTT23**, Sect. 2], the solution of Eq. (2.7) is given by

$$\ell(t) = \nu_1^{-1} E_{1-\alpha}(-\nu_1^{-1} \nu_2 t^{1-\alpha}), \tag{2.8}$$

where

$$E_{1-\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma((1-\alpha)n + 1)}, z \in \mathbf{C},$$

is the Mittag–Leffler function.

Throughout the paper, we denote $s_\alpha(\cdot, \lambda)$ and $r_\alpha(\cdot, \lambda)$ being the solutions of (2.5) and (2.6), respectively. Recall that the kernel function ℓ is completely positive iff $s_\alpha(\cdot, \lambda), r_\alpha(\cdot, \lambda)$ are nonnegative for every $\lambda > 0$. In [**GKMR14**, Proposition 3.23, p. 47], it is shown that ℓ is completely positive. Moreover, using the same arguments as in [**DTT23**, Propositions 2.1 and 2.2], we obtain the following results.

Proposition 2.1. *Let $\ell, s_\alpha(\cdot, \lambda), r_\alpha(\cdot, \lambda)$ be the solution of the equation (2.7), (2.5) and (2.6) respectively. Then the following claims hold:*

(i) $\dfrac{1}{\nu_1 + \nu_2 \Gamma(\alpha) t^{1-\alpha}} \leq \ell(t) \leq \dfrac{1}{\nu_1 + \nu_2 \Gamma(2-\alpha)^{-1} t^{1-\alpha}}$, *for all* $t \geq 0$.

(ii) $\ell(\cdot)$ *is a differentiable function on* $(0, \infty)$ *and*

$$0 \leq -\ell'(t) \leq \nu_1^{-2} \nu_2 t^{-\alpha}, \textit{for a.e. } t > 0.$$

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(iii) *For every* $\lambda > 0$, *the function* $s_\alpha(\cdot, \lambda)$ *is nonnegative and nonincreasing. Moreover,*

$$s_\alpha(t,\lambda)\left[1+\lambda\int_0^t \ell(\tau)d\tau\right] \le 1, \text{ for all } t \ge 0. \tag{2.9}$$

(iv) *For each* $t > 0$, *the functions* $\lambda \mapsto s_\alpha(t,\lambda)$ *and* $\lambda \mapsto r_\alpha(t,\lambda)$ *are nonincreasing.*

(v) *The function* $r_\alpha(\cdot,\lambda)$ *is nonnegative and the following two equalities hold*

$$s_\alpha(t,\lambda) = 1-\lambda\int_0^t r_\alpha(\tau,\lambda)d\tau = \nu_1 r_\alpha(t,\lambda) + \nu_2\big(g_\alpha * r_\alpha(\cdot,\lambda)\big)(t),\ t \ge 0. \tag{2.10}$$

In addition, for each $\lambda > 0$, *the following estimates hold*

$$\lambda r_\alpha(t,\lambda) \le \frac{1}{t}, \text{ for all } t > 0 \text{ and } r_\alpha(t,\lambda) \le \ell(t), \text{ for all } t \ge 0. \tag{2.11}$$

Remark 2.1. (i) In view of the representation (2.10) and the inequality (2.11), for each $\lambda > 0$, we have

$$0 \le -s_\alpha'(t,\lambda) = \lambda r_\alpha(t,\lambda) \le \frac{1}{t}, \text{ for all } t > 0,$$

and

$$\lambda\int_0^t r_\alpha(s,\lambda)ds \le 1, \forall t \ge 0.$$

(ii) Let $v(t) = s_\alpha(t,\lambda)v_0 + \big(r_\alpha(\cdot,\lambda) * \omega\big)(t)$, here $\omega \in L^1_{loc}(\mathbf{R}^+)$. Then, by owing to the same lines as ones in [**DTT23**, Proposition 2.3], v solves the problem

$$\nu_1 v'(t) + \nu_2\big(g_{1-\alpha} * v'\big)(t) + \lambda v(t) = \omega(t),\ v(0) = v_0. \tag{2.12}$$

It follows from Remark 2.1(ii) and Eqs. (2.3)–(2.4) that $u_n(t) = \xi(t), t \in [-q, 0]$ and

$$u_n(t) = s_\alpha(t,\lambda_n)\xi_n(0) + r_\alpha(\cdot,\lambda_n) * F_n(t).$$

We then obtain

$$u(t) = \mathcal{S}_\alpha(t)\xi(0) + \int_0^t \mathcal{R}_\alpha(t-\tau)F(\tau)d\tau, \tag{2.13}$$

where

$$\mathcal{S}_\alpha(t)v = \sum_{n=1}^\infty s_\alpha(t,\lambda_n)v_n e_n, \tag{2.14}$$

$$\mathcal{R}_\alpha(t)v = \sum_{n=1}^\infty r_\alpha(t,\lambda_n)v_n e_n. \tag{2.15}$$

It is easily seen that $\mathcal{S}_\alpha(t)$ and $\mathcal{R}_\alpha(t)$ are linear operators acting on H. Some basic properties of these operators are collected in the following lemma.

Lemma 2.1 (See [**ADT22**, Lemma 2.4])**.** *Let* $\{\mathcal{S}_\alpha(t)\}_{t\ge 0}$ *and* $\{\mathcal{R}_\alpha(t)\}_{t\ge 0}$ *be the families of linear operators defined by* (2.14) *and* (2.15)*, respectively. Then*

(i) *For each* $v \in H$ *and* $T > 0$, $\mathcal{S}_\alpha(\cdot)v \in C([0,T];H)$ *and* $A\mathcal{S}_\alpha(\cdot)v \in C((0,T];H)$. *Moreover,*

$$\|\mathcal{S}_\alpha(t)v\| \le s_\alpha(t,\lambda_1)\|v\|,\ t \in [0,T], \tag{2.16}$$

$$\|\mathcal{S}_\alpha(t)v\|_{\mathbf{V}_1} \le \frac{\|v\|}{(1*\ell)(t)},\ t \in (0,T]. \tag{2.17}$$

In addition, $\mathcal{S}_\alpha(\cdot)$ *is differentiable on* $(0,\infty)$ *and the following estimate holds*

$$\|\mathcal{S}_\alpha'(t)v\| \le \frac{\|v\|}{t}, \forall v \in H, \forall t > 0. \tag{2.18}$$

(ii) *Let $v \in H, T > 0$ and $g \in C([0,T];H)$. Then $\mathcal{R}_\alpha(\cdot)v \in C([0,T];H)$ and $\mathcal{R}_\alpha * g \in C([0,T];\mathbf{V}_{1/2})$. Furthermore,*

$$\|\mathcal{R}_\alpha(t)v\| \le r_\alpha(t,\lambda_1)\|v\|,\ t \in [0,T], \tag{2.19}$$

$$\|(\mathcal{R}_\alpha * g)(t)\| \le \int_0^t r_\alpha(t-\tau,\lambda_1)\|g(\tau)\|d\tau,\ t \in [0,T], \tag{2.20}$$

$$\|(\mathcal{R}_\alpha * g)(t)\|_{\mathbf{V}_{1/2}} \le \left(\int_0^t r_\alpha(t-\tau,\lambda_1)\|g(\tau)\|^2 d\tau\right)^{\frac{1}{2}},\ t \in [0,T]. \tag{2.21}$$

Moreover, $\mathcal{R}_\alpha(\cdot)$ is differentiable on $(0,\infty)$ and the following estimate holds

$$\|\mathcal{R}'_\alpha(t)v\| \le (\nu_1^{-1}t^{-1} + \nu_1^{-2}\nu_2 t^{-\alpha})\|v\|, \forall v \in H, \forall t > 0. \tag{2.22}$$

Remark 2.2. (i) The first statement of Lemma 2.1 guarantees that the operator $\mathcal{S}_\alpha(t) : H \to H$ is compact for any $t > 0$, due to the compactness of the embedding $\mathbf{V}_1 \hookrightarrow H$.

(ii) Utilizing the same arguments as in [**DTT23**, Lemma 2.5] and the properties of $\mathcal{R}_\alpha$ stated in Lemma 2.1, it is shown that the Cauchy operator defined by

$$\mathcal{Q}_\alpha : C([0,T];H) \to C([0,T];H),$$
$$\mathcal{Q}_\alpha(g)(t) = \big(\mathcal{R}_\alpha * g\big)(t), \tag{2.23}$$

is compact.

2.2. **Regularity of mild solution.** In this section, we prove a result on C^1-regularity of a mild solution to (2.1)–(2.2) when ξ, F are sufficiently regular functions. Then we obtain the strong solution whose definition can be given as follows.

Definition 1. A function $u \in C([-q,T];H)$ is said to be a strong solution to (2.1)–(2.2) on the interval $[-q,T]$ iff (2.1) and (2.2) hold as an equation in H.

Our main result on regularity of mild solutions for the time–delayed linear problem (2.1)–(2.2) is presented in the following theorem.

Theorem 2.1. *Consider the initial value problem (2.1)–(2.2). Let* (Ha) *hold and assume that $F \in C^\sigma([0,T];H)$, $\xi \in C^\sigma([-q,0];H)$, $\xi(0) \in \mathbf{V}_1$. Then the mild solution u of the problem obeys*

(i) *u is Hölder continuous on $[-q,T]$ with exponent σ,*
(ii) *$u \in C^1((0,T];H)$,*
(iii) *$Au \in C((0,T];H)$,*
(iv) *$g_{1-\alpha} * u' \in C((0,T];H)$ and then u is a strong solution.*

Proof. Let u be the mild solution of problem (2.1)–(2.2). We have $u(t) = \xi(t), t \in [-q,0]$ and

$$u(t) = \mathcal{S}_\alpha(t)\xi(0) + \int_0^t \mathcal{R}_\alpha(t-\tau)F(\tau)d\tau, t \in [0,T].$$

Denote C_F, C_ξ be the Hölder constants of F, ξ, respectively. We first show the proof of the assertion (i). Let $t \in [-q,T]$ and $h \in (0, T-t]$. We estimate $\|u(t+h) - u(t)\|$ in the following three distinct cases.

Case 1. $-q \le t < t+h \le 0$. Then, by assumptions, one has

$$\|u(t+h) - u(t)\| \le C_\xi h^\sigma.$$

Case 2. $-q < t \le 0 < t+h$. In this case $|t| \le h$ and

$$\begin{aligned}\|u(t+h) - u(t)\| &= \|\mathcal{S}_\alpha(t+h)\xi(0) + \int_0^{t+h} \mathcal{R}_\alpha(t+h-\tau)F(\tau)d\tau - \xi(t)\| \\ &\le \|[\mathcal{S}_\alpha(t+h) - I]\xi(0)\| + \|\xi(0) - \xi(t)\|\end{aligned}$$

$$+ \| \int_0^{t+h} \mathcal{R}_\alpha(t+h-\tau)F(\tau)d\tau\|. \tag{2.24}$$

Since

$$\begin{aligned}[\mathcal{S}_\alpha(t+h)-I]\xi(0) &= \sum_{n=1}^{\infty}\big(s_\alpha(t+h,\lambda_n)-1\big)\xi_n(0)e_n,\\ &= -\sum_{n=1}^{\infty}\lambda_n\int_0^{t+h} r_\alpha(t+h-\tau,\lambda_n)d\tau\xi_n(0)e_n,\end{aligned}$$

it turns out that

$$\begin{aligned}\|[\mathcal{S}_\alpha(t+h)-I]\xi(0)\|^2 &= \|\sum_{n=1}^{\infty}\int_0^{t+h} r_\alpha(t+h-\tau,\lambda_n)d\tau\phi_n e_n\|^2\\ &\le \left(\int_0^{t+h}\ell(t+h-\tau)d\tau\right)^2\|\phi\|^2\\ &\le \left(\int_0^{h}\nu_1^{-1}d\tau\right)^2\|\phi\|^2\\ &= h^2\nu_1^{-2}\|\phi\|^2,\end{aligned}$$

where $\phi := A\xi(0)$, thanks to Proposition 2.1(i), (v). It infers that

$$\begin{aligned}\|[\mathcal{S}(t+h)-I]\xi(0)\| &\le \nu_1^{-1}\|\phi\|h\\ &\le \nu_1^{-1}\|\phi\|(T+q)^{1-\sigma}h^\sigma.\end{aligned}$$

In addition, the second and third terms in the right hand side of (2.24) can be estimated as

$$\|\xi(0)-\xi(t)\| \le C_\xi|t|^\sigma \le C_\xi h^\sigma,$$

and

$$\begin{aligned}\|\int_0^{t+h}\mathcal{R}_\alpha(t+h-\tau)F(\tau)d\tau\| &\le \int_0^{h} r_\alpha(t+h-\tau,\lambda_1)\|F(\tau)\|d\tau\\ &\le \nu_1^{-1}\|F\|_\infty h\\ &\le \nu_1^{-1}\|F\|_\infty(T+q)^{1-\sigma}h^\sigma.\end{aligned}$$

The estimates above follow that

$$\|u(t+h)-u(t)\| \le \left[\nu_1^{-1}\|\phi\|(T+q)^{1-\sigma}+C_\xi+\nu_1^{-1}\|F\|_\infty(T+q)^{1-\sigma}\right]h^\sigma.$$

Case 3. $0<t\le T$ and $0<t+h\le T$. Then

$$\begin{aligned}\|u(t+h)-u(t)\| \le \|[\mathcal{S}_\alpha(t+h)-\mathcal{S}_\alpha(t)]\xi(0)\| &+ \|\int_0^t[\mathcal{R}_\alpha(t+h-\tau)-\mathcal{R}_\alpha(t-\tau)]F(\tau)d\tau\|\\ &+\|\int_t^{t+h}\mathcal{R}_\alpha(t+h-\tau)F(\tau)d\tau\|\\ &= \mathrm{T}_1+\mathrm{T}_2+\mathrm{T}_3.\end{aligned}$$

We first estimate T_1. Using the mean value formula, one gets

$$[\mathcal{S}_\alpha(t+h)-\mathcal{S}_\alpha(t)]\xi(0) = h\int_0^1\mathcal{S}'_\alpha(t+\theta h)\xi(0)d\theta.$$

Then

$$\begin{aligned}\mathrm{T}_1 &= h\|\int_0^1\mathcal{S}'_\alpha(t+\theta h)\xi(0)d\theta\|\\ &\le h\|\xi(0)\|\int_0^1\frac{d\theta}{t+\theta h}\\ &\le \|\xi\|_0\ln\big(1+\frac{h}{t}\big)\end{aligned}$$

$$\leq \sigma^{-1}\|\xi\|_0\left(\frac{h}{t}\right)^{\sigma}$$
$$= \sigma^{-1}\|\xi\|_0 t^{-\sigma}h^{\sigma},$$

thanks to Lemma 2.1(i) and the fact that $\ln\big(1+r\big) \leq \dfrac{r^\beta}{\beta}$ for $r > 0$ and $\beta \in (0,1)$. Regarding T_2, we recall that $\mathcal{R}_\alpha(\cdot)$ is differentiable on $(0,\infty)$. By the same arguments as above, we obtain

$$\|[\mathcal{R}_\alpha(t+h-\tau)-\mathcal{R}_\alpha(t-\tau)]F(\tau)\| \leq \sigma^{-1}\nu_1^{-1}\|F(\tau)\|h^{\sigma}\frac{1}{(t-\tau)^{\sigma}}.$$

That leads to the estimation of T_2 as follows

$$\mathrm{T}_2 \leq \sigma^{-1}\nu_1^{-1}\|F\|_\infty T^{1-\sigma}(1-\sigma)^{-1}h^{\sigma}.$$

Making use of Lemma 2.1(ii) and inequalites that $r_\alpha(t,\lambda) \leq \ell(t) \leq \nu_1^{-1}$ for $t \geq 0$, $\lambda > 0$, we have

$$\mathrm{T}_3 \leq \|F\|_\infty\nu_1^{-1}h \leq \|F\|_\infty\nu_1^{-1}T^{1-\sigma}h^{\sigma}.$$

We now arrive at

$$\|u(t+h)-u(t)\| \leq \left[\sigma^{-1}\|\xi\|_0 t^{-\sigma}+\sigma^{-1}\nu_1^{-1}\|F\|_\infty T^{1-\sigma}(1-\sigma)^{-1}+\|F\|_\infty\nu_1^{-1}T^{1-\sigma}\right]h^{\sigma}.$$

The proof of the assertions (ii)–(iv) can be finished by means of Lemma 2.1 and arguing as in the proof of Theorem 4.2 in [**DTT23**]. The proof is complete. □

3. Global existence and regularity of solutions

In this section, we show results on the global existence and regularity of mild solutions to the nonlinear problem (1.1)–(1.2) on $[-q,T]$ for any $T > 0$ by using the fixed point approach. We first need a definition of mild solution to our problem. From the definition of mild solution to the initial value problem (2.1)–(2.2), we have the following one.

Definition 2. A function $u \in C\big([-q,T];H\big)$ is said to be a mild solution of the problem (1.1)–(1.2) on $[-q,T]$ iff $u(t) = \xi(t)$ for $t \in [-q,0]$ and

$$u(t) = \mathcal{S}_\alpha(t)\xi(0) + \int_0^t \mathcal{R}_\alpha(t-\tau)f\big(\tau, u(\tau-\rho(\tau))\big)d\tau,$$

for any $t \in [0,T]$.

For our goal, we make some notations:

- $C_\xi([0,T];H) := \{u \in C([0,T];H) : u(0) = \xi(0)\}$, for given $\xi \in C([-q,0];H)$.
- $u[\xi]$ is a function which is defined as
$$u[\xi](t) = \begin{cases} u(t) & \text{if } t \in [0,T] \\ \xi(t) & \text{if } t \in [-q,0], \end{cases}$$
here $u \in C_\xi([0,T];H)$ and then we see that $u[\xi] \in C([-q,T];H)$.
- $\mathcal{G} : C_\xi([0,T];H) \to C_\xi([0,T];H)$ is an operator defined by
$$\mathcal{G}(u)(t) = \mathcal{S}_\alpha(t)\xi(0) + \int_0^t \mathcal{R}_\alpha(t-\tau)f\big(\tau, u[\xi]_\rho(\tau)\big)d\tau.$$

Apparently, the set $C_\xi([0,T];H)$ is a closed subset of $C([0,T];H)$ and

$$u[\xi]_\rho(t) = \begin{cases} u(t-\rho(t)) & \text{if } t-\rho(t) \in [0,T] \\ \xi(t-\rho(t)) & \text{if } t-\rho(t) \in [-q,0]. \end{cases}$$

The operator $\mathcal{G}$ will be referred to as *the solution operator* since the fact that u is a fixed point of $\mathcal{G}$ iff $u[\xi]$ is a mild solution of (1.1)–(1.2). This operator is continuous if f is a continuous map.

We are now in a position to state our first main result on the global existence of solutions to (1.1)–(1.2).

Theorem 3.1. *Let* (Ha) *hold and suppose that*

(**F**1) *the function f is continuous and*

$$\|f(t,v)\| \le a(t)\psi(\|v\|),\ \forall t \in [0,T], v \in H,$$

where $a \in L^1_{loc}(\mathbf{R}^+)$ is a nonnegative function, $\psi \in C(\mathbf{R}^+)$ is a nonnegative and nondecreasing function such that

$$\limsup_{r\to 0} \frac{\psi(r)}{r} \cdot \sup_{t\in[0,T]} \int_0^t r_\alpha(t-\tau,\lambda_1)a(\tau)d\tau < 1. \tag{3.1}$$

Then there exists $\delta > 0$ such that the problem (1.1)–(1.2) *has at least one mild solution on $[-q,T]$, provided $\|\xi\|_0 \le \delta$. Moreover, if f satisfies*

(**F**2) *$f(\cdot,0) = 0$ and is locally Lipschitz continuous, i.e., for each $r > 0$, there exists a nonnegative constant $L(r)$ such that*

$$\|f(t,v_1) - f(t,v_2)\| \le L(r)\|v_1 - v_2\|,$$

for all $t \in [0,T], v_i \in H$ with $\|v_i\| \le r, i \in \{1,2\}$ and $\limsup_{r\to 0} L(r) < \lambda_1$, then the mild solution to (1.1)–(1.2) *is unique.*

Proof. Since f is continuous, $\mathcal{G}$ is continuous as well. Let N_f be the Nemytskii operator, that is $N_f(u)(t) = f\big(t, u[\xi]_\rho(t)\big)$, $t \in [0,T]$. By the representation

$$\mathcal{G}(u) = \mathcal{S}_\alpha(\cdot)\xi(0) + \mathcal{Q}_\alpha \circ N_f(u),$$

we get $\mathcal{G}$ is a compact operator thanks to the compactness of $\mathcal{Q}_\alpha$ stated in Remark 2.2. Therefore, we employ the Schauder fixed point theorem to get our result. For this purpose, we show that there exists some real number $R > 0$ such that

$$\mathcal{G}(\mathsf{B}_R) \subset \mathsf{B}_R,$$

where B_R be the closed ball in $C_\xi([0,T];H)$ centered at origin with radius R. Let

$$\psi^* = \limsup_{r\to 0} \frac{\psi(r)}{r} \text{ and } \mathbf{M} = \sup_{t\in[0,T]} \int_0^t r_\alpha(t-\tau,\lambda_1)a(\tau)d\tau.$$

From (3.1), we can chose $\epsilon > 0$ such that

$$(\psi^* + \epsilon)\mathbf{M} < 1.$$

By definition of ψ^*, there exists $R > 0$ such that

$$\frac{\psi(r)}{r} \le \psi^* + \epsilon, \forall r \in (0, 2R].$$

Taking $\delta = \dfrac{1 - \mathbf{M}(\psi^* + \epsilon)}{1 + \mathbf{M}(\psi^* + \epsilon)} R$, then $\delta > 0$ and $\delta < R$.

Let $u \in \mathsf{B}_R$, we have

$$\begin{aligned}
\|\mathcal{G}(u)(t)\| &\le s_\alpha(t,\lambda_1)\|\xi(0)\| + \int_0^t r_\alpha(t-\tau,\lambda_1)\|f(\tau,u[\xi]_\rho(\tau))\|d\tau \\
&\le \|\xi(0)\| + \int_0^t r_\alpha(t-\tau,\lambda_1)a(\tau)\psi(\|u[\xi]_\rho(\tau)\|)d\tau \\
&\le \|\xi\|_0 + \int_0^t r_\alpha(t-\tau,\lambda_1)a(\tau)\psi\big(\sup_{\theta\in[0,\tau]}\|u(\theta)\| + \|\xi\|_0\big)d\tau
\end{aligned}$$

$$\begin{aligned}
&\le \delta + (\psi^* + \epsilon)\big(R+\delta\big)\int_0^t r_\alpha(t-\tau,\lambda_1)a(\tau)d\tau\\
&\le \delta + (\psi^* + \epsilon)\big(R+\delta\big)\mathbf{M}\\
&\le \big(1+(\psi^*+\epsilon)\mathbf{M}\big)\delta + \mathbf{M}(\psi^*+\epsilon)R,\\
&\le R, \forall t\in[0,T],
\end{aligned} \tag{3.2}$$

thanks to Lemma 2.1 and the formulation of δ. The inequality (3.2) implies that $\mathcal{G}(\mathsf{B}_R) \subset \mathsf{B}_R$, provided $\|\xi\|_0 \le \delta$.

We now consider $\mathcal{G} : \mathsf{B}_R \to \mathsf{B}_R$. Then Schauder fixed point theorem allow us to obtain the desired result. Now assume that the locally Lipschitz condition (**F2**) is fulfilled. In this case, the assumption (**F1**) is verified for $\psi(r) = rL(r)$, $a \equiv 1$ and for

$$\mathbf{M} = \sup_{[0,T]}\int_0^t r_\alpha(t-\tau,\lambda_1)a(\tau)d\tau = \lambda_1^{-1},$$

thank to Proposition 2.1(v). Thus, one can apply the arguments as in the case (**F1**) to get the global existence of mild solutions to (1.1)–(1.2). Let u_i, $i\in\{1,2\}$ are two solutions of (1.1)–(1.2) on $[-q,T]$. Recalling that $u_i(t)=\xi(t), t\in[-q,0]$ and

$$u_i(t) = \mathcal{S}_\alpha(t)\xi(0) + \int_0^t \mathcal{R}_\alpha(t-\tau)f(\tau,u_i[\xi]_\rho(\tau))d\tau, t\in[0,T],$$

we deduce that

$$\begin{aligned}
\|u_1(t)-u_2(t)\| &\le \int_0^t r_\alpha(t-\tau,\lambda_1)L(r^*)\|u_1[\xi]_\rho(\tau)-u_2[\xi]_\rho(\tau)\|d\tau\\
&\le \nu_1^{-1}L(r^*)\int_0^t \sup_{[0,\tau]}\|u_1(\theta)-u_1(\theta)\|d\tau,
\end{aligned}$$

where $r^* := \max\{\|u_i\|_\infty : i=1,2\}$. Here we have used the fact that $r_\alpha(t,\lambda_1)\le \ell(t) \le \nu_1^{-1}$ for all $t\ge 0$, thanks to Proposition 2.1 (i), (v). Since the last integral is nondecreasing in t, we thus get

$$\sup_{[0,t]}\|u_1(t)-u_2(t)\| \le \nu_1^{-1}L(r^*)\int_0^t \sup_{[0,\tau]}\|u_1(\theta)-u_1(\theta)\|d\tau.$$

Using the classical Gronwall inequality, we obtain $\sup\limits_{\tau\in[0,t]}\|u_1(\tau)-u_2(\tau)\| = 0, \forall t\in[0,T]$, which implies $u_1=u_2$. The proof is complete. ☐

Remark 3.1. Results about the global existence obtained in Theorem 3.1 require the initial datum and coefficients to be sufficiently small. These limitations will be removed if one of the following two cases occurs

(i) either

(**F3**) f is continuous such that $\|f(t,v)\| \le a(t)\|v\| + b(t)$, for all $t\in[0,T], v\in H$, where $a,b\in L^1_{loc}(\mathbf{R}^+)$ are nonnegative functions

(ii) or

(**F4**) f is a globally Lipschitzian function, i.e., there exists $a\in L^1_{loc}(\mathbf{R}^+)$ is a nonnegative function such that

$$\|f(t,v_1)-f(t,v_2)\| \le a(t)\|v_1-v_2\|,$$

for all $t\in[0,T], v_i\in H, i\in\{1,2\}$. More precisely, in the case (i), by the same arguments in the proof of Theorem 3.4 in [**DTT23**], we can find a closed, bounded and convex set D in $C_\xi([0,T];H)$ such that $\mathcal{G}(\mathsf{D})\subset\mathsf{D}$. And then, the global existence of mild solutions to the problem (1.1)–(1.2) is followed by using the same lines as used for the condition (**F1**) in the proof of Theorem 3.1. Besides, in the case (ii), one can employ an appropriate weighted norm on $C_\xi([0,T];H)$ and prove that the solution operator $\mathcal{G}$ is a contraction operator, see [**DTT23**, Theorem 3.2] for details.

In the rest of this section, we deal with the regularity in time of mild solutions to the nonlinear problem (1.1)–(1.2). For this aim, we need the following assumption on f:

(**F**5) f is a locally Lipschitz–Hölder function, that is, for each $r > 0$, there exists a non-negative constant $L(r)$ such that

$$\|f(t_1, v_1) - f(t_2, v_2)\| \leq L(r)\big(|t_1 - t_2|^\alpha + \|v_1 - v_2\|\big),$$

for all $t_i \in [0, T]$ and $v_i \in H$ such that $\|v_i\| \leq r, i \in \{1, 2\}$.

The second main result in this paper is as follows.

Theorem 3.2. *Assume that* (Ha) *and* (**F**5) *hold with* $\limsup_{r\to 0} L(r) < \lambda_1$ *and, additionally,* ρ *is a Hölder continuous with exponent* α*. Then there exists* $\delta > 0$ *such that if* $\|\xi\|_0 \leq \delta$ *then the unique mild solution* u *to* (1.1)–(1.2) *on* $[-q, T]$ *belongs to* $C^1((0, T]; H)$*, provided that* $\xi \in C^\alpha([-q, 0]; H)$ *with* $\xi(0) \in \mathbf{V}_1$.

Proof. It is easy to check out that, under the assumptions of Theorem 3.2, the assumptions (**F**2) of Theorem 3.1 also hold. Therefore, due to Theorem 3.1, for each $\xi \in C([-q, 0]; H)$ with $\|\xi\|_0 \leq \delta$, the problem (1.1)–(1.2) admits a unique mild solution u on $[-q, T]$ satisfying $\|u\| \leq R$, where R, δ are choosen in Theorem 3.1.

We denote $\tilde{F}(t) = f(t, u[\xi]_\rho(t)), t \in [0, T]$, where u is a mild solution of the problem (1.1)–(1.2) associated with ξ. Recalling that $u(t) = \xi(t), t \in [-q, 0]$ and

$$u(t) = \mathcal{S}_\alpha(t)\xi(0) + \int_0^t \mathcal{R}_\alpha(t - \tau)\tilde{F}(\tau)d\tau, t \in [0, T].$$

Clearly, $\tilde{F}$ is continuous on $[0, T]$ and

$$\|\tilde{F}(t)\| \leq L(\delta + R)(\delta + R) \leq 2RL(2R), \forall t \in [0, T].$$

Now, using the same lines as the ones in the proof of claim 1 in Theorem 2.1 with some minor modifications, it is shown that the mild solution u is Hölder continuous on $[-q, T]$ with exponent α.

We now show that $\tilde{F}$ is Hölder continuous on $[0, T]$. Indeed, let γ_1, γ_2 be Hölder constants of u, ρ, respectively. For $t \in (0, T]$ and $h \in (0, T - t]$, we first have

$$\begin{aligned}\|\tilde{F}(t + h) - \tilde{F}(t)\| &= \|f\big(t + h, u[\xi]_\rho(t + h)\big) - f\big(t, u[\xi]_\rho(t)\big)\| \\ &\leq L(2R)\big(h^\alpha + \|u[\xi]_\rho(t + h) - u[\xi]_\rho(t)\|\big). \end{aligned}\tag{3.3}$$

On the other hand

$$\begin{aligned}\|u[\xi]_\rho(t + h) - u[\xi]_\rho(t)\| &= \|u(t + h - \rho(t + h)) - u(t - \rho(t))\| \\ &\leq \gamma_1\big(h + \rho(t) - \rho(t + h)\big)^\alpha \\ &\leq \gamma_1\big(h + \gamma_2 h^\alpha\big)^\alpha \\ &\leq \gamma_1\big(h^\alpha T^{1-\alpha} + \gamma_2 h^\alpha\big)^\alpha \\ &\leq \gamma_1\big(T^{1-\alpha} + \gamma_2\big)^\alpha h^\alpha. \end{aligned}\tag{3.4}$$

From (3.3)–(3.4), we get

$$\|\tilde{F}(t + h) - \tilde{F}(t)\| \leq \gamma_3 h^\alpha,$$

where $\gamma_3 = L(2R)\big(1 + \gamma_1(T^{1-\alpha} + \gamma_2)^\alpha\big)$. Due to this and Theorem 2.1 we get the conclusion of this theorem. □

Remark 3.2. Let us point out that if the nonlinearity f enjoys global Lipschitz-Hölder condition, i.e.

$$\|f(t_1, v_1) - f(t_2, v_2)\| \leq L_f\big(|t_1 - t_2|^\alpha + \|v_1 - v_2\|\big), \tag{3.5}$$

for all $t_i \in [0, T]$, $v_i \in H, i \in \{1, 2\}$, then the C^1-regularity of mild solution for the problem (1.1)–(1.2) can be proved by relaxing both assumptions $L_f \in [0, \lambda_1)$ and the smallness of initial condition. In this case, the existence and uniqueness of mild solution is guaranteed by Remark 3.1 and the C^1-regularity is proved by employing the same arguments as in the proof of Theorem 3.2.

4. LARGE–TIME BEHAVIOR OF SOLUTIONS

In this section, our aim is to determine the large–time behavior of the solution of equation (1.1). To accomplish this goal, we establish a Halanay–type inequality in order to prove the existence of an absorbing set, the dissipativity and stability (in the Lyapunov sense) in the case the nonlinear function f satisfies the global Lipschitz condition. Moreover, in the case the nonlinear function f obeys a sublinear growth condition, we establish sufficient conditions to ensure the existence of polynomial decay solutions for the problem (1.1)–(1.2).

4.1. **Dissipativity and stability.** In this subsection, we deal with the dissipativity and stability of solutions to Eq. (1.1). The following Halanay–type inequality plays an important role in our argument.

Lemma 4.1. *Let z be a continuous and nonnegative function satisfying*

$$z(t) \le s_\alpha(t,\lambda)z_0 + \int_0^t r_\alpha(t-\tau,\lambda)[p(\tau) + \kappa \sup_{\theta\in[\tau-\rho(\tau),\tau]} z(\theta)]\, d\tau, t > 0, \tag{4.1}$$

$$z(s) = \psi(s), s \in [-q,0], \tag{4.2}$$

where $\kappa \in \big(0,\lambda\big)$, $\psi \in C([-q,0],\mathbf{R}^+)$ and $p \in L^1_{loc}(\mathbf{R}^+)$ which is nondecreasing. Then

$$z(t) \le \frac{\lambda}{\lambda-\kappa}\Big[z_0 + r_\alpha(\cdot,\lambda) * p(t)\Big] + \frac{\kappa}{\lambda}\sup_{\theta\in[-q,0]}\psi(\theta), \forall t > 0. \tag{4.3}$$

Furthermore, if $r_\alpha(\cdot,\lambda) * p$ is bounded on $[0,\infty)$ and $\lim_{t\to\infty}(t-\rho(t)) = \infty$ then

$$\limsup_{t\to\infty} z(t) \le \frac{\lambda}{\lambda-\kappa}\sup_{t\ge 0} r_\alpha(\cdot,\lambda) * p(t). \tag{4.4}$$

In particular, if $p = 0$ then $z(t) \to 0$ as $t \to \infty$.

Proof. By the inequality (4.1), Proposition 2.1(v) and Remark 2.1(ii), it follows that

$$\begin{aligned} z(t) &\le z_0 + r_\alpha(\cdot,\lambda) * p(t) + \kappa \sup_{\theta\in[-q,t]} z(\theta)\int_0^t r_\alpha(\tau,\lambda)d\tau \\ &= z_0 + r_\alpha(\cdot,\lambda) * p(t) + \frac{\kappa}{\lambda}\sup_{\theta\in[-q,t]} z(\theta)\big(1 - s_\alpha(t,\lambda)\big) \\ &\le z_0 + r_\alpha(\cdot,\lambda) * p(t) + \frac{\kappa}{\lambda}\sup_{\theta\in[-q,t]} z(\theta). \end{aligned} \tag{4.5}$$

Observe that, due to $p(\cdot)$ is nondecreasing, the function $t \mapsto z_0 + r_\alpha(\cdot,\lambda) * p(t)$ is nondecreasing. Based on this fact, the inequality (4.5) and Lemma 2.3 in [**PL21**], we get the inequality (4.3) as desired.

Let us now assume that $r_\alpha(\cdot,\lambda) * p$ is bounded on $[0,\infty)$ and $\lim_{t\to\infty}(t-\rho(t)) = \infty$. Using the inequality (4.3), we know that $z(\cdot)$ is bounded by

$$\bar{z} := \frac{\lambda}{\lambda-b}\Big[z_0 + \sup_{t\ge 0} r_\alpha(\cdot,\lambda) * p(t)\Big] + \frac{\kappa}{\lambda}\sup_{\theta\in[-q,0]}\psi(\theta),$$

and thus the limit $\vartheta = \lim_{t\to\infty}\sup_{\zeta\in[t,\infty)} z(\zeta)$ exists. Since $t-\rho(t) \to \infty$ as $t \to \infty$, then for any $\epsilon > 0$ one can find $T^* > 0$ such that

$$\sup_{\zeta\in[t-\rho(t),t]} z(\zeta) \le \sup_{\zeta\in[t-\rho(t),\infty]} z(\zeta) \le \vartheta + \epsilon, \forall t \ge T^*.$$

On the other hand, since for each $\lambda > 0$, $s_\alpha(t,\lambda) \to 0$ as $t \to \infty$ and $r_\alpha(\cdot,\lambda) \in L^1(\mathbf{R}^+)$, then one can choose $t > T^*$ large enough such that

$$s_\alpha(t,\lambda) \le \epsilon \text{ and } \int_{t-T^*}^t r_\alpha(\tau,\lambda)d\tau \le \epsilon.$$

From these observations above and the inequality (4.1), there holds that

$$\begin{aligned} z(t) &\le s_\alpha(t,\lambda)z_0 + r_\alpha(\cdot,\lambda) * p(t) + \kappa\left(\int_0^{T^*} + \int_{T^*}^t\right) r_\alpha(t-\tau,\lambda) \sup_{\theta\in[\tau-\rho(\tau),\tau]} z(\theta)d\tau \\ &\le s_\alpha(t,\lambda)z_0 + r_\alpha(\cdot,\lambda) * p(t) + \kappa\bar{z}\int_0^{T^*} r_\alpha(t-\tau,\lambda)ds + \kappa(\vartheta+\epsilon)\int_{T^*}^t r_\alpha(t-\tau,\lambda)ds \\ &\le \epsilon z_0 + r_\alpha(\cdot,\lambda) * p(t) + \kappa\bar{z}\int_{t-T^*}^t r_\alpha(\tau,\lambda)ds + \kappa(\vartheta+\epsilon)\int_0^t r_\alpha(t-\tau,\lambda)ds \\ &\le \epsilon z_0 + r_\alpha(\cdot,\lambda) * p(t) + \kappa\bar{z}\epsilon + \frac{\kappa(\vartheta+\epsilon)}{\lambda}. \end{aligned} \tag{4.6}$$

Inspired by (4.6), one has

$$\vartheta = \lim_{t\to\infty} \sup_{\theta\in[t,\infty)} z(\theta) \le \frac{\vartheta\kappa}{\lambda} + \sup_{t\ge 0} r_\alpha(\cdot,\lambda) * p(t) + \left(z_0 + \kappa\bar{z} + \frac{\kappa}{\lambda}\right)\epsilon.$$

This inequality entails

$$\vartheta \le \frac{\lambda}{\lambda-\kappa}\sup_{t\ge 0} r_\alpha(\cdot,\lambda) * a(t) + \frac{\lambda}{\lambda-\kappa}\left(v_0 + \kappa\bar{z} + \frac{\kappa}{\lambda}\right)\epsilon. \tag{4.7}$$

Since ϵ is an arbitrarily positive number, we conclude follows from (4.7) that

$$\limsup_{t\to\infty} z(t) \le \vartheta \le \frac{\lambda}{\lambda-\kappa}\sup_{t\ge 0} r_\alpha(\cdot,\lambda) * p(t).$$

The last inequality settles the stated results in the lemma. □

We will prove both stability and dissipativity of solutions of our system–essentially based on the Halanay–type inequality in Lemma 4.1.

Theorem 4.1. *Assume that the hypothesis* (**F**2) *of Theorem 3.1 hold for any* $T > 0$. *Then the zero solution of* (1.1) *is asymptotically stable.*

Proof. Take R, ψ^*, δ, and ϵ as in the proof of Theorem 3.1 in which for every $\xi \in C([-q,0],H)$ with $\|\xi\|_0 \le \delta$, there exists a unique mild solution to (1.1)–(1.2) such that $\|u(t)\| \le R$ for all $t \ge 0$. Due to

$$\|u[\xi]_\rho(t)\| \le \|\xi\|_0 + \|u\|_\infty \le 2R, \text{ for all } t \in [0,T],$$

it holds that

$$\begin{aligned} \|u(t)\| &\le \|\mathcal{S}_\alpha(t)\xi(0)\| + \int_0^t \|\mathcal{R}_\alpha(t-\tau)\|_{op}\|f(\tau,u[\xi]_\rho(\tau))\|d\tau \\ &\le s_\alpha(t,\lambda_1)\|\xi(0)\| + \int_0^t r_\alpha(t-\tau,\lambda_1)L(2R)\|u[\xi]_\rho(\tau)\|d\tau \\ &\le s_\alpha(t,\lambda_1)\|\xi(0)\| + \int_0^t r_\alpha(t-\tau,\lambda_1)(\psi^*+\epsilon)\sup_{\theta\in[\tau-\rho(\tau),\tau]}\|u(\theta)\|d\tau. \end{aligned}$$

Applying the Halanay–type inequality in Lemma 4.1 with $v(t) = \|u(t)\|, t \ge -q, \lambda = \lambda_1$, we obtain

$$\|u(t)\| \le \frac{\lambda_1}{\lambda_1 - \psi^* - \epsilon}\|\xi(0)\| + \frac{\psi^*+\epsilon}{\lambda_1}\|\xi\|_0, \forall t \ge 0, \tag{4.8}$$

$$\lim_{t\to\infty}\|u(t)\| = 0. \tag{4.9}$$

The inequalities (4.8), (4.9) guarantee the stability and attractivity of the zero solution, respectively. We thus finish the proof of this theorem. □

Considering the case when f is globally Lipschitzian, we have a stronger result.

Theorem 4.2. *Assume that the hypothesis* (**F**4) *holds for any* $T > 0$ *and for* $a \in L^\infty(\mathbf{R}^+;\mathbf{R}^+)$. *If* $\|a\|_{L^\infty(\mathbf{R}^+)} < \lambda_1$, *then every mild solution of* (1.1)–(1.2) *is asymptotically stable.*

Proof. Let u and v be solutions of (1.1)–(1.2). Then, by the formula of solutions and by Lemma 2.1, we get that

$$\begin{aligned}\|u(t)-v(t)\| &\le \|\mathcal{S}_\alpha(t)[u(0)-v(0)]\| \\ &\quad + \int_0^t \mathcal{R}_\alpha(t-\tau)\|_{op}\|f(\tau,u[\xi]_\rho(\tau)) - f(\tau,v[\xi]_\rho(\tau))\|d\tau \\ &\le s_\alpha(t,\lambda_1)\|u(0)-v(0)\| + \int_0^t r_\alpha(t-\tau,\lambda_1)a(\tau)\|u[\xi]_\rho(\tau)-v[\xi]_\rho(\tau)\|d\tau \\ &\le s_\alpha(t,\lambda_1)\|u(0)-v(0)\| \\ &\quad + \int_0^t r_\alpha(t-\tau,\lambda_1)\|a\|_{L^\infty(\mathbf{R}^+)} \sup_{[\tau-\rho(\tau),\tau]}\|u(\theta)-v(\theta)\|d\tau.\end{aligned}$$

Applying Lemma 4.1 leads to

$$\begin{aligned}&\|u(t)-v(t)\| \le \frac{\lambda_1\|u(0)-v(0)\|}{\lambda_1 - \|a\|_{L^\infty(\mathbf{R}^+)}} + \frac{\|a\|_{L^\infty(\mathbf{R}^+)}}{\lambda_1}\|u(0)-v(0)\|_0, \forall t\ge 0,\\ &\lim_{t\to\infty}\|u(t)-v(t)\| = 0,\end{aligned}$$

from which we obtain the conclusion of this theorem. □

In the next theorem, we establish a result on the dissipativity of solutions of our system.

Theorem 4.3. *Let the hypothesis* (**F**3) *hold for any* $T > 0$ *with* $a \in L^\infty(\mathbf{R}^+;\mathbf{R}^+)$ *satisfying* $\|a\|_{L^\infty(\mathbf{R}^+)} < \lambda_1$ *and* $b \in L^1_{loc}(\mathbf{R}^+;\mathbf{R}^+)$ *is nondecreasing such that* $r_\alpha(\cdot,\lambda_1) * b$ *is a bounded function on* $\mathbf{R}^+$. *Then there exists an absorbing set for solutions of* (1.1)–(1.2) *with arbitrary initial data. Moreover, if* $b = 0$, *then the zero solution of* (1.1) *is asymptotically stable.*

Proof. Let u be a solution of (1.1)–(1.2). Using Lemma 2.1 and the estimate of f, we obtain

$$\begin{aligned}\|u(t)\| &\le s_\alpha(t,\lambda_1)\|\xi(0)\| + \int_0^t r_\alpha(t-\tau,\lambda_1)[b(\tau) + a(\tau)\|u[\xi]_\rho(\tau)\|]d\tau \\ &\le s_\alpha(t,\lambda_1)\|\xi(0)\| + \int_0^t r_\alpha(t-\tau,\lambda_1)[b(\tau) + \|a\|_{L^\infty(\mathbf{R}^+)}\|u[\xi]_\rho(\tau)\|]\,d\tau.\end{aligned}$$

Using Lemma 4.1 again, we arrive at

$$\limsup_{t\to\infty}\|u(t)\| \le \frac{\lambda_1}{\lambda_1 - \|a\|_{L^\infty(\mathbf{R}^+)}}\sup_{t\ge 0} r_\alpha(\cdot,\lambda_1) * b(t).$$

Put

$$R = \epsilon + \frac{\lambda_1}{\lambda_1 - \|a\|_{L^\infty(\mathbf{R}^+)}}\sup_{t\ge 0} r_\alpha(\cdot,\lambda_1) * b(t)$$

for some $\epsilon > 0$, then the ball B_R is an absorbing set for solutions of (1.1)–(1.2). Finally, if $b = 0$, then (1.1) admits the zero solution and it holds that

$$\begin{aligned}&\|u(t)\| \le \frac{\lambda_1}{\lambda_1 - \|a\|_{L^\infty(\mathbf{R}^+)}}\|\xi(0)\| + \frac{\|a\|_{L^\infty(\mathbf{R}^+)}}{\lambda_1}\|\xi\|_0, \forall t\ge 0,\\ &\lim_{t\to\infty}\|u(t)\| = 0,\end{aligned}$$

thanks to Proposition 4.1 again, which ensures the asymptotic stability of the zero solution. □

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4.2. **Existence of polynomial decay mild solutions.** We start this section by recalling some facts and basic results on measure of noncompactness, and fixed point theorem for condensing maps which are employed to prove the existence of decay solutions.

Let E be a Banach space. Denote by $\mathcal{B}(E)$ the collection of nonempty bounded subsets of E. We will use the following definition of the measure of noncompactness (see, e.g. [KOZ01]).

Definition 3. A function $\psi : \mathcal{B}(E) \to \mathbf{R}^+$ is called a measure of noncompactness *(MNC)* on E if

$$\psi(\overline{\text{co}}\ D) = \psi(D) \text{ for every } D \in \mathcal{B}(E),$$

where $\overline{\text{co}}\ D$ is the closure of convex hull of D. An MNC ψ is said to be:

- (i) monotone if for each $D_0, D_1 \in \mathcal{B}(E)$ such that $D_0 \subseteq D_1$, we have $\psi(D_0) \leq \psi(D_1)$;
- (ii) nonsingular if $\psi(\{a\} \cup D) = \psi(D)$ for any $a \in E, D \in \mathcal{B}(E)$;
- (iii) invariant with respect to the union with a compact set, if $\psi(K \cup D) = \psi(D)$ for every relatively compact set $K \subset E$ and $D \in \mathcal{B}(E)$;
- (iv) algebraically semi-additive if $\psi(D_0 + D_1) \leq \psi(D_0) + \psi(D_1)$ for any $D_0, D_1 \in \mathcal{B}(E)$;
- (v) regular if $\psi(D) = 0$ is equivalent to the relative compactness of D.

A typical example on MNC satisfying all properties stated in Definition 3 is the *Hausdorff* MNC $\chi(\cdot)$ defined by

$$\chi(D) = \inf\{\varepsilon > 0 : D \text{ has a finite } \varepsilon - \text{net}\}.$$

Definition 4. A continuous map $\mathcal{F} : Z \subseteq E \to E$ is said to be condensing with respect to an MNC ψ ($\psi-$condensing) if for any bounded set $D \subset Z$, the relation

$$\psi(D) \leq \psi(\mathcal{F}(D))$$

implies the relative compactness of D.

Let ψ be a monotone and nonsingular MNC in E. We have the following fixed point principle.

Theorem 4.4 (See [KOZ01, Corollary 3.3.1])**.** *Let $\mathcal{M}$ be a bounded convex closed subset of E and let $\mathcal{F} : \mathcal{M} \to \mathcal{M}$ be a ψ-condensing map. Then* $\text{Fix}(\mathcal{F}) := \{x \in E : x = \mathcal{F}(x)\}$ *is a nonempty and compact set.*

Let D be a bounded set in $BC\big([0,\infty); H\big)$, a space of bounded continuous functions on $[0,\infty)$ taking values in H. We denote

$$BC_\xi = \{v \in BC([0,\infty); H) : v(0) = \xi(0)\ \},$$

where ξ is given in (1.2), which becomes a closed subset of $BC\big([0,\infty); H\big)$ with sup-norm denoted by $\|\cdot\|_{BC}$. Now let $\pi_T : BC\big([0,\infty); H\big) \to C\big([0,T]; H\big)$ be the restriction operator on $BC\big([0,\infty); H\big)$, i.e. $\pi_T(u)$ is the restriction of $u \in BC\big([0,\infty); H\big)$ to the interval $[0,T]$. Consider

$$d_\infty(D) = \lim_{T\to\infty} \sup_{u\in D} \sup_{t\geq T} \|u(t)\|, \tag{4.10}$$

$$\chi_\infty(D) = \sup_{T>0} \chi_T(\pi_T(D)) \tag{4.11}$$

where $\chi_T(\cdot)$ is the Hausdorff MNC in $C\big([0,T], H\big)$. Then the following MNC defined in [KL17, Lemma 2.1] (see also [HKK17, Lemma 4.1])

$$\chi^*(D) = d_\infty(D) + \chi_\infty(D), \tag{4.12}$$

satisfies all properties stated in Definition 4. In addtion, if $\chi^*(D) = 0$ then D is relatively compact in $BC\big([0,\infty); H\big)$.

Let
$$\mathbf{B}_R^\gamma(\eta) = B_R \cap \{y \in BC_\xi : \sup_{t\ge 0} t^\gamma \|y(t)\| \le \eta\}$$
where B_R is the ball in BC_ξ centered at the origin with radius $R > 0$, γ and η are positive numbers. It is easy to see that $\mathbf{B}_R^\gamma(\eta)$ is nonempty, closed and convex in BC_ξ.

Before stating our main results in this part, we will show in the following two lemmas that there exists a such $\mathbf{B}_R^\gamma(\eta)$ which is invariant under the solution operator for $0 < \gamma < \min\{\beta_1(1-\alpha), \beta_2\alpha\}$, for some $\beta_1, \beta_2 \in (0, 1]$.

Lemma 4.2. *Let* (Ha) *hold and f satisfies* (**F**3) *for all $T > 0$ with $b \equiv 0$. Then there exists $R > 0$ such that $\mathcal{G}(\mathbf{B}_R) \subset \mathbf{B}_R$ provided that*
$$\sup_{t\ge 0} \int_0^t (t-\tau)^{-\beta_1(1-\alpha)} a(\tau)\, d\tau < \nu_1^{1-\beta_1} \nu_2^{\beta_1} \Gamma(2-\alpha)^{-\beta_1}, \tag{4.13}$$
for some $\beta_1 \in (0, 1]$.

Proof. Assume to the contrary that for each $n \in \mathbb{N}^*$, we have $u_n \in BC_\xi$ such that $\|u_n\|_{BC} \le n$ but $\|\mathcal{G}(u_n)\|_{BC} > n$. From the formulation of $\mathcal{G}$, we first obtain the following bounds
$$\begin{aligned}
\|\mathcal{G}(u_n)(t)\| &\le s_\alpha(t,\lambda_1)\|\xi(0)\| + \int_0^t r_\alpha(t-\tau,\lambda_1)\|f(\tau, u_n[\xi]_\rho(\tau))\| d\tau \\
&\le \|\xi(0)\| + \int_0^t r_\alpha(t-\tau,\lambda_1) a(\tau) \|u_n[\xi]_\rho(\tau)\| d\tau \\
&\le \|\xi\|_0 + \int_0^t r_\alpha(t-\tau,\lambda_1) a(\tau)\big(\|u_n(\tau)\| + \|\xi\|_0\big) d\tau \\
&\le \|\xi\|_0 + \int_0^t r_\alpha(t-\tau,\lambda_1) a(\tau)(\|\xi\|_0 + \|u_n\|_{BC}) d\tau.
\end{aligned}$$
Taking Proposition 2.1(i), (v) into account and combining with the fact that $1 + r \ge r^{\beta_1}, \forall r \ge 0, \beta_1 \in (0, 1]$, we get the following inequalities
$$\begin{aligned}
r_\alpha(t,\lambda_1) &\le \frac{1}{\nu_1(1+\nu_1^{-1}\nu_2\Gamma(2-\alpha)t^{1-\alpha})} \\
&\le \nu_1^{\beta_1-1}\nu_2^{-\beta_1}\Gamma(2-\alpha)^{\beta_1} t^{-\beta_1(1-\alpha)}.
\end{aligned} \tag{4.14}$$
Thus
$$\begin{aligned}
\|\mathcal{G}(u_n)(t)\| &\le \|\xi\|_0 + (\|\xi\|_0 + n)\int_0^t r_\alpha(t-\tau,\lambda_1)a(\tau)\, d\tau \\
&\le \|\xi\|_0 + (\|\xi\|_0 + n)\nu_1^{\beta_1-1}\nu_2^{-\beta_1}\Gamma(2-\alpha)^{\beta_1}\int_0^t (t-\tau)^{-\beta_1(1-\alpha)}a(\tau)\, d\tau \\
&\le \|\xi\|_0 + (\|\xi\|_0 + n)\nu_1^{\beta_1-1}\nu_2^{-\beta_1}\Gamma(2-\alpha)^{\beta_1}\sup_{t\ge 0}\int_0^t (t-\tau)^{-\beta_1(1-\alpha)}a(\tau)\, d\tau.
\end{aligned}$$
From the last inequality and the fact that $\|\mathcal{G}(u_n)\|_{BC} > n$, we conclude that
$$\begin{aligned}
1 < \frac{\|\mathcal{G}(u_n)\|_{BC}}{n} &\le \frac{\|\xi\|_0\big(1 + \nu_1^{\beta_1-1}\nu_2^{-\beta_1}\Gamma(2-\alpha)^{\beta_1}\sup_{t\ge 0}\int_0^t (t-\tau)^{-\beta_1(1-\alpha)}a(\tau)\, d\tau\big)}{n} \\
&\quad + \nu_1^{\beta_1-1}\nu_2^{-\beta_1}\Gamma(2-\alpha)^{\beta_1}\sup_{t\ge 0}\int_0^t (t-\tau)^{-\beta_1(1-\alpha)}a(\tau)\, d\tau.
\end{aligned} \tag{4.15}$$
Letting $n \to \infty$ in the inequality (4.15), we get a contradiction to (4.13). □

Lemma 4.3. *Let the hypotheses of Lemma 4.2 hold and $0 \le \rho(t) \le q$ for all $t \ge 0$. Then there exists $\eta > 0$ such that $\mathcal{G}(\mathbf{B}_R^\gamma(\eta)) \subset \mathbf{B}_R^\gamma(\eta)$ provided that (4.13) and*

$$\sup_{t\ge 0} \int_0^{t/2} (t-\tau)^{\gamma-\beta_1(1-\alpha)} a(\tau)\, d\tau < \infty, \tag{4.16}$$

and

$$2^\gamma \nu_1^{\beta_1-1}\nu_2^{-\beta_1}\Gamma(2-\alpha)^{\beta_1} \sup_{t\ge 0} \int_{t/2}^{t} (t-\tau)^{\gamma-\beta_1(1-\alpha)} a(\tau) d\tau < 1 \tag{4.17}$$

are satisfied, where $0 < \gamma < \min\{\beta_1(1-\alpha), \beta_2\alpha\}$ for some $\beta_2 \in (0,1]$.

Proof. Assume to the contrary that for each $n = 1, 2, \ldots$, there exists $u_n \in \mathbf{B}_R^\gamma(n)$ with

$$\sup_{t\ge 0} t^\gamma \|\mathcal{G}(u_n)(t)\| > n. \tag{4.18}$$

Then one has

$$\begin{aligned} t^\gamma\|\mathcal{G}(u_n)(t)\| & \\ &\le t^\gamma s_\alpha(t,\lambda_1)\|\xi(0)\| + t^\gamma \int_0^t r_\alpha(t-\tau,\lambda_1)\|f(\tau, u_n[\xi]_\rho(\tau))\| d\tau \\ &\le t^\gamma s_\alpha(t,\lambda_1)\|\xi\|_0 + t^\gamma \int_0^t r_\alpha(t-\tau,\lambda_1) a(\tau)\|u_n[\xi]_\rho(\tau)\| d\tau \\ &\le I_1(t) + I_2(t) + I_3(t), \end{aligned} \tag{4.19}$$

here

$$\begin{aligned} I_1(t) &= t^\gamma s_\alpha(t,\lambda_1)\|\xi\|_0, \\ I_2(t) &= t^\gamma \int_0^{t/2} r_\alpha(t-\tau,\lambda_1) a(\tau)\|u_n[\xi]_\rho(\tau)\| d\tau, \\ I_3(t) &= t^\gamma \int_{t/2}^{t} r_\alpha(t-\tau,\lambda_1) a(\tau)\|u_n[\xi]_\rho(\tau)\| d\tau. \end{aligned}$$

We now derive estimates for I_1, I_2, I_3. Clearly, $I_1(0) = 0$. For $t > 0$, using Proposition 2.1(i), (iii), we first obtain

$$\begin{aligned} \int_0^t \ell(\tau)\, d\tau &\ge \int_0^t \frac{d\tau}{\nu_1 + \nu_2\Gamma(\alpha)\tau^{1-\alpha}} \\ &\ge \int_0^t \frac{d\tau}{\nu_1 + \nu_2\Gamma(\alpha)t^{1-\alpha}} \\ &= \frac{t}{\nu_1 + \nu_2\Gamma(\alpha)t^{1-\alpha}}. \end{aligned}$$

Therefore

$$\begin{aligned} I_1(t) &\le \frac{t^\gamma\|\xi\|_0}{1 + \lambda_1\int_0^t \ell(\tau)\, d\tau} \\ &\le \|\xi\|_0 \lambda_1^{-\beta_2} t^\gamma \Big(\int_0^t \ell(\tau)\, d\tau \Big)^{-\beta_2} \\ &\le \|\xi\|_0 \lambda_1^{-\beta_2} t^{\gamma-\beta_2\alpha} (\nu_1 t^{\alpha-1} + \nu_2\Gamma(\alpha))^{\beta_2} \\ &< \infty, \end{aligned} \tag{4.20}$$

since

$$\lim_{t\to\infty} t^{\gamma-\beta_2\alpha}(\nu_1 t^{\alpha-1} + \nu_2\Gamma(\alpha))^{\beta_2} = 0.$$

For $I_2(t)$, exploiting the inequality (4.14), we have

$$I_2(t) \le t^\gamma \int_0^{t/2} r_\alpha(t-\tau,\lambda_1) a(\tau)\Big(\sup_{\theta\in[0,\tau]} \|u_n(\theta)\| + \|\xi\|_0\Big) d\tau$$

$$\leq \nu_1^{\beta_1-1}\nu_2^{-\beta_1}\Gamma(2-\alpha)^{\beta_1}(R+\|\xi\|_0)t^\gamma \int_0^{t/2}(t-\tau)^{-\beta_1(1-\alpha)}a(\tau)d\tau,$$

due to the fact that $u_n \in B_R$. Since $0 \leq \tau \leq \frac{t}{2}$, we get $t-\tau \geq 2^{-1}t$. Therefore, we arrive at

$$\begin{aligned}
I_2(t) &\leq 2^\gamma \nu_1^{\beta_1-1}\nu_2^{-\beta_1}\Gamma(2-\alpha)^{\beta_1}(R+\|\xi\|_0)\int_0^{t/2}(t-\tau)^{\gamma-\beta_1(1-\alpha)}a(\tau)d\tau \\
&\leq 2^\gamma \nu_1^{\beta_1-1}\nu_2^{-\beta_1}\Gamma(2-\alpha)^{\beta_1}(R+\|\xi\|_0)\sup_{t\geq 0}\int_0^{t/2}(t-\tau)^{\gamma-\beta_1(1-\alpha)}a(\tau)d\tau \\
&< \infty, \qquad (4.21)
\end{aligned}$$

thanks to (4.16).

Regarding $I_3(t)$, we see that $u_n \in B_R$ and $\|u_n[\xi]_\rho(t)\| \leq \|\xi\|_0 + \sup\limits_{\theta\in[0,t]}\|u_n(\theta)\|$ for $t \geq 0$. Then

$$t^\gamma\|u_n[\xi]_\rho(t)\| \leq (q+\epsilon)^\gamma(\|\xi\|_0+R),$$

for any fixed $\epsilon > 0$ and $t \in [0; q+\epsilon]$.

In addition, we have $t^\gamma\|u_n(t)\| \leq n$ and $0 \leq \rho(t) \leq q$ for all $t \geq 0$. Consequently, for $t \geq q+\epsilon$, one has

$$\begin{aligned}
t^\gamma\|u_n[\xi]_\rho(t)\| &= t^\gamma\|u_n(t-\rho(t))\| \\
&\leq (t-\rho(t))^\gamma\|u_n(t-\rho(t))\| + \rho^\gamma(t)\|u_n(t-\rho(t))\| \\
&\leq n + q^\gamma R.
\end{aligned}$$

We obtain

$$t^\gamma\|u_n[\xi]_\rho(t)\| \leq n + q^\gamma R + (q+\epsilon)^\gamma(\|\xi\|_0+R), \forall t \geq 0.$$

Combining with the formulation of $I_3(t)$ deduces

$$\begin{aligned}
I_3(t) &= t^\gamma\int_{t/2}^t \tau^{-\gamma}r_\alpha(t-\tau,\lambda_1)a(\tau)\tau^\gamma\|u_n[\xi]_\rho(\tau)\|d\tau \\
&\leq 2^\gamma\left[n+q^\gamma R+(q+\epsilon)^\gamma(\|\xi\|_0+R)\right]\int_{t/2}^t r_\alpha(t-\tau,\lambda_1)a(\tau)d\tau \\
&\leq C_1(n+C_2)\int_{t/2}^t(t-\tau)^{\gamma-\beta_1(1-\alpha)}a(\tau)d\tau \qquad (4.22)
\end{aligned}$$

where $C_1 = 2^\gamma\nu_1^{\beta_1-1}\nu_2^{-\beta_1}\Gamma(2-\alpha)^{\beta_1}$ and $C_2 = q^\gamma R + (q+\epsilon)^\gamma(\|\xi\|_0+R)$.

By (4.19) and (4.22), we get

$$\frac{1}{n}t^\gamma\|\mathcal{G}(u_n)(t)\| \leq \frac{I_1(t)+I_2(t)}{n} + \frac{C}{n} + C_1\sup_{t\geq 0}\int_{t/2}^t(t-\tau)^{\gamma-\beta_1(1-\alpha)}a(\tau)d\tau,$$

here $C = C_1C_2\sup\limits_{t\geq 0}\int_{t/2}^t(t-\tau)^{\gamma-\beta_1(1-\alpha)}a(\tau)d\tau$.

From (4.20), (4.21) and the assumption (4.17), we take the last inequality into account to get

$$\lim_{n\to\infty}\frac{1}{n}\sup_{t\geq 0}t^\gamma\|\mathcal{G}(u_n)(t)\| < 1,$$

which is a contradiction to (4.18). The proof is complete. □

The main result of this section is stated in the following theorem.

Theorem 4.5. *Let the hypotheses of Lemma 4.3 hold. Then there exists a mild solution of the system (1.1)–(1.2) which decays with the polynomial rate.*

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Proof. Lemma 4.3 allows us to consider

$$\mathcal{G} : \mathbf{B}^{\gamma}_{R}(\eta) \to \mathbf{B}^{\gamma}_{R}(\eta).$$

We now prove that $\mathcal{G}$ is χ^*- condensing. Let $D \subset \mathbf{B}^{\gamma}_{R}(\eta)$, we have

$$\chi^* \left(\mathcal{G}(D)\right) = d_{\infty} \left(\mathcal{G}(D)\right) + \chi_{\infty} \left(\mathcal{G}(D)\right).$$

Firstly, we estimate the term $d_{\infty} \left(\mathcal{G}(D)\right)$. For any element $u \in \mathcal{G}(D)$ and $T > 0$, one has $\|u(t)\| \leq t^{-\gamma}\eta$ for all $t \geq T$. That implies

$$\sup_{t \geq T} \|u(t)\| \leq T^{-\gamma}\eta.$$

Then

$$\begin{aligned} d_{\infty} \left(\mathcal{G}(D)\right) &= \lim_{T\to\infty} \sup_{u\in\mathcal{G}(D)} \sup_{t\geq T} \|u(t)\| \\ &= 0. \end{aligned}$$

Secondly, the term $\chi_{\infty} \left(\mathcal{G}(D)\right)$ is taken into account. For each $T > 0$, we have

$$\chi_T \left(\pi_T(\mathcal{G}(D))\right) \leq \chi_T \left(\pi_T(\mathcal{Q}_{\alpha} \circ N_f(D))\right),$$

where $N_f(D) := \{f(\cdot, v(\cdot - \rho(\cdot)) \mid v \in D\}$. Since $N_f(D)$ is a bounded set and $\mathcal{Q}_{\alpha}$ is a compact operator, we get $\chi_T \left(\pi_T(\mathcal{Q}_{\alpha} \circ N_f(D))\right) = 0$. Consequently, we have

$$\chi_{\infty} \left(\mathcal{G}(D)\right) = 0.$$

Finally, we see that if $\chi^*(D) \leq \chi^* \left(\mathcal{G}(D)\right)$ then $\chi^*(D) = 0$, so D is relatively compact. That means $\mathcal{G}$ is χ^*- condensing. By Theorem 4.4, $\mathcal{G}$ possesses a fixed point. The proof is complete. □

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