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GLOBAL STRONG SOLUTIONS TO THE 3D COMPRESSIBLE NAVIER–STOKES–KORTEWEG EQUATIONS WITH LARGE INITIAL DATA

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ABSTRACT. In this paper, we consider the Cauchy problem for the compressible Navier–Stokes–Korteweg equations in $\mathbb{R}^3$ and construct the global strong solutions to the equations with a class of large initial data satisfying some special conditions.

1. Introduction

In this paper, we investigate the Cauchy problem for the Navier–Stokes–Korteweg (NSK for short) system which was first rigorously derived by Dunn–Serrin in [11]. From a physical viewpoint, it allows to describe the motion of compressible fluids with capillarity effect of material [1, 3, 12]. The conservation of mass and of momentum writes:

$$\begin{cases} \partial_t\rho+\mathrm{div}(\rho u)=0, \\ \partial_t(\rho u)+\mathrm{div}(\rho u\otimes u)-\mathrm{div}\big(2\mu(\rho)\mathbb{D}u\big)-\nabla\big(\lambda(\rho)\mathrm{div}u\big)+\nabla P(\rho)=\mathrm{div}\mathbb{K}, \\ (u,\rho)|_{t=0}=(u_0,\rho_0), \\ \lim\limits_{|x|\to\infty}\big(u(t,x),\rho(t,x)\big)=(0,1). \end{cases} \tag{1.1}$$

Here $u=u(t,x)=\big(u_1(t,x),u_2(t,x),u_3(t,x)\big)\in\mathbb{R}^3$ denotes the velocity field and $\rho=\rho(t,x)\in\mathbb{R}^+$ is the density. The density-dependent functions $\mu(\rho)$ and $\lambda(\rho)$ (the shear and bulk viscosity coefficients of the flow) are supposed to be smooth enough and to fulfill the standard strong parabolicity assumption:

$$\mu>0 \quad\text{and}\quad 2\mu+\lambda>0.$$

The strain tensor $\mathbb{D}u=(\nabla u+\nabla^\top u)/2$ is the symmetric part of the velocity gradient. The barotropic assumption means that the pressure $P(\rho)$ depends only upon the density ρ of fluid and the function P is suitably smooth in what follows. The Korteweg tensor $\mathrm{div}\mathbb{K}$ allows to describe the variation of density at the interfaces between two phases, generally a mixture liquid-vapor, which can be written as follows:

$$\mathrm{div}\mathbb{K}=\nabla\left(\rho\kappa(\rho)\Delta\rho+\frac{\kappa(\rho)+\rho\kappa'(\rho)}{2}|\nabla\rho|^2\right)-\mathrm{div}\big(\kappa(\rho)\nabla\rho\otimes\nabla\rho\big), \tag{1.2}$$

where the regular function $\kappa(\rho)$ denotes the capillary coefficient.

There have been huge amount of literature on the study of NSK by many physicists and mathematicians due to its physical importance, complexity, rich phenomena and mathematical challenges.

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Bresch-Desjardins-Lin [2] proved the existence of global weak solution and then Haspot [19] improved their result. Hattori-Li [13, 14] obtained the local existence and global existence of classical solutions to the Cauchy problem for the initial data belong to $H^{s+1}(\mathbb{R}^d)\times H^s(\mathbb{R}^d)$ with $s\geq[d/2]+3$. Danchin-Desjardins [10] improved this result by working with small initial data in the framework of critical Besov spaces. Hou-Peng-Zhu [20] showed the global well-posedness of classical solutions to the 3D compressible fluid models of Korteweg type when the initial total energy is small and improved the results obtained by Hattori-Li [13, 14]. Kotschote [22] proved the local existence of strong solutions in a bounded domain. Haspot [17] considered the cases where the viscosity coefficients $\mu(\rho),\lambda(\rho)$ and the pressure $P(\rho)$ linearly depends on the density for System (1.1)-(1.2) with $\kappa(\rho)=\frac{\kappa}{\rho}$, and obtained global solutions with suitable small initial data in the L^2 framework. Subsequently, Haspot [18] continued to investigate the Cauchy problem for System (1.1)-(1.2) with $\big(\mu(\rho),\lambda(\rho),P(\rho)\big)=(\mu\rho,0,\rho)$, and established global existence under the setting of slightly subcritical L^p type initial data, where the specific choice of the pressure is crucial since it provides a gain of integrability on the effective velocity. Following the assumptions on the viscosity coefficients of [18], Yu-Wu [27] established the global well-posedness of strong solutions to 2D NSK with nonvacuum and general pressure laws in the framework of Sobolev spaces. Chikami-Kobayashi [9] obtained global solutions to NSK under linear stability conditions in critical Besov spaces and the optimal decay rates of the global solutions in the $L^2(\mathbb{R}^d)$-framework. Kobayashi-Tsuda [21] proved the existence of global L^2 solutions for the NSK around a constant state and obtained parabolic type decay rate of the solutions. Murata-Shibata [25] proved that NSK admits a unique, global strong solutions for small initial data in $\mathbb{R}^d$ with $3\leq d\leq 7$ by the maximal L^p-L^q regularity and L^p-L^q decay properties of solutions to the linearized equations. For results on non-local capillary terms and convergence to various models, we refer to the works by Charve and Haspot [5, 6, 7, 15]

To motivate our results, we briefly review some examples of large initial data generating global strong solutions. Lei-Lin-Zhou [23] obtained the global well-posedness for incompressible Navier-Stokes equation in energy space with a class of large initial data which includes the Beltrami flow. When the Korteweg tensor $\operatorname{div}\mathbb{K}$ is neglected, System (1.1) reduces to the classical compressible Navier-Stokes (CNS) equations. Charve-Danchin [4] and Chen-Miao-Zhang [8] constructed global solutions of CNS equations with such kind of the highly oscillating initial data. Recently, Li et al. [24] constructed global smooth solutions to 3D CNS equations with a class of special initial data, where the initial velocity u_0 in $\dot{B}^{-1}_{\infty,\infty}$ can be arbitrarily large while the initial density ρ_0-1 is small in H^3. Following the assumptions on the viscosity coefficients of [18], Zhai-Li [31] proved global solutions to System (1.1) without smallness condition imposed on the vertical component of the incompressible part of the velocity by using the weighted Chemin-Lerner-norm technique. Zhang [30] constructed a class of global large solutions to the compressible NSK system with constant viscosity coefficients in critical Besov spaces. Recently, by assuming $\mu(\rho)=\mu\rho^2$ and $\lambda(\rho)=(\lambda-2\mu)\rho^2+\frac{\kappa}{\lambda}\rho$ and introducing "the effective velocity" which was successfully used in Haspot's works [16, 18], Yu-Li-Wu [28] constructed global smooth solutions to NSK with a class of special initial data, where the initial velocity in $L^\infty(\mathbb{R}^3)$ can be arbitrarily large while the initial data ρ_0-1 is small in $H^3(\mathbb{R}^3)$. Subsequently, Yu-Yang-Wu [29] proved that both large initial data (ρ_0-1,u_0) in $L^2(\mathbb{R}^3)$ can generate global classical solutions to NSK with the above assumption. We should mention that the special choice of $\lambda(\rho)$ in [28, 29] makes both the new density and velocity equations parabolic. Question appears: For general smooth functions $\mu(\rho)$ and $\lambda(\rho)$, does NSK possess global solutions with both large initial data (ρ_0-1,u_0)? In this paper, we shall construct the global strong solutions to NSK with a class of large initial data. Here the "large" means that both the L^∞-norm of initial velocity u_0 and the L^1-norm of initial data ρ_0-1 can be arbitrarily large (see Remark 1.4), or both the L^2-norm of initial data (ρ_0-1,u_0) can be arbitrarily large (see Remark 1.5). Our main idea is splitting the linearized equations from NSK and exploring the damping effect of the linearized system with initial data whose Fourier frequency is supported in the small annulus.

1.1. ***Reformulation of System.*** The main difficulties in the study of the compressible fluid flows when dealing with vacuum is that the momentum equation loses its parabolic regularizing effect, that is why in the present paper we suppose that the initial data ρ_0 is a small perturbation of an equilibrium state $\bar\rho=1$ (just for convenience). In this paper, we take the specific choice on the coefficient (assume that μ,λ,κ are positive constant)

$$\mu(\rho)=\mu\rho^2,\quad \lambda(\rho)=\lambda\rho^2\quad\text{and}\quad \kappa(\rho)=\kappa. \tag{1.3}$$

We obtain from (1.2) that

$$\mathrm{div}\mathbb{K}=\kappa\rho\nabla\Delta\rho.$$

Due to the momentum equations $(1.1)_2$, one has

$$\rho\left(\partial_t u+u\cdot\nabla u-\mu\rho(\Delta u+\nabla\mathrm{div}u)-4\mu\nabla\rho\cdot\mathbb{D}u\right)-\lambda\rho^2\nabla\mathrm{div}u-2\lambda\rho\nabla\rho\mathrm{div}u+\nabla P(\rho)=\kappa\rho\nabla\Delta\rho,$$

hence, as long as ρ does not vanish, which reduces to

$$\partial_t u+u\cdot\nabla u-\mu\rho\Delta u-\big(\mu+\lambda\big)\rho\nabla\mathrm{div}u-4\mu\nabla\rho\cdot\mathbb{D}u-2\lambda\nabla\rho\mathrm{div}u+\rho^{-1}\nabla P(\rho)=\kappa\nabla\Delta\rho.$$

Denoting $\widetilde\rho:=\rho-1$, we can reformulate system (1.1) equivalently as follows

$$\begin{cases}\partial_t\widetilde\rho+\mathrm{div}u=-\mathrm{div}(\widetilde\rho u),\\ \partial_t u+u\cdot\nabla u-\mu\Delta u-\big(\mu+\lambda\big)\nabla\mathrm{div}u-\kappa\nabla\Delta\widetilde\rho+\nabla\widetilde\rho=\mathbf{S}(\widetilde\rho,u)+K(\widetilde\rho)\nabla\widetilde\rho,\\ (\widetilde\rho,u)|_{t=0}=(\widetilde\rho_0,u_0),\\ \lim\limits_{|x|\to\infty}\big(\widetilde\rho(t,x),u(t,x)\big)=(0,0),\end{cases} \tag{1.4}$$

here and in what follows, for notational simplicity, we denote

$$K(s)=1-\frac{P'(1+s)}{1+s}\quad\text{and}\quad \mathbf{S}(\widetilde\rho,u)=4\mu\nabla\widetilde\rho\cdot\mathbb{D}u+2\lambda\nabla\widetilde\rho\mathrm{div}u+\mu\widetilde\rho\Delta u+\big(\mu+\lambda\big)\widetilde\rho\nabla\mathrm{div}u.$$

In this paper, we assume that $P'(1)=1$ without loss of generality.

The investigation with the linearization of (1.4) is given by

$$\begin{cases}\partial_t\Theta+\mathrm{div}U=0,\\ \partial_t U-\mu\Delta U-(\mu+\lambda)\nabla\mathrm{div}U-\kappa\nabla\Delta\Theta+\nabla\Theta=0,\\ (\Theta,U)|_{t=0}=(\Theta_0,u_0).\end{cases} \tag{1.5}$$

Introducing the new unknowns

$$\phi:=\widetilde\rho-\Theta\quad\text{and}\quad w:=u-U,$$

then System (1.4) can be rewritten as follows

$$\begin{cases} \partial_t \phi + \operatorname{div} w = -\operatorname{div}((\phi+\Theta)(w+U)), \\ \partial_t w - \mu\Delta w - (\lambda+\mu)\nabla \operatorname{div} w - \kappa\nabla\Delta\phi + \nabla\phi = K(\widetilde{\rho})\nabla\phi + K(\widetilde{\rho})\nabla\Theta + \mathbf{F_1} + \mathbf{F_2} + \mathbf{F_3}, \\ (\phi, w)|_{t=0} = (0,0), \end{cases} \tag{1.6}$$

where

$$\begin{aligned} \mathbf{F_1} &= -w\cdot\nabla w + \mathbf{S}(\phi,w), \\ \mathbf{F_2} &= -U\cdot\nabla w - w\cdot\nabla U + \mathbf{S}(\phi,U) + \mathbf{S}(\Theta,w), \\ \mathbf{F_3} &= -U\cdot\nabla U + \mathbf{S}(\Theta,U). \end{aligned}$$

1.2. *Statement of Main Result.* Our main goal is to establish the global strong solutions to (1.1) for a class of large initial data. Throughout the paper, when no vacuum is considered, we focus on the new system (1.6) since it is equivalent to the original system (1.1) under the assumptions (1.3). The main result of our paper reads as follows:

Theorem 1.1. *Let $\mu, \kappa > 0$ and $\nu := \mu + \lambda/2 > 0$. Assume that (Θ_0, U_0) satisfies*

$$U_0 = (\partial_2 a, -\partial_1 a, 0) \quad \textit{with} \quad \operatorname{supp}\widehat{\Theta_0},\ \operatorname{supp}\widehat{a} \subset \mathscr{C},$$

where a is scalar functions and

$$\mathscr{C} = \begin{cases} \mathscr{C}_1 := \left\{\xi\in\mathbb{R}^3 : |\xi_1-\xi_2| \le \varepsilon,\ 0<\varepsilon\ll 1,\ 1\le|\xi|\le 3\right\}, & \text{if } \nu^2 \le \kappa, \\ \mathscr{C}_2 := \left\{\xi\in\mathbb{R}^3 : |\xi_1-\xi_2| \le \varepsilon,\ 0<\varepsilon\ll 1,\ \frac{1}{\sqrt{\nu^2-\kappa}} < |\xi| \le \frac{2}{\sqrt{\nu^2-\kappa}}\right\}, & \text{if } \nu^2 > \kappa, \end{cases} \tag{1.7}$$

there exists a sufficiently small positive constant $\varepsilon_0 = \varepsilon_0(\mu,\kappa,\varepsilon)$ such that if

$$\left(\|\widehat{\Theta}_0\|_{L^1}^2 + \|\widehat{\Theta}_0, \widehat{a}\|_{L^1}\|\Theta_0\|_{L^2} + \varepsilon\|\widehat{a}\|_{L^1}\|a\|_{L^2}\right)\exp\left(C\left(\|\widehat{\Theta}_0,\widehat{a}\|_{L^1} + \|\Theta_0\|_{L^2}^2\right)\right) \le \varepsilon_0, \tag{1.8}$$

then system (1.6) has a unique global strong solution (w,ϕ) in $\mathbb{R}^3\times(0,\infty)$ satisfying that for any $0<T<\infty$

$$\begin{cases} w \in L^\infty([0,T];H^1), & \nabla w \in L^2([0,T];H^1), \\ \phi \in L^\infty([0,T];H^2), & \nabla\phi \in L^2([0,T];H^2). \end{cases} \tag{1.9}$$

Remark 1.1. *We can also have a version of Theorem 1.1 for any smooth functions $\mu(\rho)$ and $\lambda(\rho)$. Just for a clear presentation, we choose to work in the special case $\mu(\rho) = \mu\rho^2$ and $\lambda(\rho) = \lambda\rho^2$ and the divergence-free initial data in the present paper. Furthermore, the solution obtained in Theorem 1.1 indeed possess more high regularity and can be smooth.*

Remark 1.2. *We should mention that, Theorem 1.1 is different from our previous one in* [29]. *On the one hand, we do not require the strong restriction on the coefficient $\kappa \ne \lambda^2$. On the other hand, to "kill" the third-order derivative term $\nabla\Delta\rho$ in* [29], *we have to assume the algebraic relation $\lambda\big((2\mu-\lambda)\rho + \rho^{-1}\lambda(\rho)\big) = \kappa$, here we drop this special relation and prove Theorem 1.1 holds for general smooth functions $\mu(\rho)$ and $\lambda(\rho)$.*

Remark 1.3. *Compared with the previous result in [29] where the initial data both can be arbitrarily large in L^2, Theorem 1.1 also allows that, both initial velocity with $\|u_0\|_{L^\infty} \gg 1$ and density with $\|\widetilde{\rho}_0\|_{L^1} \gg 1$, generates a unique global solution to the 3D compressible Navier–Stokes–Korteweg equations. We refer to Remark 1.4 below for the new construction of initial data.*

Remark 1.4. *Theorem 1.1 implies that some initial data with $\|U_0\|_{L^\infty} \gg 1$ and $\|\Theta_0\|_{L^1} \gg 1$ can generate a unique global solution to (1.6). We just consider the case $\nu^2 \le \kappa$ since the case $\nu^2 > \kappa$ can be done by the construction in [29]. This kind of initial data can be constructed as follows. We set*

$$\Theta_0 = \varepsilon^{\frac{1}{p}} a \quad \textit{and} \quad U_0 = (\partial_2 a, -\partial_1 a, 0)$$

with $1 < p < 2, 0 < \varepsilon \ll 1$ and

$$\widehat{a}(\xi_1, \xi_2, \xi_3) = \varepsilon^{-1} \left(\log\log \frac{1}{\varepsilon}\right)^{\frac{1}{2}} \widehat{\varphi}(\xi_1, \xi_2) \widehat{\psi}(\xi_3),$$

here two even functions $\widehat{\varphi} \in \mathscr{C}_0^\infty(\mathbb{R}^2)$ and $\widehat{\psi} \in \mathscr{C}_0^\infty(\mathbb{R})$ both taking values in $[0,1]$ such that

$$\begin{aligned}
&\mathrm{supp}\ \widehat{\varphi} \subset \left\{\xi_h = (\xi_1, \xi_2) :\ |\xi_1 - \xi_2| \le \varepsilon \ll 1,\ \frac{8}{9} \le |\xi_h|^2 \le \frac{9}{8}\right\},\\
&\widehat{\varphi}(\xi_h) \equiv 1 \quad \text{for} \quad \xi_h \in \left\{|\xi_1 - \xi_2| \le \frac{\varepsilon}{2},\ \frac{17}{18} \le |\xi_h|^2 \le \frac{17}{16}\right\},\\
&\mathrm{supp}\ \widehat{\psi} \subset \left\{\xi_3 :\ \frac{8}{9} \le |\xi_3|^2 \le \frac{9}{8}\right\} \quad \text{and} \quad \widehat{\psi}(\xi_3) \equiv 1,\ |\xi_3|^2 \in \left[\frac{17}{18}, \frac{17}{16}\right].
\end{aligned}$$

By simple calculations, one has

$$\int_{\mathbb{R}^2} \widehat{\varphi}(\xi_1, \xi_2) \mathrm{d}\xi_h \approx \varepsilon \quad \text{and} \quad \int_{\mathbb{R}} \widehat{\psi}(\xi_3) \mathrm{d}\xi_3 \approx 1,$$

which in turn gives that

$$\|\widehat{U}_0\|_{L^1} \approx \|\widehat{a}\|_{L^1} \approx \left(\log\log \frac{1}{\varepsilon}\right)^{\frac{1}{2}} \quad \text{and} \quad \|U_0\|_{L^2} \approx \|a\|_{L^2} \approx \varepsilon^{-\frac{1}{2}} \left(\log\log \frac{1}{\varepsilon}\right)^{\frac{1}{2}}.$$

Equivalently,

$$\|\widehat{\Theta}_0\|_{L^1} \approx \varepsilon^{\frac{1}{p}} \left(\log\log \frac{1}{\varepsilon}\right)^{\frac{1}{2}} \quad \text{and} \quad \|\Theta_0\|_{L^2} \approx \varepsilon^{\frac{2-p}{2p}} \left(\log\log \frac{1}{\varepsilon}\right)^{\frac{1}{2}}.$$

Also, by Hausdorff-Young's inequality, we have

$$\|\Theta_0\|_{L^p} \gtrsim \|\widehat{\Theta}_0\|_{L^{\frac{p}{p-1}}} \gtrsim \left(\log\log \frac{1}{\varepsilon}\right)^{\frac{p-1}{2p}}$$

and (for more details see [24])

$$\|U_0\|_{L^\infty} \gtrsim \left(\log\log \frac{1}{\varepsilon}\right)^{\frac{1}{2}}.$$

By the classical interpolation inequality $\|\Theta_0\|_{L^p}^p \lesssim \|\Theta_0\|_{L^1}^{2-p}\|\Theta_0\|_{L^2}^{2(p-1)}$ *for* $p\in(1,2)$*, we have*

$$\|\Theta_0\|_{L^1} \gtrsim \varepsilon^{\frac{1-p}{p}}\left(\log\log\frac{1}{\varepsilon}\right)^{\frac{p-1}{2(p-2)}} \to +\infty \quad as \quad \varepsilon\to 0^+.$$

Furthermore, we have

$$\text{LHS of (1.8)} \approx \left(\varepsilon^{\frac{2}{p}}+\varepsilon^{\frac{1}{p}-\frac{1}{2}}+\varepsilon^{\frac{1}{2}}\right)\left(\log\log\frac{1}{\varepsilon}\right)\exp\left(C\left(\log\log\frac{1}{\varepsilon}\right)^{\frac{1}{2}}\right).$$

Therefore, choosing ε *small enough, we deduce that the 3D compressible Navier–Stokes–Korteweg equations* (1.6) *has a unique global solution.*

Remark 1.5. *Theorem 1.1 also implies that some initial data with* $\|U_0\|_{L^2}\gg 1$ *and* $\|\Theta_0\|_{L^2}\gg 1$ *can generate a unique global solution to* (1.6)*. We set*

$$\Theta_0 = a \quad and \quad U_0 = (\partial_2 a, -\partial_1 a, 0) \quad with \quad \widehat{a}(\xi_1,\xi_2,\xi_3) = \varepsilon^{-\frac{1}{2}}\left(\log\log\frac{1}{\varepsilon}\right)^{\frac{1}{2}}\widehat{\varphi}(\xi_1,\xi_2)\widehat{\psi}(\xi_3),$$

here two even functions $\widehat{\varphi}\in\mathscr{C}_0^\infty(\mathbb{R}^2)$ *and* $\widehat{\psi}\in\mathscr{C}_0^\infty(\mathbb{R})$ *are defined as above.*

Following the above argument, then one has

$$\|\widehat{\Theta}_0\|_{L^1} \approx \|\widehat{U}_0\|_{L^1} \approx \|\widehat{a}\|_{L^1} \approx \varepsilon^{\frac{1}{2}}\left(\log\log\frac{1}{\varepsilon}\right)^{\frac{1}{2}}$$

and

$$\|\Theta_0\|_{L^2} \approx \|U_0\|_{L^2} \approx \|a\|_{L^2} \approx \left(\log\log\frac{1}{\varepsilon}\right)^{\frac{1}{2}}.$$

Furthermore, we have

$$\text{LHS of (1.8)} \approx \varepsilon^{\frac{1}{2}}\left(\log\log\frac{1}{\varepsilon}\right)\exp\left(C\log\log\frac{1}{\varepsilon}\right).$$

Therefore, choosing ε *small enough, we deduce that the 3D compressible Navier–Stokes–Korteweg equations* (1.6) *has a unique global solution.*

Remark 1.6. *We also emphasize that Theorem 1.1 holds for the case* $\kappa=0$*, that is,* $\mathrm{div}\mathbb{K}=0$*. As mentioned above,* (1.1) *with* $\kappa=0$ *becomes the CNS equations, thus Theorem 1.1 with minor modifications holds for the 3D compressible Navier-Stokes equations.*

Remark 1.7. *When assuming that* $\kappa\neq 0$ *and* $P'(1)=0$*, we still can explore the damping effect for the* Θ*-equation by following this present method. However, we will encounter some difficulties. Particularly, to establish the desired a prior bounds, we have no way to cancel the term* $\mathrm{div}w$ *in the* ϕ*-equation. Thus, we have to leave it as an interesting problem and consider it in the future.*

1.3. *Organization of the Paper.* The rest of this paper is organized as follows: In Section 2, we establish the key exponential decay in time for (U,Θ) which will play a crucial role in the proof of our main theorem. In Section 3, we obtain the global-in-time a priori estimates which are sufficient to prove Theorem 1.1.

2. Preliminaries

2.1. ***Notation.*** Firstly, we introduce some notations and conventions which will be used throughout this paper. $a \approx b$ means $C^{-1}b \leq a \leq Cb$ for some positive harmless constants C. We will use the simplified notation $\|f_1, \cdots, f_n\|_X = \|f_1\|_X + \cdots + \|f_n\|_X$ for some Banach space X. $\langle f, g\rangle$ denotes the inner product in $L^2(\mathbb{R}^3)$, namely, $\langle f, g\rangle = \int_{\mathbb{R}^3} f \cdot g \mathrm{d}x$. The Fourier transform of f with respect to the space variable is given by $\mathscr{F} f(\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}^3} e^{-\mathrm{i}x\cdot\xi} f(x)\mathrm{d}x$. Let $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3$ be a multi-index and $D^\alpha = \partial^{|\alpha|}/\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{x_3}^{\alpha_3}$ with $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$. For $m \in \mathbb{N}$, the norms of the integer order Sobolev space $H^m(\mathbb{R}^3)$ and $W^{m,\infty}(\mathbb{R}^3)$ are defined by $\|f\|_{H^m(\mathbb{R}^3)} \approx \|f\|_{\dot{H}^m(\mathbb{R}^3)} + \|f\|_{L^2(\mathbb{R}^3)}$ and $\|f\|_{W^{m,\infty}(\mathbb{R}^3)} := \sum_{0\leq i\leq m} \|\nabla^i f\|_{L^\infty(\mathbb{R}^3)}$.

2.2. ***Exponential decay.*** Setting

$$W := \mathrm{div} U \quad \text{with} \quad W_0 = \mathrm{div} U_0 = 0,$$

then we deduce from (1.5) that

$$\begin{cases} \Theta_t + W = 0, \\ W_t - 2\nu\Delta W - \kappa\Delta^2\Theta + \Delta\Theta = 0, \\ (\Theta, W)|_{t=0} = (\Theta_0, 0). \end{cases} \tag{2.1}$$

Next, motivated the idea from [8], we shall give the explicit expression of $(\widehat{W}, \widehat{\Theta})$.

Lemma 2.1. *Assume that* (Θ, W) *solves* (2.1), *for* $\xi \in \mathscr{C}$ *given by* (1.7), *we have*

$$\widehat{\Theta}(\xi, t) = \left(\frac{\lambda_+ e^{\lambda_- t} - \lambda_- e^{\lambda_+ t}}{\lambda_+ - \lambda_-}\right)\widehat{\Theta}_0(\xi),$$

$$\widehat{W}(\xi, t) = (\kappa|\xi|^4 + |\xi|^2)\left(\frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-}\right)\widehat{\Theta}_0(\xi),$$

where $\lambda_\pm$ *are given by* (2.5) *below.*

Proof. Applying the operator Δ to $(2.1)_1$ gives

$$(\Delta\Theta)_t = -\Delta W. \tag{2.2}$$

From (2.2) and $(2.1)_1$, we get

$$\begin{cases} \Theta_{tt} - 2\nu(\Delta\Theta)_t + \kappa\Delta^2\Theta - \Delta\Theta = 0, \\ W_{tt} - 2\nu(\Delta W)_t + \kappa\Delta^2 W - \Delta W = 0. \end{cases} \tag{2.3}$$

Taking the Fourier transform of $(2.3)_1$ and $(2.3)_2$, we have

$$\begin{cases} \widehat{\Theta}_{tt} + 2\nu|\xi|^2\widehat{\Theta}_t + (\kappa|\xi|^4 + |\xi|^2)\widehat{\Theta} = 0, \\ \widehat{W}_{tt} + 2\nu|\xi|^2\widehat{W}_t + (\kappa|\xi|^4 + |\xi|^2)\widehat{W} = 0, \\ \widehat{\Theta}(\xi, 0) = \widehat{\Theta}_0(\xi), \quad \widehat{\Theta}_t(\xi, 0) = 0, \\ \widehat{W}(\xi, 0) = 0, \quad \widehat{W}_t(\xi, 0) = (\kappa|\xi|^4 + |\xi|^2)\widehat{\Theta}_0(\xi). \end{cases} \tag{2.4}$$

Straightforward calculations give two roots of the corresponding characteristic equations as follows

$$\lambda_{\pm} = -\nu|\xi|^2 \pm \mathrm{i}|\xi|\alpha(\xi), \tag{2.5}$$

where

$$\alpha(\xi) := \begin{cases} \sqrt{1-(\nu^2-\kappa)|\xi|^2}, & \text{if} \quad \nu^2 < \kappa, \\ 1, & \text{if} \quad \nu^2 = \kappa, \\ \sqrt{(\nu^2-\kappa)|\xi|^2 - 1}, & \text{if} \quad \nu^2 > \kappa. \end{cases}$$

We should mention that $\alpha(\xi)$ may not be real if $\nu^2 > \kappa$, which is the reason why we require that $|\xi| > \frac{1}{\sqrt{\nu^2-\kappa}}$.

Thus, the solution of (2.4) has the form

$$\begin{cases} \widehat{\Theta}(\xi,t) = A_1(\xi)e^{\lambda_- t} + B_1(\xi)e^{\lambda_+ t}, \\ \widehat{W}(\xi,t) = A_2(\xi)e^{\lambda_- t} + B_2(\xi)e^{\lambda_+ t}. \end{cases} \tag{2.6}$$

Using the initial conditions, we obtain

$$A_1 = \frac{\lambda_+}{\lambda_+ - \lambda_-}\widehat{\Theta}_0 \quad \text{and} \quad B_1 = -\frac{\lambda_-}{\lambda_+ - \lambda_-}\widehat{\Theta}_0,$$
$$A_2 = -\frac{\kappa|\xi|^4 + |\xi|^2}{\lambda_+ - \lambda_-}\widehat{\Theta}_0 \quad \text{and} \quad B_2 = \frac{\kappa|\xi|^4 + |\xi|^2}{\lambda_+ - \lambda_-}\widehat{\Theta}_0.$$

Plugging the above into (2.6) yields the desired results of Lemma 2.1.

Applying the operator curl to $(1.5)_2$, we also have

$$\partial_t \mathrm{curl} U - \mu\Delta \mathrm{curl} U = 0,$$

which gives that

$$\mathrm{curl} U = e^{\mu t\Delta}\mathrm{curl} U_0. \tag{2.7}$$

Due to the basic vector identity

$$\Delta U = \nabla \mathrm{div} U - \mathrm{curlcurl} U = \nabla W - e^{\mu t\Delta}\mathrm{curlcurl} U_0,$$

this gives

$$U = -(-\Delta)^{-1}\nabla W + (-\Delta)^{-1}e^{\mu t\Delta}\mathrm{curlcurl} U_0 = -(-\Delta)^{-1}\nabla W + e^{\mu t\Delta}U_0. \tag{2.8}$$

Therefore, we deduce

Lemma 2.2. *Let $\lambda_{\pm}$ be given by (2.5). Assume that (U,Θ) solves (1.5) with $\mathrm{div} U_0 = 0$, for $\xi \in \mathscr{C}$ given by (1.7), we have*

$$\widehat{U}(t,\xi) = -(\kappa|\xi|^2+1)\left(\frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-}\right)\widehat{\nabla\Theta}_0(\xi) + \widehat{V}(t,\xi), \tag{2.9}$$

$$\widehat{\Theta}(t,\xi) = \left(\frac{\lambda_+ e^{\lambda_- t} - \lambda_- e^{\lambda_+ t}}{\lambda_+ - \lambda_-}\right)\widehat{\Theta}_0(\xi), \tag{2.10}$$

here and in what follows we denote $V(t,x) := e^{\mu\Delta t}U_0(x)$.

By restricting the Fourier frequency to the annulus given by (1.7), we can obtain

Lemma 2.3. *Let $\lambda_\pm$ be given by (2.5) and $\xi \in \mathscr{C}$ be given by (1.7). Then*

$$\left|\frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-}\right| + \left|\frac{\lambda_+ e^{\lambda_- t} - \lambda_- e^{\lambda_+ t}}{\lambda_+ - \lambda_-}\right| \le C(1+t)e^{-c_0 t},$$

where C and c_0 are two positive constants which depend on κ and ν.

Proof. Straightforward calculations yields

$$\left|\frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-}\right| = e^{-\nu t|\xi|^2} \cdot \left|\frac{\sin\big(t|\xi|\alpha(\xi)\big)}{2|\xi|\alpha(\xi)}\right| \le t e^{-\nu t|\xi|^2} \le Cte^{-c_0 t}.$$

Due to (2.5), one has

$$|\lambda_-|^2 = \nu^2|\xi|^4 + |\xi|^2\alpha^2(\xi).$$

Thus

- if $\nu^2 \le \kappa$ and $\xi \in \mathscr{C}_1$, then $|\lambda_-|^2 = |\xi|^2 + \kappa|\xi|^4 \le C(\kappa)$;
- if $\nu^2 > \kappa$ and $\xi \in \mathscr{C}_2$, then $|\lambda_-|^2 = (2\nu^2 - \kappa)|\xi|^4 - |\xi|^2 \le C(\kappa, \nu)$.

In summary, in either case above, we obtain

$$\begin{aligned}\left|\frac{\lambda_+ e^{\lambda_- t} - \lambda_- e^{\lambda_+ t}}{\lambda_+ - \lambda_-}\right| &\le \left|e^{\lambda_- t}\right| + |\lambda_-|\left|\frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-}\right| \\ &\le e^{-\nu t|\xi|^2} + Cte^{-\nu t|\xi|^2} \\ &\le C(1+t)e^{-c_0 t}.\end{aligned}$$

This ends the proof of Lemma 2.3.

We should emphasize that, although the Θ-equation has no dissipation, we still can explore the damping effect for the Θ-equation. Based on the above Lemma, we can establish the exponential decay in time for (U, Θ) which will play a crucial role in the proof of Theorem 1.1.

Lemma 2.4. *Under the assumptions of Theorem 1.1, for all $m \in \mathbb{N}$, there exists positive constants C, c_0, μ_0 such that*

$$\begin{aligned}&\|\nabla^m U\|_{L^\infty} \le C\|\widehat{U}\|_{L^1} \le C(1+t)(e^{-c_0 t} + e^{-\mu_0 t})\|\widehat{\Theta}_0, \widehat{a}\|_{L^1},\\ &\|\nabla^m \Theta\|_{L^\infty} \le C\|\widehat{\Theta}\|_{L^1} \le C(1+t)e^{-c_0 t}\|\widehat{\Theta}_0\|_{L^1},\\ &\|\nabla^m U\|_{L^2} \le C\|U\|_{L^2} \le C(1+t)(e^{-c_0 t} + e^{-\mu_0 t})\|\Theta_0, a\|_{L^2},\\ &\|\nabla^m \Theta\|_{L^2} \le C\|\Theta\|_{L^2} \le C(1+t)e^{-c_0 t}\|\Theta_0\|_{L^2},\end{aligned}$$

where

$$\mu_0 = \begin{cases} \frac{\mu}{\nu^2 - \kappa}, & \textit{if} \quad \nu^2 > \kappa, \\ \mu, & \textit{if} \quad \nu^2 \le \kappa. \end{cases}$$

Proof. Due to (2.9), then using the fact $\|\mathbf{f}\|_{L^\infty} \le C\|\widehat{\mathbf{f}}\|_{L^1}$ and the support condition of $(\widehat{\Theta}_0, \widehat{a}_0)$, by Lemma 2.3 we have

$$\begin{aligned} \|\nabla^m U\|_{L^\infty} &\lesssim \big\||\xi|^m \widehat{U}(t,\xi)\big\|_{L^1} \lesssim \big\|\widehat{U}(t,\xi)\big\|_{L^1} \\ &\lesssim \left\|\frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-}\widehat{\Theta}_0\right\|_{L^1} + \left\|e^{-\mu|\xi|^2 t}\widehat{U}_0\right\|_{L^1} \\ &\lesssim (1+t)(e^{-c_0 t} + e^{-\mu_0 t})\|\widehat{\Theta}_0, \widehat{a}\|_{L^1}. \end{aligned} \tag{2.11}$$

Thanks to similar argument, we obtain the rest of the estimates and end the proof of Lemma 2.4.

Lemma 2.5. *Under the assumptions of Theorem 1.1, we have*

$$\|U\cdot\nabla U\|_{H^3} \le C\varepsilon e^{-\mu_0 t}\|\widehat{a}\|_{L^1}\|a\|_{L^2} + C(1+t)e^{-\mu_0 t}\|\widehat{a}, \widehat{\Theta}_0\|_{L^1}\|\Theta_0\|_{L^2}.$$

Proof. Direct calculations show that

$$\begin{aligned} &U\cdot\nabla U = V\cdot\nabla V + V\cdot\nabla(U-V) + (U-V)\cdot\nabla U, \\ &V\cdot\nabla V^1 = V^1\partial_1 V^1 + V^2\partial_2 V^1 = (V^1+V^2)\partial_1 V^1 - V^2(\partial_1-\partial_2)V^1, \\ &V\cdot\nabla V^2 = V^1\partial_1 V^2 + V^2\partial_2 V^2 = (V^1+V^2)\partial_2 V^2 + V^1(\partial_1-\partial_2)V^2, \\ &V\cdot\nabla V^3 = 0. \end{aligned}$$

Note that $U_0 = (\partial_2 a, -\partial_1 a, 0)$ and $V = e^{\mu\Delta t}U_0$, using Hölder's inequality yields

$$\begin{aligned} \|V\cdot\nabla V\|_{H^3} &\le \big\|(V^1+V^2)\partial_1 V^1,\, (V^1+V^2)\partial_2 V^2\big\|_{H^3} + \big\|V^2(\partial_1-\partial_2)V^1,\, V^1(\partial_1-\partial_2)V^2\big\|_{H^3} \\ &\le \|V^1+V^2\|_{W^{3,\infty}}\|\partial_1 V^1, \partial_2 V^2\|_{H^3} + \|(\partial_1-\partial_2)(V^1,V^2)\|_{W^{3,\infty}}\|V^1, V^2\|_{H^3} \\ &\le C\big\||\xi_1-\xi_2|e^{-\mu|\xi|^2 t}\widehat{a}(\xi)\big\|_{L^1}\big\|e^{-\mu|\xi|^2 t}\widehat{a}(\xi)\big\|_{L^2} \\ &\le C\varepsilon e^{-\mu_0 t}\|\widehat{a}\|_{L^1}\|a\|_{L^2}, \end{aligned}$$

where we have used the condition (1.7) in the last step.

By Lemmas 2.2-2.4, we have

$$\|V\cdot\nabla(U-V)\|_{H^3} + \|(U-V)\cdot\nabla U\|_{H^3} \le C(1+t)e^{-\mu_0 t}\|\widehat{a}\|_{L^1}\|\Theta_0\|_{L^2},$$

where we have used that

$$\sum_{i=0}^{4}\|\nabla^i(U-V)\|_{L^2} \le C\left\|\left(\frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-}\right)\widehat{\nabla\Theta_0}\right\|_{L^2} \le C(1+t)\|\Theta_0\|_{L^2}.$$

Thus, we end the proof of Lemma 2.5.

3. Proof of Theorem 1.1

By the standard local well-posedness theory (see [22, 13, 14] for example), we can obtain that there exists a unique strong solution to (1.6) on some time interval $[0, T^*)$, where T^* is the lifespan of solution. We shall prove $T^* = \infty$, which is enough to prove Theorem 1.1. For the sake of implicity, we will introduce the following notations

$$A(t) := \|w(t)\|_{H^1}^2 + \|\phi(t)\|_{H^2}^2 \quad \text{and} \quad B(t) := \|\nabla w(t)\|_{H^1}^2 + \|\nabla\phi(t)\|_{H^2}^2$$

and define

$$\Gamma := \sup\Big\{t \in [0,T^*) : \sup_{\tau\in[0,t]} A(\tau) \le \eta \ll 1\Big\},$$

which together with Sobolev's inequality imply that

$$\sup_{\tau\in[0,\Gamma]} \|\phi(\tau)\|_{L^\infty} \le C\eta \le \frac{1}{4}. \tag{3.12}$$

Using Lemma 2.4 tells us that

$$\sup_{\tau\in[0,\Gamma]} \|\Theta(\tau)\|_{L^\infty} \le C\|\widehat{\Theta}_0\|_{L^1} \le C\varepsilon_0 \le \frac{1}{4}. \tag{3.13}$$

We should notice that ε_0 and η are two small enough positive constants which will be determined later on. Thus we have

$$\sup_{\tau\in[0,\Gamma]} \|\widetilde{\rho}(\tau)\|_{L^\infty} \le \frac{1}{2}. \tag{3.14}$$

We recall the following composition estimate (see [26]):

Let $m\in\mathbb{N}$. Assume that $f \in \dot{H}^m \cap L^\infty$ and $F \in W^{m+2,\infty}_{\rm loc}$ with $F(0) = 0$, we have, for some constant $C(M)$ depending on $M = \sup_{k\le m+2,|t|\le\|f\|_{L^\infty}} \|F^{(k)}(t)\|_{L^\infty}$ that

$$\|F(f)\|_{\dot{H}^m} \le C(M)\|f\|_{\dot{H}^m},$$

Combing this composition estimate, we emphasize that this fact shall be used in the sequel: for $p\in[1,\infty]$ and $m\in\mathbb{N}$

$$\|K(\widetilde{\rho})\|_{L^p} \le C\|\widetilde{\rho}\|_{L^p} \quad \text{and} \quad \|K(\widetilde{\rho})\|_{\dot{H}^m} \le C\|\widetilde{\rho}\|_{\dot{H}^m}.$$

Step 1: Estimation of $\|\phi\|_{H^2}$.

Taking the inner product of Eqs.$(1.6)_1$ with ϕ yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\phi\|_{L^2}^2 + \langle \mathrm{div} w, \phi\rangle = -\langle w\cdot\nabla\phi, \phi\rangle - \langle \phi\,\mathrm{div} w, \phi\rangle \tag{3.15}$$

$$+\langle \Theta w + \phi U, \nabla\phi\rangle \tag{3.16}$$

$$-\langle \Theta\,\mathrm{div} U + U\cdot\nabla\Theta, \phi\rangle. \tag{3.17}$$

By Hölder's inequality and Lemma 2.4, we obtain

$$|(3.15)| \lesssim \Big(\|w\|_{L^3}\|\nabla\phi\|_{L^2} + \|\phi\|_{L^3}\|\nabla\phi\|_{L^2}\Big)\|\phi\|_{L^6} \lesssim A^{\frac{1}{2}}(t)B(t),$$

$$|(3.16)| \lesssim \Big(\|\Theta\|_{L^\infty}\|w\|_{L^2} + \|U\|_{L^\infty}\|\phi\|_{L^2}\Big)\|\nabla\phi\|_{L^2} \lesssim \|\widehat{U},\widehat{\Theta}\|_{L^1}A(t),$$

$$|(3.17)| \lesssim \|\Theta\,\mathrm{div} U + U\cdot\nabla\Theta\|_{L^2}\|\phi\|_{L^2} \lesssim \|\Theta\|_{L^2}\|\widehat{U}\|_{L^1}A^{\frac{1}{2}}(t).$$

Thus

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\phi\|_{L^2}^2+\langle \mathrm{div} w,\phi\rangle \lesssim A^{\frac{1}{2}}(t)B(t)+\|\widehat{U},\widehat{\Theta}\|_{L^1}A(t)+\|\Theta\|_{L^2}\|\widehat{U}\|_{L^1}A^{\frac{1}{2}}(t). \tag{3.18}$$

Applying $-\Delta$ to both sides of Eqs.(1.6)$_1$, taking the inner product with $-\Delta\phi$ yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta\phi\|_{L^2}^2+\langle \Delta\mathrm{div} w,\Delta\phi\rangle := I_1+\cdots+I_4, \tag{3.19}$$

where

$$\begin{aligned}
I_1 &= -\langle \Delta(w\cdot\nabla\phi),\Delta\phi\rangle-\langle \Delta(\phi\mathrm{div} w),\Delta\phi\rangle,\\
I_2 &= -\langle \Delta(U\cdot\nabla\phi),\Delta\phi\rangle,\\
I_3 &= -\langle \Delta(w\cdot\nabla\Theta),\Delta\phi\rangle-\langle \Delta(\Theta\mathrm{div} w),\Delta\phi\rangle-\langle \Delta(\phi\mathrm{div} U),\Delta\phi\rangle,\\
I_4 &= -\langle \Delta(\Theta\mathrm{div} U+U\cdot\nabla\Theta),\Delta\phi\rangle.
\end{aligned}$$

Integrating by parts, we can rewrite first three terms as follows

$$\begin{aligned}
I_1 &= \langle \partial_i w\cdot\nabla\phi,\partial_i\Delta\phi\rangle+\langle w\cdot\nabla\partial_i\phi,\partial_i\Delta\phi\rangle+\langle \partial_i\phi\mathrm{div} w,\partial_i\Delta\phi\rangle+\langle \phi\partial_i\mathrm{div} w,\partial_i\Delta\phi\rangle,\\
I_2 &= -2\langle \partial_i U\cdot\nabla\partial_i\phi,\Delta\phi\rangle-\langle \Delta U\cdot\nabla\phi,\Delta\phi\rangle-\langle U\cdot\nabla\Delta\phi,\Delta\phi\rangle,\\
I_3 &= -\langle \partial_i(\partial_i w\cdot\nabla\Theta+w\cdot\nabla\partial_i\Theta),\Delta\phi\rangle+\langle \partial_i(\Theta\mathrm{div} w),\partial_i\Delta\phi\rangle-\langle \Delta(\phi\mathrm{div} U),\Delta\phi\rangle.
\end{aligned}$$

By Hölder's inequality, the facts $\|\mathbf{f}\|_{L^6}\lesssim\|\nabla\mathbf{f}\|_{L^2}$, $\|\mathbf{f}\|_{L^\infty}\lesssim\|\nabla\mathbf{f}\|_{H^1}$ and $\|\nabla^m\Theta\|_{L^\infty}\lesssim\|\Theta\|_{L^\infty}$, we obtain

$$\begin{aligned}
|I_1| &\lesssim \Big(\|\nabla\phi\|_{L^3}\|\nabla w\|_{L^6}+\|w\|_{L^3}\|\nabla^2\phi\|_{L^6}+\|\phi\|_{L^\infty}\|\nabla\mathrm{div} w\|_{L^2}\Big)\|\nabla\Delta\phi\|_{L^2}\lesssim A^{\frac{1}{2}}(t)B(t),\\
|I_2| &\lesssim \Big(\|\nabla U\|_{L^\infty}+\|\Delta U\|_{L^\infty}+\|\mathrm{div} U\|_{L^\infty}\Big)\|\nabla\phi\|_{H^1}^2\lesssim\|\widehat{U}\|_{L^1}A(t),\\
|I_3| &\lesssim \|\Theta\|_{L^\infty}\big(\|\nabla w\|_{H^1}\|\Delta\phi\|_{L^2}+\|w\|_{L^2}\|\Delta\phi\|_{L^2}+\|\mathrm{div} w\|_{H^1}\|\nabla\Delta\phi\|_{L^2}\big)+\|\mathrm{div} U\|_{W^{2,\infty}}\|\phi\|_{H^2}^2\\
&\lesssim \|\widehat{U},\widehat{\Theta}\|_{L^1}A(t)+\|\widehat{\Theta}\|_{L^1}B(t),\\
|I_4| &\lesssim \|\Theta\mathrm{div} U+U\cdot\nabla\Theta\|_{H^2}\|\Delta\phi\|_{L^2}\lesssim\|\Theta\|_{L^2}\|\widehat{U}\|_{L^1}A^{\frac{1}{2}}(t).
\end{aligned}$$

Inserting the above into (3.19) yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta\phi\|_{L^2}^2+\langle \Delta\mathrm{div} w,\Delta\phi\rangle \lesssim \big(A^{\frac{1}{2}}(t)+\|\widehat{\Theta}\|_{L^1}\big)B(t)+\|\widehat{U},\widehat{\Theta}\|_{L^1}A(t)+\|\Theta\|_{L^2}\|\widehat{U}\|_{L^1}A^{\frac{1}{2}}(t). \tag{3.20}$$

Step 2: Estimation of $\|w\|_{H^1}$.

Due to (1.6)$_1$, that is, $-\mathrm{div} w=\partial_t\phi+\mathrm{div}((\phi+\Theta)(w+U))$, then integrating by parts yields

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\phi\|_{L^2}^2+\langle \nabla\Delta\phi,w\rangle=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\phi\|_{L^2}^2-\langle \Delta\phi,\mathrm{div} w\rangle\\
&=\langle \Delta\phi,\mathrm{div}((\phi+\Theta)(w+U))\rangle\\
&=-\langle \nabla\Delta\phi,\phi w\rangle+\langle \Delta\phi,U\cdot\nabla\phi+w\cdot\nabla\Theta+\phi\mathrm{div} U+\Theta\mathrm{div} w+\Theta\mathrm{div} U+U\cdot\nabla\Theta\rangle\\
&\le\|\nabla\Delta\phi\|_{L^2}\|\phi\|_{L^6}\|w\|_{L^3}+\|\Delta\phi\|_{L^2}\big(\|\widehat{U},\widehat{\Theta}\|_{L^1}\|\phi,w,\nabla\phi,\nabla w\|_{L^2}+\|\Theta\mathrm{div} U,U\cdot\nabla\Theta\|_{L^2}\big)\\
&\lesssim A^{\frac{1}{2}}(t)B(t)+\|\widehat{U},\widehat{\Theta}\|_{L^1}A(t)+\|\Theta\|_{L^2}\|\widehat{U}\|_{L^1}A^{\frac{1}{2}}(t).
\end{aligned}\tag{3.21}$$

Doting Eqs.(1.6)$_2$ with $w-\Delta w$ gives

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|_{H^1}^2+\mu\|\nabla w\|_{H^1}^2+\big(\mu+\lambda\big)\|\mathrm{div} w\|_{H^1}^2\\ &\quad-(\kappa+1)\langle\nabla\Delta\phi,w\rangle+\langle\nabla\phi,w\rangle+\kappa\langle\nabla\Delta\phi,\Delta w\rangle:=\sum_{i=1}^{4}J_i,\end{aligned}\tag{3.22}$$

where

$$\begin{aligned}J_1&=-\langle w\cdot\nabla w,w\rangle+\langle w\cdot\nabla w,\Delta w\rangle+\langle\mathbf{S}(\phi,w),w\rangle-\langle\mathbf{S}(\phi,w),\Delta w\rangle,\\ J_2&=-\langle U\cdot\nabla w,w\rangle-\langle w\cdot\nabla U,w\rangle+\langle U\cdot\nabla w,\Delta w\rangle+\langle w\cdot\nabla U,\Delta w\rangle\\ &\quad+\langle\mathbf{S}(\phi,U),w\rangle-\langle\mathbf{S}(\phi,U),\Delta w\rangle+\langle\mathbf{S}(\Theta,w),w\rangle-\langle\mathbf{S}(\Theta,w),\Delta w\rangle,\\ J_3&=-\langle U\cdot\nabla U,w\rangle+\langle U\cdot\nabla U,\Delta w\rangle+\langle\mathbf{S}(\Theta,U),w\rangle-\langle\mathbf{S}(\Theta,U),\Delta w\rangle,\\ J_4&=\big\langle K(\widetilde{\rho})\nabla\phi,w\big\rangle+\big\langle K(\widetilde{\rho})\nabla\Theta,w\big\rangle-\big\langle K(\widetilde{\rho})\nabla\phi,\Delta w\big\rangle-\big\langle K(\widetilde{\rho})\nabla\Theta,\Delta w\big\rangle.\end{aligned}$$

Taking similar arguments as for I_1-I_3, we obtain

$$\begin{aligned}|J_1|&\lesssim A^{\frac12}(t)B(t),\\ |J_2|&\lesssim\|\widehat{U},\widehat{\Theta}\|_{L^1}A(t)+\|\widehat{\Theta}\|_{L^1}B(t),\\ |J_3|&\lesssim\Big(\|U\cdot\nabla U\|_{H^1}+\|\Theta\|_{L^2}\|\widehat{U}\|_{L^1}\Big)A^{\frac12}(t).\end{aligned}$$

For the term J_4, we rewrite it as

$$\begin{aligned}J_4&=\underbrace{\big\langle K(\widetilde{\rho})\nabla\phi,w\big\rangle+\big\langle K(\widetilde{\rho})\nabla\partial_i\phi,\partial_i w\big\rangle+\big\langle\partial_i K(\widetilde{\rho})\nabla\phi,\partial_i w\big\rangle}_{=J_{4,1}}\\ &\quad+\underbrace{\big\langle K(\widetilde{\rho})\nabla\Theta,w\big\rangle+\big\langle K(\widetilde{\rho})\nabla\partial_i\Theta,\partial_i w\big\rangle+\big\langle\partial_i K(\widetilde{\rho})\nabla\Theta,\partial_i w\big\rangle}_{=J_{4,2}}.\end{aligned}$$

By the Hölder inequality, ε-Young inequality (ε shall be fixed later) and Lemma 2.4, we have

$$\begin{aligned}|J_{4,1}|&\lesssim\|K(\widetilde{\rho})\|_{L^\infty}\|\nabla\phi,\nabla^2\phi\|_{L^2}\|w,\nabla w\|_{L^2}+\|K(\widetilde{\rho})\|_{\dot H^1}\|\nabla\phi\|_{L^6}\|\nabla w\|_{L^3}\\ &\lesssim\|\phi,\Theta\|_{L^\infty}A(t)+\|\nabla\phi,\Theta\|_{L^2}\|\nabla^2\phi\|_{L^2}\|\nabla w\|_{H^1}\\ &\lesssim A^{\frac12}(t)B(t)+\Big(\|\widehat{\Theta}\|_{L^1}+\|\Theta\|_{L^2}^2\Big)A(t)+\varepsilon B(t),\\ |J_{4,2}|&\lesssim\|K(\widetilde{\rho})\|_{L^2}\|\Theta\|_{L^\infty}\|w,\nabla w\|_{L^2}+\|K(\widetilde{\rho})\|_{\dot H^1}\|\Theta\|_{W^{1,\infty}}\|\nabla w\|_{L^2}\\ &\lesssim\|\widehat{\Theta}\|_{L^1}A(t)+\|\widehat{\Theta}\|_{L^1}\|\Theta\|_{L^2}A^{\frac12}(t)\\ &\lesssim\Big(\|\widehat{\Theta}\|_{L^1}+\|\Theta\|_{L^2}^2\Big)A(t)+\|\widehat{\Theta}\|_{L^1}^2,\end{aligned}$$

then we can deduce that

$$|J_4|\le C\Big(\|\widehat{\Theta}\|_{L^1}+\|\Theta\|_{L^2}^2\Big)A(t)+C\|\widehat{\Theta}\|_{L^1}^2+\Big(CA^{\frac12}(t)+\varepsilon\Big)B(t).\tag{3.23}$$

Gathering the above estimations together, we deduce that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|_{H^1}^2+\min(\mu,\nu)\|\nabla w\|_{H^1}^2-(\kappa+1)\langle\nabla\Delta\phi,w\rangle+\langle\nabla\phi,w\rangle+\kappa\langle\nabla\Delta\phi,\Delta w\rangle\\&\le C\left(\|\widehat{U},\widehat{\Theta}\|_{L^1}+\|\Theta\|_{L^2}^2\right)A(t)+C\left(\|U\cdot\nabla U\|_{H^2}+\|\Theta\|_{L^2}\|\widehat{U}\|_{L^1}\right)A^{\frac{1}{2}}(t)\\&\quad+C\|\widehat{\Theta}\|_{L^1}^2+\left(CA^{\frac{1}{2}}(t)+C\|\widehat{\Theta}\|_{L^1}+\varepsilon\right)B(t).\end{aligned}\tag{3.24}$$

Performing (3.18) $+\kappa\times$ (3.20) $+(\kappa+1)\times$ (3.21) $+$ (3.24) yields

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|w\|_{H^1}^2+\|\phi\|_{L^2}^2+(\kappa+1)\|\nabla\phi\|_{L^2}^2+\kappa\|\Delta\phi\|_{L^2}^2\right)+\min(\mu,\nu)\|\nabla w\|_{H^1}^2\\&\le C\left(\|\widehat{U},\widehat{\Theta}\|_{L^1}+\|\Theta\|_{L^2}^2\right)A(t)+C\left(\|U\cdot\nabla U\|_{H^2}+\|\Theta\|_{L^2}\|\widehat{U}\|_{L^1}\right)A^{\frac{1}{2}}(t)\\&\quad+C\|\widehat{\Theta}\|_{L^1}^2+\left(CA^{\frac{1}{2}}(t)+C\|\widehat{\Theta}\|_{L^1}+\varepsilon\right)B(t).\end{aligned}\tag{3.25}$$

Step 3: Estimation of $\sum_{|\alpha|\le1}\langle D^\alpha w,D^\alpha\nabla\phi\rangle$.
Next, we will find the dissipation of ϕ via the estimation of the crossing term $\sum_{|\alpha|\le1}\langle D^\alpha w,D^\alpha\nabla\phi\rangle$. To achieve this goal, performing direct calculations gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{|\alpha|\le1}\langle D^\alpha w,D^\alpha\nabla\phi\rangle+\kappa\|\Delta\phi\|_{H^1}^2+\|\nabla\phi\|_{H^1}^2-\|\mathrm{div}w\|_{H^1}^2+\nu\sum_{|\alpha|\le1}\langle D^\alpha\nabla w,D^\alpha\Delta\phi\rangle:=\sum_{i=1}^4K_i,\tag{3.26}$$

where

$$\begin{aligned}K_1=&\sum_{|\alpha|\le1}\langle D^\alpha(w\cdot\nabla\phi),D^\alpha\mathrm{div}w\rangle+\sum_{|\alpha|\le1}\langle D^\alpha(\phi\mathrm{div}w),D^\alpha\mathrm{div}w\rangle-\sum_{|\alpha|\le1}\langle D^\alpha(w\cdot\nabla w),D^\alpha\nabla\phi\rangle\\&+\langle\mathbf{S}(\phi,w),\nabla\phi\rangle+\sum_{0<|\alpha|\le1}\langle D^{\alpha-e_i}\mathbf{S}(\phi,w),D^\alpha\partial_{x_i}\nabla\phi\rangle,\\K_2=&-\sum_{|\alpha|\le1}\langle D^\alpha\nabla(U\cdot\nabla\phi),D^\alpha w\rangle-\sum_{|\alpha|\le1}\langle D^\alpha(U\cdot\nabla w),D^\alpha\nabla\phi\rangle-\sum_{|\alpha|\le1}\langle D^\alpha(w\cdot\nabla U),D^\alpha\nabla\phi\rangle\\&-\sum_{|\alpha|\le1}\langle D^\alpha\nabla(w\cdot\nabla\Theta),D^\alpha w\rangle-\sum_{|\alpha|\le1}\langle D^\alpha\nabla(\phi\mathrm{div}U),D^\alpha w\rangle+\sum_{|\alpha|\le1}\langle D^\alpha(\Theta\mathrm{div}w),D^\alpha\mathrm{div}w\rangle\\&+\sum_{|\alpha|\le1}\langle D^\alpha\mathbf{S}(\phi,U),D^\alpha\nabla\phi\rangle+\langle\mathbf{S}(\Theta,w),\nabla\phi\rangle+\sum_{0<|\alpha|\le1}\langle D^{\alpha-e_i}\mathbf{S}(\Theta,w),D^\alpha\partial_{x_i}\nabla\phi\rangle,\\K_3=&-\sum_{|\alpha|\le1}\langle D^\alpha(U\cdot\nabla U),D^\alpha\nabla\phi\rangle-\sum_{|\alpha|\le1}\langle D^\alpha\nabla(\Theta\mathrm{div}U),D^\alpha w\rangle-\sum_{|\alpha|\le1}\langle D^\alpha\nabla(U\cdot\nabla\Theta),D^\alpha w\rangle\\&+\sum_{|\alpha|\le1}\langle D^\alpha\mathbf{S}(\Theta,U),D^\alpha\nabla\phi\rangle,\\K_4=&\langle K(\widetilde{\rho})\nabla\Theta,\nabla\phi\rangle+\langle K(\widetilde{\rho})\nabla\phi,\nabla\phi\rangle+\sum_{0<|\alpha|\le1}\langle D^\alpha\big(K(\widetilde{\rho})\nabla\Theta\big),D^\alpha\nabla\phi\rangle\\&+\sum_{0<|\alpha|\le1}\langle D^\alpha\big(K(\widetilde{\rho})\nabla\phi\big),D^\alpha\nabla\phi\rangle.\end{aligned}$$

By Hölder's inequality, we get

$$|K_1| \lesssim A^{\frac{1}{2}}(t)B(t), \tag{3.27}$$

$$|K_2| \lesssim \|\widehat{U},\widehat{\Theta}\|_{L^1}A(t)+\|\widehat{\Theta}\|_{L^1}B(t), \tag{3.28}$$

$$|K_3| \lesssim (\|U\cdot\nabla U\|_{H^2}+\|\Theta\|_{L^2}\|\widehat{U}\|_{L^1})A^{\frac{1}{2}}(t). \tag{3.29}$$

Taking similar arguments as for J_4, we obtain

$$|K_4| \leq C\left(\|\widehat{\Theta}\|_{L^1}+\|\Theta\|_{L^2}^2\right)A(t)+C\|\widehat{\Theta}\|_{L^1}^2+\left(CA^{\frac{1}{2}}(t)+\varepsilon\right)B(t). \tag{3.30}$$

Putting (3.27)-(3.30) together with (3.26) implies

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\sum_{|\alpha|\leq 1}\langle D^\alpha w, D^\alpha\nabla\phi\rangle+\kappa\|\Delta\phi\|_{H^1}^2+\|\nabla\phi\|_{H^1}^2-\|\mathrm{div} w\|_{H^1}^2+\nu\sum_{|\alpha|\leq 1}\langle D^\alpha\nabla w, D^\alpha\Delta\phi\rangle\\
&\lesssim \left(A^{\frac{1}{2}}(t)+\|\widehat{\Theta}\|_{L^1}+\varepsilon\right)B(t)+\left(\|\widehat{U},\widehat{\Theta}\|_{L^1}+\|\Theta\|_{L^2}^2\right)A(t)\\
&\quad+\left(\|U\cdot\nabla U\|_{H^2}+\|\Theta\|_{L^2}\|\widehat{U}\|_{L^1}\right)A^{\frac{1}{2}}(t)+\|\widehat{\Theta}\|_{L^1}^2.
\end{aligned} \tag{3.31}$$

Step 4: Closure of The A Priori Estimates.
Now, we need to close the above all estimates from **Step 1-Step 3**.

Fundamental observations give that for some suitable positive constant γ

$$\begin{aligned}
\|\phi\|_{H^2}^2+\|w\|_{H^1}^2+\gamma\langle\Delta w,\nabla\phi\rangle &\approx A(t),\\
\frac{\gamma}{2}\|\nabla\phi\|_{H^2}^2+\min(\mu,\nu)\|\nabla w\|_{H^1}^2-\gamma\|\nabla\mathrm{div} w\|_{L^2}^2-\gamma\nu\langle\Delta w,\nabla\Delta\phi\rangle &\approx B(t).
\end{aligned}$$

Performing $\gamma\times(3.31)+(3.25)$, then integrating the resulting in time yields

$$\begin{aligned}
A(t)+\int_0^t B(s)\mathrm{d}s &\lesssim \int_0^t\left(A^{\frac{1}{2}}(s)+\|\widehat{\Theta}\|_{L^1}+\varepsilon\right)B(s)\mathrm{d}s\\
&\quad+\int_0^t\left(\|\widehat{\Theta}\|_{L^1}^2+(\|U\cdot\nabla U\|_{H^3}+\|\widehat{U}\|_{L^1}\|\Theta\|_{L^2})A^{\frac{1}{2}}(s)\right)\mathrm{d}s\\
&\quad+\int_0^t\left(\|\widehat{\Theta},\widehat{U}\|_{L^1}+\|\Theta\|_{L^2}^2\right)A(s)\mathrm{d}s.
\end{aligned} \tag{3.32}$$

Taking η and ε small enough and absorbing the first term of RHS of (3.32), then for all $t\in[0,\Gamma]$, from (3.32), using Gronwall's inequality and Lemmas 2.4-2.5, we obtain that

$$\begin{aligned}
A(t) &\leq C\int_0^t\left(\|\widehat{\Theta}\|_{L^1}^2+\|\widehat{U}\|_{L^1}\|\Theta\|_{L^2}+\|U\cdot\nabla U\|_{H^3}\right)\mathrm{d}s\exp\left(C\int_0^t(\|\widehat{\Theta},\widehat{U}\|_{L^1}+\|\Theta\|_{L^2}^2)\mathrm{d}s\right)\\
&\leq C\left(\|\widehat{\Theta}_0\|_{L^1}^2+\|\widehat{\Theta}_0,\widehat{a}\|_{L^1}\|\Theta_0\|_{L^2}+\varepsilon\|\widehat{a}\|_{L^1}\|a\|_{L^2}\right)\exp\left(C\left(\|\widehat{\Theta}_0,\widehat{a}\|_{L^1}+\|\Theta_0\|_{L^2}^2\right)\right)\\
&\leq C\varepsilon_0,
\end{aligned} \tag{3.33}$$

where we have used the smallness condition (1.8) in the last step.

Choosing $\eta=2C\varepsilon_0$, thus we can get

$$\sup_{\tau\in[0,t]}A(\tau)\leq\frac{\eta}{2}\quad\text{for}\quad t\leq\Gamma.$$

Hence, if $\Gamma < T^*$, due to the continuity of the solutions, we can obtain that there exists $0 < \varepsilon \ll 1$ such that

$$\sup_{\tau\in[0,t]} A(\tau) \leq \eta \quad \text{for} \quad t \leq \Gamma + \varepsilon < T^*,$$

which is contradiction with the definition of Γ. Thus, we can conclude $\Gamma = T^*$ and

$$\sup_{\tau\in[0,t]} A(\tau) \leq C < \infty \quad \text{for all} \quad t \in (0, T^*),$$

which implies that $T^* = +\infty$. This completes the proof of Theorem 1.1.

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Declarations

Data Availability No data was used for the research described in the article.

Conflict of interest The authors declare that they have no conflict of interest.

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