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ENTIRE SOLUTIONS OF THE GENERALIZED HESSIAN INEQUALITY

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ABSTRACT. In this paper, we study the more general Hessian inequality $\sigma_k^{\frac{1}{k}}(\lambda(D_i(A(|Du|)D_ju))) \geq f(u)$ including the Laplacian, p-Laplacian, mean curvature, k-mean curvature and Hessian operators. We give a nonexistence result and provide a sufficient and necessary condition on the global solvability, which is a generalized Keller-Osserman condition. We also discuss the regularity of solutions.

1. Introduction and the statement of results

In this paper, we discuss the solvability of the generalized Hessian inequality

$$\sigma_k^{\frac{1}{k}}\left(\lambda\left(D_i\left(A\left(|Du|\right)D_ju\right)\right)\right) \geq f(u) \text{ in } \mathbb{R}^n, \tag{1.1}$$

where

$$\sigma_k(\lambda) = \sum_{1\leq i_1<\cdots<i_k\leq n} \lambda_{i_1}\cdots\lambda_{i_k},\ \lambda = (\lambda_1,\lambda_2,\cdots,\lambda_n) \in \mathbb{R}^n,\ k = 1,2,\cdots,n$$

is the k-th elementary symmetric function, $\lambda\left(D_i\left(A\left(|Du|\right)D_ju\right)\right)$ denotes the eigenvalues of the symmetric matrix of $\left(D_i\left(A\left(|Du|\right)D_ju\right)\right)$, and A, f are two given positive continuous functions in $(0,+\infty)$.

The generalized Hessian operator $\sigma_k\left(\lambda\left(D_i\left(A\left(|Du|\right)D_ju\right)\right)\right)$, introduced by many authors [1, 6, 15, 19], is an important class of fully nonlinear operator. It is a generalization of some typical operators we shall be interested in as follows: the m-k-Hessian operator for the case $A(p) = p^{m-2}$, $m > 1$ is treated by Trudinger and Wang [21]; the k-mean curvature operator for the case $A(p) = \left(1+p^2\right)^{-\frac{1}{2}}$ is treated by Concus and Finn [5] and Peletier and Serrin [17]; the generalized k-mean curvature operator for the case $A(p) = \left(1+p^2\right)^{-\alpha}$, $\alpha < \frac{1}{2}$ is treated by Kusano and Swanson [11]. See [14, 19] for more operators.

In particular, (1.1) is the k-Hessian innequality for the case $A(p) = 1$. For $k = 1$, Wittich [23] ($n = 2$), Haviland [8] ($n = 3$), Walter [22] ($n \geq 2$) proved the Laplacian equation

$$\Delta u = f(u) \text{ in } \mathbb{R}^n$$

has no solution if and only if

$$\int^\infty \left(\int^s f(t)dt\right)^{-\frac{1}{2}} ds < \infty.$$

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Here and after, we omit the lower limit to admit an arbitrary positive constant. Keller [10] and Osserman [16] showed that the Laplacian inequality

$$\Delta u \geq f(u) \text{ in } \mathbb{R}^n$$

has a positive solution $u \in C^2(\mathbb{R}^n)$ if and only if f satisfies the Keller-Osserman condition

$$\int^{\infty} \left(\int^{s} f(t)dt \right)^{-\frac{1}{2}} ds = \infty. \tag{1.2}$$

The condition (1.2) is often used to study the boundary blow-up (explosive, large) solutions (see [12, 13, 18]). Ji and Bao [9] extended the above results from $k = 1$ to $1 \leq k \leq n$, which can be regardes as the generalized Keller-Osserman condition. Naito and Usami [14] extended the above results from $A(p) = 1$ to the generalized Hessian inequality (1.1) for $k = 1$ and got similar results.

In this paper, we shall extend this result from $k = 1$ to $1 \leq k \leq n$ for the generaralized Hessian inequality (1.1) and develop existence and nonexistence conditions of entire solutions for (1.1). To state our results, we define a generalized k-convex entire solution of (1.1) to be a function $u \in \Phi^k(\mathbb{R}^n)$ which satisfies (1.1) at each $x \in \mathbb{R}^n$, where

$$\Phi^k(\mathbb{R}^n) = \left\{ u \in C^1(\mathbb{R}^n) : A(|Du|)Du \in C^1(\mathbb{R}^n),\ \lambda\left(D_i\left(A(|Du|)D_j u\right)\right) \in \Gamma_k \text{ in } \mathbb{R}^n \right\},$$

and

$$\Gamma_k := \{\lambda \in \mathbb{R}^n : \sigma_l(\lambda) > 0,\ l = 1, 2, \cdots, k\}.$$

In (1.1), we assume that the positive function $A \in C^1(0,\infty)$ satisfies

$$pA(p) \in C[0,\infty) \text{ is strictly monotone increasing in } (0,\infty), \tag{1.3}$$

and the positive function $f \in C(0,\infty)$ satisfies

$$f \text{ is monotone non-decreasing in } (0,\infty). \tag{1.4}$$

First, we discuss the situation

$$\lim_{p\to\infty} pA(p) < \infty. \tag{1.5}$$

A nonexistence theorem for the global solvability of the inequality (1.1) is as follows.

Theorem 1.1. *Assume that A satisfies (1.3), (1.5) and f satisfies (1.4), then the inequality (1.1) has no positive solution $u \in C^2(\mathbb{R}^n \backslash \{0\}) \cap \Phi^k(\mathbb{R}^n)$.*

Remark 1.2. *The k-mean curvature inequality (1.1) for the case $A(p) = \left(1 + p^2\right)^{-\frac{1}{2}}$ satisfies the Theorem 1.1, and the corresponding results were obtained by Cheng and Yau* [3] *and Tkachev* [20].

Next, we discuss the situation

$$\lim_{p\to\infty} pA(p) = \infty. \tag{1.6}$$

Now we define a continuous function $\Psi : [0,\infty) \to [0,\infty)$ that satisfies

$$\Psi(p) := p\left(pA(p)\right)^k - \int_0^p \left(tA(t)\right)^k dt,\ p \geq 0. \tag{1.7}$$

It follows from the condition (1.3) that the inverse function of Ψ exists in $[0,\infty)$, denoted by Ψ^{-1}. For example, if $A(p)=p^{m-2}$, $m>1$, then

$$\Psi(p)=\frac{(m-1)k}{(m-1)k+1}p^{(m-1)k+1} \text{ and } \Psi^{-1}(p)=\left(\frac{(m-1)k+1}{(m-1)k}p\right)^{\frac{1}{(m-1)k+1}}.$$

A sufficient and necessary condition for the global solvability of the inequality (1.1) is as follows.

Theorem 1.3. *Assume that A satisfies (1.3), (1.6) and f satisfies (1.4), then the inequality (1.1) has a positive solution $u\in C^2(\mathbb{R}^n\backslash\{0\})\cap\Phi^k(\mathbb{R}^n)$ if and only if*

$$\int^{\infty}\left(\Psi^{-1}\left(\int^s f^k(t)dt\right)\right)^{-1}ds=\infty. \tag{1.8}$$

For $k=1$, $A(p)=1$, (1.8) is exactly the Keller-Osserman condition (1.2). Thus we can regard (1.8) as a generalized Keller-Osserman condition.

If we strengthen the case (1.6) to

$$0<\liminf_{p\to\infty}\frac{A(p)}{p^{m-2}}\le\limsup_{p\to\infty}\frac{A(p)}{p^{m-2}}<\infty \text{ for some } m>1. \tag{1.9}$$

As a consequence of Theorems 1.3, we obtain the following corollary.

Corollary 1.4. *Assume that A satisfies (1.3), (1.9) and f satisfies (1.4), then the inequality (1.1) has a positive solution $u\in C^2(\mathbb{R}^n\backslash\{0\})\cap\Phi^k(\mathbb{R}^n)$ if and only if*

$$\int^{\infty}\left(\int^s f^k(t)dt\right)^{-\frac{1}{k(m-1)+1}}ds=\infty. \tag{1.10}$$

Remark 1.5. *Corollary 1.4 holds for the cases $A(p)=1$, $m=2$ which was obtained by Ji and Bao [9]; $A(p)=p^{m-2}$, $m>1$ which was obtained by Feng and Bao [2]. As for $A(p)=\left(1+p^2\right)^{-\alpha}$, $m=2-2\alpha>1$, $A(p)=p^{2m-2}\left(1+p^{2m}\right)^{-\frac{1}{2}}$, $m>1$ and more cases in [14, 19] are first obtained by authors of this paper.*

Remark 1.6. *Under the assumption of Corollary 1.4, if $f(u)=u^\gamma$, $\gamma\ge 0$, then the inequality (1.1) has a positive solution $u\in C^2(\mathbb{R}^n\backslash\{0\})\cap\Phi^k(\mathbb{R}^n)$ if and only if $\gamma\le m-1$.*

If we strengthen the condition of f from (1.4) to the positive function $f\in C(\mathbb{R})$ satisfying

$$f \text{ is monotone non-decreasing in } \mathbb{R}, \tag{1.11}$$

then we have the similar corollary which does not require the solution of (1.1) to be positive.

Corollary 1.7. *Assume that A satisfies (1.3) and f satisfies (1.11). If (1.5) holds, then the inequality (1.1) has no solution $u\in C^2(\mathbb{R}^n\backslash\{0\})\cap\Phi^k(\mathbb{R}^n)$; if (1.6) holds, then the inequality (1.1) has a solution $u\in C^2(\mathbb{R}^n\backslash\{0\})\cap\Phi^k(\mathbb{R}^n)$ if and only if (1.8) holds, in particular, if (1.9) holds, then the inequality (1.1) has a solution $u\in C^2(\mathbb{R}^n\backslash\{0\})\cap\Phi^k(\mathbb{R}^n)$ if and only if (1.10) holds.*

Remark 1.8. *Under the assumption of Corollary 1.7, if $f(u)=e^u$, then the inequality (1.1) has no solution $u\in C^2(\mathbb{R}^n\backslash\{0\})\cap\Phi^k(\mathbb{R}^n)$.*

In particular, we will get a better regularity of solutions $u \in C^2(\mathbb{R}^n \backslash \{0\}) \cap \Phi^k(\mathbb{R}^n)$. If $A \in C^1[0,\infty)$, $A(0) \neq 0$, then $u \in C^2(\mathbb{R}^n) \cap \Phi^k(\mathbb{R}^n)$. If A does not satisfy the above conditions, we only consider the condition

$$0 < \liminf_{p\to 0} \frac{A(p)}{p^{l-2}} \leq \limsup_{p\to 0} \frac{A(p)}{p^{l-2}} < \infty \text{ for some } l > 2, \tag{1.12}$$

then $u \in W^{2,nq}_{loc}(\mathbb{R}^n)$, $1 < q < \frac{l-1}{l-2}$, by embedding theorem, we have $u \in C^{1,\alpha}(\mathbb{R}^n) \cap \Phi^k(\mathbb{R}^n)$ for some $\alpha \in (0,1)$. See Remarks 2.2 and 2.4 for details.

The rest of our paper is organized as follows. In Section 2, we give some properties of radial solutions and the local existence of the Cauchy problem associated to (1.1) as preliminaries. In Section 3, we give the comparison principle and prove Theorems 1.1, 1.3 and Corollaries 1.4, 1.7.

2. Preliminary results of radial solutions

To prove Theorems 1.1 and 1.3, we need to get some properties of radial solutions in $B_R := \{x \in \mathbb{R}^n : |x| < R\}$, $R > 0$.

Lemma 2.1. *For any constant $a > 0$, assume that $\varphi(r) \in C[0,R) \cap C^1(0,R)$ is the positive solution of the Cauchy problem to the implicit equation*

$$\begin{cases} A\left(|\varphi'(r)|\right)\varphi'(r) = \left(\dfrac{nr^{k-n}}{C_n^k}\displaystyle\int_0^r s^{n-1} f^k(\varphi(s))ds\right)^{\frac{1}{k}} =: F(r,\varphi),\ r > 0, \\ \varphi(0) = a. \end{cases} \tag{2.1}$$

Then $\varphi'(0) = 0$, $\varphi'(r) > 0$ in $(0,R)$, and it satisfies $\varphi(r) \in C^1[0,R) \cap C^2(0,R)$ with $A(\varphi'(r))\varphi'(r) \in C^1[0,R)$, and the ordinary differential equation

$$\begin{aligned} &C_{n-1}^{k-1}\left(A\left(\varphi'(r)\right)\varphi'(r)\right)'\left(\frac{A\left(\varphi'(r)\right)\varphi'(r)}{r}\right)^{k-1} + C_{n-1}^k\left(\frac{A\left(\varphi'(r)\right)\varphi'(r)}{r}\right)^k \\ &= \frac{C_n^k r^{1-n}}{n}\left(r^{n-k}\left(A\left(\varphi'(r)\right)\varphi'(r)\right)^k\right)' = f^k\left(\varphi(r)\right). \end{aligned} \tag{2.2}$$

Proof. We define

$$h(r) := \int_0^r A\left(|\varphi'(s)|\right)\varphi'(s)ds,$$

then it satisfies $h(0) = 0$ and

$$h'(r) = A\left(|\varphi'(r)|\right)\varphi'(r) = \left(\frac{nr^{k-n}}{C_n^k}\int_0^r s^{n-1}f^k(\varphi(s))ds\right)^{\frac{1}{k}} > 0,\ 0 < r < R.$$

It is easy to see that $h(r) \in C^2(0,R)$. By (1.3) and (2.1), we know $\varphi'(r) > 0$ in $(0,R)$.

$$\lim_{r\to 0}\frac{h(r)-h(0)}{r-0} = \lim_{r\to 0} h'(\xi) = \lim_{\xi\to 0}\left(\frac{n\xi^{k-n}}{C_n^k}\int_0^{\xi} s^{n-1}f^k(\varphi(s))ds\right)^{\frac{1}{k}} = 0,$$

where $\xi=\xi(r)\in(0,r)$. Therefore $h'(0)=0$ and $h(r)\in C^1[0,R)$, which implies that $\varphi'(0)=0$ and $\varphi(r)\in C^1[0,R)$. One can see that

$$\lim_{r\to 0}\frac{h'(r)-h'(0)}{r-0}=\lim_{r\to 0}\left(\frac{n\int_0^r s^{n-1}f^k(\varphi(s))ds}{C_n^k r^n}\right)^{\frac{1}{k}}=\left(\frac{f^k(a)}{C_n^k}\right)^{\frac{1}{k}}. \tag{2.3}$$

Consequently, we get $h(r)\in C^2[0,R)$, which implies that $A(\varphi'(r))\varphi'(r)\in C^1[0,R)$.

By using (2.1) to calculate directly, we can derive

$$\begin{aligned} h''(r)&=\frac{(h'(r))^{1-k}}{k}\left(\frac{n(k-n)r^{k-n-1}}{C_n^k}\int_0^r s^{n-1}f^k(\varphi(s))ds+\frac{nr^{k-1}}{C_n^k}f^k(\varphi(r))\right)\\ &\geq\frac{1}{C_n^k}\left(\frac{h'(r)}{r}\right)^{1-k}f^k(\varphi(r))>0, \end{aligned} \tag{2.4}$$

then $h'(r)\in C^1[0,R)$ is a strictly monotone increasing function of r, and by (1.3), $g(\varphi'):=A(\varphi')\varphi'\in C^1(0,\varphi'(R))$ is a strictly monotone increasing function of φ', then there exists inverse function $\varphi'(r)=g^{-1}(h'(r))\in C^1(0,R)$, which implies $\varphi(r)\in C^2(0,R)$.

By (2.4) and (2.1), we have

$$h''(r)=\frac{k-n}{k}\frac{h'(r)}{r}+\frac{n}{kC_n^k}\left(\frac{h'(r)}{r}\right)^{1-k}f^k(\varphi(r)),$$

it is easy to verify that $\varphi(r)$ satisfies the ODE equation (2.2). □

Remark 2.2. *In particular, if $A\in C^1[0,R)$, $A(0)\neq 0$, consider the function $H(r,\varphi'):=h'(r)-g(\varphi')=0$, then $H_{\varphi'}(0,0)=A(0)\neq 0$, hence we know from the implicit function theorem that there exists $\varphi'(r)\in C^1[0,R)$, then we can strengthen the regularity to $\varphi(r)\in C^2[0,R)$.*

If A does not satisfy the above conditions, we only consider the condition (1.12) and then by (2.3), we have

$$\left(\frac{f^k(a)}{C_n^k}\right)^{\frac{1}{k}}=\lim_{r\to 0}\frac{h'(r)}{r}=\lim_{r\to 0}\frac{(\varphi'(r))^{l-1}}{r}=\lim_{r\to 0}\frac{\varphi''(r)}{r^{-\frac{l-2}{l-1}}}, \tag{2.5}$$

for $1<q<\frac{l-1}{l-2}$, we can strengthen the regularity to $\varphi(r)\in C^2(0,R)\cap W^{2,q}(0,R)$.

Lemma 2.3. *For any constant $a>0$, assume that $\varphi(r)\in C[0,R)\cap C^1(0,R)$ is the positive solution of the Cauchy problem (2.1). Then $u(x)=\varphi(|x|)\in C^2(B_R\backslash\{0\})\cap\Phi^k(B_R)$, $|x|=r<R$ satisfies*

$$\begin{aligned} &\lambda(D_i(A(|Du|)D_ju))\\ &=\left(\left(A(\varphi'(r))\varphi'(r)\right)',\frac{A(\varphi'(r))\varphi'(r)}{r},\cdots,\frac{A(\varphi'(r))\varphi'(r)}{r}\right),\ r\in[0,R), \end{aligned} \tag{2.6}$$

and it is the positive solution of

$$\sigma_k(\lambda(D_i(A(|Du|)D_ju)))=\frac{C_n^k r^{1-n}}{n}\left(r^{n-k}\left(A(\varphi'(r))\varphi'(r)\right)^k\right)'=f^k(u). \tag{2.7}$$

Proof. By Lemma 2.1, we have $\varphi(r) \in C^1[0,R) \cap C^2(0,R)$ with $A(\varphi'(r))\varphi'(r) \in C^1[0,R)$, and it satisfies $\varphi'(0) = 0$, $\varphi'(r) > 0$ in $(0,R)$. For $u(x) = \varphi(r)$, $0 < r < R$, $i, j = 1, \cdots, n$, we have

$$u_i(x) = \varphi'(r)\frac{x_i}{r}, \tag{2.8}$$

$$|Du| = \left|\varphi'(r)\frac{x}{r}\right| = \varphi'(r), \tag{2.9}$$

$$u_{ij}(x) = \frac{\varphi''(r)}{r^2}x_ix_j - \frac{\varphi'(r)}{r^3}x_ix_j + \frac{\varphi'(r)}{r}\delta_{ij}.$$

Then by (2.8) and (2.9), we have

$$\begin{aligned} D_i\left(A\left(|Du|\right)D_ju\right) &= D_i\left(A\left(\varphi'(r)\right)\varphi'(r)\frac{x_j}{r}\right) \\ &= \left(A\left(\varphi'(r)\right)\varphi'(r)\right)'\frac{x_ix_j}{r^2} + A\left(\varphi'(r)\right)\varphi'(r)\frac{\delta_{ij}r - x_j\frac{x_i}{r}}{r^2} \\ &= \left(\left(A\left(\varphi'(r)\right)\varphi'(r)\right)' - \frac{A\left(\varphi'(r)\right)\varphi'(r)}{r}\right)\frac{x_ix_j}{r^2} + \frac{A\left(\varphi'(r)\right)\varphi'(r)}{r}\delta_{ij}. \end{aligned} \tag{2.10}$$

By (2.8) and $\varphi'(0) = 0$, we have

$$0 \le \lim_{x\to 0}|u_i(x)| = \lim_{x\to 0}|\varphi'(r)||\frac{x_i}{r}| \le \lim_{r\to 0}\varphi'(r) = 0,$$

which means

$$\lim_{x\to 0}u_i(x) = 0.$$

Similarly, by (2.10), we have

$$\begin{aligned} &\lim_{x\to 0}D_i\left(A\left(|Du|\right)D_ju\right) \\ &= \lim_{x\to 0}\left(\left(\left(A\left(\varphi'(r)\right)\varphi'(r)\right)' - \frac{A\left(\varphi'(r)\right)\varphi'(r)}{r}\right)\frac{x_ix_j}{r^2} + \frac{A\left(\varphi'(r)\right)\varphi'(r)}{r}\delta_{ij}\right) \\ &= \left(A\left(\varphi'(0)\right)\varphi'(0)\right)'\delta_{ij}. \end{aligned}$$

Here, we define

$$u_i(0) = 0,\ D_i\left(A\left(|Du|\right)D_ju\right)(0) = \left(A\left(\varphi'(0)\right)\varphi'(0)\right)'\delta_{ij}.$$

then $u(x) \in C^1(B_R) \cap C^2(B_R\backslash\{0\})$, with $A(|Du|)Du \in C^1(B_R)$.

It is easy to see that for $r \in [0,R)$, the matrix

$$D_i\left(A\left(|Du|\right)D_ju\right) = ax^Tx + bI,$$

where

$$a := \begin{cases} \frac{\left(A\left(\varphi'(r)\right)\varphi'(r)\right)'}{r^2} - \frac{A\left(\varphi'(r)\right)\varphi'(r)}{r^3},\ r \in (0,R), \\ 0,\ r = 0, \end{cases} \quad b := \begin{cases} \frac{A\left(\varphi'(r)\right)\varphi'(r)}{r},\ r \in (0,R), \\ \left(A\left(\varphi'(0)\right)\varphi'(0)\right)',\ r = 0. \end{cases}$$

By the calculation of linear algebra, we know that the eigenvalues of the symmetric matrix $D_i\left(A\left(|Du|\right)D_ju\right)$ is $\left(ar^2+b,b,\cdots,b\right)$. Therefore we have

$$\lambda\left(D_i\left(A\left(|Du|\right)D_ju\right)\right)=\begin{cases}\left(\left(A\left(\varphi'(r)\right)\varphi'(r)\right)',\frac{A\left(\varphi'(r)\right)\varphi'(r)}{r},\cdots,\frac{A\left(\varphi'(r)\right)\varphi'(r)}{r}\right),r\in(0,R),\\ \left(\left(A\left(\varphi'(0)\right)\varphi'(0)\right)',\cdots,\left(A\left(\varphi'(0)\right)\varphi'(0)\right)'\right),r=0.\end{cases}$$

Since

$$\lim_{r\to 0}\frac{A\left(\varphi'(r)\right)\varphi'(r)}{r}=\left(A\left(\varphi'(0)\right)\varphi'(0)\right)',$$

we can always think that (2.6) holds, and the equation (2.7) can be obtained easily by the definition of σ_k.

Since f and φ are both monotone non-decreasing, we have

$$f(\varphi(r))\geq f(\varphi(0))=f(a)>0,\ r\in[0,R).$$

Then we get $\frac{A\left(\varphi'(r)\right)\varphi'(r)}{r}>0$ and

$$\begin{aligned}&\sigma_k\left(\lambda\left(D_i\left(A\left(|Du|\right)D_ju\right)\right)\right)\\ &=C_{n-1}^{k-1}\left(\frac{A\left(\varphi'(r)\right)\varphi'(r)}{r}\right)^{k-1}\left(\left(A\left(\varphi'(r)\right)\varphi'(r)\right)'+\frac{n-k}{k}\frac{A\left(\varphi'(r)\right)\varphi'(r)}{r}\right)\\ &=f^k(\varphi(r))>0,\end{aligned}$$

which leads to

$$\left(A\left(\varphi'(r)\right)\varphi'(r)\right)'+\frac{n-k}{k}\frac{A\left(\varphi'(r)\right)\varphi'(r)}{r}>0.$$

And for $1\leq l\leq k$, we have

$$\begin{aligned}&\sigma_l\left(\lambda\left(D_i\left(A\left(|Du|\right)D_ju\right)\right)\right)\\ &=C_{n-1}^{l-1}\left(\frac{A\left(\varphi'(r)\right)\varphi'(r)}{r}\right)^{l-1}\left(\left(A\left(\varphi'(r)\right)\varphi'(r)\right)'+\frac{n-l}{l}\frac{A\left(\varphi'(r)\right)\varphi'(r)}{r}\right)\\ &\geq C_{n-1}^{l-1}\left(\frac{A\left(\varphi'(r)\right)\varphi'(r)}{r}\right)^{l-1}\left(\left(A\left(\varphi'(r)\right)\varphi'(r)\right)'+\frac{n-k}{k}\frac{A\left(\varphi'(r)\right)\varphi'(r)}{r}\right)\\ &>0.\end{aligned}$$

Therefore $\lambda\left(D_i\left(A\left(|Du|\right)D_ju\right)\right)\in\Gamma_k$ holds in B_R. □

Obviously for $u(x)=\varphi(r)$, we can see that $u(x)\in C^2\left(B_R\backslash\{0\}\right)\cap C^1\left(B_R\right)$, with $A\left(|Du|\right)Du\in C^1\left(B_R\right)$ is a solution of (2.7) if and only if $\varphi(r)\in C^1[0,R)\cap C^2(0,R)$ with $A\left(\varphi'(r)\right)\varphi'(r)\in C^1[0,R)$ is a solution of (2.2).

Remark 2.4. *In particular, if $A\in C^1[0,\infty)$, $A(0)\neq 0$, by Remark 2.2, we have $\varphi(r)\in C^2[0,R)$, and*

$$\lim_{x\to 0}u_{ij}(x)=\lim_{x\to 0}\left(\left(\varphi''(r)-\frac{\varphi'(r)}{r}\right)\frac{x_ix_j}{r^2}+\left(\frac{\varphi'(r)}{r}\right)\delta_{ij}\right)=\varphi''(0)\delta_{ij}.$$

We define $u_{ij}(0)=\varphi''(0)\delta_{ij}$, then it is straightforward to show that $u(x)\in C^2\left(R_R\right)$.

If A does not satisfy the above conditions, we only consider the condition (1.12), then by (2.5) *and* $\left|\frac{x_i x_j}{r^2}\right| \leq 1$, *we have*

$$\begin{aligned}\limsup_{r\to 0}\frac{|u_{ij}(x)|}{r^{-\frac{l-2}{l-1}}} &= \limsup_{r\to 0}\frac{\left|\left(\varphi''(r)-\frac{\varphi'(r)}{r}\right)\frac{x_i x_j}{r^2}+\left(\frac{\varphi'(r)}{r}\right)\delta_{ij}\right|}{r^{-\frac{l-2}{l-1}}}\\ &\leq \lim_{r\to 0}\frac{|\varphi''(r)|+\left|\frac{\varphi'(r)}{r}\right|+\left|\frac{\varphi'(r)}{r}\right|}{r^{-\frac{l-2}{l-1}}}\\ &= 3\left(\frac{f^k(a)}{C_n^k}\right)^{\frac{1}{k}}.\end{aligned}$$

For $\frac{l-2}{l-1}q < 1$, *we have* $D^2u(x) \in L^{nq}(B_R)$. *Then it is straightforward to see* $u(x) \in W^{2,nq}(B_R)$.

Next, we will use the Euler's break line and dicuss the local existence of the Cauchy problem (2.1) near $r = 0$. The method is similar to proving the existence theorem of ordinary differential equations (see [4]).

Lemma 2.5. *For any constant* $a > 0$, *there exists a constant* $R > 0$, *such that the Cauchy problem (2.1) has a positive solution in* $[0,R]$.

Proof. By Lemma 2.1, we know that $\varphi'(r) = g^{-1}(F(r,\varphi)) \in C[0,R) \cap C^1(0,R)$ is a strictly monotone increasing function of r. For $R > 0$ sufficiently small, We define a functional $G(\cdot,\cdot)$ in

$$\mathscr{R} := [0,R] \times \{\varphi \in C[0,R] : a \leq \varphi < 2a\},$$

which satisfies

$$G(r,\varphi) := g^{-1}(F(r,\varphi)).$$

Therefore (2.1) can be rewritten as

$$\varphi'(r) = G(r,\varphi) > 0,\ r > 0.$$

For any $m \in \mathbb{N}$ and $0 = r_0 < r_1 < \cdots < r_m = R$, We construct a Euler's break line ψ in $[0,R]$ as follows,

$$\begin{cases}\psi(r) = a,\ 0 \leq r \leq r_1,\\ \psi(r) = \psi(r_{i-1}) + G(r_{i-1},\psi)(r - r_{i-1}),\ r_{i-1} < r \leq r_i,\ i = 2,3,\cdots,m.\end{cases}$$

Step 1. We will show that $(r,\psi) \in \mathscr{R}$, which means $a \leq \psi(r) < 2a$ for any $r \in [0,R]$. Notice that

$$\begin{aligned}G(r,\psi) &\leq g^{-1}\left(\left(\frac{nr^{k-n}}{C_n^k}\int_0^r s^{n-1}ds f^k(\psi(r))\right)^{\frac{1}{k}}\right)\\ &\leq g^{-1}\left(\left(\frac{1}{C_n^k}\right)^{\frac{1}{k}} Rf(\psi(R))\right) < \infty,\end{aligned} \tag{2.11}$$

then for the break line (r,ψ), we have

$$a \le \psi(r) \le a + g^{-1}\left(\left(\frac{1}{C_n^k}\right)^{\frac{1}{k}} Rf(\psi(R))\right) r \le a + g^{-1}\left(\left(\frac{1}{C_n^k}\right)^{\frac{1}{k}} Rf(\psi(R))\right) R.$$

Therefore we choose $R>0$ sufficiently small, such that $\psi(r) < 2a$.

Step 2. We will prove that the Euler's break line ψ is an ε-appromation solution of (2.1), which means for any $\varepsilon > 0$ sufficiently small, we need to choose some appropriate points $\{r_i\}_{i=1,\cdots,m}$, such that the break line ψ satisfies

$$\left|\psi'(r) - G(r,\psi)\right| < \varepsilon,\ r \in [0,R], \tag{2.12}$$

where $\psi(r)$ is continuously differentiable a.e. in $[0,R]$.

By (2.11), we find that

$$\lim_{r\to 0} G(r,\psi) = 0$$

holds uniformly for any $(r,\psi) \in \mathscr{R}$. Therefore for any $\varepsilon > 0$, there exists $\bar{r} \in (0,R)$, such that

$$G(r,\psi) < \varepsilon,\ 0 \le r < \bar{r}.$$

We now assume that $r_1 = \bar{r}$, then for $0 < r < \bar{r}$, we have

$$\left|\psi'(r) - G(r,\psi)\right| = |G(r,\psi)| < \varepsilon,$$

which satisfies (2.12).

And then for $\bar{r} \le r \le R$, by the proof of Lemma 2.1, we know that $g^{-1} \in C[0,F(R,\psi)] \cap C^1(0,F(R,\psi)]$, then g^{-1} is Liptchitz continuous in $[F(\bar{r},\psi),F(R,\psi)]$. Let $r_{i-1} < r \le r_i$, we have

$$\begin{aligned}
&\left|\psi'(r) - G(r,\psi)\right| \le C\left|F(r_{i-1},\psi) - F(r,\psi)\right| \\
\le& C\left(\frac{n}{C_n^k}\right)^{\frac{1}{k}}\left(r^{k-n}\int_0^r s^{n-1} f^k(\psi(s))ds - r_{i-1}^{k-n}\int_0^{r_{i-1}} s^{n-1} f^k(\psi(s))ds\right)^{\frac{1}{k}} \\
\le& C\left(\frac{n}{C_n^k}\right)^{\frac{1}{k}}\left(\left(r_{i-1}^{k-n} - r^{k-n}\right)\int_0^r s^{n-1} f^k(\psi(s))ds + r_{i-1}^{k-n}\int_{r_{i-1}}^r s^{n-1} f^k(\psi(s))ds\right)^{\frac{1}{k}} \\
\le& C\left(\frac{1}{C_n^k}\right)^{\frac{1}{k}}\left(\left(r_{i-1}^{k-n} - r^{k-n}\right)R^n f^k(2a) + \bar{r}^{k-n}\left(r^n - r_{i-1}^n\right) f^k(2a)\right)^{\frac{1}{k}}.
\end{aligned}$$

Since r^{k-n} and r^n are both Liptchitz continuous functions in $[\bar{r},R]$, for the above ε, there exists $\delta(\varepsilon) > 0$ satisfying

$$\max_{2\le i\le m} |r_{i-1} - r_i| < \delta(\varepsilon),$$

and then we have

$$\left|\psi'(r) - G(r,\psi)\right| \le \tilde{C}|r_{i-1} - r| \le \tilde{C}|r_{i-1} - r_i| < \varepsilon,$$

which also satisfies (2.12). Therefore the Euler's break line ψ is an ε-appromation solution of (2.1).

Step 3. We will construct a solution of (2.1) in $[0,R]$. By Step 2, for any positive constant sequence $\{\varepsilon_j\}_{j=1}^{\infty}$ with $\varepsilon_j \to 0$ as $j \to \infty$, we have the Euler break line $\{\psi_j\}_{j=1}^{\infty}$ as a sequence of ε_j-appromation solutions in $[0,R]$. And by Step 1, for (r', ψ_j), $(r'', \psi_j) \in \mathscr{R}$, we find that

$$\left|\psi_j(r') - \psi_j(r'')\right| = G(r_{i-1}, \psi_j)|r' - r''| \le M\left|r' - r''\right|,$$

which means $\{\psi_j\}_{j=1}^{\infty}$ is equicontinuous and uniformaly bounded ($r''=0$). Therefore by the Ascoli-Arzela Theorem, there exists a uniformly convergent subsequence, still denoted as $\{\psi_j\}_{j=1}^{\infty}$, such that

$$\lim_{j\to\infty} \psi_j = \varphi.$$

Since $\psi_j \in C[0,R]$ and $\psi_j(0) = a$, obviously we have $\varphi \in C[0,R]$ and $\varphi(0) = a$. Next, by using the method similar to [9], we can get the solution of (2.1)

$$\varphi(r) = a + \int_0^r G(s,\varphi)ds. \tag{2.13}$$

By (2.13) and $\varphi \in C[0,R]$, we find $\varphi \in C^1(0,R]$. Then we can differentiate (2.13) easily and get $\varphi'(r) = G(r,\varphi)$ in $(0,R]$ and $\varphi(0) = a$. Obviously (2.1) holds for $r \in [0,R]$. We complete the proof. $\square$

It is easy to find that the Cauchy problem (2.1) always has a solution in $[0,R]$ for any constant a. In particular, when the initial value $a > 0$, the monotonicity of $\varphi(r)$ guarantees the solution of (2.1) is always positive.

3. Proof of main results

We will prove the main results by the comparison principle as follows.

Lemma 3.1. *Assume that $\varphi(r) \in C^1[0,R) \cap C^2(0,R)$ with $A(\varphi'(r))\varphi'(r) \in C^1[0,R)$ satisfying (2.2), with $\varphi'(0) = 0$ and $\varphi(r) \to \infty$ as $r \to R$. If $u(x) \in C^2(\mathbb{R}^n \backslash \{0\}) \cap \Phi^k(\mathbb{R}^n)$ is a positive solution of the inequality (1.1), then we have $u(x) \le \varphi(|x|)$ in B_R.*

Proof. By Lemma 2.3, we know that $v(x) = \varphi(|x|) \in C^2(B_R \backslash \{0\}) \cap C^1(B_R)$ is a solution of (2.7). We want to prove $u(x) \le v(x)$ for any $x \in B_R$. Suppose to the contrary that $u(x) > v(x)$ somewhere, then there exist $a > 0$ and $x_0 \in B_R$, such that $u(x) - a \le v(x)$ in B_R and $u(x_0) - a = v(x_0)$. Notice that $v(x) = \varphi(|x|) \to \infty$ as $x \to \partial B_R$ and u is bounded in B_R, then there exists $R_0 \in (0,R)$, such that $x_0 \in B_{R_0}$. Now we can assume that $\sup_{\partial B_{R_0}}(u - a - v) < 0$.

For $x \in B_{R_0}$, we define an operator

$$L[w] := \sigma_k^{\frac{1}{k}}(\lambda(D_i(A(|Dw|)D_jw))) - f(w).$$

It is obvious that $L[v] = 0$. And by (1.4), we have

$$\begin{aligned} L[u-a] &= \sigma_k^{\frac{1}{k}}(\lambda(D_i(A(|D(u-a)|)D_j(u-a)))) - f(u-a) \\ &= \left(\sigma_k^{\frac{1}{k}}(\lambda(D_i(A(|Du|)D_ju))) - f(u)\right) + (f(u) - f(u-a)) \\ &\ge 0. \end{aligned}$$

Since $u-a$ and v are the subsolution and supersolution of the operator L respectively, by the maximum principle, we have

$$0=\sup_{B_{R_0}}(u-a-v)=\sup_{\partial B_{R_0}}(u-a-v)<0,$$

we obtain a contradiction. $\square$

By the comparison principle 3.1, we can get the relationship between the solvability of the inequality (1.1) and the solvability of the Cauchy problem (2.1).

Lemma 3.2. *The inequality (1.1) has a positive solution* $u\in C^2\left(\mathbb{R}^n\backslash\{0\}\right)\cap\Phi^k\left(\mathbb{R}^n\right)$ *if and only if the Cauchy problem (2.1) has a positive solution* $\varphi(r)\in C^2(0,\infty)\cap C^1[0,\infty)$ *with* $A\left(\varphi'\left(r\right)\right)\varphi'\left(r\right)\in C^1[0,\infty)$ *for some constant* $a>0$.

Proof. First, we will prove the sufficient condition. If the Cauchy problem (2.1) has a positive entire solution $\varphi(r)$ for $R=+\infty$, we consider $u(x)=\varphi(|x|)$. By Lemma 2.3 and Lemma 2.1, we know that $u(x)$ satisfies (2.7) and $\lambda\left(D_i\left(A\left(|Du|\right)D_ju\right)\right)\in\Gamma_k$ for $x\in\mathbb{R}^n$. Then $u(x)\in C^2\left(\mathbb{R}^n\backslash\{0\}\right)\cap\Phi^k\left(\mathbb{R}^n\right)$ satisfies the inequality (1.1).

Next, we will prove the necessary condition. Suppose to the contrary that there is no entire solution $\varphi(r)$ of the Cauchy problem (2.1). Then by Lemma 2.5, we know that the Cauchy problem (2.1) has a positive local solution $\varphi(r)$, but no positive entire solution for any constant $a>0$. Here, we can assume that $[0,R)$ is the maximal existence interval of the local solution. Since $\varphi'(r)>0$ for $r>0$, we have $\varphi(r)\to\infty$ as $r\to R$. By Lemma 2.1, we know $\varphi(|x|)$ satisfies (2.2). Then by the comparison principle 3.1, any positive solution $u(x)\in C^2\left(\mathbb{R}^n\backslash\{0\}\right)\cap\Phi^k\left(\mathbb{R}^n\right)$ of (1.1) satisfies $u(x)\le\varphi(|x|)$ for $x\in B_R$. Therefore we have $u(0)\le\varphi(0)=a$. Notice that a is arbitrary, we can take $a=\frac{u(0)}{2}$ and then get a contradiction. $\square$

Proof of Theorem 1.1. On the contrary, suppose that (1.1) has a positive solution $u\in C^2\left(\mathbb{R}^n\backslash\{0\}\right)\cap\Phi^k\left(\mathbb{R}^n\right)$. Then, by Lemma 3.2, the Cauchy problem (2.1) has a positive solution $\varphi(r)\in C^2(0,\infty)\cap C^1[0,\infty)$ with $A\left(\varphi'\left(r\right)\right)\varphi'\left(r\right)\in C^1[0,\infty)$ for some constant $a>0$. Since f and φ are both monotone non-decreasing, by (2.1), we find

$$A\left(\varphi'(r)\right)\varphi'(r)=\left(\frac{nr^{k-n}}{C_n^k}\int_0^r s^{n-1}f^k(\varphi(s))ds\right)^{\frac{1}{k}}\le\left(\frac{1}{C_n^k}\right)^{\frac{1}{k}}rf\left(\varphi(r)\right),\ r>0. \tag{3.1}$$

Substituting (3.1) in (2.2), we get

$$C_{n-1}^{k-1}\left(A\left(\varphi'(r)\right)\varphi'(r)\right)'\left(\frac{A\left(\varphi'(r)\right)\varphi'(r)}{r}\right)^{k-1}\ge\frac{k}{n}f^k\left(\varphi\left(r\right)\right),\ r>0,$$

which comes to

$$\left(\left(A\left(\varphi'(r)\right)\varphi'(r)\right)^k\right)'\ge\frac{k}{C_n^k}r^{k-1}f^k\left(\varphi\left(r\right)\right),\ r>0. \tag{3.2}$$

Integrating (3.2) from 0 to r, we get

$$\left(A\left(\varphi'(r)\right)\varphi'(r)\right)^k\ge\frac{k}{C_n^k}\int_0^r s^{k-1}f^k\left(\varphi\left(s\right)\right)ds\ge\frac{1}{C_n^k}r^kf^k\left(a\right),\ r>0,$$

which leads to

$$A\left(\varphi'(r)\right)\varphi'(r) \geq \left(\frac{1}{C_n^k}\right)^{\frac{1}{k}} rf(a),\ r>0. \tag{3.3}$$

By (1.3), we can see that

$$\left(\frac{1}{C_n^k}\right)^{\frac{1}{k}} rf(a) \leq A\left(\varphi'(r)\right)\varphi'(r) \leq \lim_{p\to\infty} pA(p) < \infty,\ r>0.$$

Let $r \to \infty$ and we get a contradiction. □

To prove Theorem 1.3, we consider some properties of the function (1.7). By (1.3), we know

$$\Psi'(p) = p\left((pA(p))^k\right)' > 0,\ p>0,$$

then Ψ is strictly monotone increasing in $(0,\infty)$ and $\Psi(0)=0$. By

$$\Psi(p) + \int_0^1 (tA(t))^k\,dt = p\,(pA(p))^k - \int_1^p (tA(t))^k\,dt > (pA(p))^k,\ p>1,$$

we have $\lim_{p\to\infty}\Psi(p)=\infty$. Thus the inverse function of Ψ exists in $[0,\infty)$, denoted by Ψ^{-1}. It is obvious that Ψ^{-1} is also a strictly monotone increasing function and satisfies $\lim_{p\to\infty}\Psi^{-1}(p)=\infty$.

Now we will prove the nonexistence lemma, which is also the necessary condition of Theorem 1.3.

Lemma 3.3. *Assume that A satisfies (1.3), (1.6) and f satisfies (1.4). If*

$$\int^{\infty}\left(\Psi^{-1}\left(\int^{s} f^k(t)\,dt\right)\right)^{-1} ds < \infty, \tag{3.4}$$

then the inequality (1.1) has no positive solution $u \in C^2\left(\mathbb{R}^n\backslash\{0\}\right)\cap\Phi^k\left(\mathbb{R}^n\right)$.

Proof. On the contrary, suppose that (1.1) has a solution $u \in C^2\left(\mathbb{R}^n\backslash\{0\}\right)\cap\Phi^k\left(\mathbb{R}^n\right)$. Then, by Lemma 3.2, the Cauchy problem (2.1) has a positive solution $\varphi(r)\in C^2(0,\infty)\cap C^1[0,\infty)$ with $A\left(\varphi'(r)\right)\varphi'(r)\in C^1[0,\infty)$ for some constant $a>0$. Notice that (3.2) and (3.3) still hold. Let $r\to\infty$ in (3.3) and by (1.3), we have $\lim_{r\to\infty}\varphi'(r)=\infty$. Then $\lim_{r\to\infty}\varphi(r)=\infty$. Multiplying (3.2) by $\varphi'>0$, we have

$$\Psi'\left(\varphi'(r)\right) = \varphi'(r)\left(\left(A\left(\varphi'(r)\right)\varphi'(r)\right)^k\right)' \geq \frac{k}{C_n^k} f^k\left(\varphi(r)\right)\varphi'(r),\ r>1,$$

and then integrating the above from 1 to r, we have

$$\Psi\left(\varphi'(r)\right) \geq \frac{k}{C_n^k}\int_{\varphi(1)}^{\varphi(r)} f^k(s)\,ds,\ r>1.$$

Hence we get

$$\left(\Psi^{-1}\left(\frac{k}{C_n^k}\int_{\varphi(1)}^{\varphi(r)} f^k(s)\,ds\right)\right)^{-1}\varphi'(r) \geq 1,\ r>1.$$

Integrating the above from 1 to r again, we have

$$\int_{\varphi(1)}^{\varphi(r)}\left(\Psi^{-1}\left(\frac{k}{C_n^k}\int_{\varphi(1)}^{s} f^k(t)\,dt\right)\right)^{-1} ds \geq r-1,\ r>1. \tag{3.5}$$

Let $r \to \infty$ in (3.5), we get

$$\int_{\varphi(1)}^{\infty} \left(\Psi^{-1} \left(\frac{k}{C_n^k} \int_{\varphi(1)}^{s} f^k(t)\, dt \right) \right)^{-1} ds = \infty,$$

which contradicts (3.4). □

Next, we will prove the following lemma, which is the sufficient condition of Theorem 1.3.

Lemma 3.4. *Assume that A satisfies (1.3), (1.6) and f satisfies (1.4). If (1.8) holds, then the inequality (1.1) has a positive solution $u \in C^2(\mathbb{R}^n \backslash \{0\}) \cap \Phi^k(\mathbb{R}^n)$.*

Proof. By Lemma 3.2, we only need to prove that the Cauchy problem (2.1) has a positive solution $\varphi(r) \in C^2(0,\infty) \cap C^1[0,\infty)$ with $A(\varphi'(r))\varphi'(r) \in C^1[0,\infty)$ for some constant $a > 0$. Suppose to the contrary that no such solution of (2.1) exists. As in the proof of Lemma 3.2, the Cauchy problem (2.1) has a positive local solution $\varphi(r)$ in the maximal existence interval $[0,R)$. And by Lemma 2.1, we know that $\varphi(r)$ satisfies (2.2).

Next, we will show that

$$\varphi(R) = \lim_{r \to R} \varphi(r) = \infty, \; r \in [0,R).$$

Suppose to the contrary that $\varphi(R) < \infty$. Then by (2.1), $\varphi'(R) < \infty$ exists. By the continuation theorem of the Cauchy problem (2.1), $\varphi(r)$ as a solution of (2.1) can be extended to the right beyond R, which contradicts the definition that $[0,R)$ is the maximum existence interval. Therefore we have $\varphi(R) = \infty$.

Since $\varphi'(r) > 0$ for $0 < r < R$, then by (2.2) and (1.3), we have

$$C_{n-1}^{k-1} \left(A(\varphi'(r))\varphi'(r) \right)' \left(\frac{A(\varphi'(r))\varphi'(r)}{r} \right)^{k-1} \le f^k(\varphi(r)), \; 0 < r < R.$$

which comes to

$$\left(\left(A(\varphi'(r))\varphi'(r) \right)^k \right)' \le \frac{n}{C_n^k} r^{k-1} f^k(\varphi(r)), \; 0 < r < R.$$

Multiplying the above by $\varphi' > 0$, we have

$$\Psi'(\varphi'(r)) = \varphi'(r) \left(\left(A(\varphi'(r))\varphi'(r) \right)^k \right)' \le \frac{nR^{k-1}}{C_n^k} f^k(\varphi(r))\varphi'(r), \; 0 < r < R,$$

and then integrating from 0 to r, we get

$$\Psi(\varphi'(r)) \le \frac{nR^{k-1}}{C_n^k} \int_a^{\varphi(r)} f^k(s)\, ds, \; 0 < r < R,$$

which means

$$\left(\Psi^{-1} \left(\frac{nR^{k-1}}{C_n^k} \int_a^{\varphi(r)} f^k(s)\, ds \right) \right)^{-1} \varphi'(r) \le 1, \; 0 < r < R.$$

Integrating from 0 to r again, we have

$$\int_a^{\varphi(r)} \left(\Psi^{-1} \left(\frac{nR^{k-1}}{C_n^k} \int_a^{\varphi(r)} f^k(s)\, ds \right) \right)^{-1} ds \le r, \; 0 < r < R.$$

Let $r \to R$ and we get

$$\int_a^\infty \left(\Psi^{-1}\left(\frac{nR^{k-1}}{C_n^k} \int_a^s f^k(t)\,dt \right) \right)^{-1} ds \leq R < \infty,$$

which contradicts (1.8). □

Combining Lemma 3.3 and Lemma 3.4, we complete the proof of Theorem 1.3 immediately.

If we strengthen the condition of the operator A from (1.6) to (1.9), we need some properties of the function (1.7) to prove Corollary 1.4.

Lemma 3.5. *Assume that A satisfies (1.9), then we have*

$$0 < \liminf_{p\to\infty} \frac{\Psi^{-1}(p)}{p^{\frac{1}{k(m-1)+1}}} \leq \limsup_{p\to\infty} \frac{\Psi^{-1}(p)}{p^{\frac{1}{k(m-1)+1}}} < \infty.$$

Proof. To prove the result, we only need to prove

$$0 < \liminf_{p\to\infty} \frac{\Psi(p)}{p^{k(m-1)+1}} \leq \limsup_{p\to\infty} \frac{\Psi(p)}{p^{k(m-1)+1}} < \infty.$$

By (1.7) and (1.9), it is easy to see that

$$\limsup_{p\to\infty} \frac{\Psi(p)}{p^{k(m-1)+1}} \leq \limsup_{p\to\infty} \frac{p\,(pA(p))^k}{p^{k(m-1)+1}} < \infty.$$

Next, we will prove

$$\liminf_{p\to\infty} \frac{\Psi(p)}{p^{k(m-1)+1}} > 0,$$

which implies that there exist positive constants P and C, such that

$$\Psi(p) \geq Cp^{k(m-1)+1}, \ p \geq P. \tag{3.6}$$

By (1.9), we know that there exist positive constants P_1, C_1 and C_2, such that

$$C_1 p^{k(m-1)+1} \leq p\,(pA(p))^k \leq C_2 p^{k(m-1)+1}, \ p \geq P_1. \tag{3.7}$$

By (3.7), we can choose $\theta > 0$ sufficiently small, such that

$$\frac{C_2\theta^{k(m-1)}}{C_1} < \frac{1}{2},$$

then we have

$$\frac{\theta p\,(\theta pA(\theta p))^k}{p\,(pA(p))^k} \leq \frac{C_2(\theta p)^{k(m-1)+1}}{C_1 p^{k(m-1)+1}} = \frac{C_2\theta^{k(m-1)+1}}{C_1} < \frac{\theta}{2}, \ p \geq P, \tag{3.8}$$

where $P = \frac{P_1}{\theta}$. We get

$$\begin{aligned} \int_0^p (tA(t))^k\,dt &= \int_0^{\theta p} (tA(t))^k\,dt + \int_{\theta p}^p (tA(t))^k\,dt \\ &\le \theta p\,(\theta pA(\theta p))^k + (p-\theta p)\,(pA(p))^k \\ &= p\,(pA(p))^k \left(1-\theta+\frac{\theta p\,(\theta pA(\theta p))^k}{p\,(pA(p))^k}\right),\ p\ge P. \end{aligned} \tag{3.9}$$

Therefore by (3.9), (3.8) and (3.7), we have

$$\begin{aligned} \Psi(p) = p\,(pA(p))^k \left(1-\frac{\int_0^p (tA(t))^k\,dt}{p\,(pA(p))^k}\right) &\ge p\,(pA(p))^k \left(\theta-\frac{\theta p\,(\theta pA(\theta p))^k}{p\,(pA(p))^k}\right) \\ &> \frac{1}{2}\theta p\,(pA(p))^k \\ &\ge \frac{1}{2}\theta C_1 p^{k(m-1)+1},\ p\ge P, \end{aligned}$$

which gives (3.6). We complete the proof. □

By Theorem 1.3 and Lemma 3.5, we can get Corollary 1.4 immediately.

Proof of Corollary 1.7. We can prove the Corollary 1.7 in a similar way to above. Here, most of the properties we need are almost identical to those we have proved. By conditions (1.4) and (1.11), we know f is now a function defined on $\mathbb{R}$ instead of $(0,\infty)$. We do not need to consider the constant $a>0$ in Lemma 2.1, Lemma 2.5 and Lemma 3.2. The solution of (1.1) is not also required to be positive. □

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