Revisiting (3+1)-dimensional fractional Burgers equation: Lie symmetries, conservation laws, optimal system and power series solutions

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ABSTRACT

This paper applies Lie symmetry analysis method (LSAM) to high dimensional fractional Burgers equation. We obtained all the Lie symmetries and the corresponding conserved vectors for the equation by this method. Then we acquired a one-dimensional optimal system, which is utilized to reduce the aimed equation with Riemann-Liouville fractional derivative (RLFD) to a low dimensional fractional partial differential equation (FPDE) with Erdélyi-Kober fractional derivative (EKFD). Finally, we obtained the power series solution (PSS) of the reduced equation and provided its convergence proof. Moreover, we obtained some other low dimensional reduced FPDEs with RLFD, which can be solved by different methods in the literatures herein.

1. Introduction

As a generalization of the classical calculus, fractional calculus is believed to have originated from L'Hôspital's letter to Leibniz in 1695. Since then, it has gradually attracted the research interest of numerous mathematicians and physicists. Especially in the past half century, it has developed rapidly and achieved success in various fields of engineering and technology [1–4]. Thus it is very meaningful to solve fractional differential equation (FDE). There are currently some methods for solving FDEs, such as Adomian decomposition method [5], finite difference method [6], homotopy perturbation method [7], the sub-equation method [8], the variational iteration method [9], Lie symmetry analysis method [10], invariant subspace method [11], the reduced differential transform method [12–14] and so on. However, most of them are numerical methods. The sub-equation method is an ad hoc solving technique for FDEs with the modified RLFD. The invariant subspace method is applicable to some equations that are easy to find their invariant subspaces. While the LSAM, as a universal and effective modern method to obtain analytical solutions of FDEs, has received an increasing attention.

The LSAM was founded by the famous Norwegian mathematician Sophus Lie in the late 19th century. Then some other mathematicians further developed this method, such as Ovsiannikov [15], Olver [16], Ibragimov [17–19] and so on. As a modern analytical method, it has been extended to FDEs by Gazizov et al. [10] in 2007. Subsequently, this method was effectively applied to the (1+1)-dimensional FDEs models [20–30] and the (2+1)-dimensional FDEs models [31–34] that have emerged in various application fields.

In this paper, the LSAM is used to study the following (3+1)-dimensional time fractional Burgers equation with nonlinear term:

$$D_t^\alpha u + auu_x = u_{xx} + u_{yy} + u_{zz}, \ \ 0 < \alpha < 1, \tag{1.1}$$

where $u = u(t, x, y, z)$ is the flow velocity, t is the time, x, y, z are the spatial coordinates, and a is an arbitrary constant. The classic Burgers equation has widely used in physical and engineering fields, such as nonlinear acoustics, aerodynamics, fluid mechanics, etc. In recent years, fractional Burgers-type equations have been introduced and studied, as the additive effect of wall friction through the bounding layers can be modeled using fractional derivatives [35]. Various analytical methods and numerical techniques have been used for solving different fractional Burgers-type equations

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[35–45]. Wherein Lie symmetry analysis is an effective analytical method. In [37], Inc et al. used it to research fractional Burgers-Huxley equation, which describes the mutual influence among diffusion transport, convective effects and reaction mechanisms. In [38], Saqib et al. used it to study the time-fractional inviscid Burgers equation for non-viscous fluids. In [39], Zhang used it to study the time-fractional coupled Burgers equation, which is a evolution or sedimentation model of two types of particles in a colloid or fluid suspension at a proportional volume concentration under the action of gravity. In [45], Yu used it to study some fractional Burgers-type equations with delays.

The aim of this paper is to find all Lie symmetries of Eq. (1.1) and construct the corresponding conserved vector for each symmetry by the generalization of Noether operator and the new conservation theorem. The one-dimensional optimal system obtained through Olver's method [16] is used to reduce the dimensionality of Eq. (1.1) by one or two, where the reduced equation is then solved simultaneously to obtain the PSS. To our knowledge, it is difficult to obtain the PSS for high dimensional FPDEs, while the PSS of the reduced low dimensional equations gained by the LSAM can be easily obtained.

As we all know, there are many types of definitions for fractional derivative. This paper adopts Riemann-Liouville type [1–4]

$$
{}_aD_t^\alpha f(t,x) = D_t^n\, {}_aI_t^{n-\alpha} f(t,x) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(s,x)}{(t-s)^{\alpha-n+1}} \mathrm{d}s, & n-1 < \alpha < n, n \in \mathbb{N} \\ D_t^n f(t,x), & \alpha = n \in \mathbb{N} \end{cases}
$$

for $t > a$. For convenience, ${}_0D_t^\alpha$ is denoted as D_t^α in the following text.

The following is the arrangement of this paper. Section 2 introduces the LSAM for Eq. (1.1). Section 3 calculates the conserved vectors for all the obtained symmetries. Section 4 constructs a one-dimensional optimal system of the symmetry group. Section 5 presents the reduced equations, the PSS and the proof of its convergence. Section 6 gives the conclusion.

2. Lie symmetry analysis

Consider Eq. (1.1), which is assumed to be invariant under the following group of one-parameter (ϵ) continuous transformations:

$$
\begin{aligned}
&t^* = t + \epsilon\tau(t,x,y,z,u) + o(\epsilon), && x^* = x + \epsilon\xi(t,x,y,z,u) + o(\epsilon), \\
&y^* = y + \epsilon\zeta(t,x,y,z,u) + o(\epsilon), && z^* = z + \epsilon\theta(t,x,y,z,u) + o(\epsilon), \\
&u^* = u + \epsilon\eta(t,x,y,z,u) + o(\epsilon), && D_{t^*}^\alpha u^* = D_t^\alpha u + \epsilon\eta^{\alpha,t} + o(\epsilon), \\
&D_{x^*}u^* = D_x u + \epsilon\eta^x + o(\epsilon), && D_{y^*}u^* = D_y u + \epsilon\eta^y + o(\epsilon), \\
&D_{z^*}u^* = D_z u + \epsilon\eta^z + o(\epsilon), && D_{x^*}^2 u^* = D_x^2 u + \epsilon\eta^{xx} + o(\epsilon), \\
&D_{y^*}^2 u^* = D_y^2 u + \epsilon\eta^{yy} + o(\epsilon), && D_{z^*}^2 u^* = D_z^2 u + \epsilon\eta^{zz} + o(\epsilon),
\end{aligned}
\tag{2.1}
$$

where τ, ξ, ζ, θ, η are infinitesimals and $\eta^{\alpha,t}$, η^x, η^y, η^z, η^{xx}, η^{yy}, η^{zz} are the corresponding extensions of orders α, 1 and 2, respectively. Then the group generator is

$$
X = \tau\frac{\partial}{\partial t} + \xi\frac{\partial}{\partial x} + \zeta\frac{\partial}{\partial y} + \theta\frac{\partial}{\partial z} + \eta\frac{\partial}{\partial u}, \tag{2.2}
$$

and its corresponding prolongation is

$$
Pr^{(\alpha,2)}X = X + \eta^{\alpha,t}\frac{\partial}{\partial u_t^\alpha} + \eta^x\frac{\partial}{\partial u_x} + \eta^{xx}\frac{\partial}{\partial u_{xx}} + \eta^{yy}\frac{\partial}{\partial u_{yy}} + \eta^{zz}\frac{\partial}{\partial u_{zz}}, \tag{2.3}
$$

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where

$$
\begin{aligned}
\eta^x =& D_x\eta - u_t D_x\tau - u_x D_x\xi - u_y D_x\zeta - u_z D_x\theta,\\
\eta^y =& D_y\eta - u_t D_y\tau - u_x D_x\xi - u_y D_y\zeta - u_z D_y\theta,\\
\eta^z =& D_z\eta - u_t D_z\tau - u_x D_z\xi - u_y D_z\zeta - u_z D_z\theta,\\
\eta^{xx} =& D_x\eta^x - u_{xt} D_x\tau - u_{xx} D_x\xi - u_{xy} D_x\zeta - u_{xz} D_x\theta,\\
\eta^{yy} =& D_y\eta^y - u_{yt} D_y\tau - u_{xy} D_y\xi - u_{yy} D_y\zeta - u_{yz} D_y\theta,\\
\eta^{zz} =& D_z\eta^z - u_{zt} D_z\tau - u_{xz} D_z\xi - u_{yz} D_z\zeta - u_{zz} D_z\theta,
\end{aligned}
\tag{2.4}
$$

and

$$
\begin{aligned}
\eta^{\alpha,t} =& \frac{\partial^\alpha \eta}{\partial t^\alpha} + (\eta_u - \alpha D_t\tau)\frac{\partial^\alpha u}{\partial t^\alpha} - u\frac{\partial^\alpha \eta_u}{\partial t^\alpha} - \sum_{n=1}^{\infty}\binom{\alpha}{n} D_t^n\xi D_t^{\alpha-n}u_x\\
&- \sum_{n=1}^{\infty}\binom{\alpha}{n} D_t^n\zeta D_t^{\alpha-n}u_y - \sum_{n=1}^{\infty}\binom{\alpha}{n} D_t^n\theta D_t^{\alpha-n}u_z\\
&+ \sum_{n=1}^{\infty}\left[\binom{\alpha}{n}\frac{\partial^n \eta_u}{\partial t^n} - \binom{\alpha}{n+1} D_t^{n+1}\tau\right] D_t^{\alpha-n}u + \mu,
\end{aligned}
\tag{2.5}
$$

with

$$
\mu = \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{\alpha}{n}\binom{n}{m}\binom{k}{r}\frac{t^{n-\alpha}(-u)^r}{k!\Gamma(n+1-\alpha)}\frac{\partial^m u^{k-r}}{\partial t^m}\frac{\partial^{n-m+k}\eta}{\partial t^{n-m}\partial u^k}.
$$

Note that D_i means taking the total derivative of i $(i = t, x, y, z)$.

Remark 2.1. *From the definition of RLFD, the lower limit of the integral should be fixed under the infinitesimal transformations (2.1), that is, $t = 0$ should be invariant, i.e.*

$$
\tau(t, x, y, z, u)|_{t=0} = 0. \tag{2.6}
$$

Remark 2.2. *From the expression of μ, it can be obtained that $\mu = 0$ when η is a linear function of variable u, that is,*

$$
\frac{\partial^2 \eta}{\partial u^2} = 0. \tag{2.7}
$$

The equation (1.1) is considered to admit the one-parameter Lie symmetry group (2.1), if it satisfies the following invariance criterion:

$$
Pr^{(\alpha,2)}X\left(D_t^\alpha u + auu_x - u_{xx} - u_{yy} - u_{zz}\right)|_{(1.1)} = 0, \tag{2.8}
$$

which can be rewritten as

$$
\left(\eta^{\alpha,t} + a\eta u_x + au\eta^x - \eta^{xx} - \eta^{yy} - \eta^{zz}\right)|_{(1.1)} = 0. \tag{2.9}
$$

Putting $\eta^{\alpha,t}$, η^x, η^{xx}, η^{yy}, η^{zz} into (2.9) and setting the coefficients of different derivatives of u to zero, with the conditions (2.6) and (2.7), we can obtain the following infinitesimals:

$$
\begin{aligned}
&\tau = c_1 t,\ \ \xi = \frac{\alpha}{2}c_1 x + c_3,\ \ \zeta = \frac{\alpha}{2}c_1 y + c_2 z + c_4,\\
&\theta = \frac{\alpha}{2}c_1 z - c_2 y + c_5,\ \ \eta = -\frac{\alpha}{2}c_1 u,
\end{aligned}
\tag{2.10}
$$

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with the constants c_i $(i = 1,2,3,4,5)$. That is to say, Eq. (1.1) admits a five-dimension Lie algebra L^5 to be spanned by the following generators:

$$\begin{aligned}
&X_1 = t\frac{\partial}{\partial t} + \frac{\alpha}{2}x\frac{\partial}{\partial x} + \frac{\alpha}{2}y\frac{\partial}{\partial y} + \frac{\alpha}{2}z\frac{\partial}{\partial z} - \frac{\alpha}{2}u\frac{\partial}{\partial u},\\
&X_2 = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z},\ \ X_3 = \frac{\partial}{\partial x},\ \ X_4 = \frac{\partial}{\partial y},\ \ X_5 = \frac{\partial}{\partial z}.
\end{aligned} \tag{2.11}$$

We represent the corresponding one-parameter (ϵ) continuous transformation groups as

$$h_i\ :\ (t,x,y,z,u) \to (t^*,x^*,y^*,z^*,u^*),\ \ i = 1,2,3,4,5. \tag{2.12}$$

By solving the following Lie equations:

$$\begin{aligned}
&\frac{\mathrm{d}(t^*,x^*,y^*,z^*,u^*)}{\mathrm{d}\epsilon} = (\tau,\xi,\zeta,\theta,\eta),\\
&(t^*,x^*,y^*,z^*,u^*)|_{\epsilon=0} = (t,x,y,z,u),
\end{aligned} \tag{2.13}$$

we can get the following symmetry transformation groups corresponding to X_i $(i = 1,2,3,4,5)$:

$$\begin{aligned}
&h_1\ :\ (t,x,y,z,u) \to (e^{\epsilon}t, e^{\frac{\alpha}{2}\epsilon}x, e^{\frac{\alpha}{2}\epsilon}y, e^{\frac{\alpha}{2}\epsilon}z, e^{-\frac{\alpha}{2}\epsilon}u),\\
&h_2\ :\ (t,x,y,z,u) \to (t,x,y+\epsilon z, z-\epsilon y, u),\\
&h_3\ :\ (t,x,y,z,u) \to (t,x+\epsilon,y,z,u),\\
&h_4\ :\ (t,x,y,z,u) \to (t,x,y+\epsilon,z,u),\\
&h_5\ :\ (t,x,y,z,u) \to (t,x,y,z+\epsilon,u),
\end{aligned} \tag{2.14}$$

where ϵ is any small real parameter.

3. Conservation laws

The conservation vectors with respect to the Lie symmetries (2.11) can be obtained by means of the generalized conservation theorem [46, 47].

The equation (1.1) are denoted as

$$F = D_t^{\alpha}u + auu_x - u_{xx} - u_{yy} - u_{zz} = 0, \tag{3.1}$$

of which the formal Lagrangian is given by

$$\mathcal{L} = v(t,x,y,z)F = v(t,x,y,z)(D_t^{\alpha}u + auu_x - u_{xx} - u_{yy} - u_{zz}), \tag{3.2}$$

where $v(t,x,y,z)$ is a new function to be determined by Eq. (3.4). The Euler-Lagrange operator is

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} + (D_t^{\alpha})^*\frac{\partial}{\partial(D_t^{\alpha}u)} + \sum_{s=1}^{\infty}(-1)^s D_{i_1}\cdots D_{i_s}\frac{\partial}{\partial u_{i_1\cdots i_s}}, \tag{3.3}$$

where $(D_t^{\alpha})^*$ is the right-hand of Caputo fractional derivative as follows:

$$(D_t^{\alpha})^* f(t,x) \equiv {}_t^cD_T^{\alpha} f(t,x) = \begin{cases} \frac{1}{\Gamma(n-\alpha)}\int_t^T \frac{1}{(t-s)^{\alpha-n+1}}\frac{\partial^n}{\partial s^n}f(s,x)\mathrm{d}s, & n-1<\alpha<n, n\in\mathbb{N}\\ D_t^n f(t,x), & \alpha = n \in \mathbb{N}.\end{cases}$$

The following is the adjoint equation of (3.1):

$$F^* = \frac{\delta\mathcal{L}}{\delta u} = (D_t^{\alpha})^*v - auv_x - v_{xx} - v_{yy} - v_{zz} = 0. \tag{3.4}$$

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The new conservation theorem and Eq. (3.4) are utilized to construct conservation vectors of Eq. (1.1). According to the classical definition of the conservation laws, if $C = (C^t, C^x, C^y, C^z)$ satisfies the conservation equation $[D_t C^t + D_x C^x + D_y C^y + D_z C^z]_{F=0} = 0$, we call it a conserved vector for the given equation. Next we use Noether operators to calculate the components of C. From the basic operator equation

$$Pr^{(\alpha,2)}X + D_t\tau \cdot \mathcal{I} + D_x\xi \cdot \mathcal{I} + D_y\zeta \cdot \mathcal{I} + D_z\theta \cdot \mathcal{I} = W \cdot \frac{\delta}{\delta u} + D_t\mathcal{N}^t + D_x\mathcal{N}^x + D_y\mathcal{N}^y + D_z\mathcal{N}^z, \tag{3.5}$$

where $Pr^{(\alpha,2)}X$ is given in (2.3), $\mathcal{I}$ is an identity operator, and $W = \eta - \tau u_t - \xi u_x - \zeta u_y - \theta u_z$ is the characteristic of X, the following Noether operators are obtained:

$$\begin{aligned}
\mathcal{N}^t &= \tau\mathcal{I} + \sum_{k=0}^{n-1}(-1)^k D_t^{\alpha-1-k}(W) D_t^k \frac{\partial}{\partial(D_t^\alpha u)} - (-1)^n J(W, D_t^n \frac{\partial}{\partial(D_t^\alpha u)}),\\
\mathcal{N}^x &= \xi\mathcal{I} + W\Big(\frac{\partial}{\partial u_x} - D_x\frac{\partial}{\partial u_{xx}}\Big) + D_x(W)\frac{\partial}{\partial u_{xx}},\\
\mathcal{N}^y &= \zeta\mathcal{I} - W D_y\frac{\partial}{\partial u_{yy}} + D_y(W)\frac{\partial}{\partial u_{yy}},\\
\mathcal{N}^z &= \theta\mathcal{I} - W D_z\frac{\partial}{\partial u_{zz}} + D_z(W)\frac{\partial}{\partial u_{zz}},
\end{aligned} \tag{3.6}$$

where $n = 1 + [\alpha]$, and the operator J is

$$J(f, g) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \int_t^T \frac{f(\tau, x, y, z) g(\theta, x, y, z)}{(\theta - \tau)^{\alpha+1-n}} \mathrm{d}\theta \mathrm{d}\tau. \tag{3.7}$$

The conserved vector for each generator X is defined by

$$C = (C^t, C^x, C^y, C^z) = (\mathcal{N}^t\mathcal{L}, \mathcal{N}^x\mathcal{L}, \mathcal{N}^y\mathcal{L}, \mathcal{N}^z\mathcal{L}). \tag{3.8}$$

Case 1: $X_1 = t\frac{\partial}{\partial t} + \frac{\alpha}{2}x\frac{\partial}{\partial x} + \frac{\alpha}{2}y\frac{\partial}{\partial y} + \frac{\alpha}{2}z\frac{\partial}{\partial z} - \frac{\alpha}{2}u\frac{\partial}{\partial u}$

The characteristic of X_1 is

$$W = -\frac{\alpha}{2}u - tu_t - \frac{\alpha}{2}xu_x - \frac{\alpha}{2}yu_y - \frac{\alpha}{2}zu_z. \tag{3.9}$$

Therefore, for $0 < \alpha < 1$,

$$\begin{aligned}
C^t =& vD_t^{\alpha-1}(W) + J(W, v_t) = vD_t^{\alpha-1}(-\frac{\alpha}{2}u - tu_t - \frac{\alpha}{2}xu_x - \frac{\alpha}{2}yu_y - \frac{\alpha}{2}zu_z)\\
&+ J(-\frac{\alpha}{2}u - tu_t - \frac{\alpha}{2}xu_x - \frac{\alpha}{2}yu_y - \frac{\alpha}{2}zu_z, v_t),\\
C^x =& (auv - v_x)W + vD_x(W) = (auv - v_x)(-\frac{\alpha}{2}u - tu_t - \frac{\alpha}{2}xu_x - \frac{\alpha}{2}yu_y\\
&- \frac{\alpha}{2}zu_z) + v(-\frac{\alpha}{2}u_x - tu_{xt} - \frac{\alpha}{2}u_x - \frac{\alpha}{2}xu_{xx} - \frac{\alpha}{2}yu_{xy} - \frac{\alpha}{2}zu_{xz}),\\
C^y =& -v_yW + vD_y(W) = -v_y(-\frac{\alpha}{2}u - tu_t - \frac{\alpha}{2}xu_x - \frac{\alpha}{2}yu_y - \frac{\alpha}{2}zu_z)\\
&+ v(-\frac{\alpha}{2}u_y - tu_{yt} - \frac{\alpha}{2}xu_{xy} - \frac{\alpha}{2}u_y - \frac{\alpha}{2}yu_{yy} - \frac{\alpha}{2}zu_{yz}),\\
C^z =& -v_zW + vD_z(W) = -v_z(-\frac{\alpha}{2}u - tu_t - \frac{\alpha}{2}xu_x - \frac{\alpha}{2}yu_y - \frac{\alpha}{2}zu_z)\\
&+ v(-\frac{\alpha}{2}u_z - tu_{zt} - \frac{\alpha}{2}xu_{xz} - \frac{\alpha}{2}yu_{yz} - \frac{\alpha}{2}u_z - \frac{\alpha}{2}zu_{zz}).
\end{aligned} \tag{3.10}$$

Case 2: $X_2 = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}$

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The characteristic of X_2 is

$$W = -zu_y + yu_z. \tag{3.11}$$

Therefore, for $0 < \alpha < 1$,

$$\begin{aligned}
C^t &= vD_t^{\alpha-1}(W) + J(W, v_t) = vD_t^{\alpha-1}(-zu_y + yu_z) + J(-zu_y + yu_z, v_t),\\
C^x &= (auv - v_x)W + vD_x(W) = (auv - v_x)(-zu_y + yu_z) + v(-zu_{xy} + yu_{xz}),\\
C^y &= -v_yW + vD_y(W) = -v_y(-zu_y + yu_z) + v(-zu_{yy} + u_z + yu_{yz}),\\
C^z &= -v_zW + vD_z(W) = -v_z(-zu_y + yu_z) + v(-u_y - zu_{yz} + yu_{zz}).
\end{aligned} \tag{3.12}$$

Case 3: $X_3 = \frac{\partial}{\partial x}$

The characteristic of X_3 is

$$W = -u_x. \tag{3.13}$$

Therefore, for $0 < \alpha < 1$,

$$\begin{aligned}
C^t &= vD_t^{\alpha-1}(W) + J(W, v_t) = -vD_t^{\alpha-1}(u_x) - J(u_x, v_t),\\
C^x &= (auv - v_x)W + vD_x(W) = -auvu_x + u_xv_x - vu_{xx},\\
C^y &= -v_yW + vD_y(W) = u_xv_y - vu_{xy},\\
C^z &= -v_zW + vD_z(W) = u_xv_z - vu_{xz}.
\end{aligned} \tag{3.14}$$

Case 4: $X_4 = \frac{\partial}{\partial y}$

The characteristic of X_4 is

$$W = -u_y. \tag{3.15}$$

Therefore, for $0 < \alpha < 1$,

$$\begin{aligned}
C^t &= vD_t^{\alpha-1}(W) + J(W, v_t) = -vD_t^{\alpha-1}(u_y) - J(u_y, v_t),\\
C^x &= (auv - v_x)W + vD_x(W) = -auvu_y + u_yv_x - vu_{xy},\\
C^y &= -v_yW + vD_y(W) = u_yv_y - vu_{yy},\\
C^z &= -v_zW + vD_z(W) = u_yv_z - vu_{yz}.
\end{aligned} \tag{3.16}$$

Case 5: $X_5 = \frac{\partial}{\partial z}$

The characteristic of X_5 is

$$W = -u_z. \tag{3.17}$$

Therefore, for $0 < \alpha < 1$,

$$\begin{aligned}
C^t &= vD_t^{\alpha-1}(W) + J(W, v_t) = -vD_t^{\alpha-1}(u_z) - J(u_z, v_t),\\
C^x &= (auv - v_x)W + vD_x(W) = -auvu_z + u_zv_x - vu_{xz},\\
C^y &= -v_yW + vD_y(W) = u_zv_y - vu_{yz},\\
C^z &= -v_zW + vD_z(W) = u_zv_z - vu_{zz}.
\end{aligned} \tag{3.18}$$

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Table 1
The Commutation Table.

$[X_i, X_j]$	X_1	X_2	X_3	X_4	X_5
X_1	0	0	$-\frac{\alpha}{2}X_3$	$-\frac{\alpha}{2}X_4$	$-\frac{\alpha}{2}X_5$
X_2	0	0	0	X_5	$-X_4$
X_3	$\frac{\alpha}{2}X_3$	0	0	0	0
X_4	$\frac{\alpha}{2}X_4$	$-X_5$	0	0	0
X_5	$\frac{\alpha}{2}X_5$	X_4	0	0	0

Table 2
The Adjoint Table.

Ad	X_1	X_2	X_3	X_4	X_5
X_1	X_1	X_2	$e^{\frac{\alpha}{2}\varepsilon}X_3$	$e^{\frac{\alpha}{2}\varepsilon}X_4$	$e^{\frac{\alpha}{2}\varepsilon}X_5$
X_2	X_1	X_2	X_3	$\cos\varepsilon\ X_4 - \sin\varepsilon\ X_5$	$\cos\varepsilon\ X_5 + \sin\varepsilon\ X_4$
X_3	$X_1 - \frac{\alpha}{2}\varepsilon X_3$	X_2	X_3	X_4	X_5
X_4	$X_1 - \frac{\alpha}{2}\varepsilon X_4$	$X_2 + \varepsilon X_5$	X_3	X_4	X_5
X_5	$X_1 - \frac{\alpha}{2}\varepsilon X_5$	$X_2 - \varepsilon X_4$	X_3	X_4	X_5

4. Optimal system

Next we use Olver's method [16] to construct an optimal system for the symmetry group admited by Eq. (1.1). Firstly, in Table 1, the commutation relationships of the group generators in (2.11) can be obtained under the Lie bracket defined by

$$[X_i, X_j] = X_i X_j - X_j X_i, \ \ (i, j = 1, 2, 3, 4, 5). \tag{4.1}$$

The following is Lie series

$$Ad(exp(\varepsilon X_i))X_j = X_j - \varepsilon[X_i, X_j] + \frac{\varepsilon^2}{2}[X_i, [X_i, X_j]] - \cdots, \tag{4.2}$$

where ε is an arbitrary parameter. According to (4.2), we can obtain the adjoint representations of all the group generators in (2.11) and include them in Table 2.

Assuming $X = a_1X_1 + a_2X_2 + a_3X_3 + a_4X_4 + a_5X_5$, from Table 2, we can get the following expression:

$$Ad(exp(\varepsilon_1 X_1))X = a_1X_1 + a_2X_2 + a_3e^{\frac{\alpha}{2}\varepsilon}X_3 + a_4e^{\frac{\alpha}{2}\varepsilon}X_4 + a_5e^{\frac{\alpha}{2}\varepsilon}X_5, \tag{4.3}$$

which can be written as

$$Ad(exp(\varepsilon_1 X_1))X = (a_1, a_2, a_3, a_4, a_5)A_1(X_1, X_2, X_3, X_4, X_5)^T, \tag{4.4}$$

where

$$A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & e^{\frac{\alpha}{2}\varepsilon_1} & 0 & 0 \\ 0 & 0 & 0 & e^{\frac{\alpha}{2}\varepsilon_1} & 0 \\ 0 & 0 & 0 & 0 & e^{\frac{\alpha}{2}\varepsilon_1} \end{pmatrix}.$$

Table 3
The Construction Table.

	Coeff. X_1	Coeff. X_2	Coeff. X_3	Coeff. X_4	Coeff. X_5
P_1	a_1	a_2	$e^{\frac{\alpha}{2}\varepsilon}a_3$	$e^{\frac{\alpha}{2}\varepsilon}a_4$	$e^{\frac{\alpha}{2}\varepsilon}a_5$
P_2	a_1	a_2	a_3	$\cos\varepsilon\ a_4+\sin\varepsilon\ a_5$	$\cos\varepsilon\ a_5-\sin\varepsilon\ a_4$
P_3	a_1	a_2	$a_3-\frac{\alpha}{2}\varepsilon a_1$	a_4	a_5
P_4	a_1	a_2	a_3	$a_4-\frac{\alpha}{2}\varepsilon a_1$	$a_5+\varepsilon a_2$
P_5	a_1	a_2	a_3	$a_4-\varepsilon a_2$	$a_5-\frac{\alpha}{2}\varepsilon a_1$

Similar to A_1, we get

$$A_2=\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&\cos\varepsilon_2&-\sin\varepsilon_2\\0&0&0&\sin\varepsilon_2&\cos\varepsilon_2\end{pmatrix},\quad A_3=\begin{pmatrix}1&0&-\frac{\alpha}{2}\varepsilon_3&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix},$$

$$A_4=\begin{pmatrix}1&0&0&-\frac{\alpha}{2}\varepsilon_4&0\\0&1&0&0&\varepsilon_4\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix},\quad A_5=\begin{pmatrix}1&0&0&0&-\frac{\alpha}{2}\varepsilon_5\\0&1&0&-\varepsilon_5&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}.$$

Then we can construct the general adjoint matrix

$$A=A_1A_2A_3A_4A_5=\begin{pmatrix}1&0&-\frac{\alpha}{2}\varepsilon_3&-\frac{\alpha}{2}\varepsilon_4&-\frac{\alpha}{2}\varepsilon_5\\0&1&0&-\varepsilon_5&\varepsilon_4\\0&0&1&0&0\\0&0&0&e^{\frac{\alpha}{2}\varepsilon_1}\cos\varepsilon_2&-e^{\frac{\alpha}{2}\varepsilon_1}\sin\varepsilon_2\\0&0&0&e^{\frac{\alpha}{2}\varepsilon_1}\sin\varepsilon_2&e^{\frac{\alpha}{2}\varepsilon_1}\cos\varepsilon_2\end{pmatrix}.$$

To conveniently derive invariant functions, we use these matrixes (A_1,A_2,A_3,A_4,A_5) to construct Table 3, where $Ad(exp(\varepsilon X_i))X$ is marked P_i $(i=1,2,\cdots,5)$.

Theorem 4.1. *For vector $X=\sum_{i=1}^5 a_iX_i$ with $a_i\in\mathbf{R}$, the invariant function to Lie symmetry algebra L^5 is obtained as $\Xi=F(a_1,a_2)$, where F is an arbitrary function.*

Proof. Consider $g=\exp(\varepsilon Y)$ with $Y=\sum_{i=1}^5 b_iX_i$ is any element from Lie group G created by L^5. We call the real function Ξ an invariant when the following condition holds:

$$\Xi[Ad(g)X]=\Xi(X)\quad\text{for all}\quad X\in L^5. \tag{4.5}$$

That is,

$$\begin{aligned}Ad(exp(\varepsilon Y))X=&e^{-\varepsilon Y}Xe^{\varepsilon Y}=X-\varepsilon[Y,X]+\frac{\varepsilon^2}{2}[Y,[Y,X]]-\cdots\\=&(a_1X_1+a_2X_2+a_3X_3+a_4X_4+a_5X_5)-(\vartheta_1X_1\\&+\vartheta_2X_2+\vartheta_3X_3+\vartheta_4X_4+\vartheta_5X_5)\varepsilon+O(\varepsilon^2),\end{aligned}\tag{4.6}$$

where

$$\begin{aligned}&\vartheta_1=0,\ \ \vartheta_2=0,\ \ \vartheta_3=\frac{\alpha}{2}(b_3a_1-b_1a_3),\ \ \vartheta_4=\frac{\alpha}{2}(b_4a_1-b_1a_4)+(b_5a_2-b_2a_5),\\&\vartheta_5=\frac{\alpha}{2}(b_5a_1-b_1a_5)+(b_2a_4-b_4a_2).\end{aligned}\tag{4.7}$$

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Then the condition (4.5) arrives at

$$\Xi(a_1,a_2,a_3,a_4,a_5)=\Xi(a_1-\varepsilon\vartheta_1,a_2-\varepsilon\vartheta_2,a_3-\varepsilon\vartheta_3,a_4-\varepsilon\vartheta_4,a_5-\varepsilon\vartheta_5).$$

By differentiating the function Ξ with respect to parameter ε, and letting ε=0, the following system of first order linear PDEs can be obtained by collecting the coefficients of b_i:

$$\begin{aligned}
&\frac{\alpha}{2}a_3\frac{\partial\Xi}{\partial a_3}+\frac{\alpha}{2}a_4\frac{\partial\Xi}{\partial a_4}+\frac{\alpha}{2}a_5\frac{\partial\Xi}{\partial a_5}=0,\\
&a_5\frac{\partial\Xi}{\partial a_4}-a_4\frac{\partial\Xi}{\partial a_5}=0,\\
&-\frac{\alpha}{2}a_1\frac{\partial\Xi}{\partial a_3}=0,\\
&-\frac{\alpha}{2}a_1\frac{\partial\Xi}{\partial a_4}+a_2\frac{\partial\Xi}{\partial a_5}=0,\\
&-a_2\frac{\partial\Xi}{\partial a_4}+\frac{\alpha}{2}a_1\frac{\partial\Xi}{\partial a_5}=0.
\end{aligned}\tag{4.8}$$

By solving the system of equations above, we obtain the general invariant function of L^5 with the form $\Xi=F(a_1,a_2)$, of which F is an arbitrary function. □

Theorem 4.2. *The Killing form $K\langle X,X\rangle=\frac{3\alpha^2}{4}a_1^2$ is also an invariant function to Lie symmetry algebra* L^5.

Proof. The Killing form of L^5 is defined by

$$K\langle X,X\rangle=\mathrm{Trace}(ad X\cdot ad X),\tag{4.9}$$

where

$$ad X=\begin{pmatrix}0&0&\frac{\alpha}{2}a_3&\frac{\alpha}{2}a_4&\frac{\alpha}{2}a_5\\0&0&0&a_5&-a_4\\0&0&-\frac{\alpha}{2}a_1&0&0\\0&0&0&-\frac{\alpha}{2}a_1&a_2\\0&0&0&-a_2&\frac{\alpha}{2}a_1\end{pmatrix}.$$

Therefore, by calculation, we can easily get $K\langle X,X\rangle=\frac{3\alpha^2}{4}a_1^2$. □

Theorem 4.3. *Based on Theorems 4.1-4.2, the one-dimensional optimal system for Lie symmetry algebra* L^5 *admitted by Eq. (1.1) can be spanned by*

$$X_1,\ X_2,\ X_3,\ X_4,\ X_5,\ bX_3+dX_4,\ bX_3+dX_5,\ bX_4+dX_5,\ bX_3+cX_4+dX_5,\tag{4.10}$$

where b, c and d are free parameter.

Proof. Similar to the proofs in literatures [16, 48–52]. □

5. Reduction and solution

In what follows, we can reduce Eq. (1.1) to some different (2+1)-dimensional time FPDEs and (1+1)-dimensional time FPDEs by the obtained one-dimensional optimal system.

Case 1: X_1

The characteristic equation of X_1 is

$$\frac{\mathrm{d}t}{t}=\frac{2\mathrm{d}x}{\alpha x}=\frac{2\mathrm{d}y}{\alpha y}=\frac{2\mathrm{d}z}{\alpha z}=\frac{-2\mathrm{d}u}{\alpha u}.\tag{5.1}$$

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By solving (5.1), we can obtain the similarity variables $xt^{-\frac{\alpha}{2}}$, $yt^{-\frac{\alpha}{2}}$, $zt^{-\frac{\alpha}{2}}$ and $ut^{\frac{\alpha}{2}}$. Then the invariant solution can be constructed as follows:

$$u = t^{-\frac{\alpha}{2}} f(\omega_1,\omega_2,\omega_3),\ \ \omega_1 = xt^{-\frac{\alpha}{2}},\ \ \omega_2 = yt^{-\frac{\alpha}{2}},\ \ \omega_3 = zt^{-\frac{\alpha}{2}}. \tag{5.2}$$

Theorem 5.1. *The similarity transformation* $u = t^{-\frac{\alpha}{2}} f(\omega_1,\omega_2,\omega_3)$ *with the similarity variables* $\omega_1 = xt^{-\frac{\alpha}{2}}$, $\omega_2 = yt^{-\frac{\alpha}{2}}$, $\omega_3 = zt^{-\frac{\alpha}{2}}$ *reduce Eq. (1.1) to the (2+1)-dimensional time FPDE given by*

$$(\mathcal{P}^{1-\frac{3\alpha}{2},\alpha}_{\frac{2}{\alpha},\frac{2}{\alpha},\frac{2}{\alpha}} f)(\omega_1,\omega_2,\omega_3) + aff_{\omega_1} = f_{\omega_1\omega_1} + f_{\omega_2\omega_2} + f_{\omega_3\omega_3}, \tag{5.3}$$

where $(\mathcal{P}^{\iota,\kappa}_{\delta_1,\delta_2,\delta_3})$ *is the following left-sided EKFD operator:*

$$\begin{aligned}(\mathcal{P}^{\iota,\kappa}_{\delta_1,\delta_2,\delta_3}\psi)(\omega_1,\omega_2,\omega_3) :=& \prod_{j=0}^{m-1}(\iota + j - \frac{1}{\delta_1}\omega_1\frac{\mathrm{d}}{\mathrm{d}\omega_1} - \frac{1}{\delta_2}\omega_2\frac{\mathrm{d}}{\mathrm{d}\omega_2} - \frac{1}{\delta_2}\omega_3\frac{\mathrm{d}}{\mathrm{d}\omega_3})\\ &\times(\mathcal{K}^{\iota+\kappa,m-\kappa}_{\delta_1,\delta_2,\delta_3}\psi)(\omega_1,\omega_2,\omega_3),\ m = \begin{cases}[\kappa]+1, & \text{if } \kappa\notin\mathbb{N},\\ \kappa, & \text{if } \kappa\in\mathbb{N},\end{cases}\end{aligned} \tag{5.4}$$

with the corresponding left-sided Erdélyi-Kober fractional integral

$$(\mathcal{K}^{\iota,\kappa}_{\delta_1,\delta_2,\delta_3}\psi)(\omega_1,\omega_2,\omega_3) := \begin{cases}\frac{1}{\Gamma(\kappa)}\int_1^\infty (s-1)^{\kappa-1}s^{-(\iota+\kappa)}\psi(\omega_1 s^{\frac{1}{\delta_1}},\omega_2 s^{\frac{1}{\delta_2}},\omega_3 s^{\frac{1}{\delta_3}})\mathrm{d}s, & \kappa>0,\\ \psi(\omega_1,\omega_2,\omega_3), & \kappa=0.\end{cases} \tag{5.5}$$

Proof. When $0<\alpha<1$, from the definition of RLFD, we can obtain

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^\alpha}{\partial t^\alpha}(t^{-\frac{\alpha}{2}}f(\omega_1,\omega_2,\omega_3)) = \frac{\partial}{\partial t}\Big[\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}s^{-\frac{\alpha}{2}}f(xs^{-\frac{\alpha}{2}},ys^{-\frac{\alpha}{2}},zs^{-\frac{\alpha}{2}})\mathrm{d}s\Big].$$

Assuming $r=\frac{t}{s}$, we have

$$\begin{aligned}\frac{\partial^\alpha u}{\partial t^\alpha} =& \frac{\partial}{\partial t}\Big[\frac{t^{1-\frac{3\alpha}{2}}}{\Gamma(1-\alpha)}\int_1^\infty (r-1)^{-\alpha}r^{\frac{3\alpha}{2}-2}f(\omega_1 r^{\frac{\alpha}{2}},\omega_2 r^{\frac{\alpha}{2}},\omega_3 r^{\frac{\alpha}{2}})\mathrm{d}r\Big]\\ =&\frac{\partial}{\partial t}\Big[t^{1-\frac{3\alpha}{2}}(\mathcal{K}^{1-\frac{\alpha}{2},1-\alpha}_{\frac{2}{\alpha},\frac{2}{\alpha},\frac{2}{\alpha}}f)(\omega_1,\omega_2,\omega_3)\Big].\end{aligned}$$

Because of $\omega_1 = xt^{-\frac{\alpha}{2}}$, $\omega_2 = yt^{-\frac{\alpha}{2}}$ and $\omega_3 = zt^{-\frac{\alpha}{2}}$, the following relation holds:

$$t\frac{\partial}{\partial t}\psi(\omega_1,\omega_2,\omega_3) = -\frac{\alpha}{2}\omega_1\frac{\mathrm{d}}{\mathrm{d}\omega_1}\psi - \frac{\alpha}{2}\omega_2\frac{\mathrm{d}}{\mathrm{d}\omega_2}\psi - \frac{\alpha}{2}\omega_3\frac{\mathrm{d}}{\mathrm{d}\omega_3}\psi.$$

Hence, we arrive at

$$\begin{aligned}\frac{\partial^\alpha u}{\partial t^\alpha} =& t^{-\frac{3\alpha}{2}}\Big[(1-\frac{3\alpha}{2} - \frac{\alpha}{2}\omega_1\frac{\mathrm{d}}{\mathrm{d}\omega_1} - \frac{\alpha}{2}\omega_2\frac{\mathrm{d}}{\mathrm{d}\omega_2} - \frac{\alpha}{2}\omega_3\frac{\mathrm{d}}{\mathrm{d}\omega_3})(\mathcal{K}^{1-\frac{\alpha}{2},1-\alpha}_{\frac{2}{\alpha},\frac{2}{\alpha},\frac{2}{\alpha}}f)(\omega_1,\omega_2,\omega_3)\Big]\\ =& t^{-\frac{3\alpha}{2}}(\mathcal{P}^{1-\frac{3\alpha}{2},\alpha}_{\frac{2}{\alpha},\frac{2}{\alpha},\frac{2}{\alpha}}f)(\omega_1,\omega_2,\omega_3).\end{aligned}$$

Meanwhile,

$$auu_x = at^{-\frac{3\alpha}{2}}ff_{\omega_1},\ \ u_{xx}+u_{yy}+u_{zz} = t^{-\frac{3\alpha}{2}}(f_{\omega_1\omega_1} + f_{\omega_2\omega_2} + f_{\omega_3\omega_3}).$$

This completes the proof. □

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Next we apply the method introduced in [29] to get the PSS of (5.3). Assuming

$$f(\omega_1,\omega_2,\omega_3)=f(\omega)=\sum_{k=0}^{\infty}a_k\omega^k,\ \ \omega=C_1\omega_1+C_2\omega_2+C_3\omega_3, \tag{5.6}$$

with infinite number of undetermined constants a_k, one can get

$$f'(\omega)=\sum_{k=0}^{\infty}(k+1)a_{k+1}\omega^k,\ \ f''(\omega)=\sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}\omega^k. \tag{5.7}$$

From [29], we have

$$(\mathcal{P}_{\frac{2}{\alpha},\frac{2}{\alpha},\frac{2}{\alpha}}^{1-\frac{3\alpha}{2},\alpha}f)(\omega_1,\omega_2,\omega_3)=(\mathcal{P}_{\frac{2}{\alpha}}^{1-\frac{3\alpha}{2},\alpha}f)(\omega)=\sum_{k=0}^{\infty}\frac{\Gamma(1-\frac{(k+1)\alpha}{2})}{\Gamma(1-\frac{(k+3)\alpha}{2})}a_k\omega^k. \tag{5.8}$$

Substituting (5.6)-(5.8) into (5.3) arrives at the following equation:

$$\begin{aligned}&\sum_{k=0}^{\infty}\frac{\Gamma(1-\frac{(k+1)\alpha}{2})}{\Gamma(1-\frac{(k+3)\alpha}{2})}a_k\omega^k+aC_1\sum_{k=0}^{\infty}\Big(\sum_{i+j=k}(j+1)a_ia_{j+1}\Big)\omega^k\\&=(C_1^2+C_2^2+C_3^2)\sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}\omega^k.\end{aligned} \tag{5.9}$$

By comparing the coefficients of the same powers of ω on both sides of (5.9), we can derive the explicit expressions of a_k as $a_0=f(0)$, $a_1=f'(0)$ and

$$a_{k+2}=\frac{1}{(k+2)(k+1)(C_1^2+C_2^2+C_3^2)}\Big[\frac{\Gamma(1-\frac{(k+1)\alpha}{2})}{\Gamma(1-\frac{(k+3)\alpha}{2})}a_k+aC_1\Big(\sum_{i+j=k}(j+1)a_ia_{j+1}\Big)\Big],\ \ k\geq 0. \tag{5.10}$$

Therefore, we can obtain the following PSS of Eq. (1.1):

$$\begin{aligned}u(t,x,y,z)=t^{-\frac{\alpha}{2}}\Big\{&a_0+a_1(C_1x+C_2y+C_3z)t^{-\frac{\alpha}{2}}+\frac{1}{2(C_1^2+C_2^2+C_3^2)}\Big[\frac{\Gamma(1-\frac{\alpha}{2})}{\Gamma(1-\frac{3\alpha}{2})}a_0\\&+aC_1a_0a_1\Big](C_1x+C_2y+C_3z)^2t^{-\alpha}+\sum_{k=1}^{\infty}\frac{1}{(k+2)(k+1)(C_1^2+C_2^2+C_3^2)}\\&\times\Big[\frac{\Gamma(1-\frac{(k+1)\alpha}{2})}{\Gamma(1-\frac{(k+3)\alpha}{2})}a_k+aC_1\Big(\sum_{i+j=k}(j+1)a_ia_{j+1}\Big)\Big](C_1x+C_2y+C_3z)^{k+2}t^{-\frac{(k+2)\alpha}{2}}\Big\}.\end{aligned} \tag{5.11}$$

Theorem 5.2. *In the neighborhood of the point $(0,|a_0|)$, the obtained PSS (5.11) is convergent.*

Proof. From the coefficient expressions (5.10), taking absolute values for them, we can obtain

$$|a_{k+2}|\leq\frac{1}{C_1^2+C_2^2+C_3^2}\Big[\frac{|\Gamma(1-\frac{(k+1)\alpha}{2})|}{|\Gamma(1-\frac{(k+3)\alpha}{2})|}|a_i|+|aC_1|\sum_{i+j=k}|a_i||a_{j+1}|\Big],\ \ k\geq 0. \tag{5.12}$$

From the property of Γ function, one can easily find that the inequality $\frac{|\Gamma(1-\frac{(k+1)\alpha}{2})|}{|\Gamma(1-\frac{(k+3)\alpha}{2})|}\leq 1$ holds for an arbitrary natural number k. Thus, (5.12) can be written as

$$|a_{k+2}|\leq M\big(|a_k|+\sum_{i+j=k}|a_i||a_{j+1}|\big),\ \ k\geq 0, \tag{5.13}$$

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Table 4
Some of a_n for different fractional orders when $a_0 = a_1 = 1$

	a_0	a_1	a_2	a_3	a_4	a_5
$\alpha = 0.4$	1	1	0.2541438214	0.1018150429	0.02966323869	0.009579915765
$\alpha = 0.6$	1	1	0.1894072333	0.05543125540	0.00628336655	0.006247295912
$\alpha = 0.8$	1	1	0.1240292550	0.00034780256	0.01036475480	0.001218887081

where $M = \max\{\frac{1}{C_1^2+C_2^2+C_3^2}, \frac{|aC_1|}{C_1^2+C_2^2+C_3^2}\}$.

Consider another power series

$$B(\omega) = \sum_{k=0}^{\infty} b_k \omega^k, \tag{5.14}$$

where $b_0 = |a_0|$, $b_1 = |a_1|$ and

$$b_{k+2} = M(b_k + \sum_{i+j=k} b_i b_{j+1}), \quad k \geq 0. \tag{5.15}$$

Therefore, it can be easily derived that $|a_k| \leq b_k$ for $k = 0, 1, 2, \ldots$, that is, the power series (5.14) is called the majorant series of (5.6). The next step is to prove its convergence. By simple calculation, we can get

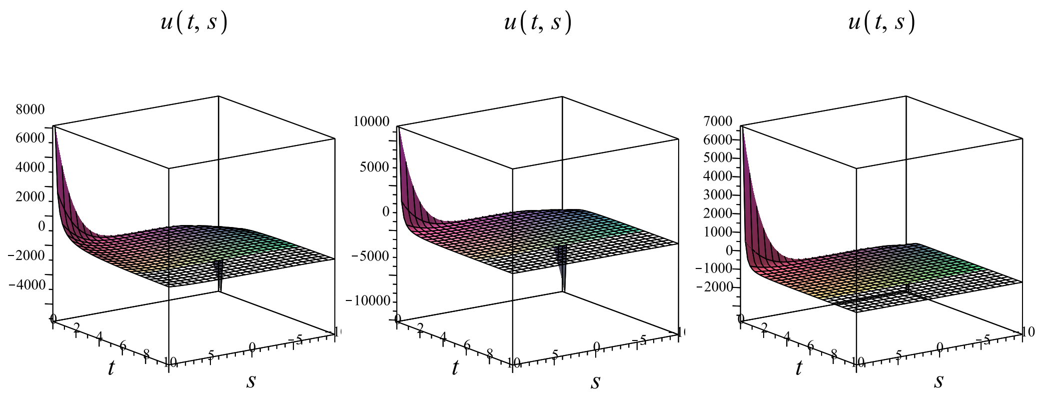
$$B(\omega) = b_0 + b_1\omega + M\left(B(\omega)\omega^2 + B(\omega)(B(\omega) - b_0)\omega\right). \tag{5.16}$$

For the independent variable ω, the following function is viewed as an implicit function:

$$\Psi(\omega, B) = B - b_0 - b_1\omega - M\left(B\omega^2 + B(B - b_0)\omega\right). \tag{5.17}$$

It can be seen that $\Psi(0, b_0) = 0$, $\frac{\partial}{\partial B}\Psi(0, b_0) \neq 0$, and $\Psi(\omega, B)$ is an analytic function in a neighborhood of $(0, b_0)$. From implicit function theorem, the power series (5.14) is analytic in the same neighborhood, that is to say, the PSS (5.11) is convergent in a neighborhood of $(0, |a_0|)$. □

Assuming $a = 1$ in Eq. (1.1), $C_1 = C_2 = C_3 = 1$ and $s = x + y + z$ in (5.11), Tabs.4-5 show some values of a_n and α, while Figs.1-2 illustrate the dynamical behavior of the solution (5.11) for different parameters.



(a) α=0.4 (b) α=0.6 (c) α=0.8

Figure 1: Dynamical profiles of the truncated power series solution (5.11) with $a = 1$, $C_1 = C_2 = C_3 = 1$, $a_0 = a_1 = 1$ and $s = x + y + z$.

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Table 5
Some of a_n for different fractional orders when $a_0 = a_1 = 0.1$

	a_0	a_1	a_2	a_3	a_4	a_5
$\alpha = 0.4$	0.1	0.1	0.01041438214	0.002473399409	0.000107398179	-0.00001169010880
$\alpha = 0.6$	0.1	0.1	0.003940723327	-0.001517613459	-0.0002735785966	-0.00003708124004
$\alpha = 0.8$	0.1	0.1	-0.002597074506	-0.006372178961	-0.00007474377889	0.0001048892399

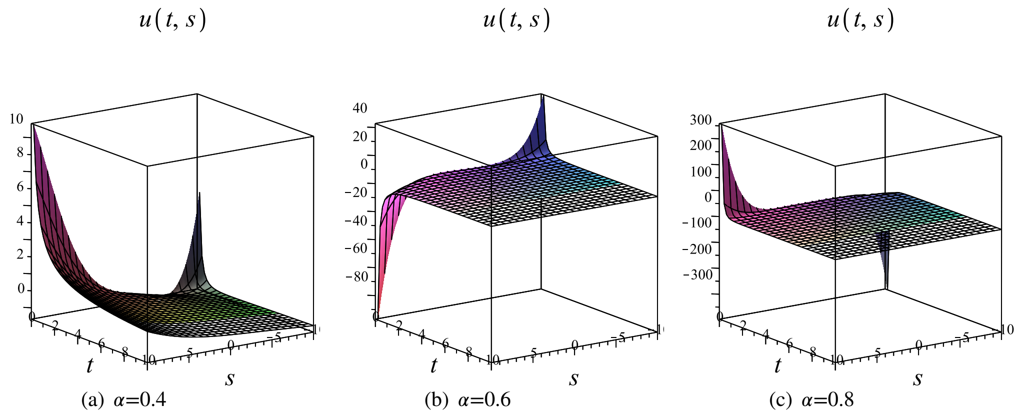

Figure 2: Dynamical profiles of the truncated power series solution (5.11) with $a = 1$, $C_1 = C_2 = C_3 = 1$, $a_0 = a_1 = 0.1$ and $s = x + y + z$.

Case 2: X_2

The characteristic equation of X_2 is

$$\frac{\mathrm{d}t}{0} = \frac{\mathrm{d}x}{0} = \frac{\mathrm{d}y}{z} = \frac{\mathrm{d}z}{-y} = \frac{\mathrm{d}u}{0}. \tag{5.18}$$

By solving (5.18), we can obtain the similarity variables t, x, $y^2 + z^2$ and u. Then the invariant solution can be constructed as follows:

$$u = f(\omega_1, \omega_2, \omega_3), \ \ \omega_1 = t, \ \ \omega_2 = x, \ \ \omega_3 = y^2 + z^2. \tag{5.19}$$

Finally, we can obtain the following reduced equation by substituting (5.19) into Eq. (1.1):

$$D^{\alpha}_{\omega_1} f + a f f_{\omega_2} = f_{\omega_2\omega_2} + 4 f_{\omega_3} + 4\omega_3 f_{\omega_3\omega_3}. \tag{5.20}$$

Case 3: X_3

The characteristic equation of X_3 is

$$\frac{\mathrm{d}t}{0} = \frac{\mathrm{d}x}{1} = \frac{\mathrm{d}y}{0} = \frac{\mathrm{d}z}{0} = \frac{\mathrm{d}u}{0}. \tag{5.21}$$

We can obtain the similarity variables t, y, z, u by solving (5.21) and construct the following invariant solution:

$$u = f(\omega_1, \omega_2, \omega_3), \ \ \omega_1 = t, \ \ \omega_2 = y, \ \ \omega_3 = z. \tag{5.22}$$

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Finally, we can obtain the following reduced equation by substituting (5.22) into Eq. (1.1):

$$D^{\alpha}_{\omega_1} f = f_{\omega_2\omega_2} + f_{\omega_3\omega_3}. \tag{5.23}$$

Case 4: X_4

The characteristic equation of X_4 is

$$\frac{\mathrm{d}t}{0} = \frac{\mathrm{d}x}{0} = \frac{\mathrm{d}y}{1} = \frac{\mathrm{d}z}{0} = \frac{\mathrm{d}u}{0}. \tag{5.24}$$

We can obtain the similarity variables t, x, z, u by solving (5.24) and construct the following invariant solution:

$$u = f(\omega_1, \omega_2, \omega_3), \;\; \omega_1 = t, \;\; \omega_2 = x, \;\; \omega_3 = z. \tag{5.25}$$

Finally, we can obtain the following reduced equation by substituting (5.25) into Eq. (1.1):

$$D^{\alpha}_{\omega_1} f + a f f_{\omega_2} = f_{\omega_2\omega_2} + f_{\omega_3\omega_3}. \tag{5.26}$$

Case 5: X_5

The characteristic equation of X_5 is

$$\frac{\mathrm{d}t}{0} = \frac{\mathrm{d}x}{0} = \frac{\mathrm{d}y}{0} = \frac{\mathrm{d}z}{1} = \frac{\mathrm{d}u}{0}. \tag{5.27}$$

We can obtain the similarity variables t, x, y, u by solving (5.27) and construct the following invariant solution:

$$u = f(\omega_1, \omega_2, \omega_3), \;\; \omega_1 = t, \;\; \omega_2 = x, \;\; \omega_3 = y. \tag{5.28}$$

Finally, we can obtain the following reduced equation by substituting (5.28) into Eq. (1.1):

$$D^{\alpha}_{\omega_1} f + a f f_{\omega_2} = f_{\omega_2\omega_2} + f_{\omega_3\omega_3}. \tag{5.29}$$

Case 6: $bX_3 + dX_4$

The characteristic equation of $bX_3 + dX_4$ is

$$\frac{\mathrm{d}t}{0} = \frac{\mathrm{d}x}{b} = \frac{\mathrm{d}y}{d} = \frac{\mathrm{d}z}{0} = \frac{\mathrm{d}u}{0}. \tag{5.30}$$

By solving (5.30), we can obtain the similarity variables t, $dx - by$, z and u. Then the invariant solution can be constructed as follows:

$$u = f(\omega_1, \omega_2, \omega_3), \;\; \omega_1 = t, \;\; \omega_2 = dx - by, \;\; \omega_3 = z. \tag{5.31}$$

Finally, we can obtain the following reduced equation by substituting (5.31) into Eq. (1.1):

$$D^{\alpha}_{\omega_1} f + ad f f_{\omega_2} = (b^2 + d^2) f_{\omega_2\omega_2} + f_{\omega_3\omega_3}. \tag{5.32}$$

Case 7: $bX_3 + dX_5$

The characteristic equation of $bX_3 + dX_5$ is

$$\frac{\mathrm{d}t}{0} = \frac{\mathrm{d}x}{b} = \frac{\mathrm{d}y}{0} = \frac{\mathrm{d}z}{d} = \frac{\mathrm{d}u}{0}. \tag{5.33}$$

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By solving (5.33), we can obtain the similarity variables t, $dx - bz$, y and u. Then the invariant solution can be constructed as follows:

$$u = f(\omega_1, \omega_2, \omega_3),\ \ \omega_1 = t,\ \ \omega_2 = dx - bz,\ \ \omega_3 = y. \tag{5.34}$$

Finally, we can obtain the following reduced equation by substituting (5.34) into Eq. (1.1):

$$D^{\alpha}_{\omega_1} f + ad f f_{\omega_2} = (b^2 + d^2) f_{\omega_2\omega_2} + f_{\omega_3\omega_3}. \tag{5.35}$$

Case 8: $bX_4 + dX_5$

The characteristic equation of $bX_4 + dX_5$ is

$$\frac{\mathrm{d}t}{0} = \frac{\mathrm{d}x}{0} = \frac{\mathrm{d}y}{b} = \frac{\mathrm{d}z}{d} = \frac{\mathrm{d}u}{0}. \tag{5.36}$$

By solving (5.36), we can obtain the similarity variables t, $dy - bz$, x and u. Then the invariant solution can be constructed as follows:

$$u = f(\omega_1, \omega_2, \omega_3),\ \ \omega_1 = t,\ \ \omega_2 = dy - bz,\ \ \omega_3 = x. \tag{5.37}$$

Finally, we can obtain the following reduced equation by substituting (5.37) into Eq. (1.1):

$$D^{\alpha}_{\omega_1} f + a f f_{\omega_3} = (b^2 + d^2) f_{\omega_2\omega_2} + f_{\omega_3\omega_3}. \tag{5.38}$$

Case 9: $bX_3 + cX_4 + dX_5$

The characteristic equation of $bX_3 + cX_4 + dX_5$ is

$$\frac{\mathrm{d}t}{0} = \frac{\mathrm{d}x}{b} = \frac{\mathrm{d}y}{c} = \frac{\mathrm{d}z}{d} = \frac{\mathrm{d}u}{0}. \tag{5.39}$$

By solving (5.39), we can obtain the similarity variables t, $\frac{1}{b}x + \frac{1}{c}y - \frac{2}{d}z$ and u. Then the invariant solution can be constructed as follows:

$$u = f(\omega_1, \omega_2),\ \ \omega_1 = t,\ \ \omega_2 = \frac{1}{b}x + \frac{1}{c}y - \frac{2}{d}z. \tag{5.40}$$

Finally, we can obtain the following reduced equation by substituting (5.40) into Eq. (1.1):

$$D^{\alpha}_{\omega_1} f + \frac{a}{b} f f_{\omega_2} = (\frac{1}{b^2} + \frac{1}{c^2} + \frac{4}{d^2}) f_{\omega_2\omega_2}. \tag{5.41}$$

Note that Eqs. (5.20), (5.23), (5.26), (5.29), (5.32) , (5.35), (5.38) and (5.41) are some (2+1)- and (1+1)-dimensional fractional generalized Burgers equations with Riemann-Liouville fractional derivative, respectively. These reduced equations were studied by means of different numerical and analytical methods in [35–38, 40–44].

6. Conclusion

This paper extends the (1+1)-dimensional and the (2+1)-dimensional FPDEs to the (3+1)-dimensional FPDEs. The LSAM is successfully used for reducing high dimensional time FPDE to some low dimensional equations and obtaining the corresponding PSSs. This indicates that the LSAM can be effectively applied to some higher dimensional time FPDEs in physical science and engineering. However, using the LSAM to solve high dimensional FPDEs has its demerits, such as the need to repeatedly use the LSAM to reduce the dimensionality of the equation, which increases computational complexity. The equation after dimensionality reduction can also be solved by using other numerical and analytical methods, which requires us to perfectly combine the LSAM with other methods. These issues will promote the improvement and development of the LSAM for FPDEs. In addition to high dimensional time FPDEs, we will also apply the LSAM to study some high dimensional space-time FPDEs, fractional difference differential equations, fractional stochastic differential equations, etc., which have appeared in many important fields of science and technology.

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References

[1] Samko SG, Kilbas AA, Marichev OI. Fractional integrals and derivatives: theory and applications. Yverdon: Gordon and Breach Science Publishers; 1993.

[2] Podlubny I. Fractional differential equations. San Diego: Academic Press; 1999.

[3] Hilfer R. Applications of fractional calculus in physics. Singapore: World Scientific; 2000.

[4] Kilbas AA, Srivastava HM, Trujillo JJ. Theory and applications of fractional differential equations. New York: Elsevier; 2006.

[5] Daftardar-Gejji V, Jafari H. Adomian decomposition: a tool for solving a system of fractional differential equations. J Math Anal Appl 2005; 301: 508-518.

[6] Meerschaert MM, Scheffler HP, Tadjeran C. Finite difference methods for two-dimensional fractional dispersion equation. J Comput Phys 2006; 211: 249-261.

[7] Momani S, Odibat Z. Homotopy perturbation method for nonlinear partial differential equations of fractional order. Phys Lett A 2007; 365: 345-350.

[8] Zhang S, Zhang HQ. Fractional sub-equation method and its applications to nonlinear fractional PDEs. Phys Lett A 2011; 375: 1069-1073.

[9] Momani S, Odibat Z. Numerical comparison of methods for solving linear differential equations of fractional order. Chaos Solitons and Fractals 2007; 31: 1248-1255.

[10] RGazizov RK, Kasatkin AA, Lukashchuk SY. Continuous transformation groups of fractional differential equations. Vestnik USATU 2007; 9: 125-135.

[11] Gazizov RK, Kasatkin AA. Construction of exact solutions for fractional order differential equations by the invariant subspace method. Comput Math Appl 2013; 66: 576-584.

[12] Tamboli VK, Tandel PV. Reduced differential transform method for the treatment of internal atmospheric waves phenomenon. Int J Appl Comput Math 2022; 8: 174.

[13] Tamboli VK, Tandel PV. Solution of the time-fractional generalized Burger-Fisher equation using the fractional reduced differential transform method. J Ocean Eng Sci 2022; 7: 399-407.

[14] Tamboli VK, Tandel PV. Solution of the non-linear fractional sir epidemic model using fractional reduced differential transform method. VNSGU J Res Innov 2023; 2: 50-73.

[15] Ovsiannikov LV. Group analysis of differential equations. New York: Academic Press; 1982.

[16] Olver PJ. Applications of lie groups to differential equations. Heidelberg: Springer; 1986.

[17] Ibragimov NH. CRC handbook of lie group analysis of differentia equations, Volume 1, Symmetries, exact solutions and conservation laws. Boca Raton, FL: CRC Press; 1993.

[18] Ibragimov NH. CRC handbook of lie group analysis of differential equations, Volume 2, Applications in engineering and physical sciences. Boca Raton, FL: CRC Press; 1994.

[19] Ibragimov NH. CRC handbook of lie group analysis of differential equations, Volume 3, New trends in theoretical developments and computational methods. Boca Raton, FL: CRC Press; 1995.

[20] RGazizov RK, Kasatkin AA, Lukashchuk SY. Symmetry properties of fractional diffusion equations. Phys Scr 2009; T136: 014016.

[21] El Kinani EH, Ouhadan A. Lie symmetry analysis of some time fractional partial differential equations. Int J Mod Phys Conf Ser 2015; 38: 1560075.

[22] Yourdkhany M, Nadjafikhah M. Symmetries, similarity invariant solution, conservation laws and exact solutions of the time-fractional Harmonic Oscillator equation. J Geom Phys 2020; 153: 103661.

[23] Zhang ZY. Symmetry determination and nonlinearization of a nonlinear time-fractional partial differential equation. Proc R Soc A 2020; 476: 20190564.

[24] Zhang ZY, Li GF. Lie symmetry analysis and exact solutions of the time-fractional biological population model. Phys A 2020; 540: 123134.

[25] Feng YQ, Yu JC. Lie symmetry analysis of fractional ordinary differential equation with neutral delay. AIMS Mathematics 2021; 6: 3592-3605.

[26] Nass AM. Symmetry analysis of space-time fractional Poisson equation with a delay. Quaest Math 2019; 42: 1221-1235.

[27] Mpungu K, Nass AM. Symmetry analysis of time fractional convection-reaction-diffusion equation with a delay. Results Nonlinear Anal 2019; 2: 113-124.

[28] Yu JC, Feng YQ, Wang XJ. Lie symmetry analysis and exact solutions of time fractional Black-Scholes equation. Int J Financ Eng 2022; 9: 2250023.

[29] Yu JC, Feng YQ. Lie symmetry analysis and exact solutions of space-time fractional cubic Schrödinger equation. Int J Geom Methods M 2022; 19: 2250077.

[30] Yu JC, Feng YQ. Lie symmetry, exact solutions and conservation laws of some fractional partial differential equations. J Appl Anal Comput 2023; 13: 1872-1889.

[31] Saha Ray S. Invariant analysis and conservation laws for the time fractional (2+1)-dimensional Zakharov-Kuznetsov modified equal width equation using Lie group analysis. Comput Math Appl 2018; 76: 2110-2118.

[32] Ren RC, Zhang SL. Invariant analysis, conservation laws, and some exact solutions for (2+1)-dimension fractional long-wave dispersive system. Comp Appl Math 2020; 39: 249.

[33] Cheng XY, Wang LZ. Invariant analysis, exact solutions and conservation laws of (2+1)-dimensional time fractional Navier-Stokes equations. Proc R Soc A 2021; 477: 20210220.

[34] Zhu HM, Zhang ZY, Zheng J. The time-fractional (2+1)-dimensional Hirota-Satsuma-Ito equations: Lie symmetries, power series solutions and conservation laws. Commun Nonlinear Sci 2022; 115: 106724.

[35] Yang M. Numerical approximation of time-fractional Burgers-type equation. Adv Differ Equ 2020; 2020: 182.

[36] Selvaraj R, Venkatraman S, Ashok DD, Krishnakumar K. Exact solutions of time-fractional generalised Burgers-Fisher equation using generalised Kudryashov method. Pramana-J Phys 2020; 94: 137.

[37] Inc M, Yusuf A, Aliyu AI, Baleanu D. Lie symmetry analysis and explicit solutions for the time fractional generalized Burgers-Huxley equation.

Opt Quant Electron 2018; 50: 94.
[38] Saqib M, Hussain Q, Kara AH, Zaman F. On Lie symmetry analysis of nonhomogeneous generalized inviscid and fractional Burgers' equation. Math Meth Appl Sci 2021; 44: 8726-8738.
[39] Zhang HY, Zhang YF. On the time-fractional coupled Burger equation: Lie symmetry reductions, approximate solutions and conservation laws. Wave Random Complex 2022; 32: 2297-2312.
[40] Gaur M, Singh K. Symmetry analysis of time-fractional potential Burgers' equation. Math Commun 2017; 22: 1-11.
[41] Wang GW, Kara AH. Group analysis, fractional explicit solutions and conservation laws of time fractional generalized Burgers equation. Commun Theor Phys 2018; 69: 5-8.
[42] Syam MI, Obayda DA, Alshamsi W, et al. Generalized solutions of the fractional Burger's equation. Results Phys 2019; 15: 102525.
[43] Ishag KA, Ahmed AM, Ali AE. The techniques for solving fractional Burger's equations. Asian J Pure Appl Math 2022; 4: 60-73.
[44] Uddin MH, Arefin MA, Akbar MA, Inc M. New explicit solutions to the fractional-order Burgers? equation. Math Probl Eng 2021; 2021: 6698028.
[45] Yu JC. Lie symmetry analysis of time fractional Burgers equation, Korteweg-de Vries equation and generalized reaction-diffusion equation with delays. Int J Geom Methods M 2022; 19: 2250219.
[46] Ibragimov NH. Nonlinear self-adjointness and conservation laws. J Phys A-Math Theor 2011; 44: 432002.
[47] Ibragimov NH. A new conservation theorem. J Math Anal Appl 2007; 333: 311-328.
[48] Chou KS, Li GX. A Note on optimal systems for the heat equation. J Math Anal Appl 2001; 261: 741-751.
[49] Hu XR, Li YQ, Chen Y. A direct algorithm of one-dimensional optimal system for the group invariant solutions. J Math Phys 2015; 56: 053504.
[50] Tanwar MV. Optimal system, symmetry reductions and group-invariant solutions of (2+1)-dimensional ZK-BBM equation. Phys Scr 2021; 96: 065215.
[51] Liu JG, Yang XJ, Geng LL, Fan YR. Group analysis of the time fractional (3+1)-dimensional KdV-type equation. Fractals 2021; 29: 2150169.
[52] Simon SG, Bira B, Zeidan D. Optimal systems, series solutions and conservation laws for a time fractional cancer tumor model. Chaos Soliton Fract 2023; 169: 113311.

21 Dec 2023 18:53:43 PST
230906-Feng Version 3 - Submitted to Rocky Mountain J. Math.