Semi-analytical solution for two-dimensional coupled nonlinear Burgers' equations using homotopy analysis method

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Abstract

This study derives a semi-analytical solution for the two-dimensional (2D) coupled nonlinear Burgers' equations using the homotopy analysis method (HAM). Two test problems in a square space domain with different forms of boundary and initials conditions are taken to show the effectiveness of the HAM method. The convergence behavior of the HAM-based solution is shown through the squared residual error approach. It is demonstrated that there is a good match between the HAM-based solution and the exact solution of the problem.

Keywords: 2D Burgers' equations; Nonlinear PDEs; Homotopy analysis method; Analytical solution.

1 Introduction

Systems of nonlinear partial differential equations are used to simulate physical processes in science and engineering. Burgers' equation is a crucial and fundamental nonlinear partial differential equation in fluid mechanics, which is used to explain how convection and diffusion interact. Bateman [1] was the first who proposed Burgers' equation. After that, Burgers [2, 3] utilized this equation to explain how the combination of the opposing actions of convection and diffusion causes turbulent flow in a channel. As a result, it is known as the "Burgers' equation". Burgers' equation is a simplified version of the Navier-Stokes equation. It arises in a wide range of applied mathematics, including gas dynamics [4], traffic flow [5], jet flows [6], and shock waves [7]. According to the independent works done by Hopf [8] and Cole [9], Burgers' equation can be converted into a linear heat equation using the given initial condition. Due to its broad applicability, many academicians have been keen to derive its solution analytically and numerically using different tools.

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The numerical methods used to solve Burgers' equations typically include finite element, finite difference, and spectral methods. Using these numerical methods, a lot of work [10, 11, 12, 13, 14] has been done to solve one-dimensional (1D) and 2D coupled Burgers' equations. However, studies to solve coupled 2D viscous Burgers' equations analytically are very limited in the literature. Fletcher [15] first derived an exact solution for the 2D Burgers' equations utilizing a 2D Hopf-Cole transformation. After that, Biazar and Aminikhah [16], Liao [17], and Gao and Zou [18] derived analytical expressions for the 2D coupled Burgers' equations. Gao and Zou [18] considered a particular type of initial condition containing sine and cosine functions. They expressed the exact solution of the 2D coupled Burgers' equations in terms of the quotient of two infinite series involving exponential, trigonometric, and Bessel functions.

In this paper, the 2D coupled Burgers' equations have been solved using the homotopy analysis method (HAM). This method was invented by Liao [19] and is founded on the theory of homotopic deformation in topology. The approach of HAM is a unified one that logically includes the Adomian decomposition method [20], Lyapunov's method [21], and the homotopy perturbation method [22] as particular cases. It has several advantages over the other methods as it is independent of small and large parameters and can adjust the solution's convergence region. The HAM has received significant attention in applied mathematics and has been used to solve various nonlinear science and engineering problems. Unlike the old h-curve approach, a new technique known as the squared residual error method is adopted to find the value of the auxiliary parameters. Researchers [23, 24, 25, 26, 27] have successfully applied this approach to find the optimum value of convergence control parameters.

2 Mathematical Model

A semi-analytical model for 2D coupled nonlinear Burgers' equations with initial conditions is solved in this study. The 2D coupled nonlinear Burgers' equations are given as:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0, \tag{2.1}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} - \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) = 0, \tag{2.2}$$

where $\nu = \frac{1}{Re}$ is the kinematic viscosity, Re represents the Reynolds number and u and v are the velocities of the fluid along x and y directions. The time and space domains are considered as $t > 0$ and $\Omega = \{(x, y) : d_1 \le x \le d_2, d_3 \le y \le d_4\}$. The set of corresponding conditions is defined as:

$$\left.\begin{aligned} u(x, y, t = 0) &= f_1(x, y) \\ v(x, y, t = 0) &= f_2(x, y) \end{aligned}\right\} \text{for } (x, y) \in \Omega, \tag{2.3}$$

and

$$\left.\begin{aligned} u(x = d_1, y, t) &= g_1(y, t), & u(x = d_2, y, t) &= g_2(y, t), \\ u(x, y = d_3, t) &= g_3(x, t), & u(x, y = d_4, t) &= g_4(x, t), \\ v(x = d_1, y, t) &= h_1(y, t), & v(x = d_2, y, t) &= h_2(y, t), \\ v(x, y = d_3, t) &= h_3(x, t), & v(x, y = d_4, t) &= h_4(x, t), \end{aligned}\right\} \text{for } t > 0, \tag{2.4}$$

where f_1, f_2, g_1, g_2, g_3, g_4, h_1, h_2, h_3 and h_4 are the known functions.

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3 Homotopy Analysis Method (HAM)

In the area of applied mathematics, the HAM has drawn a lot of interest. This method was proposed by Liao [19]. The idea of homotopic deformation in topology is the source of the general strategy. The fundamental idea behind HAM is to use a homotopy function, which includes an embedding parameter, an auxiliary linear operator, and an initial guess of the exact solution. According to Liao [28], this method has a great choice to guess initial estimates for converging to the exact solution. In this section, the methodology of HAM to solve coupled equations given by Eqs. (2.1) and (2.2) is discussed. To apply HAM, Eqs. (2.1) and (2.2) can be rewritten as:

$$\mathscr{N}_i[u_1(x,y,t), u_2(x,y,t)] = 0, \quad \text{for} \quad i = 1, 2. \tag{3.1}$$

where $\mathscr{N}_1$ and $\mathscr{N}_2$ are the nonlinear operators which are defined as:

$$\mathscr{N}_1[u,v] = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \tag{3.2}$$

$$\mathscr{N}_2[u,v] = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} - \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right). \tag{3.3}$$

Instead of using the conventional homotopy, Liao [19] generalized it to generate the so-called zeroth-order deformation equation which is given as:

$$(1-p)\mathscr{L}_i[\Phi_i(x,y,t;p) - u_{i,0}(x,y,t)] - ph_i\mathscr{H}_i(x,y,t)\mathscr{N}_i[\Phi_1(x,y,t;p), \Phi_2(x,y,t;p)] = 0, \tag{3.4}$$

subjected to the initial conditions

$$\Phi_i(x,y,t=0;p) = f_i, \tag{3.5}$$

where $\mathscr{L}_i$ are auxiliary linear operators, $p \in [0,1]$ is the embedding parameter, $u_{i,0}(x,y,t)$ are initial estimates of u and v, $\Phi_i(x,y,t;p)$ are unknown functions, h_i are non-zero auxiliary parameters or convergence control parameters and $\mathscr{H}_i(x,y,t)$ are known as non-zero auxiliary functions. It is observed from Eq. (3.4) that

$$\Phi_i(x,y,t;p=0) = u_{i,0} \tag{3.6}$$

and

$$\Phi_i(x,y,t;p=1) = u_i(t,x,y) \tag{3.7}$$

i.e., as p is increasing from 0 to 1, the unknown functions Φ_i are varying from initial estimates $u_{i,0}$ to the exact solutions $u_i(t,x,y)$ of the original nonlinear problems. Such kind of variation is known as deformation in Topology. HAM offers great flexibility to select the parameters h_i, the auxiliary functions $\mathscr{H}_i(x,y,t)$, the linear operators $\mathscr{L}_i$ and the initial estimates $u_{i,0}$ [23, 28]. Assuming that all of them are appropriately selected such that the solutions $\Phi_i(x,y,t;p)$ of deformation equation Eq. (3.4) and its derivatives with respect to p, i.e. $\frac{\partial^r \Phi_i(x,y,t;p)}{\partial p^r}\Big|_{p=0}$, which are known as r-th order deformation derivatives, exist. Expanding $\Phi_i(x,y,t;p)$ about $p=0$ using Taylor's series, one gets

$$\Phi_i(x,y,t;p) = \Phi_i(x,y,t;0) + \sum_{r=1}^{\infty} \frac{1}{r!}\frac{\partial^r \Phi_i(x,y,t;p)}{\partial p^r}\Bigg|_{p=0} p^r. \tag{3.8}$$

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Eq. (3.8) can be rewritten as

$$\Phi_i(x,y,t;p) = \Phi_i(x,y,t;0) + \sum_{r=1}^{\infty} u_{i,r}(x,y,t)p^r, \tag{3.9}$$

where

$$u_{i,r} = \frac{1}{r!}\frac{\partial^r \Phi_i(x,y,t;p)}{\partial p^r}\bigg|_{p=0}. \tag{3.10}$$

Assume that the initial approximations $u_{i,0}$, h_i, $\mathscr{H}_i$ and $\mathscr{L}_i$ are appropriately selected such that the power series given by Eq. (3.9) is convergent at $p = 1$. Then the series at $p = 1$ becomes

$$\Phi_i(x,y,t;p=1) = \Phi_i(x,y,t;0) + \sum_{r=1}^{\infty} u_{i,r}(x,y,t). \tag{3.11}$$

Using Eqs. (3.6-3.7), the above-defined power series Eq. (3.11) becomes

$$u_i(x,y,t) = u_{i,0} + \sum_{r=1}^{\infty} u_{i,r}. \tag{3.12}$$

The aforementioned expression Eq. (3.12) established a connection between the initial approximations $u_{i,0}$ and the exact solutions $u_i(x,y,t)$ through the terms $u_{i,r}(x,y,t)$, which are known as higher order terms. To obtain $u_{i,r}(x,y,t)$ for $r \geq 1$, differentiating Eqs. (3.4) and (3.5) r times with respect to parameter p, setting $p = 0$, and then dividing them by $r!$, the rth-order deformation equations are obtained as follows

$$\mathscr{L}_i\Big[u_{i,r} - \chi_r u_{i,r-1}\Big] = h_i \mathscr{H}_i \mathscr{R}_{i,r}\big(\vec{u}_{i,r-1}\big), \tag{3.13}$$

subject to

$$u_{i,r}(x,y,t) = 0 \tag{3.14}$$

where the vector

$$\vec{u}_{i,r} = \Big\{u_{i,0},\ u_{i,1},\ u_{i,2},\ u_{i,3}, ...,\ u_{i,r}\Big\}, \tag{3.15}$$

$$\chi_r = \begin{cases} 1 & \text{if } r > 1, \\ 0 & \text{if } r \leq 1, \end{cases} \tag{3.16}$$

and

$$\mathscr{R}_{i,r}\big(\vec{u}_{i,r-1}\big) = \frac{1}{(r-1)!}\frac{\partial^{r-1}\mathscr{N}_i\big[\Phi_1(x,y,t;p), \Phi_2(x,y,t;p)\big]}{\partial p^{r-1}}\bigg|_{p=0}. \tag{3.17}$$

At last, the n-th order HAM-based solution of $u_i(x,y,t)$ is given as

$$u_i(x,y,t) \approx \sum_{r=0}^{n} u_{i,r}(x,y,t). \tag{3.18}$$

In the present study, we have applied HAM to solve two coupled nonlinear PDEs given by Eqs. (2.1) and (2.2) together with the conditions given by Eqs. (2.3) and (2.4). For that purpose, the auxiliary linear operators are selected as

$$\mathscr{L}_1\left[\Phi_1\right] = \frac{\partial \Phi_1}{\partial t}, \tag{3.19}$$

$$\mathscr{L}_2\left[\Phi_2\right] = \frac{\partial \Phi_2}{\partial t}, \tag{3.20}$$

and the initial approximations are simply chosen as

$$u_0 = f_1 \quad \text{and} \quad v_0 = f_2, \tag{3.21}$$

i.e., the initial conditions given by Eq. (2.3) are selected as initial approximations. To reduce the complexity of computing, the auxiliary functions $\mathscr{H}_i$ are taken as $\mathscr{H}_i = 1$ for $i = 1, 2$ according to Vajravelu and Van Gorder [29]. The higher order terms $u_{1,r} = u_r(x, y, t)$ and $u_{2,r} = v_r(x, y, t)$ for $r \geq 1$ can now be derived according to Eq. (3.13) using the initial conditions Eq. (3.14) and the inverse of the linear operators defined by Eqs. (3.19) and (3.20) as shown below:

$$u_r = \chi_r u_{r-1}(t, x, y) + h_1 \int_0^t \mathscr{R}_{1,r}\left(\vec{u}_{r-1}, \vec{v}_{r-1}\right) dt, \tag{3.22}$$

and

$$v_r = \chi_r v_{r-1}(t, x, y) + h_2 \int_0^t \mathscr{R}_{2,r}\left(\vec{u}_{r-1}, \vec{v}_{r-1}\right) dt, \tag{3.23}$$

where

$$\begin{aligned} \mathscr{R}_{1,r}\left(\vec{u}_{r-1}, \vec{v}_{r-1}\right) =& \frac{\partial u_{r-1}}{\partial t} + \sum_{j=0}^{r-1} u_{r-1} \frac{\partial u_{r-1-j}}{\partial x} + \sum_{k=0}^{r-1} v_{r-1} \frac{\partial u_{r-1-k}}{\partial y} \\ & - \nu \left(\frac{\partial^2 u_{r-1}}{\partial x^2} + \frac{\partial^2 u_{r-1}}{\partial y^2} \right), \end{aligned} \tag{3.24}$$

and

$$\begin{aligned} \mathscr{R}_{2,r}\left(\vec{u}_{r-1}, \vec{v}_{r-1}\right) =& \frac{\partial v_{r-1}}{\partial t} + \sum_{j=0}^{r-1} u_{r-1} \frac{\partial v_{r-1-j}}{\partial x} + \sum_{k=0}^{r-1} v_{r-1} \frac{\partial v_{r-1-k}}{\partial y} \\ & - \nu \left(\frac{\partial^2 v_{r-1}}{\partial x^2} + \frac{\partial^2 v_{r-1}}{\partial y^2} \right). \end{aligned} \tag{3.25}$$

Therefore, n-th order HAM solution of $u(x, y, t)$ and $v(x, y, t)$ can be expressed as

$$u(x, y, t) \approx \sum_{r=0}^{n} u_r(x, y, t), \tag{3.26}$$

and

$$v(x, y, t) \approx \sum_{r=0}^{n} v_r(x, y, t), \tag{3.27}$$

where $u_r(x, y, t)$ and $v_r(x, y, t)$ are defined by Eqs. (3.22) and (3.23).

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4 Convergence Analysis

It can be noted from the solution expressions of $u(x,y,t)$ and $v(x,y,t)$, obtained using HAM given by Eqs. (3.26) and (3.27), depend on the convergence control parameters h_1 and h_2. According to Liao [28], these auxiliary parameters control the rate of solution series and convergence region; therefore, proper values of these parameters are needed to get a convergent solution. Earlier, Liao [28] introduced the h-curve approach to find the ideal values of these parameters. However, this approach is ineffective for finding the optimal parameter values to speed up the series convergence. Later, Liao [23] introduced a new technique known as the square residual error method, which minimizes the square residual error to obtain the suitable value of the auxiliary parameters, which is a good technique for determining the convergence control parameters. In the present study, this approach is used to find the suitable values of the convergence parameters. To that end, the averaged squared residual error is defined as

$$E_n[h_1,h_2] = \frac{1}{T}\frac{1}{(d_2-d_1)}\frac{1}{(d_4-d_3)}\int_0^T\int_{d_1}^{d_2}\int_{d_3}^{d_4}\bigg[\Big(\mathscr{N}_1[u,v]\Big)^2 + \Big(\mathscr{N}_2[u,v]\Big)^2\bigg]dydxdt, \tag{4.1}$$

where $u(x,y,t)$ and $v(x,y,t)$ are n-th order approximations given by Eqs. (3.26) and (3.27). One may face difficulty in the integration provided by Eq. (4.1). Therefore, for the sake of computational efficiency, one can calculate the discrete form of the averaged squared residual error numerically, which is defined as follows:

$$E_n[h_1,h_2] \approx \frac{1}{(L+1)}\frac{1}{(M+1)}\frac{1}{(N+1)}\sum_{j=0}^{L}\sum_{k=0}^{M}\sum_{l=0}^{N}\bigg[\Big(\mathscr{N}_1[u(x_j,y_k,t_l),v(x_j,y_k,t_l)]\Big)^2 + \Big(\mathscr{N}_2[u(x_j,y_k,t_l),v(x_j,y_k,t_l)]\Big)^2\bigg], \tag{4.2}$$

where $x_j = d_1 + \frac{j(d_2-d_1)}{L}$, $y_k = d_3 + \frac{k(d_4-d_3)}{M}$ and $t_l = \frac{lT}{N}$; L, M and N are integers. It was proved by Liao [23] that the homotopy series converges to the final solution when averaged squared residual error approaches zero. Therefore, it is enough to examine the error given by Eq. (4.2). In this study, we have assumed that the convergence control parameters are the same, i.e., $h_1 = h_2 = h$.

5 Test Problems

In this section, two test problems of 2D coupled nonlinear Burgers' equations with different sets of initial and boundary conditions are considered to illustrate the homotopy analysis method's effectiveness in finding a semi-analytical solution.

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5.1 Problem 1

Let us consider 2D coupled viscous Burgers' equations given by Eqs. (2.1) and (2.2) subjected to the initial conditions

$$\left.\begin{aligned} u(x,y,0) &= \frac{3}{4} - \frac{1}{4\left[1+\exp\left(\frac{Re(-4x+4y)}{32}\right)\right]}, \\ v(x,y,0) &= \frac{3}{4} + \frac{1}{4\left[1+\exp\left(\frac{Re(-4x+4y)}{32}\right)\right]}, \end{aligned}\right\} \text{for } (x,y) \in \Omega_1. \tag{5.1}$$

Here, the boundary conditions for u and v are considered according to Eq. (5.1) for computational domain $\Omega_1 = \{(x,y) : 0 \le x \le 1,\ 0 \le y \le 1\}$ and $t > 0$. The analytical solution to this problem using Hopf-Cole transformation was derived by Fletcher [15] and is given as follows:

$$\begin{aligned} u(x,y,t) &= \frac{3}{4} - \frac{1}{4\left[1+\exp\left(\frac{Re(-4x+4y-t)}{32}\right)\right]}, \\ v(x,y,t) &= \frac{3}{4} + \frac{1}{4\left[1+\exp\left(\frac{Re(-4x+4y-t)}{32}\right)\right]}. \end{aligned} \tag{5.2}$$

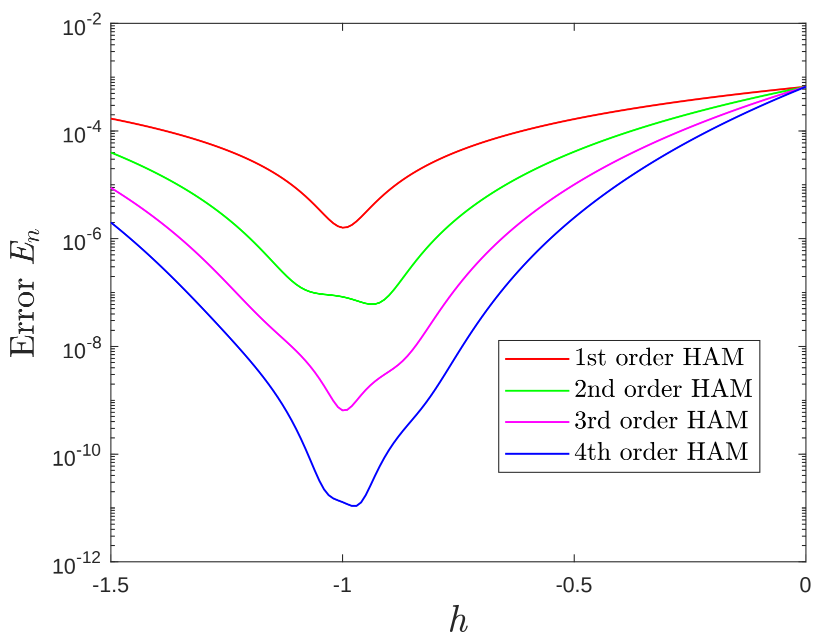
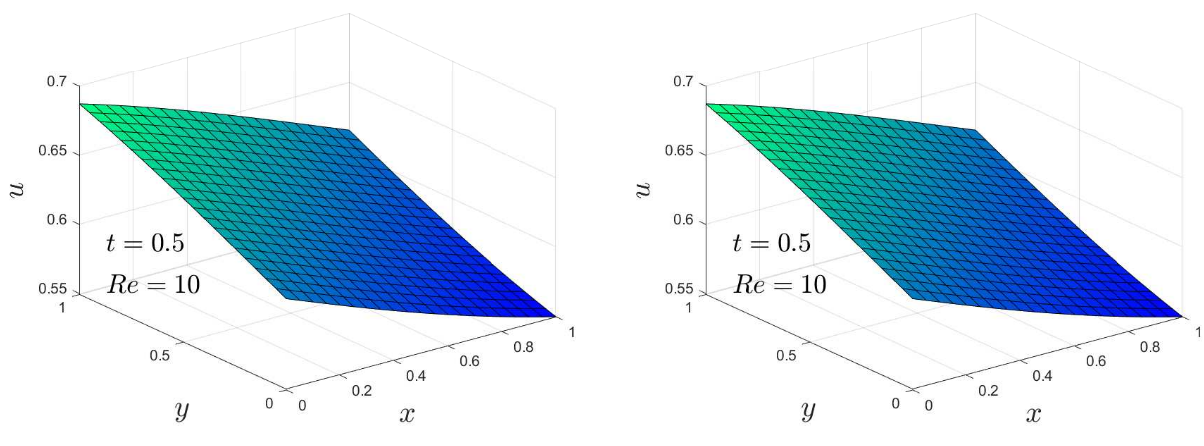


Figure 1: Averaged squared residual error E_n according to Eq. (4.2) with $Re = 10$ for Problem 1.

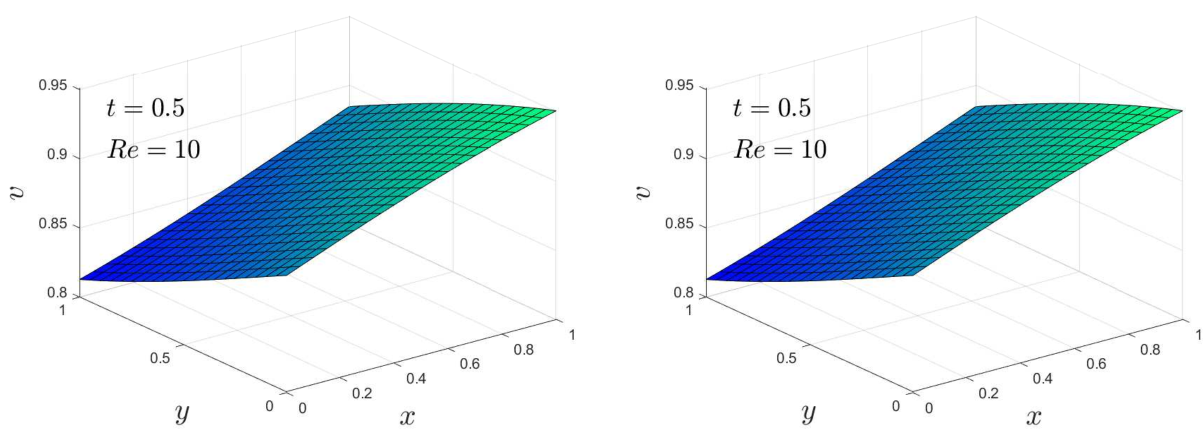
HAM has been used to solve this problem. Here, initial approximations for $u(x,y,t)$ and $v(x,y,t)$ are chosen as $u_0 = \frac{3}{4} - \frac{1}{4\left[1+\exp\left(\frac{(-4x+4y)Re}{32}\right)\right]}$ and $v_0 = \frac{3}{4} + \frac{1}{4\left[1+\exp\left(\frac{(-4x+4y)Re}{32}\right)\right]}$, respectively.

To find the optimum convergence parameter h, the averaged squared residual error E_n according to Eq. (4.2) is plotted in Fig. 1 for different orders of approximations of HAM. The computational domain for calculating E_n is considered as Ω_1 and $t \in (0, 1]$. The number of node points to discretize the error E_n are taken as $L = M = N = 10$. It is clear from Fig. 1 that the error is decreasing as the order of the HAM-based solution is increasing and tending to zero. It indicates the adequacy of the selected initial approximations and linear operators and, consequently, the method's convergence behavior. Here, the optimum convergence parameter h is found as $h = -0.9746$ for the 4th order solution.



(a) (b)

Figure 2: (a) 5th order HAM solution and (b) Exact solution for $u(x, y, t)$ at $t = 0.5$ and $h = -0.9746$ for Problem 1.



(a) (b)

Figure 3: (a) 5th order HAM solution and (b) Exact solution for $v(x, y, t)$ at $t = 0.5$ and $h = -0.9746$ for Problem 1.

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Table 1: Validation of the 5th order HAM solution with the exact solution for $u(x, y, t)$ at $Re = 10$ for Problem 1.

(x, y)	$t = 0.05$		$t = 0.25$		$t = 0.5$		$t = 2.0$	
	Exact	HAM	Exact	HAM	Exact	HAM	Exact	HAM
(0,0.1)	0.6318	0.6318	0.6279	0.6279	0.6230	0.6230	0.5944	0.5944
(0,0.4)	0.6547	0.6547	0.6510	0.6510	0.6463	0.6463	0.6172	0.6172
(0,0.7)	0.6756	0.6756	0.6723	0.6723	0.6681	0.6681	0.6405	0.6406
(0,1.0)	0.6936	0.6936	0.6909	0.6909	0.6871	0.6873	0.6628	0.6628
(0.25,0.1)	0.6123	0.6123	0.6085	0.6085	0.6037	0.6037	0.5768	0.5768
(0.25,0.4)	0.6357	0.6357	0.6318	0.6318	0.6270	0.6270	0.5981	0.5981
(0.25,0.7)	0.6584	0.6584	0.6547	0.6547	0.6500	0.6500	0.6211	0.6211
(0.25,1.0)	0.6789	0.6789	0.6756	0.6756	0.6715	0.6715	0.6444	0.6444
(0.5,0.1)	0.5935	0.5935	0.5898	0.5898	0.5854	0.5854	0.5613	0.5613
(0.5,0.4)	0.6162	0.6162	0.6123	0.6123	0.6075	0.6075	0.5802	0.5802
(0.5,0.7)	0.6396	0.6396	0.6357	0.6357	0.6309	0.6309	0.6018	0.6018
(0.5,1.0)	0.6619	0.6619	0.6584	0.6584	0.6538	0.6538	0.6250	0.6250
(0.75,0.1)	0.5760	0.5760	0.5727	0.5727	0.5688	0.5688	0.5480	0.5480
(0.75,0.4)	0.5972	0.5972	0.5935	0.5935	0.5889	0.5889	0.5642	0.5642
(0.75,0.7)	0.6201	0.6201	0.6162	0.6162	0.6114	0.6114	0.5836	0.5837
(0.75,1.0)	0.6434	0.6434	0.6396	0.6396	0.6347	0.6347	0.6056	0.6056
(1.0,0.1)	0.5606	0.5606	0.5577	0.5577	0.5543	0.5543	0.5370	0.5370
(1.0,0.4)	0.5794	0.5794	0.5760	0.5760	0.5719	0.5719	0.5505	0.5504
(1.0,0.7)	0.6009	0.6009	0.5972	0.5972	0.5926	0.5926	0.5672	0.5672
(1.0,1.0)	0.6240	0.6240	0.6201	0.6201	0.6153	0.6153	0.5872	0.5872

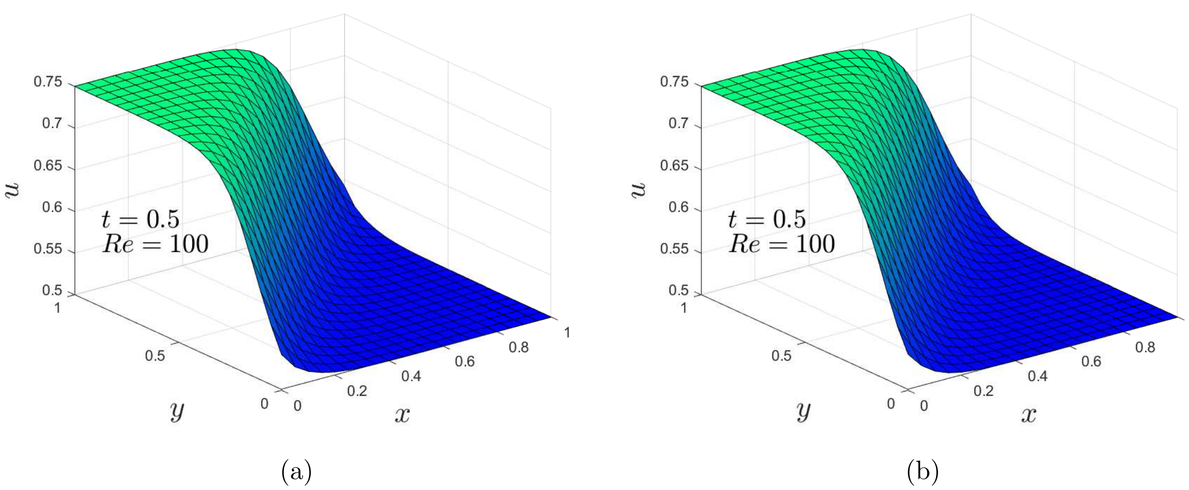
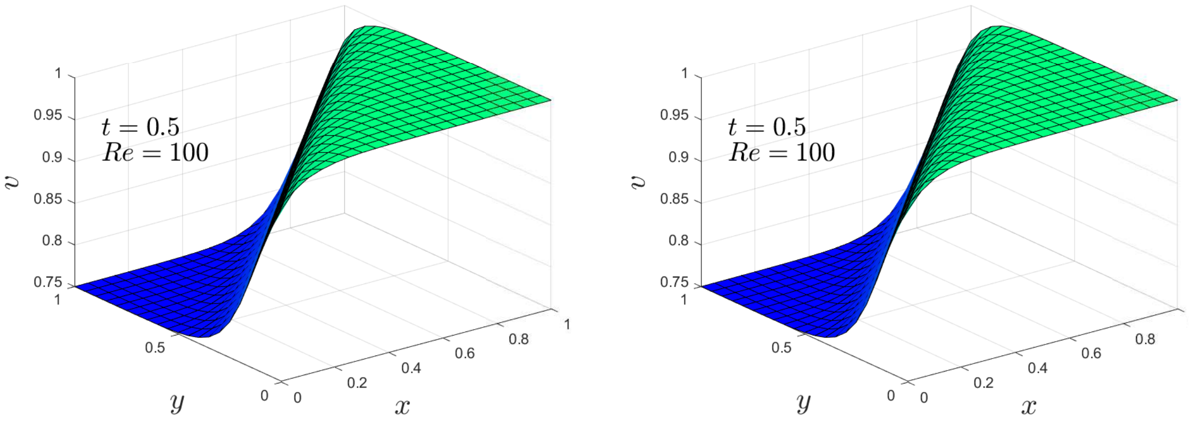


Figure 4: (a) 5th order HAM solution and (b) Exact solution for $u(x, y, t)$ at $t = 0.5$ and $h = -0.9746$ for Problem 1.



(a) (b)

Figure 5: (a) 5th order HAM solution and (b) Exact solution for $v(x, y, t)$ at $t = 0.5$ and $h = -0.9746$ for Problem 1.

Table 2: Validation of the 5th order HAM solution with the exact solution for $v(x, y, t)$ at $Re = 10$ for Problem 1.

(x, y)	$t = 0.05$		$t = 0.25$		$t = 0.5$		$t = 2.0$	
	Exact	HAM	Exact	HAM	Exact	HAM	Exact	HAM
(0,0.1)	0.8682	0.8682	0.8721	0.8721	0.8770	0.8770	0.9056	0.9056
(0,0.4)	0.8453	0.8453	0.8490	0.8490	0.8537	0.8537	0.8828	0.8828
(0,0.7)	0.8244	0.8244	0.8277	0.8277	0.8319	0.8319	0.8595	0.8594
(0,1.0)	0.8064	0.8064	0.8091	0.8091	0.8127	0.8127	0.8372	0.8372
(0.25,0.1)	0.8877	0.8877	0.8915	0.8915	0.8963	0.8963	0.9232	0.9232
(0.25,0.4)	0.8643	0.8643	0.8682	0.8682	0.8730	0.8730	0.9019	0.9019
(0.25,0.7)	0.8416	0.8416	0.8453	0.8453	0.8500	0.8500	0.8789	0.8789
(0.25,1.0)	0.8211	0.8211	0.8244	0.8244	0.8285	0.8285	0.8556	0.8556
(0.5,0.1)	0.9065	0.9065	0.9102	0.9102	0.9146	0.9146	0.9387	0.9387
(0.5,0.4)	0.8838	0.8838	0.8877	0.8877	0.8925	0.8925	0.9198	0.9198
(0.5,0.7)	0.8604	0.8604	0.8643	0.8643	0.8691	0.8691	0.8982	0.8982
(0.5,1.0)	0.8381	0.8381	0.8416	0.8416	0.8462	0.8462	0.8750	0.8750
(0.75,0.1)	0.9240	0.9240	0.9273	0.9273	0.9312	0.9312	0.9520	0.9520
(0.75,0.4)	0.9028	0.9028	0.9065	0.9065	0.9111	0.9111	0.9358	0.9358
(0.75,0.7)	0.8799	0.8799	0.8838	0.8838	0.8886	0.8886	0.9164	0.9163
(0.75,1.0)	0.8566	0.8566	0.8604	0.8604	0.8653	0.8653	0.8944	0.8944
(1.0,0.1)	0.9394	0.9394	0.9423	0.9423	0.9457	0.9457	0.9630	0.9630
(1.0,0.4)	0.9206	0.9206	0.9240	0.9240	0.9281	0.9281	0.9495	0.9496
(1.0,0.7)	0.8991	0.8991	0.9028	0.9028	0.9074	0.9074	0.9328	0.9328
(1.0,1.0)	0.8760	0.8760	0.8799	0.8799	0.8847	0.8847	0.9128	0.9128

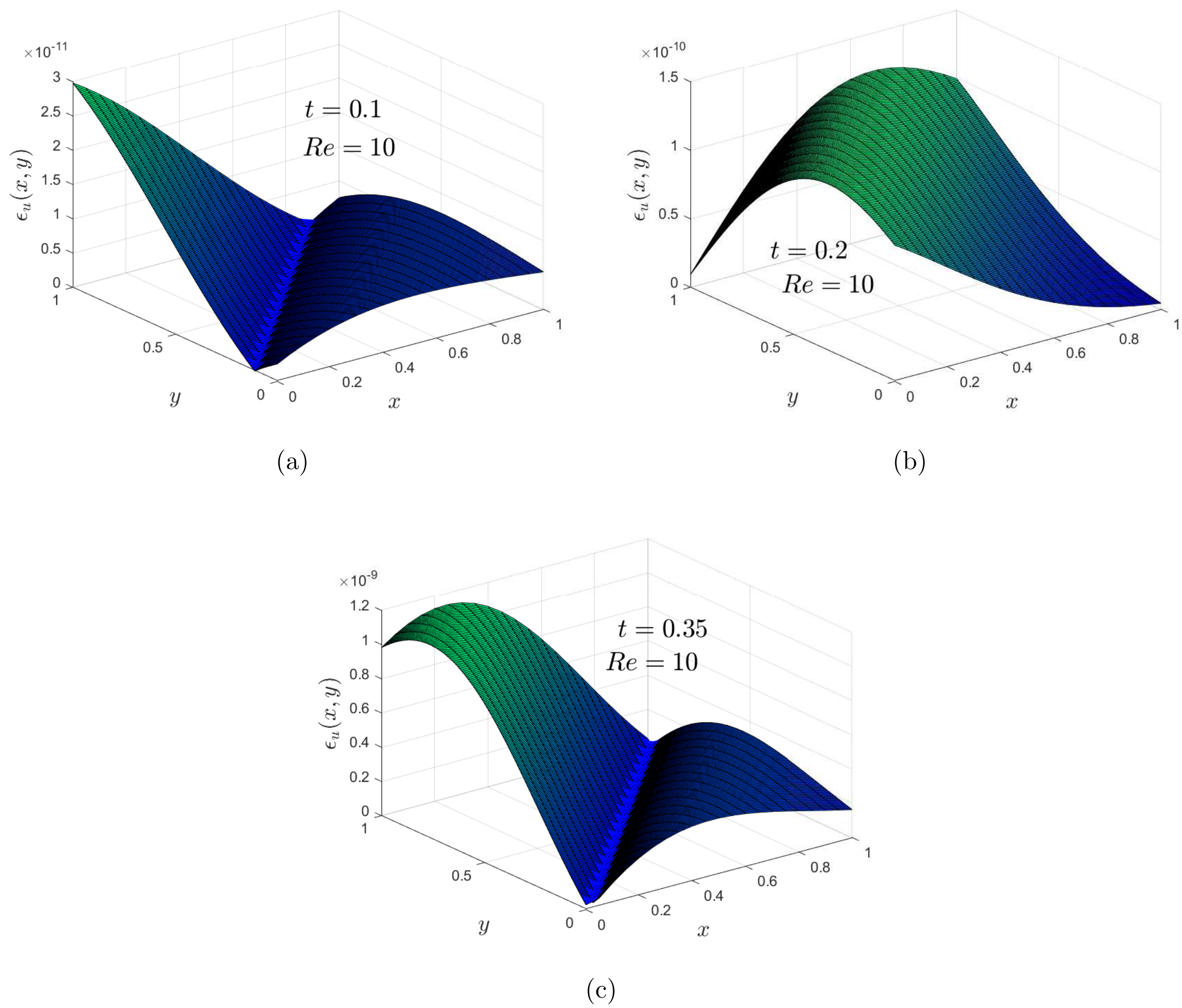


Figure 6: Point-wise error $\epsilon_u(x, y)$ for $u(x, y)$ over the domain (x, y) at different times (a) $t = 0.1$, (b) $t = 0.2$ and (c) $t = 0.35$ for Problem 1.

Fig. 2 and Fig. 3 compare the obtained 5th order HAM-based solution with the exact solution given by Eq. (5.2) for $u(x, y, t)$ and $v(x, y, t)$, respectively, at time $t = 0.5$ and $\nu = 0.1$. On the other hand, Fig. 4 and Fig. 5 compare the obtained 5th order HAM-based solution with the exact solution given by Eq. (5.2) for u and v, respectively, at time $t = 0.5$ and $\nu = 0.01$. These figures show a good agreement between the exact solution and the derived HAM solution. Aside from the visual solutions, Table 1 and Table 2 provide quantitative assessments for a few selected random points from the domain, ensuring the stability of the adopted method.
For more clarifications, the pointwise errors which are defined as

$$\epsilon_u(x, y) = ||u_{ham} - u_{exact}|| \tag{5.3}$$

for $u(x, y)$ and

$$\epsilon_v(x, y) = ||v_{ham} - v_{exact}|| \tag{5.4}$$

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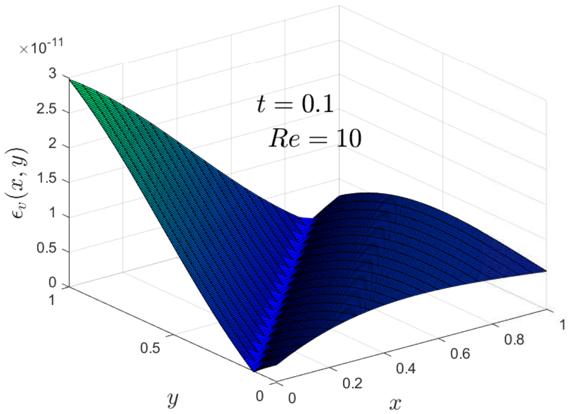
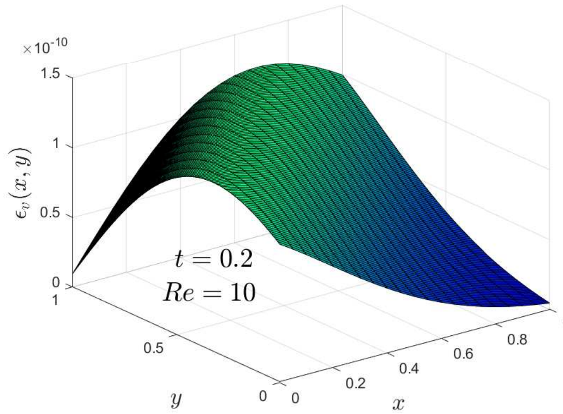
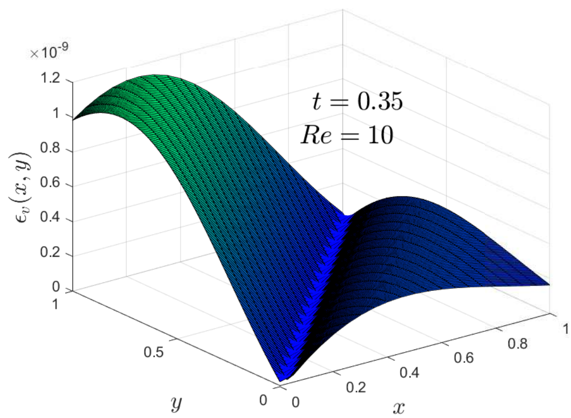


(a)



(b)



(c)

Figure 7: Point-wise error $\epsilon_v(x, y)$ for $v(x, y)$ over the domain (x, y) at different times (a) $t = 0.1$, (b) $t = 0.2$ and (c) $t = 0.35$ for Problem 1.

for $v(x, y)$ over the computational domain $(x, y) \in \Omega_1$ are plotted in Fig. 6 and Fig. 7, respectively, at different times for Problem 1. One can see from these figures that the errors are very less over the entire domain (x, y).

5.2 Problem 2

Now, let us consider 2D coupled viscous nonlinear Burgers' equations given by Eqs. (2.1) and (2.2) with the initial conditions

$$\left.\begin{array}{l} u(x, y, 0) = cos(y)sin(x) \\ v(x, y, 0) = cos(x)sin(y) \end{array}\right\} \text{for } (x, y) \in \Omega_2, \tag{5.5}$$

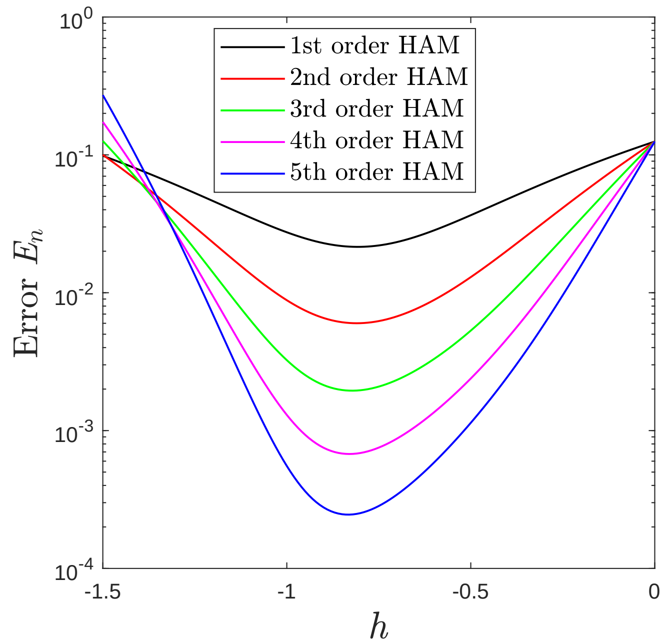


Figure 8: Averaged squared residual error according to Eq. (4.2) with $Re = 100$ for Problem 2.

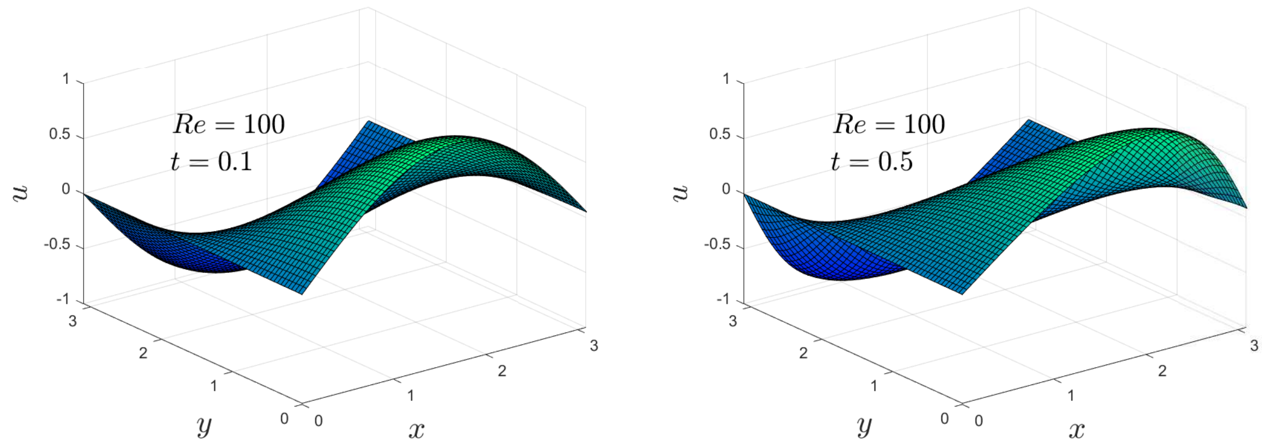
and the boundary conditions

$$\left.\begin{aligned} u(x=0,y,t)=0, \quad u(x=\pi,y,t)=0,\\ u(x,y=0,t)=sin(x), \quad u(x,y=\pi,t)=-sin(x),\\ v(x=0,y,t)=sin(y), \quad v(x=\pi,y,t)=-sin(y),\\ v(x,y=0,t)=0, \quad v(x,y=\pi,t)=0. \end{aligned}\right\} \text{for } t>0. \tag{5.6}$$

Here, space domain is considered as $\Omega_2 = \{(x, y) : 0 \leq x \leq \pi,\ 0 \leq y \leq \pi\}$ and time domain is considered as $t > 0$.

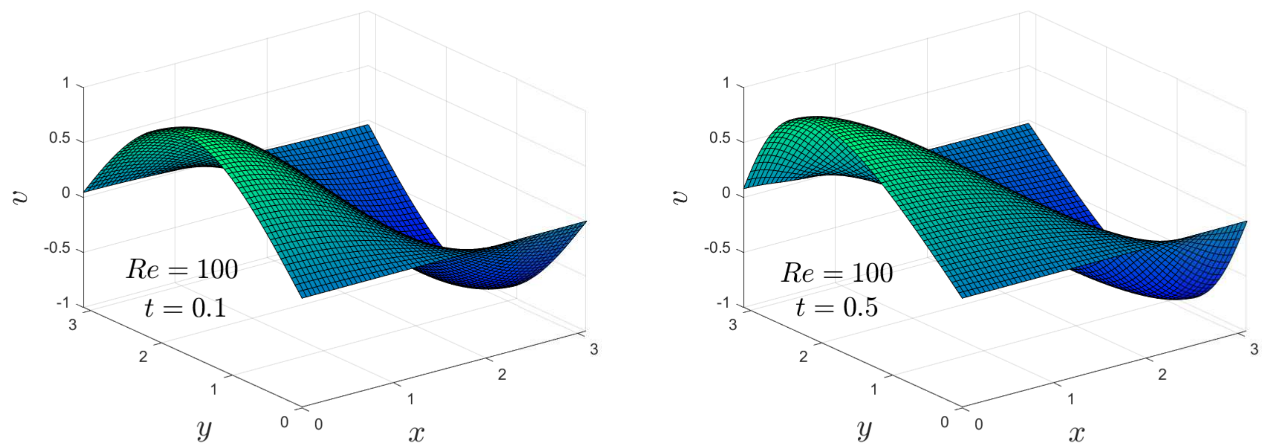
In this example, initial approximations for u and v are taken as $u_0 = cos(y)sin(x)$ and $v_0 = cos(x)sin(y)$. It can be seen that these initial approximations satisfy initial and boundary conditions given by Eqs. (5.5) and (5.6). According to Liao [23], it is sufficient to analyze the residual error only to know the convergence behavior of the HAM-based solution. Therefore, the averaged squared residual error E_n according to Eq. (4.2) is plotted in Fig. 8 for different orders of approximations. The computational domain for calculating E_n is considered as Ω_2 and $t \in (0, 0.5]$. The number of node points to compute the error E_n are taken as $L = M = N = 10$. Fig. 8 shows that the residual error E_n decreases and tends to zero as the order of the HAM-based solution increases. It shows that the selected initial approximations and linear operators are appropriate, and the obtained solution is convergent. Here, the optimum value of h is found as $h = -0.8333$ for the 5th order solution.

Figs. 9 and 10 show the 8th order HAM-based solution for $u(x, y, t))$ and $v(x, y, t)$, respectively, at two different times for Problem 2.



(a) (b)

Figure 9: 8th order HAM solution for $u(x, y, t)$ at (a) $t = 0.1$ and (b) $t = 0.5$ for Problem 2.



(a) (b)

Figure 10: 8th order HAM solution for $v(x, y, t)$ at (a) $t = 0.1$ and (b) $t = 0.5$ for Problem 2.

6 Conclusions

This work provides a semi-analytical solution for 2D coupled nonlinear Burgers' equations with initial conditions using the homotopy analysis method. The averaged squared residual error approach is used instead of the h-curve method to determine the suitable optimum value of the auxiliary parameters. Two test problems have been considered to demonstrate the usefulness of the chosen methodology. The derived HAM solutions are validated against the problems' exact solutions and exhibit excellent agreement with it. The 2D Burgers' equations can be solved using the suggested method regardless of the initial and boundary conditions.

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Conflict of Interest

There is no conflict of interest among the authors.

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