

Several problems on reduced spherical polygons of thickness less than $\pi/2$ ¹

Cen Liu and Yanxun Chang

Department of Mathematics, Beijing Jiaotong University, Beijing 100044, China
cenliumath@foxmail.com, yxchang@bjtu.edu.cn

Abstract: The present paper aims to solve some problems proposed by Lassak about the reduced spherical polygons. The main result is to show that the regular spherical n -gon has the minimal perimeter among all reduced spherical polygons of fixed thickness less than $\pi/2$ and with at most n vertices. In addition, we determine the maximal diameter of every reduced spherical polygons with a fixed thickness less than $\pi/2$. We also find the smallest spherical radius that contains every reduced spherical polygons with a fixed thickness less than $\pi/2$.

Keywords: reduced spherical polygon, thickness, perimeter, diameter

Mathematics Subject Classification (2010): 52A55

1 Introduction

Reduced convex bodies have been investigated in Euclidean space, normed space and spherical space, respectively, one can refer to [7, 9–11]. The present paper focus on the reduced polygons on the sphere. Particularly, the investigation of reduced spherically convex polygon was first introduced by Lassak in [8].

One can check that, the last paragraph of the article [8] establishes four conjectures about the perimeter and area of reduced spherically convex polygons with a fixed thickness less than $\pi/2$. In particular, the four conjectures have been solved in [1] with a given thickness $\pi/2$. Furthermore, two conjectures about the area of reduced spherical polygon of thickness less than $\pi/2$ have been solved in [4]. The main result of this paper confirms one of the remaining conjectures which shows that the regular spherical n -gon has the minimal perimeter among all reduced spherical n -gons of a fixed thickness less than $\pi/2$. In addition, there are still some other problems about the reduced spherical polygons proposed by Lassak in [8] and [9], respectively. We consider these problems in the following.

- P1. Is it true that the regular spherical n -gon has minimal perimeter among all reduced spherical polygons of the same thickness less than $\pi/2$ and with at most n vertices?
- P2. Is it true that the equality in Theorem 4.2 [8] holds only for regular triangle?
- P3. What is the smallest radius of a disk which contains every reduced polygon of a given thickness on S^2 ?

In Section 2, we review some results from the literature and several useful lemmas are established. Section 3 confirms the Problem P1. Actually, we use a similar method

¹Supported by NSFC under Grant 11971053.

to solve this problem as the proof process in [4]. Although the answer to Problem P2 in Section 4 is negative. We determine the supremum of the diameter of any reduced spherically convex polygons and confirm P2 in our investigation. In Section 5, based on the consideration of P2, we get the smallest radius of a disk that contains every reduced spherical polygons of a fixed thickness less than $\pi/2$, which answers the Problem P3.

2 Preliminaries

For the notions of the reduced convex polygons on S^2 , one can see [1, 4, 8, 9]. Let S^2 be the unit sphere of E^3 . The intersection of S^2 with any two-dimensional subspace of E^3 is called a *great circle*. A pair of *antipodes* are the intersection of S^2 with any one-dimensional subspace of E^3 . If $a, b \in S^2$ are different points and not antipodes, then there is exactly one great circle containing them; Denote by *arc* ab , shortly ab , the shorter part of the great circle containing them. The *distance* $|ab|$ is the length of ab .

Let C be a spherical set, for any two points $a, b \in C$, if we have $ab \subseteq C$ and $|ab| < \pi$, then C is a *convex* set. The *convex body* is a closed convex set with non-empty interior.

The *spherical disk* of radius $r \in (0, \pi/2]$ and *center* $k \in S^2$ is the set of points having distance at most r from k ; the *spherical circle* is the boundary of the spherical disk. Spherical disks of radius $\pi/2$ are called *hemispheres*. If G and H are different hemispheres and their centers are not antipodes, then $L = G \cap H$ is called a *lune* of S^2 . The parts of $\text{bd}(G)$ and $\text{bd}(H)$ contained in $G \cap H$ are denoted by G/H and H/G , respectively. We define the *thickness* $\Delta(L)$ of the lune $L = G \cap H$ as the distance between the centers of G/H and H/G . The thickness of a spherically convex polygon C is the minimal thickness of a lune which contains this polygon, and we denote it by $\Delta(C)$. A convex body $C \subset S^2$ is said to be *reduced* if $\Delta(R) < \Delta(C)$ for each convex body R being a proper subset of C .

Some useful definitions given in [8] are established here. If C is a subset of a convex set of S^2 , then the intersection of all convex sets containing C is called a *convex hull* of C . The convex hull of $k \geq 3$ points on S^2 such that each of them does not belong to the convex hull of the remaining points is called a *spherically convex k -gon*. If V is a spherically convex k -gon, we denote by v_1, \dots, v_k the vertices of V in the counterclockwise order. We define the counterclockwise direction as the positive orientation. A spherically convex polygon with sides of equal length and interior angles of equal measure is called a *regular spherical polygon*.

Let p be a point in a hemisphere different from its center and let l be the great circle bounding this hemisphere. The *projection* of p on l is the point t such that $|pt| = \min\{|pc| : c \in l\}$. For a spherically convex odd-gon $V = v_1v_2 \cdots v_n$, by the *opposite side to the vertex v_i* we mean the side $v_{i+(n-1)/2}v_{i+(n+1)/2}$, where the indices are taken modulo n .

A few formulas of spherical trigonometry in [14] are useful for our research. For a right spherical triangle with hypotenuse c and legs a, b , denote by A, B and C the corresponding angles of edges a, b and c , respectively. Then we have

$$\cos A = \tan b \cot c, \tag{1}$$

$$\cos B = \tan a \cot c, \tag{2}$$

$$\sin b = \sin c \sin B, \tag{3}$$

$$\cos c = \cos a \cos b, \tag{4}$$

$$\cos c = \cot A \cot B, \tag{5}$$

$$\cos B = \cos b \sin A. \tag{6}$$

For any spherical triangle, by the sine theorem on sphere, we have

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}. \tag{7}$$

Our paper is based on the following results given in [8] and [4], respectively.

Lemma 2.1 [8, Theorem 3.2] *Every reduced spherical polygon is an odd-gon of thickness at most $\frac{\pi}{2}$. A spherically convex odd-gon V with $\Delta(V) < \frac{\pi}{2}$ is reduced if and only if the projection of every its vertices on the great circle containing the opposite side belongs to the relative interior of this side and the distance of this vertex from this side is $\Delta(V)$.*

Lemma 2.2 [8, Corollary 3.3] *Every spherical regular odd-gon of thickness at most $\frac{\pi}{2}$ is reduced.*

Lemma 2.3 [8, Corollary 3.6] *For every reduced odd-gon $V = v_1v_2 \cdots v_n$ with $\Delta(V) < \frac{\pi}{2}$, we have $|v_it_{i+(n+1)/2}| = |t_iv_{i+(n+1)/2}|$, for $i = 1, 2, \dots, n$, where t_i denotes the projection of v_i on the opposite side.*

Lemma 2.4 [8, Corollary 3.9] *If $V = v_1v_2 \cdots v_n$ is a reduced spherical polygon with $\Delta(V) < \frac{\pi}{2}$, then $\beta_i \leq \alpha_i$ for every $i \in \{1, \dots, n\}$.*

We use the similar method as in [4] to prove the Problem P1, then the notations given in [4, Section 3] are applicable and we rewrite it here. In a reduced spherical polygon $V = v_1v_2 \cdots v_n$, let t_i be the projection of v_i on the opposite side $v_{i+(n-1)/2}v_{i+(n+1)/2}$. Let o_i be the intersection point of v_it_i and $v_{i+(n+1)/2}t_{i+(n+1)/2}$. Put $\alpha_i = \angle v_{i+1}v_it_i$, $\beta_i = \angle t_iv_iv_{i+(n+1)/2}$, and $\varphi_i = \angle v_i o_i t_{i+(n+1)/2} = \angle t_i o_i v_{i+(n+1)/2}$, where $i \in \{1, 2, \dots, n\}$. Fig. 1 presents some notations in a reduced spherical pentagon.

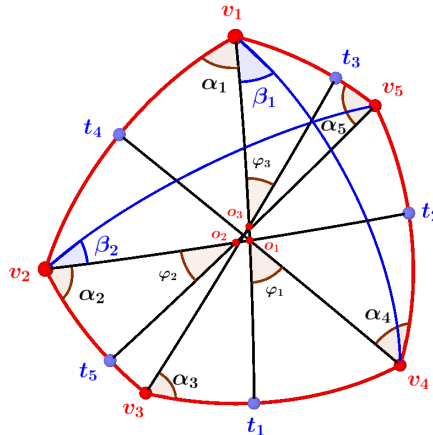


Figure 1: Related notations in a reduced spherical pentagon

Lemma 2.5 [4, Lemma 3.2] *For every reduced spherical odd-gon $V = v_1v_2 \cdots v_n$ with $\Delta(V) < \frac{\pi}{2}$, the spherical triangles $v_i o t_{i+(n+1)/2}$ and $v_{i+(n+1)/2} o t_i$ are congruent, where $i \in \{1, 2, \dots, n\}$.*

Lemma 2.6 [4, Fact 3.4] *For every reduced spherical polygon $V = v_1v_2 \cdots v_n$ with a thickness $\Delta(V) < \frac{\pi}{2}$, we have $0 < \varphi_i < \frac{\pi}{2}$, where $i \in \{1, \dots, n\}$.*

Lemma 2.7 [4, Lemma 3.5] *For any reduced spherical polygon $V = v_1v_2 \cdots v_n$ of thickness less than $\frac{\pi}{2}$, we have $\sum_{i=1}^n \varphi_i \geq \pi$. Moreover,*

- (1) *if V is a non-regular reduced spherical polygon, then $\sum_{i=1}^n \varphi_i \geq \pi$;*
- (2) *if V is a regular spherical polygon, then $\sum_{i=1}^n \varphi_i = \pi$ and $\varphi_i = \frac{\pi}{n}$, where $i \in \{1, 2, \dots, n\}$.*

In the following, we investigate the length of side, interior angle and circumradius of regular spherical triangle with a given thickness. Since different formulas of spherical trigonometry derive different expressions, here we establish one of them.

Lemma 2.8 *Let $V = v_1v_2v_3$ be a regular spherical triangle with thickness less than $\frac{\pi}{2}$. Then we have*

- (1) *the interior angle of V is 2γ ;*
- (2) *the length of one side of V is $2 \arccos(\frac{1}{2\sin\gamma})$;*
- (3) *the circumradius of V is $\arcsin(\frac{2}{\sqrt{3}}\sqrt{1 - \frac{1}{4\sin^2\gamma}})$,*

where $\gamma = \arcsin \frac{-\cos\Delta(V) + \sqrt{\cos^2\Delta(V) + 8}}{4}$.

Proof Lemma 2.2 permits that V is reduced. Hence we apply the above notations to V .

From the knowledge of regular spherical trigonometry, we know that $\alpha_i = \beta_i$ and t_i is the midpoint of $v_{i+1}v_{i+2}$, where $i = 1, 2, 3$ and the indices are taken modulo 3. Consider the right spherical triangle $v_i t_i v_{i+2}$, then we have $|v_i t_i| = \Delta(V)$. Let $|t_i v_{i+2}| = a$, thus $|v_i v_{i+2}| = 2a$. Denote by γ the angle $\angle t_i v_i v_{i+2}$. By Equation (4), we get that

$$\cos 2a = \cos \Delta(V) \cos a, \tag{8}$$

and thus $\cos a = \frac{\cos \Delta(V) + \sqrt{\cos^2 \Delta(V) + 8}}{4}$. By Equation (7), we get that

$$\frac{\sin \gamma}{\sin a} = \frac{\sin \frac{\pi}{2}}{\sin 2a},$$

from this and (8), we obtain that

$$\gamma = \arcsin \frac{-\cos \Delta(V) + \sqrt{\cos^2 \Delta(V) + 8}}{4}.$$

The interior angle of V is as desired. Since $\Delta(V) \in (0, \frac{\pi}{2})$, it follows that $\gamma \in (\frac{\pi}{6}, \frac{\pi}{4})$ and $2\gamma \in (\frac{\pi}{3}, \frac{\pi}{2})$.

Consider the right spherical triangle $v_1 t_2 o$ in V , where o is the center point of the circumcircle of V . By using (6), we obtain that the length of $v_1 t_2$ is $\arccos(\frac{1}{2\sin\gamma})$. Hence the length of one side of V is $2 \arccos(\frac{1}{2\sin\gamma})$. From Equation (3), it follows that the circumradius of V is $\arcsin(\frac{2}{\sqrt{3}}\sqrt{1 - \frac{1}{4\sin^2\gamma}})$. \square

In fact, γ is half of the interior angle of a regular triangle of thickness $\omega \in (0, \frac{\pi}{2})$. For convenient, we denote the number $\arcsin \frac{-\cos \omega + \sqrt{\cos^2 \omega + 8}}{4}$ by γ in the whole research, where $\gamma \in (\frac{\pi}{6}, \frac{\pi}{4})$.

By Lemma 2.4, for each reduced spherical n -gon V , it is true that $\beta_i \leq \alpha_i$, (if V is a reduced polygon in the plane, the relation is $\beta_i \leq \frac{\pi}{6} \leq \alpha_i$, one can refer [5, Theorem 8]). Here we intend to find a similar relation between β_i and α_i as in the plane.

In a spherically convex polygon V , for arbitrary two points $x, y \in \text{bd}(V)$, denote by \widehat{xy} the arcs of $\text{bd}(V)$ from x to y in a positive orientation. By $|\widehat{xy}|$ we mean the length of \widehat{xy} .

Lemma 2.9 *For each reduced spherical polygon $V = v_1v_2 \cdots v_n$ of thickness less than $\frac{\pi}{2}$, we have $\beta_i \leq \gamma \leq \alpha_i$, where $i \in \{1, \dots, n\}$. Especially, β_i and α_i attain γ at the same time and only when V is a regular triangle, we can get that $\beta_i = \gamma = \alpha_i$.*

Proof Consider the right spherical triangle $v_it_iv_{i+(n+1)/2}$ in V . Lemma 2.1 implies that $|v_it_i| = \Delta(V)$. By Lemma 2.5, we have $\angle v_iv_{i+(n+1)/2}t_i = \alpha_i + \beta_i$. By Equation (7), we have

$$\frac{\sin \frac{\pi}{2}}{\sin |v_iv_{i+(n+1)/2}|} = \frac{\sin(\alpha_i + \beta_i)}{\sin \Delta(V)},$$

then $\Delta(V) \leq \alpha_i + \beta_i \leq \frac{\pi}{2}$. By Equation (6), we have

$$\cos(\alpha_i + \beta_i) = \cos \Delta(V) \sin \beta_i. \tag{9}$$

Lemma 2.4 permits that $\cos(\alpha_i + \beta_i) \leq \cos 2\beta_i$, then (9) satisfies

$$\cos \Delta(V) \sin \beta_i \leq \cos 2\beta_i. \tag{10}$$

As a result, we get $\sin \beta_i \leq \sin \gamma$ and thus $\beta_i \leq \gamma$.

Lemma 2.4 also permits that $\cos(\alpha_i + \beta_i) \geq \cos 2\alpha_i$, then (9) satisfies

$$\cos \Delta(V) \sin \alpha_i \geq \cos 2\alpha_i. \tag{11}$$

Therefore, we get the inequality $\sin \alpha_i \geq \sin \gamma$ and thus $\alpha_i \geq \gamma$.

Consequently, the inequality $\beta_i \leq \gamma \leq \alpha_i$ holds.

Claim that β_i and α_i attain γ at the same time. For some $i \in \{1, \dots, n\}$, if $\beta_i = \gamma$, we can check that (10) becomes the following equality

$$\cos \Delta(V) \sin \gamma = \cos 2\gamma,$$

from this, Equation (9) becomes

$$\cos(\alpha_i + \gamma) = \cos 2\gamma.$$

Hence we obtain $\alpha_i = \gamma = \beta_i$. For the case when $\alpha_i = \gamma$, we can also get the same conclusion.

With the assumption that $\alpha_i = \gamma = \beta_i$, we will prove that V degenerates into a regular spherical triangle. Denote by u_i the intersection point of the arcs which contains v_iv_{i+1} and $v_{i+(n-1)/2}v_{i+(n+1)/2}$, respectively. Fig. 2 illustrates the corresponding notations on a reduced spherical pentagon.

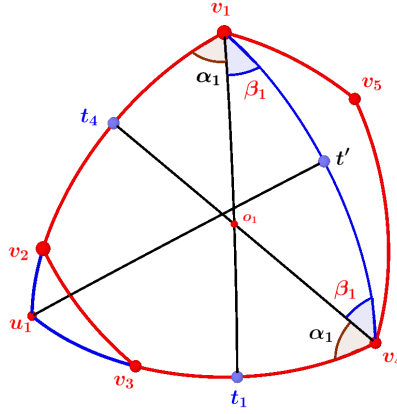


Figure 2: Some proof details shown in a reduced spherical pentagon

Claim that $v_i u_i v_{i+(n+1)/2}$ is a regular spherical triangle. By the definition of regular triangle, we only need to show that the lengths of the three sides of $v_i u_i v_{i+(n+1)/2}$ are the same with each other. Since $\angle u_i v_i v_{i+(n+1)/2} = \angle v_i v_{i+(n+1)/2} u_i = \alpha_i + \beta_i = 2\gamma$, it follows that $|u_i v_i| = |v_i v_{i+(n+1)/2}|$. Let $c = |u_i v_i|$ and $c' = |v_i v_{i+(n+1)/2}|$. Then we intend to show $c = c'$.

Let t' be the projection of u_i on the arc $v_i v_{i+(n+1)/2}$. Since $|u_i v_i| = |v_i v_{i+(n+1)/2}|$, t' is the mid-point of $v_i v_{i+(n+1)/2}$ and $u_i v_i t'$ is a right spherical triangle. Then by Equation (2), we obtain $\cos 2\gamma = \tan \frac{c'}{2} \cot c$, and thus

$$\tan c = \frac{\tan \frac{c'}{2}}{\cos 2\gamma} = \frac{\sin c'}{(1 + \cos c') \cos 2\gamma}. \quad (12)$$

In the right spherical triangle $v_i t' v_{i+(n+1)/2}$. Equation (5) implies that $\cos c' = \cot \gamma \cot 2\gamma$, thus we get

$$\frac{\cos c'}{1 + \cos c'} = \frac{1}{1 + \tan \gamma \tan 2\gamma} = \cos 2\gamma.$$

Consequently, we obtain $(1 + \cos c') \cos 2\gamma = \cos c'$. From this and Equation (12), it follows that $\tan c = \tan c'$. So $c = c'$, and hence $v_i u_i v_{i+(n+1)/2}$ is a regular triangle.

The following discussion is similar as that in [5, Theorem 8]. From the above analysis, we get

$$|\widehat{t_{i+(n+1)/2} t_i}| \leq |t_{i+(n+1)/2} u_i| + |u_i t_i| = |v_i v_{i+(n+1)/2}| \leq |\widehat{v_{i+(n+1)/2} v_i}|. \quad (13)$$

By Lemma 2.3, we note that

$$|\widehat{t_{i+(n+1)/2} t_i}| = |\widehat{v_{i+(n+1)/2} v_i}|.$$

Hence the inequality (13) becomes

$$|\widehat{t_{i+(n+1)/2} t_i}| = |t_{i+(n+1)/2} u_i| + |u_i t_i| = |v_i v_{i+(n+1)/2}| = |\widehat{v_{i+(n+1)/2} v_i}|.$$

This means that (13) attains the equation only when V degenerates into a regular spherical triangle. This completes the proof. \square

The above lemma plays an important role in our consideration of the maximal diameter of reduced spherical polygons in Section 4.2. The following discussions are very useful for Section 3.

Lemma 2.10 *Let $f_1(x) = \arccos \frac{1+\lambda x}{\sqrt{1+\lambda^2}}$ and $f_2(x) = \arccos \frac{x(1+\lambda x)}{\lambda-x}$. Then $\frac{f_1(x)}{f_2(x)}$ is a decreasing function of x , where $\lambda \in (0, +\infty)$ and $x \in (0, \frac{-1+\sqrt{1+\lambda^2}}{\lambda})$.*

Proof We use a series of elementary calculation methods to solve this problem. Let $f(x) = \frac{f_1(x)}{f_2(x)}$, then the first derivative of $f(x)$ is

$$f'(x) = -\frac{\lambda \cdot h(x)}{(\lambda-x)\sqrt{-\lambda(1+x^2)(2x+\lambda x^2-\lambda)}f_2^2(x)},$$

where

$$h(x) = (\lambda-x)\sqrt{1+x^2}f_2(x) + (x^2-2\lambda x-1)f_1(x).$$

Here we have $x < \frac{-1+\sqrt{1+\lambda^2}}{\lambda} < \lambda$, hence $\lambda-x > 0$. If $h(x) > 0$ in the domain $x \in (0, \frac{-1+\sqrt{1+\lambda^2}}{\lambda})$, then we confirm the result $f'(x) < 0$.

By a simple calculation, we get that the first derivative of $h(x)$ is

$$h'(x) = \frac{(-2x^2 + \lambda x - 1)f_2(x)}{\sqrt{1+x^2}} + 2(x-\lambda)f_1(x),$$

and the second derivative of $h(x)$ is

$$h''(x) = \frac{\lambda(\lambda x^3 + 3x^2 - 3\lambda x + 2\lambda^2 + 1)}{(1+x^2)(\lambda-x)\sqrt{-\lambda(2x+\lambda x^2-\lambda)}} + \frac{\lambda - \frac{3x}{2} - \frac{x^3}{2}}{(1+x^2)^{3/2}}f_2(x) + t(x),$$

where $t(x) = -\frac{3x}{2\sqrt{1+x^2}}f_2(x) + 2f_1(x)$. According to the root discrimination of quadratic function, we find that

$$3x^2 - 3\lambda x + 2\lambda^2 + 1 > 0. \quad (14)$$

Since $\sqrt{-\lambda(2x+\lambda x^2-\lambda)} > 0$ and $0 < x < \frac{-1+\sqrt{1+\lambda^2}}{\lambda} < 1$, it follows that

$$\lambda > 2x + \lambda x^2 > \frac{3x}{2} + \frac{x^3}{2} > 0. \quad (15)$$

From the inequalities (14), (15), $\lambda-x > 0$ and if $t(x) > 0$, then we obtain $h''(x) > 0$.

By a simple calculation, we check that the first derivative of $t(x)$ is

$$t'(x) = \frac{\lambda(x^3 + 2\lambda x^2 + 7x - 4\lambda)}{2(1+x^2)(\lambda-x)\sqrt{-\lambda(\lambda x^2 + 2x - \lambda)}} - \frac{3}{2(1+x^2)^{3/2}}f_2(x).$$

From $\sqrt{-\lambda(\lambda x^2 + 2x - \lambda)} > 0$ and $0 < x < \frac{-1+\sqrt{1+\lambda^2}}{\lambda} < 1$, we get

$$4\lambda > 4\lambda x^2 + 8x > x^3 + 2\lambda x^2 + 7x.$$

Then $t'(x) < 0$, thus we obtain $t(x) > t(\frac{-1+\sqrt{1+\lambda^2}}{\lambda}) = 0$.

Consequently, the above analysis show us $h''(x) > 0$ in the domain $x \in (0, \frac{-1+\sqrt{1+\lambda^2}}{\lambda})$. From $h''(x) > 0$, we find that $h'(x) < h'(\frac{-1+\sqrt{1+\lambda^2}}{\lambda}) = 0$, hence $h'(x) < 0$ and thus $h(x) > h(\frac{-1+\sqrt{1+\lambda^2}}{\lambda}) = 0$. By $h(x) > 0$ and $\lambda-x > 0$, they confirm $f'(x) < 0$, which completes the proof. \square

Lemma 2.11 Let $F(x) = \arccos \frac{1+\lambda g(x)}{\sqrt{1+\lambda^2}}$, where $g(x) = \frac{-(1+\cos x) + \sqrt{(1+\cos x)^2 + 4\lambda^2 \cos x}}{2\lambda}$, $\lambda \in (0, +\infty)$ and $x \in (0, \frac{\pi}{2})$. Then $F'(x) > 0$ and $F''(x) > 0$.

Proof For convenience, set $r(x) = \sqrt{(1 + \cos x)^2 + 4\lambda^2 \cos x}$. We find the first derivative of $F(x)$ is

$$F'(x) = \frac{\cos \frac{x}{2} \sqrt{1 + 2\lambda^2 + \cos x - r(x)}}{r(x)}.$$

Therefore, we obtain $F'(x) > 0$. The second derivative of $F(x)$ is

$$F''(x) = \frac{\lambda^2 \sin \frac{x}{2} (3 + 4\lambda^2 + \cos x - (1 - \cos x)(r(x) - \cos x))}{r^3(x) \sqrt{1 + 2\lambda^2 + \cos x - r(x)}}.$$

One can check that in the domain $x \in (0, \frac{\pi}{2})$, the following inequality holds

$$3 + 4\lambda^2 + \cos x - (1 - \cos x)(r(x) - \cos x) > 0.$$

Hence $F''(x) > 0$ and thus $F(x)$ is a convex function of x . □

3 The perimeter of reduced spherical polygons

In this section, we consider the Problem P1 proposed by Lassak in [8]. Let V be a spherically convex n -gon, for simplicity, we use ω to replace $\Delta(V)$ in this section, where $\omega \in (0, \pi/2)$. Let $\lambda = \tan \omega$, then $\lambda \in (0, +\infty)$. Denote by $\text{perim}(V)$ the perimeter of V . Let us define several functions which are needed in the following theorems. Set

$$f_1(x) = \arccos \frac{1 + \lambda x}{\sqrt{1 + \lambda^2}}$$

and

$$f_2(x) = \arccos \frac{x(1 + \lambda x)}{\lambda - x},$$

where $x \in (0, \frac{-1 + \sqrt{1 + \lambda^2}}{\lambda})$. Set

$$g(\varphi) = \frac{-(1 + \cos \varphi) + \sqrt{(1 + \cos \varphi)^2 + 4\lambda^2 \cos \varphi}}{2\lambda},$$

where $\varphi \in (0, \frac{\pi}{2})$. And thus $g(\varphi) \in (0, \frac{-1 + \sqrt{1 + \lambda^2}}{\lambda})$.

Lemma 3.1 For a reduced spherical polygon $V = v_1 v_2 \cdots v_n$ with thickness less than $\frac{\pi}{2}$, the perimeter is $\text{perim}(V) = 2 \sum_{i=1}^n f_1(y_i)$, where $y_i = g(\varphi_i)$.

Proof For each $i \in \{1, 2, \dots, n\}$, we focus on the right spherical triangle $o_i t_i v_{i+(n+1)/2}$ in V . Let $|t_i v_{i+(n+1)/2}| = a_i$, $|o_i t_i| = b_i$ and $|o_i v_{i+(n+1)/2}| = c_i$. By Lemma 2.5, we obtain $\text{perim}(V) = 2 \sum_{i=1}^n |v_i t_{i+(n+1)/2}| = 2 \sum_{i=1}^n a_i$. Moreover, we have $|o_i t_i| + |o_i v_{i+(n+1)/2}| = b_i + c_i = \omega$. Note that $b_i < \omega$. From Equation (1), we obtain

$$\cos \varphi_i = \frac{\tan b_i}{\tan c_i} = \frac{\tan b_i}{\tan(\omega - b_i)} = \frac{\tan b_i (1 + \tan \omega \tan b_i)}{\tan \omega - \tan b_i}.$$

Let $\tan b_i = y_i$, from this and $\lambda = \tan \omega$, the above equation becomes

$$\cos \varphi_i = \frac{y_i(1 + \lambda y_i)}{\lambda - y_i}, \quad (16)$$

thus

$$\varphi_i = \arccos \frac{y_i(1 + \lambda y_i)}{\lambda - y_i}. \quad (17)$$

Moreover, we have

$$\tan c_i = \frac{\tan b_i}{\cos \varphi_i} = \frac{\lambda - y_i}{1 + \lambda y_i},$$

and thus $c_i = \arctan \frac{\lambda - y_i}{1 + \lambda y_i}$. By Equation (4), we get that

$$\cos a_i = \frac{\cos c_i}{\cos b_i} = \frac{1 + \lambda y_i}{\sqrt{1 + \lambda^2}},$$

hence $a_i = \arccos \frac{1 + \lambda y_i}{\sqrt{1 + \lambda^2}} = f_1(y_i)$. Therefore, $\text{perim}(V) = 2 \sum_{i=1}^n f_1(y_i)$, where

$$y_i = \frac{-(1 + \cos \varphi_i) + \sqrt{(1 + \cos \varphi_i)^2 + 4\lambda^2 \cos \varphi_i}}{2\lambda} = g(\varphi_i),$$

which can be calculated by Equation (16). \square

Theorem 3.2 *The regular spherical n -gon has the minimum perimeter among all regular spherical k -gons of fixed thickness $\omega \in (0, \frac{\pi}{2})$, where k, n are odd and $3 \leq k \leq n$.*

Proof Let $V_k = v_1 v_2 \cdots v_k$ be a regular spherical odd-gon. Lemma 2.2 shows that V_k is reduced, then by Lemma 3.1, we have

$$\text{perim}(V_k) = 2 \sum_{i=1}^k f_1(y_i),$$

where $y_i = g(\varphi_i)$. V_k is a regular polygon, then Lemma 2.7 (2) shows that $\varphi_1 = \cdots = \varphi_k = \frac{\pi}{k}$. Thus we have $y_1 = \cdots = y_k = g(\frac{\pi}{k})$, where

$$g\left(\frac{\pi}{k}\right) = \frac{-(1 + \cos \frac{\pi}{k}) + \sqrt{(1 + \cos \frac{\pi}{k})^2 + 4\lambda^2 \cos \frac{\pi}{k}}}{2\lambda}.$$

For brevity, we denote φ_i and y_i by φ and y , respectively, where $i = 1, 2, \dots, k$. Then $\varphi = \frac{\pi}{k}$ and $y = g(\varphi)$, where $y \in (0, \frac{-1 + \sqrt{1 + \lambda^2}}{\lambda})$. Equation (17) shows that

$$\varphi = \arccos \frac{y(1 + \lambda y)}{\lambda - y} = f_2(y),$$

then $k = \frac{\pi}{\varphi} = \frac{\pi}{f_2(y)}$. As a result,

$$\text{perim}(V_k) = 2k f_1(y) = 2\pi \frac{f_1(y)}{f_2(y)}, \quad (18)$$

where $y \in (0, \frac{-1 + \sqrt{1 + \lambda^2}}{\lambda})$.

Lemma 2.10 permits that $\frac{f_1(y)}{f_2(y)}$ is a decreasing function of y . Since

$$y = g\left(\frac{\pi}{k}\right) = \frac{-(1 + \cos \frac{\pi}{k}) + \sqrt{(1 + \cos \frac{\pi}{k})^2 + 4\lambda^2 \cos \frac{\pi}{k}}}{2\lambda},$$

it follows that y is an increasing function of k . Thus the perimeter of the regular spherical polygon decreases with the increase of k . The conclusion is as desired. \square

Theorem 3.3 *The regular spherical n -gon has the minimum perimeter among all reduced spherical n -gons with the same thickness $\omega \in (0, \frac{\pi}{2})$.*

Proof Let $V = v_1v_2 \cdots v_n$ be an arbitrary reduced spherical n -gon, then by Lemma 3.1, the perimeter of V is $\text{perim}(V) = 2 \sum_{i=1}^n f_1(y_i)$, where $y_i = g(\varphi_i)$ and

$$g(\varphi_i) = \frac{-(1 + \cos \varphi_i) + \sqrt{(1 + \cos \varphi_i)^2 + 4\lambda^2 \cos \varphi_i}}{2\lambda}.$$

Let $F(\varphi_i) = f_1(g(\varphi_i)) = \arccos \frac{1+\lambda g(\varphi_i)}{\sqrt{1+\lambda^2}}$, where $\varphi_i \in (0, \frac{\pi}{2})$ and $\lambda = \tan \omega \in (0, +\infty)$. Thus $\text{perim}(V) = 2 \sum_{i=1}^n F(\varphi_i)$. Lemma 2.11 states that $F(x)$ is a convex function of x . Thus from Jensen's inequality [3], we obtain that

$$\frac{F(\varphi_1) + \cdots + F(\varphi_n)}{n} \geq F\left(\frac{\varphi_1 + \cdots + \varphi_n}{n}\right),$$

the equality holds when $\varphi_1 = \cdots = \varphi_n$. Then the perimeter of V satisfies

$$\text{perim}(V) = 2n\left(\frac{F(\varphi_1) + \cdots + F(\varphi_n)}{n}\right) \geq 2nF\left(\frac{\varphi_1 + \cdots + \varphi_n}{n}\right).$$

Case 1. In the case V is a regular spherical polygon. By Equation (18), it follows that $\text{perim}(V) = 2nF(\frac{\pi}{n})$.

Case 2. In the case V is a non-regular spherical polygon. Lemmas 2.6 and 2.7 imply that

$$\pi \leq \varphi_1 + \cdots + \varphi_n < \frac{n\pi}{2}.$$

Since $F(x)$ is an increasing function of x within the domain $x \in (0, \frac{\pi}{2})$, it follows that

$$F\left(\frac{\varphi_1 + \cdots + \varphi_n}{n}\right) \geq F\left(\frac{\pi}{n}\right).$$

Consequently, we get

$$\text{perim}(V) = 2n\left(\frac{F(\varphi_1) + \cdots + F(\varphi_n)}{n}\right) \geq 2nF\left(\frac{\varphi_1 + \cdots + \varphi_n}{n}\right) \geq 2nF\left(\frac{\pi}{n}\right).$$

The conclusion follows by the above two cases. □

By Theorems 3.2 and 3.3, we confirm the Problem P1.

4 The maximal diameter of reduced spherical polygons

Recall that the Problem P2 is proposed by Lassak after the discussion of [8, Theorem 4.2]. In the following, we establish some related conclusions. Denote by $\text{diam}(V)$ the diameter of a spherical polygon V .

Lemma 4.1 [8, Proposition 4.1] *The diameter of any reduced spherical n -gon is realized only for some pairs of vertices whose indices (mod n) differ by $\frac{n-1}{2}$ or $\frac{n+1}{2}$.*

Lemma 4.2 [8, Theorem 4.2] *For every reduced spherical polygon V on the sphere, we have*

$$\text{diam}(V) \leq \arccos \left(\cos \Delta(V) \sqrt{1 - \frac{\sqrt{2}}{2} \sin \Delta(V)} \right), \quad (19)$$

with equality for the regular spherical triangle in the part of V .

The Problem P2 asks if the equality in Lemma 4.2 holds only for regular triangle? The answer is negative, so we derive a new inequality different from (19) and confirm the Problem P2 within our investigation. The following statement is analogous to the second part of [5, Theorem 9]. Although the proof process is trivial and the method is the same as [8, Theorem 4.2], for the completeness, we restate as follows.

Theorem 4.3 *For every reduced spherical polygon $V = v_1 \cdots v_n$ of thickness less than $\frac{\pi}{2}$, we have*

$$\text{diam}(V) \leq 2 \arccos \left(\frac{1}{2 \sin \gamma} \right), \quad (20)$$

where $\gamma = \arcsin \frac{-\cos \Delta(V) + \sqrt{\cos^2 \Delta(V) + 8}}{4}$, and the equality holds only for the regular spherical triangle.

Proof By Lemma 4.1, we assume that $\text{diam}(V) = |v_i v_{i+(n+1)/2}|$ for some $i \in \{1, 2, \dots, n\}$ (in the case $\text{diam}(V) = |v_i v_{i+(n-1)/2}|$, the following is similar).

Consider the spherical triangle $v_i t_i v_{i+(n+1)/2}$. Put

$$a_i = |t_i v_{i+(n+1)/2}|, \quad c_i = |v_i v_{i+(n+1)/2}|, \quad \text{and} \quad \gamma_i = \angle v_i v_{i+(n+1)/2} t_i.$$

Lemmas 2.4 and 2.5 (resp. [8, Corollary 3.7]) imply that $\gamma_i = \alpha_i + \beta_i \geq 2\beta_i$, where $0 < \gamma_i \leq \frac{\pi}{2}$ (for the range of γ_i , we can refer to the proof process of Lemma 2.9). From $|v_i t_i| = \Delta(V)$ and Equation (7), we obtain

$$\frac{\sin \beta_i}{\sin a_i} = \frac{\sin \gamma_i}{\sin \Delta(V)}.$$

Thus from $\gamma_i \geq 2\beta_i$, we acquire $\frac{\sin \beta_i}{\sin a_i} \geq \frac{\sin 2\beta_i}{\sin \Delta(V)}$ and hence

$$\sin a_i \leq \frac{\sin \Delta(V)}{2 \cos \beta_i}. \quad (21)$$

Moreover, by Lemma 2.9, we have $\beta_i \leq \gamma$ (In the proof process of Lemma 4.2, Lassak use a slight rough upper bound $\beta_i \leq \frac{\pi}{4}$ to obtain the final inequality. This is the main difference between our proof with Lassak's). Hence we get $\sin a_i \leq \frac{\sin \Delta(V)}{2 \cos \gamma}$ and then

$$\cos a_i \geq \sqrt{1 - \frac{\sin^2 \Delta(V)}{4 \cos^2 \gamma}}. \quad (22)$$

Applying the Pythagorean theorem (i.e. Equation (4)) on sphere to (22), we obtain

$$\cos c_i \geq \cos \Delta(V) \sqrt{1 - \frac{\sin^2 \Delta(V)}{4 \cos^2 \gamma}},$$

as a result,

$$\text{diam}(V) = c_i \leq \arccos \left(\cos \Delta(V) \sqrt{1 - \frac{\sin^2 \Delta(V)}{4 \cos^2 \gamma}} \right).$$

Observe that the equality holds only when (21) and (22) attain the equality, which requires $\gamma_i = 2\beta_i$ and $\beta_i = \gamma$. This leads to the equality $\gamma = \beta_i = \alpha_i$ and thus V is a regular spherical triangle by Lemma 2.9. From the above discussions and Lemma 2.8 (2), we obtain the following equality

$$\arccos \left(\cos \Delta(V) \sqrt{1 - \frac{\sin^2 \Delta(V)}{4 \cos^2 \gamma}} \right) = 2 \arccos \left(\frac{1}{2 \sin \gamma} \right).$$

This completes the proof. \square

Lemma 2.2 implies that the thickness of every reduced spherical polygon is at most $\frac{\pi}{2}$. Then in Lemma 4.2, the thickness of V satisfies $\Delta(V) \leq \frac{\pi}{2}$, while there exists an error in this lemma. For clarity, we establish the following version.

Proposition 4.4 *For every reduced spherical polygon V of thickness at most $\frac{\pi}{2}$, we have*

$$\text{diam}(V) \leq \arccos \left(\cos \Delta(V) \sqrt{1 - \frac{\sqrt{2}}{2} \sin \Delta(V)} \right), \quad (23)$$

the equality holds only when the thickness of V is $\frac{\pi}{2}$.

Proof Since (23) has been proved in [8, Theorem 4.2], we omit it.

In the case $\Delta(V) < \frac{\pi}{2}$, by Theorem 4.3, it follows that $\text{diam}(V) \leq 2 \arccos(\frac{1}{2 \sin \gamma})$. One can check that

$$2 \arccos \left(\frac{1}{2 \sin \gamma} \right) < \arccos \left(\cos \Delta(V) \sqrt{1 - \frac{\sqrt{2}}{2} \sin \Delta(V)} \right).$$

In the case $\Delta(V) = \frac{\pi}{2}$, [12, Proposition 2] implies that V is a body of constant width, thus $\text{diam}(V) = \Delta(V) = \frac{\pi}{2}$. One can also calculate that the right hand side of the inequality (23) equals $\frac{\pi}{2}$. \square

By the above investigation, Problem P2 holds only when considering Theorem 4.3.

5 The smallest radius of a spherical disk covering reduced spherical polygons

The problems about covering the reduced convex bodies by a disk in E^2 and S^2 can be found in [6] and [13], respectively. For the planar case [6], every reduced body R (resp. reduced convex polygon) is contained in a disk of radius $\frac{\sqrt{2}}{2} \Delta(R)$ (resp. $\frac{2}{3} \Delta(R)$).

In the sphere, [13, Theorem 2] shows that every reduced spherical body R of thickness at most $\frac{\pi}{2}$ is contained in a disk of radius $\arctan(\sqrt{2} \tan \frac{\Delta(R)}{2})$. And we can not improve the above estimate when considering all reduced spherical bodies. Now we concentrate on the Problem P3 to find the smallest radius of a disk that contains every reduced polygon of a given thickness on S^2 .

In [2], Dekster extends the Jung Theorem to the n -dimensional hyperbolic space and the n -dimensional sphere ($n \geq 2$). We only exhibit the conclusions we need.

Lemma 5.1 [2, Theorem 2] *Let S be a compact set in S^n of diameter D and circumradius r . Let B be a metric ball of radius r containing S . Then*

$$D \geq 2 \arcsin\left(\sqrt{\frac{n+1}{2n}} \sin r\right), \quad r \in [0, \pi], \quad \text{for } S \subseteq S^n.$$

This version applies to compact sets. The reduced spherically convex polygon is exactly a compact (bounded and closed) set in S^2 . So we could get the following statement.

Theorem 5.2 *Every reduced spherically convex polygon V of thickness $\Delta(V) \in (0, \frac{\pi}{2})$ is contained in a disk of radius $\arcsin(\frac{2}{\sqrt{3}} \sqrt{1 - \frac{1}{4 \sin^2 \gamma}})$, where*

$$\gamma = \arcsin \frac{-\cos \Delta(V) + \sqrt{\cos^2 \Delta(V) + 8}}{4}.$$

Proof Denote by r the circumradius of V . In S^2 , by Lemma 5.1, we have $\text{diam}(V) \geq 2 \arcsin(\frac{\sqrt{3}}{2} \sin r)$, thus

$$r \leq \arcsin\left(\frac{2}{\sqrt{3}} \sin \frac{\text{diam}(V)}{2}\right).$$

Combining this with (20) of Theorem 4.3, it follows that

$$r \leq \arcsin\left(\frac{2}{\sqrt{3}} \sin \frac{2 \arccos(\frac{1}{2 \sin \gamma})}{2}\right) = \arcsin\left(\frac{2}{\sqrt{3}} \sqrt{1 - \frac{1}{4 \sin^2 \gamma}}\right).$$

The conclusion is as desired. □

Note that the inequality (20) of Theorem 4.3 attains equality only when V is a regular triangle, so the estimate $\arcsin(\frac{2}{\sqrt{3}} \sqrt{1 - \frac{1}{4 \sin^2 \gamma}})$ is sharp only for the regular triangle, and the estimate can not be improved.

The following table lists some values of the radius under specific thickness.

radius \ thickness	$\frac{\pi}{8}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
$\arcsin(\frac{2}{\sqrt{3}} \sqrt{1 - \frac{1}{4 \sin^2 \gamma}})$	0.260304...	0.345523...	0.511669...	0.670020...

Table 1: The values of radius under specific thickness

Data Availability Statement

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

References

- [1] Chang Y., Liu C., Su Z. The perimeter and area of reduced spherical polygons of thickness $\frac{\pi}{2}$, Results Math., 2020, 75: 135.
- [2] Dekster B. V. The Jung theorem for spherical and hyperbolic spaces, Acta Math. Hungar., 1995, 67(4): 315-331.

- [3] Jensen J. L. W. V. Sur les fonctions convexes et les inégalités entre les valeurs moyennes, *Acta Math.*, 1906, 30: 175-193.
- [4] Liu C., Chang Y., Su Z. The area of reduced spherical polygons, arXiv: 2009.13268v1.
- [5] Lassak M. Reduced convex bodies in the plane. *Israel J. Math.*, 1990, 70: 365-379.
- [6] Lassak M. On the smallest disk containing a planar reduced convex body, *Arch. Math.*, 2003, 80: 553-560.
- [7] Lassak M. Width of spherical convex bodies. *Aequationes Math.*, 2015, 89: 555-567.
- [8] Lassak M. Reduced spherical polygons. *Colloq. Math.*, 2015, 138: 205-216.
- [9] Lassak M. Spherical geometry – a survey on width and thickness of convex bodies. arXiv: 2012.13652v3.
- [10] Lassak M., Martini H. Reduced convex bodies in Euclidean space - a survey. *Expo. Math.*, 2011, 29: 204-219.
- [11] Lassak M., Martini H. Reduced convex bodies in finite dimensional normed spaces: a survey. *Results Math.*, 2014, 66: 405-426.
- [12] Lassak M. Diameter, width and thickness of spherical reduced convex bodies with an application to Wulff shapes. *Beiträge Alg. Geom.*, 2020, 61: 369-378 .
- [13] Musielak M. Covering a reduced spherical body by a disk. *Ukr. Math. J.*, 2021, 72: 1613-1624.
- [14] Murray D. A. *Spherical Trigonometry*. Longmans Green and CO, London, Bombay and Calcuta, 1900.