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AN ARGUMENT PRINCIPLE FOR REAL FUNCTIONS AND NON-HARMONIC SINGULARITIES IN THE PLANE

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ABSTRACT. A real, rational function $r(t)$ determines an *argument function* $\vartheta(t)$, whose increment $\Delta\vartheta(t)$ relates the numbers of distinct, real zeros and poles of $r(t)$. The present expository note derives such a result, and explains its relevance to the *index* for a certain class of planar singularities.

1. Introduction

If $p(t)$ is a polynomial of degree $d \geq 1$ with real coefficients, its number of distinct, real zeros $\mathscr{Z}[p] \leq d$ may be determined by the classical method of Sturm sequences (which we briefly recall in Remark 2.2). On the other hand, $\mathscr{Z}[p] = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{(p')^2 - pp''}{(p')^2 + p^2} dt$, as shown in §2.

This note concerns the geometrical interpretation of such a formula. Actually, the topological meaning will be more fully realized for $r(t) = \frac{p(t)}{q(t)}$, a real rational function. We may associate with r two smooth functions of t (see §2 for details): The *complex indicatrix* $\hat{r} := \frac{r - ir'}{r + ir'}$, and its argument $\vartheta(t) := \arg \hat{r}(t)$. Then the numbers $\mathscr{Z}[r]$, $\mathscr{P}[r]$ of distinct real zeros and poles of r satisfy:

$$\mathscr{Z}[r] - \mathscr{P}[r] = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{(r')^2 - rr''}{(r')^2 + r^2} dt = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\hat{r}'}{\hat{r}} dt = \frac{\Delta\vartheta}{2\pi}, \tag{1}$$

where $\Delta\vartheta = \vartheta(\infty) - \vartheta(-\infty)$.

We call Equation 1 the *real argument principle* to invite comparison with the argument principle in complex variables: $Z_f - P_f = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)} dz = \frac{\Delta\theta}{2\pi}$; here, $\Delta\theta$ is the net change in $\theta = \arg f(z)$ over the closed contour γ. But as usual in complex analysis, Z_f and P_f count zeros and poles of the meromorphic function $f(z)$ *with multiplicities*, and the zeros/poles lie *inside* (not *on*) the contour of integration γ. So the comparison is a curious one.

That being the case, we devote §3 and §4 to a related topic which gives the real argument principle much of the same geometric flavor and intuitive appeal of its complex namesake. Namely, we discuss *non-harmonic singularities*—a class of planar foliation singularities directly generalizing the zeros and poles of meromorphic functions. Theorem 3.1 gives the index of such a singularity by a simple formula which closely resembles Equation 1. To come full circle, we conclude that the index of a

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non-harmonic singularity may be computed by the method of Sturm sequences (Corollary 3.2). The concluding section §4 is a brief recapitulation of §3 via the *Poincaré-Hopf Theorem* and foliations of the real projective plane by pencils of algebraic curves.

2. The integral formula and real argument principle

We list notation for a real, rational function r, its logarithmic derivative l and complex indicatrix $\hat{r}$; also for the above integrand $R[r]$:

$$r = \frac{p}{q}, \quad l := \frac{r'}{r}, \quad \hat{r} = \frac{r - ir'}{r + ir'}, \quad R[r] := \frac{(r')^2 - rr''}{(r')^2 + r^2}.$$

One verifies the following additional expressions for $R[r]$:

$$R[r] = -\frac{d}{dt}\arctan l = \frac{1}{2i}\frac{d}{dt}\log\hat{r} = \frac{q^2((p')^2 - pp'') - p^2((q')^2 - qq'')}{(pq' - qp')^2 + p^2q^2} =: \frac{P}{Q}. \tag{2}$$

From the first expression, $R[r]$ is evidently bounded on $\mathbb{R}$, so its real singularities are removable. Further, the indicated polynomials P, Q satisfy $\deg Q \geq \deg P + 2$, so the improper integral $\int_{-\infty}^{\infty} R[r]dt$ converges.

To evaluate this integral, let $t_1 < \cdots < t_m$ be the sequence of real zeros and poles of $r(t)$. Then $\int_{-\infty}^{\infty} R[r]dt$ may be computed as a sum of integrals over subintervals $(-\infty, t_1], [t_1, t_2], \ldots, [t_m, \infty)$. Write $r = (t - t_1)^{n_1} \ldots (t - t_m)^{n_m} s(t)$, where $s(t)$ is a rational function with no real zeros or poles. The poles of the logarithmic derivative $l = \frac{r'}{r} = \frac{s'}{s} + \sum_{j=1}^{m} \frac{n_j}{t - t_j}$ are the zeros and poles of r.

Now consider an interval (a, b) not containing any pole of l; that is, for some j, $t_j < a < b < t_{j+1}$ (we allow $t_j = t_0 = -\infty$ or $t_{j+1} = t_{m+1} = \infty$). Then we may evaluate $\int_{t_j}^{t_{j+1}} R[r]dt$ by taking one-sided limits $a \downarrow t_j$, $b \uparrow t_{j+1}$ in the integral

$$\int_a^b R[r]dt = -\arctan l\Big|_a^b = -\arctan\left(\frac{s'}{s} + \sum_{k=1}^{m} \frac{n_k}{t - t_k}\right)\Bigg|_a^b.$$

E.g., if t_j and t_{j+1} are both zeros of r, the exponents n_j, n_{j+1} are positive, so $\lim_{t\downarrow t_j} \frac{n_j}{t - t_j} = \infty$ and, likewise, $\lim_{t\uparrow t_j} \frac{n_{j+1}}{t - t_{j+1}} = -\infty$, while the remaining terms in l have finite limits. Thus, $\int_{t_j}^{t_{j+1}} R[r]dt = -\arctan u|_{\infty}^{-\infty} = \pi$—each endpoint contributing $\frac{\pi}{2}$. On the other hand, if one or the other endpoint t_k is a pole of r ($n_k < 0$), it contributes $-\frac{\pi}{2}$, while an "endpoint" $\pm\infty$ contributes nothing, since $l(\infty) = 0$.

The upshot is that $\int_{-\infty}^{\infty} R[r]dt$, as a sum over subintervals, may be reorganized as a sum over poles of l, each contributing $\pm\pi$: $\int_{-\infty}^{\infty} R[r]dt = \pi(\mathscr{Z}[r] - \mathscr{P}[r])$. Using Equation 2, the second integral in Equation 1 now follows as well.

Remark 2.1. Applying Equation 1 to polynomials p and q, an alternative integral formula now follows at once: $\mathscr{Z}[r]-\mathscr{P}[r]=\mathscr{Z}[p]-\mathscr{Z}[q]=\dfrac{1}{\pi}\displaystyle\int_{-\infty}^{\infty}(R[p]-R[q])dt$ (even though $R[p]-R[q]\neq R[\frac{p}{q}]$, as can be seen from from Equation 2).

Now we turn to the real argument principle for $r(t)=\frac{p(t)}{q(t)}$. Let $t_0\in\mathbb{R}$. Unless t_0 is a non-simple zero of r, the indicatrix $\hat{r}=\frac{r-ir'}{r+ir'}$ is a well-defined complex unit. But then even if $r(t_0)=r'(t_0)=0$, $\hat{r}$ cannot have a pole at t_0 (t_0 must be a removable singularity). Thus, $\hat{r}$ maps the real line smoothly to the unit circle $\hat{r}:\mathbb{R}\to S^1\subset\mathbb{C}$. Consequently, there is also a smooth argument function $\vartheta(t)$ along $\hat{r}(t)$. In fact, it is given by the indefinite integral: $\vartheta(t)=\int_{-\infty}^{t}2R[r]d\tau$. Namely, in view of Equation 2 (and the fact that $|\hat{r}|=1$), we have: $\frac{d}{dt}\vartheta=2R[r]=-i\frac{d}{dt}\log\hat{r}=\frac{d}{dt}\arg\hat{r}$; further $\vartheta(-\infty)=0$ matches $\arg\hat{r}(-\infty)=\arg\frac{1-il(\infty)}{1+il(\infty)}=0$. In particular, $\Delta\vartheta=2\int_{-\infty}^{\infty}R[r]dt=2\pi(\mathscr{Z}[r]-\mathscr{P}[r])$.

Remark 2.2. It is interesting to consider the integral formula and argument principle in contrast with the very different, algebraic method of Sturm sequences. Here we briefly describe the method, whose full justification may be found in [3, §5.2].

To count zeros of $p(x)=x^d+c_{d-1}x^{d-1}+\cdots+c_0$, $c_j\in\mathbb{R}$, first assume $p(x)$ has only simple zeros; thus, p and its derivative p' have greatest common divisor $(p,p')=1$. A sequence of polynomials of decreasing degree, $p_0=p$, $p_1=p'$, $p_2,\dots,p_n$, $n\leq d$, is defined by a variant of the Euclidean algorithm: $p_0=Q_1p_1-p_2$ with $\deg p_2<\deg p_1$; $p_1=Q_2p_2-p_3$, $\deg p_3<\deg p_2$, etc.; until $p_{n-2}=Q_{n-1}p_{n-1}-p_n$, and finally, $p_{n-1}=Q_np_n$. Here, p_n is a g.c.d of p and p' (as for the usual Euclidean algorithm $p_{i-1}=Q_ip_i+p_{i+1}$). But p_n is therefore a constant, since we assume $(p,p')=1$.

For x not a zero of p, consider the sequence of real numbers $\Sigma(x):=\{p_0(x),p_1(x),\dots,p_n(x)\}$. Let $\sigma(x)$ count the number of sign changes in $\Sigma(x)$ (ignoring zeros among the $p_i(x)$). Assume $a<b$ are not zeros of p. Then it can be shown that the number of zeros of p in (a,b) is given by: $\mathscr{Z}_a^b[p]=\sigma(a)-\sigma(b)$. (The idea is this: As x increases from a to b, $\sigma(x)$ steps down by *one* each time x passes a zero of p, and $\sigma(x)$ is otherwise constant.)

If p has non-simple zeros, one can always divide out by the g.c.d. of p and p' and apply the above to $\tilde{p}=p/(p,p')$—which has the same zeros as p, but all simple. Finally, to count all real zeros of p, one may take $(a,b)=(-M,M)$ for M large enough. For $M=\max\{1,|c_{n-1}|,\dots,|c_0|\}$, say, it may be shown that $|p(x)|>0$ for $|x|>M$, so $\mathscr{Z}[p]=\sigma(-M)-\sigma(M)$.

3. Planar foliations and the index of a non-harmonic singularity

Figure 1 shows standard planar vector field singularities of index n. Namely, $w_n(z)=u+iv=z^n=(x+iy)^n$ defines a vector field on the plane $\mathbb{C}\simeq\mathbb{R}^2$, for $n=0,1,2,\dots$, and on the punctured plane $\mathbb{C}\setminus\{0\}$, for $n=-1,-2,\dots$. As the point $z=\varepsilon e^{i\theta}$ makes one counterclockwise turn around the

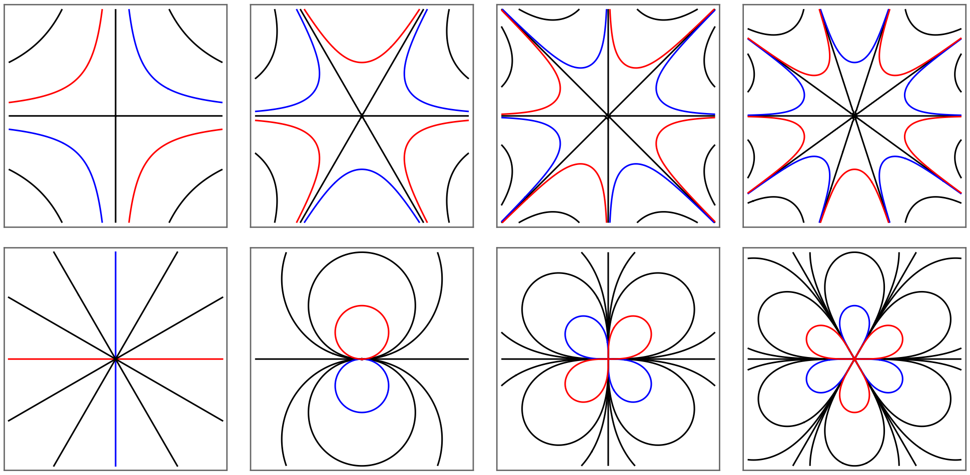

FIGURE 1. Phase portraits of $\frac{dz}{dt} = z^n$ have indices $\mathscr{I}_0(z^n) = n$ at $z = 0$. Top: $n = -1,-2,-3,-4$. Bottom: $n = 1,2,3,4$.

circle $|z| = \varepsilon > 0$, the vector $w_n = \varepsilon^n e^{in\theta}$ completes n rotations (clockwise, in case $n < 0$). Thus, the singularity at the origin has index $\mathscr{I}_0(w_n) = n$.

For our purposes, a somewhat more general construction is required. To explain the idea, we first consider the trajectories of w_n. Let $z = x(t) + iy(t)$ solve the ODE: $\frac{dz}{dt} = z^n$. Separation of variables yields the implicit solution in terms of the complex potential $\Phi(z) = h + ik = \int \frac{dz}{z^n} = \frac{1}{1-n} z^{1-n}$ $(n \neq 1)$. Then we have $\frac{d}{dt}\Phi(z(t)) = z^{-n}(t)z'(t) = 1$, so, $k(x,y) = \Im(\Phi(z))$ is constant along trajectories.

Thus, for $n \neq 1$, the phase portrait of the ODE consists of algebraic curves arising as level sets $k(x,y) = k_0$ of a homogeneous, harmonic polynomial or rational function, $k(x,y) = \frac{g(x,y)}{h(x,y)}$. These well-known curves are the *sinus spirals* $r^m \sin m\theta = A$, which have many interesting properties [7].

We now proceed to define the larger class of singularities of principal interest here. Let $g(x,y) = \Pi_j(A_j x + B_j y)$ and $h(x,y) = \Pi_k(C_k x + D_k y)$ be a pair of real, binary forms with no common factors. (Here we depend on the Fundamental Theorem of Algebra, and note that some linear factors may come in complex conjugate pairs.) Also, we allow h or g to be constant.

Then the level set diagram of $k(x,y) = \frac{g(x,y)}{h(x,y)}$ (including $k = \infty$) gives a planar foliation $\mathscr{F}$; that is, $\mathscr{F}$ is the pencil of algebraic curves $\mathscr{F} : \alpha g + \beta h,\ \alpha, \beta \in \mathbb{R}$. (Strictly speaking, we distinguish the pencil from its foliation; but we use $\mathscr{F}$ to denote both.) When g and h are forms, as they are here, we refer to $\mathscr{F} : \alpha g + \beta h$ as a *tangent pencil*, with the following interpretation in mind: If we start with a general pencil of algebraic curves $\tilde{\mathscr{F}} : \alpha \tilde{g} + \beta \tilde{h}$ with singularity at p, we can form a new pencil $\mathscr{F} : \alpha g + \beta h$ by replacing $\tilde{g}, \tilde{h}$ by all their (real and complex) tangent lines at p. Then $\mathscr{F}$ is a kind of "first approximation" of $\tilde{\mathscr{F}}$ at p.

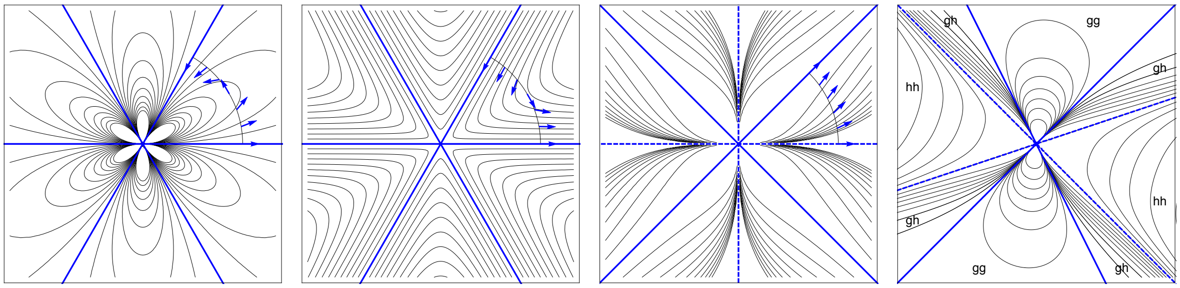

FIGURE 2. Indices: $\mathscr{I}_p(\mathscr{F}_1) = 4$, $\mathscr{I}_p(\mathscr{F}_2) = -2$, $\mathscr{I}_p(\mathscr{F}_3) = \mathscr{I}_p(\mathscr{F}_4) = 1$.

Figure 2 shows tangent pencils $\mathscr{F}_1, \ldots, \mathscr{F}_4$ for the four rational functions:

$$k_1 = \frac{(3x^2y - y^3)}{(x^2+y^2)^2},\; k_2 = \frac{x^2+y^2}{3x^2y - y^3},\; k_3 = \frac{xy}{(x^2-y^2)(x^2+y^2)},\; k_4 = \frac{(y-x)(2x+y)}{(y+x)(3y-x)(x^2+y^2)}$$

Here we use *proper* rational functions $k = g/h$ ($\deg g < \deg h$), noting k and $1/k$ define the same foliation. (We allow $\deg g = \deg h$; but then $\mathscr{F}$ is indistinguishable, as a foliation, from the standard pencil of lines $\alpha x + \beta y$.)

Each such foliation $\mathscr{F}$ has one finite singularity $p = (0,0)$ (along with singularities "at infinity"—see §4). Further, $\mathscr{F}$ is divided into *sectors* of various types by the *real lines* of g and h. E.g., in Figure 2, p is seen to be the meeting point of: The $\mathscr{E} = 6$ *elliptic sectors* in $\mathscr{F}_1$; the $\mathscr{H} = 6$ *hyperbolic sectors* in $\mathscr{F}_2$; the $\mathscr{P} = 8$ *parabolic sectors* in $\mathscr{F}_3$; sectors of all three types in $\mathscr{F}_4$. (The nomenclature is standard [1], but Figure 2 should suffice to indicate the meanings of the italicized terms.)

Relative to Figure 1, what's new here is that p is a *non-harmonic singularity*, in general. We note that $\mathscr{F}_1$ and $\mathscr{F}_2$ are qualitatively similar to the respective phase portraits for $\frac{dz}{dt} = z^4$ and $\frac{dz}{dt} = z^{-2}$. On the other hand, $\mathscr{F}_3$ and $\mathscr{F}_4$ show behaviors which do not occur above; in particular, when $k(x,y)$ is harmonic, all sectors at a point p are of the same type, and are never parabolic. By comparison, the non-harmonic singularities of tangent pencils exhibit greater variety.

Nevertheless, we can still write down simple formulas for index $\mathscr{I}_p(\mathscr{F})$. To make the most explicit connection to the ideas of §1–2, we use the following terminology. The *zeros* of forms g and h are the corresponding real or complex lines through the origin—respectively, $(A_jx + B_jy) = 0$ and $(C_kx + D_ky) = 0$. We will be especially interested in the numbers of *distinct real zeros* of g and h—their *visible lines*. These numbers will be denoted $\mathscr{Z}[g]$ and $\mathscr{Z}[h]$. Further, the *poles* of $k(x,y) = \frac{g(x,y)}{h(x,y)}$ are the zeros of h; so k has $\mathscr{P}[k] = \mathscr{Z}[h]$ *distinct real poles*.

To discuss the index $\mathscr{I}_p(\mathscr{F})$, we remark that $\mathscr{F}$ is the phase portrait of a polynomial vector field $\mathbf{V} = u\mathbf{i} + v\mathbf{j}$ with unique zero p, as is not hard to show. Thus we can define $\mathscr{I}_p(\mathscr{F}) := \frac{\Delta\phi_\varepsilon}{2\pi}$, where $\phi_\varepsilon(\theta)$, $0 \le \theta < 2\pi$ is a continuous argument function for $\mathbf{V}$ along a circle $x^2 + y^2 = \varepsilon^2$:

$\mathbf{V}(\varepsilon\cos\theta, \varepsilon\sin\theta) = r(\theta)(\cos\phi\mathbf{i} + \sin\phi\mathbf{j})$. Although the argument function $\phi_\varepsilon(\theta)$ obviously depends on the radius ε, it will be seen that $\Delta\phi_\varepsilon = \phi_\varepsilon(2\pi) - \phi_\varepsilon(0)$ does not, so $\mathscr{I}_p(\mathscr{F})$ is well-defined.

Theorem 3.1. *Let $\mathscr{F}$ be the planar foliation defined by a homogeneous, proper rational function $k(x,y) = \frac{g(x,y)}{h(x,y)}$. Then the index of $\mathscr{F}$ at $p = (0,0)$ is given by:*

$$\mathscr{I}_p(\mathscr{F}) := \frac{\Delta\phi_\varepsilon}{2\pi} = 1 + \mathscr{Z}[k] - \mathscr{P}[k] \tag{3}$$

Proof. It might seem natural to consider the standard integral formula for index of a vector field $\mathbf{V} = u\mathbf{i} + v\mathbf{j}$ at a point p: $\mathscr{I}_p(\mathbf{V}) = \dfrac{1}{2\pi}\displaystyle\int_{C_\varepsilon} d\phi = \dfrac{1}{2\pi}\displaystyle\int_{C_\varepsilon} \dfrac{udv - vdu}{u^2 + v^2}$. But it turns out that this leads to a different, more complicated, integral than in Equation 1. Instead, we invoke the classical formula for index in terms of elliptic and hyperbolic sectors: $\mathscr{I}_p(\mathscr{F}) = 1 + \frac{\mathscr{E} - \mathscr{H}}{2}$ (see [1]).

We can easily explain the heuristics of the latter formula by referring to the arrows in Figure 2. Along an arc of C_ε in an elliptic sector, the net rotation of $\mathbf{V}$ *relative to the standard pencil of lines* $\mathscr{L} : \sin\theta x - \cos\theta y$ is $\Delta(\phi - \theta) = \pi$; the corresponding increments for hyperbolic and parabolic sectors are $\Delta(\phi - \theta) = -\pi$ and $\Delta(\phi - \theta) = 0$. If there are $\mathscr{E}$ elliptic sectors and $\mathscr{H}$ hyperbolic sectors, the net relative increment is $\Delta(\phi - \theta) = \pi(\mathscr{E} - \mathscr{H})$. Thus, $\Delta\phi = \Delta\theta + \pi(\mathscr{E} - \mathscr{H}) = 2\pi + \pi(\mathscr{E} - \mathscr{H})$ as claimed. E.g., $\mathscr{I}_p(\mathscr{F}_1) = 1 + \frac{6-0}{2} = 4$, $\mathscr{I}_p(\mathscr{F}_2) = 1 + \frac{0-6}{2} = -2$, $\mathscr{I}_p(\mathscr{F}_3) = 1$, and $\mathscr{I}_p(\mathscr{F}_4) = 1 + \frac{2-2}{2} = 1$.

Now we turn to a brief explanation of sector types. Let sectors of a tangent pencil $\mathscr{F} : \alpha g + \beta h$ be labelled gg, hh, and gh, according to their bounding line pairs—see Figure 2 (right). Since $d < e$, the shape of $\mathscr{F}$ is dominated by g near p and by h far from p. In fact, the real lines of h (dashed) are asymptotes for curves in hh and gh sectors, while curves in gh and gg sectors approach p (in one or both directions), where they are asymptotically tangent to lines of g.

The last equality in Equation 3 now follows by the same logic as employed to prove the first equality in §1. Namely, we have regarded $\Delta\phi_\varepsilon$ as a sum over sectors, each contributing $\pi, -\pi$ or 0; but this sum may be re-organized as a sum over real lines of g and h. Each line borders *four* sectors, and altogether contributes $\pm 4\frac{\pi}{2} = \pm 2\pi$—depending on whether the line belongs to g or h. □

Corollary 3.2. *The index of a tangent pencil foliation at $p = (0,0)$ may be computed algorithmically by the method of Sturm sequences.*

Proof. We dehomogenize binary forms g, h to get polynomials in one variable $t = \frac{y}{x}$: $g(x,y) = x^d g(1,t)$, $h(x,y) = x^e h(1,t)$, where $d = \deg g$, $e = \deg h$. We use the Sturm method, as in Remark 2.2, to find the numbers of distinct real zeros $\mathscr{Z}[g(1,t)]$ and $\mathscr{Z}[h(1,t)]$. Also, we check if $g(0,1) = 0$ or $h(0,1) = 0$. Then $\mathscr{I}_p(\mathscr{F}) = 1 + \mathscr{Z}[g(1,t)] - \mathscr{Z}[h(1,t)] + \sigma$, where $\sigma = 1, -1$, or 0, depending on whether $g(0,1) = 0$, $h(0,1) = 0$, or neither. □

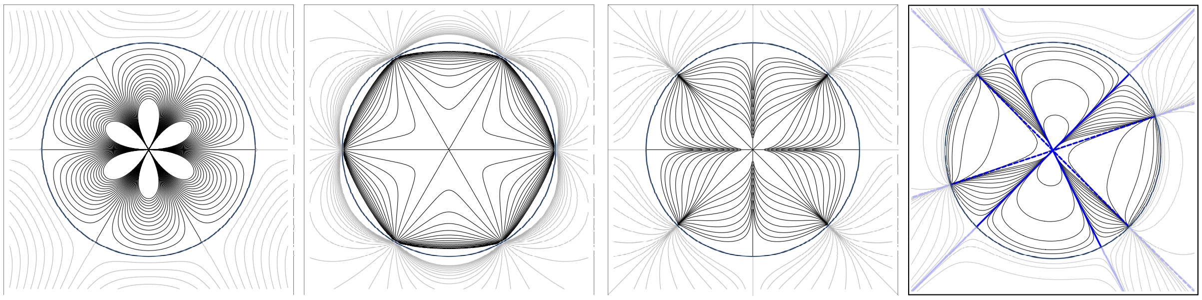

FIGURE 3. Tangent pencils $\mathscr{F}_1, \mathscr{F}_2, \mathscr{F}_3$ and $\mathscr{F}_4$ in the disk $\mathbb{D}$ (and in $\hat{\mathbb{C}}$).

4. Tangent pencil foliations of the real projective plane

In this section, we place Equation 3 into a larger context. Figure 3 shows the same pencils $\mathscr{F}_1, \ldots, \mathscr{F}_4$ as above, but in the disk model of the real projective plane, $\mathbb{D} \simeq \mathbb{R}\mathrm{P}^2$. By applying the *Poincaré-Hopf Theorem* to such foliations of $\mathbb{R}\mathrm{P}^2$, we obtain an alternate proof and a visually appealing interpretation of Equation 3.

The *Poincaré-Hopf Theorem* [1, 6] relates indices of a singular foliation $\mathscr{F}$ on a compact surface S to its Euler characteristic: $\sum \mathscr{I}_{p_j}(\mathscr{F}) = \chi(S)$. E.g., if $\mathscr{F}$ is the foliation of the sphere by parallels of latitude (or by meridians), the polar singularities have index *one*, and $\sum \mathscr{I}_{p_j}(\mathscr{F}) = \chi(S^2) = 2$. On the other hand, the theorem does not directly apply, say, to the phase portrait of a planar dynamical system, since $\mathbb{R}^2$ is not compact. Hence the idea of *Poincaré compactification* of the phase plane to capture "behavior at infinity" [1].

Thus, we consider pencil foliations of the projective plane $\mathbb{R}\mathrm{P}^2 = \mathbb{R}^2 \cup \mathbb{R}\mathrm{P}^1$—the compact space obtained from $\mathbb{R}^2$ by adding *ideal points*. $\mathbb{R}\mathrm{P}^2$ may be defined as the set of lines through the origin in $\mathbb{R}^3 = \{(x_1, x_2, x_3)\}$; for any non-zero $\lambda \in \mathbb{R}$, $(\lambda x_1 : \lambda x_2 : \lambda x_3) \in \mathbb{R}\mathrm{P}^2$ represents the same point in homogeneous coordinates. Non-horizontal lines correspond to points in the affine plane $x_3 = 1$ (by intersection); horizontal lines represent ideal points $(x_1 : x_2 : 0)$.

Now we consider a pencil $\mathscr{F} : \alpha G + \beta H$ of projective curves of degree m. By Bézout's Theorem, there are at most m^2 intersection points q—where $G(q) = H(q) = 0$—assuming G, H have no common factors [2]; these are the *base point singularities* of $\mathscr{F}$. Any other point q lies on a unique curve $F_q := H(q)G - G(q)H = 0$. Such a point may also be a singularity of $\mathscr{F}$—e.g., if q is a node or cusp on F_q. Typically, the latter singularities are also isolated—e.g., if $\mathscr{F}$ is a *regular pencil*—so Poincaré-Hopf may be applied [4].

Likewise for a tangent pencil $\mathscr{F} : \alpha g + \beta h$ (which is regular only for $d \leq 1$). We introduce homogenous coordinates to consider the corresponding pencil of projective curves:

$$\mathscr{F} : \alpha x_3^{e-d} \tilde{G} + \beta H, \quad \tilde{G}(x_1,x_2,x_3) := x_3^d g(\frac{x_1}{x_3},\frac{x_2}{x_3}), \quad H(x_1,x_2,x_3) := x_3^e h(\frac{x_1}{x_3},\frac{x_2}{x_3})$$

Thus, to form the "first curve" $G = x_3^{e-d}\tilde{G}$, in the projective setting, amounts to augmenting the d (real and complex) lines of g by ($e-d$ copies of) the ideal line x_3.

Now let us consider how the Poincaré-Hopf Theorem applies to Figure 3. Here we use the disk model $\mathbb{D} \simeq \mathbb{R}\mathrm{P}^2$ merely as a visual aid; we simply note that $\mathbb{D} = D \cup (S^1/\pm)$ contains points in the interior of the unit disk $D \subset \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, as well as antipodal pairs $\pm e^{i\theta}$ on the unit circle S^1, representing ideal points. Besides the singularity of $\mathscr{F}_k$ at $p_0 = p$, as in Figure 2, "new" singularities appear at ideal points $p_j \in S^1/\pm$. (The *lighter* foliation extends $\mathscr{F}$ to the disk exterior E, by antipodal symmetry on $\hat{\mathbb{C}}$, and helps to identify *types* of such ideal singularities.)

Of course, each example must satisfy: $\sum \mathscr{I}_{p_j}(\mathscr{F}) = \chi(\mathbb{D}) = \chi(\mathbb{R}\mathrm{P}^2) = 1$. On $S^1/\pm$, $\mathscr{F}_1$ has three *saddles* (self-intersections of G), $\mathscr{F}_2$ has three *sources* (base points in $G \cap H$), $\mathscr{F}_3$ has two of each, and $\mathscr{F}_4$ has four of each. Using also the known values of $\mathscr{I}_p(\mathscr{F}_j)$, the full index sums $S_k = \sum \mathscr{I}_{p_j}(\mathscr{F}_k)$ are seen to be: $S_1 = 4-3 = 1$, $S_2 = -2+3 = 1$, $S_3 = 1+0 = 1$, and $S_4 = 1+2-2 = 1$.

In these examples, the pattern of simple singularities on the ideal line $x_3 = 0$ reflects the more complicated singularity at p, in a predictable way. Namely, $\mathscr{F}$ has $\mathscr{Z}[h]$ ideal base points in $\mathbb{R}\mathrm{P}^2$—the intersections of the real lines of h with $x_3 = 0$; These simple base points of $\mathscr{F}$ are singularities of index $\mathscr{I}(\mathscr{F}) = 1$. Further, there are $\mathscr{Z}[g]$ ideal *self intersections* of G—where the real lines of g meet $x_3 = 0$. These nodes of G give saddle singularities of index $\mathscr{I}(\mathscr{F}) = -1$.

Turning things around, if we had not already known $\mathscr{I}_p(\mathscr{F})$, the index of the more complicated singularity at the origin, we could have solved for it in the Poincaré-Hopf formula: $S = \sum \mathscr{I}_{p_j}(\mathscr{F}) = \mathscr{I}_p(\mathscr{F}) + \mathscr{Z}[h] - \mathscr{Z}[g] = \chi(\mathbb{D}) = 1$. That is, we recover Equation 3, $\mathscr{I}_p(\mathscr{F}) = 1 + \mathscr{Z}[g] - \mathscr{Z}[h]$, by a totally different route. It is perhaps unusual to derive a local formula from a global theorem; but the graphics in Figure 3 reveal this relationship almost at a glance.

Finally, it is tempting to ask how far such ideas could be pushed for other classes of pencil singularities. We note that even for regular pencils of *cubics*, e.g., the theory of pencil singularities is more complicated; these and their *configurations* have been extensively investigated in modern algebraic geometry because of their connection to the theory of rational elliptic surfaces [5]. But the *index* of a singularity in the real setting—as discussed here and in [4]—is a coarser invariant. For this, a *Sturm-like algorithm* might suffice.

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