ROCKY MOUNTAIN JOURNAL OF MATHEMATICS
Vol. , No. , YEAR
https://doi.org/rmj.YEAR..PAGE

GENERALIZED n-FRACTIONAL POLYNOMIAL P-FUNCTIONS WITH SOME RELATED INEQUALITIES AND THEIR APPLICATIONS

SERAP ÖZCAN

ABSTRACT. In this work we introduce a novel class of convex functions called generalized n-fractional polynomial P-function and investigate some of its algebraic properties. We establish Hermite–Hadamard inequality for the newly defined class of functions. In addition, we obtain some refinements of the right side of Hermite–Hadamard type inequalities for functions whose derivatives in absolute value at certain powers are generalized n-fractional polynomial P-function. Some applications to special means and new error estimates for the trapezoidal formula are given. The results obtained in this work are generalizations of some known results in the literature.

1. Introduction

Convex functions play a fundamental role in the realm of mathematical analysis, offering valuable insights into the principles of optimization, approximation, and variational calculus. Their significance extends across a multitude of scientific and engineering disciplines, where they serve as indispensable tools for modeling complex phenomena and solving challenging problems. Rooted in the elegant properties of convexity, the Hermite-Hadamard inequality emerges as a potent mathematical tool, providing essential bounds on the integral mean values of convex functions. This inequality is stated as follows [11, 12]:

Let $\varphi : \Im \to \mathbb{R}$ be a convex function and $\varphi \in L[\upsilon, \omega]$. Then the inequality

$$\varphi\left(\frac{\upsilon+\omega}{2}\right) \leq \frac{1}{\omega-\upsilon}\int_{\upsilon}^{\omega} \varphi(\chi)d\chi \leq \frac{\varphi(\upsilon)+\varphi(\omega)}{2} \tag{1.1}$$

holds for all $\upsilon, \omega \in \Im$ with $\upsilon < \omega$. The H–H inequality is one of the most celebrated investigated findings involving convex functions. It shows a fundamental connection between convexity and inequalities, offering valuable insights into the nature of convex functions. This result offers us a necessary and sufficient condition for a function to be convex.

Named in honor of Charles Hermite and Jacques Hadamard, this inequality finds wide-ranging applications, spanning from mathematical physics to economics, where it contributes to the development of robust analytical frameworks and facilitates the exploration of intricate mathematical relationships.

The study of convex functions and the Hermite-Hadamard inequality has garnered significant attention in recent decades, with researchers exploring their theoretical foundations, analytical properties, and practical implications across various domains. For instance, in the realm of economics, convex functions play a pivotal role in modeling utility functions, production functions, and cost functions,

2020 *Mathematics Subject Classification.* 26D15, 26A51, 26d10.
Key words and phrases. Convex function, Generalized n-fractional polynomial P-function, integral inequalities, Hermite-Hadamard inequality.

providing insights into consumer behavior, market equilibrium, and resource allocation [1]. Moreover, convex optimization techniques have been extensively utilized in finance for portfolio optimization, risk management, and option pricing, where convexity assumptions underpin many quantitative models [5].

In addition to these traditional applications, convexity concepts have found diverse and innovative applications in emerging fields such as quantum convexity and multiplicative convexity. In quantum information theory, for example, quantum convexity theory provides a framework for understanding the geometry of quantum states and operations, with applications in quantum communication, cryptography, and computing [20, 28]. Similarly, in the realm of multiplicative calculus, convexity theory offers novel perspectives on the behavior of multiplicative convex functions and their derivatives, leading to applications in various fields of pure and applied sciences [10, 17, 25, 26, 29, 30]. In optimization theory, for instance, quasi-convex functions arise naturally in problems involving constraints that are not necessarily convex but possess certain desirable properties, leading to efficient algorithms for solving non-convex optimization problems [4]. Moreover, in image processing and computer vision, pseudo-convex optimization techniques are employed for image denoising, segmentation, and reconstruction, leveraging the robustness and stability properties of pseudo-convex functions [21].

In recent years, there has been considerable interest in the generalization of convex functions and the H–H inequality. Researchers in this field are focusing on expanding the boundaries of traditional convex functions and related inequalities to discover new mathematical insights and relationships. Various articles published in recent years reflect the increasing interest and research efforts in this area. Some refinements, generalizations and enhancements of convex functions and inequality (1.1) can be found in the recent papers [2, 3, 6, 8, 15, 16, 13, 19, 23, 24, 31].

These examples highlight the growing interest and importance of generalizing convex functions and related inequalities in contemporary mathematical research. By exploring new avenues of generalization, researchers aim to enhance our understanding of fundamental mathematical concepts and develop more powerful analytical tools with applications across various scientific disciplines.

2. Fundamentals

In this section, we will provide some fundamental definitions and theorems.

In [9], the authors introduced the class of P-functions and established related H–H inequality as follows:

Definition 2.1. A non-negative function $\varphi : \Im \subseteq \mathbb{R} \to \mathbb{R}$ is said to be P-function if

$$\varphi(\lambda\upsilon + (1-\lambda)\omega) \leq \varphi(\upsilon) + \varphi(\omega) \tag{2.1}$$

holds for all $\upsilon, \omega \in \Im$ and $\lambda \in [0,1]$.

Theorem 2.2. *Let $\varphi : \Im \to \mathbb{R}$ be a P-function and $\varphi \in L[\upsilon,\omega]$ Then*

$$\varphi\left(\frac{\upsilon+\omega}{2}\right) \leq \frac{2}{\omega-\upsilon}\int_{\upsilon}^{\omega}\varphi(\chi)d\chi \leq 2\left[\varphi(\upsilon)+\varphi(\omega)\right] \tag{2.2}$$

holds for all $\upsilon, \omega \in \Im$ with $\upsilon < \omega$.

In [27], the concept of h-convexity is introduced as follows:

Definition 2.3. Let $h : K \to \mathbb{R}$ be a non-negative function, $h \neq 0$. One says that $\varphi : \Im \to \mathbb{R}$ is an h-convex function, if φ is non-negative and the inequality

$$\varphi(\eta\upsilon + (1-\eta)\omega) \leq h(\eta)\varphi(\upsilon) + h(1-\eta)\varphi(\omega)$$

holds for all $\upsilon, \omega \in \Im$, $\eta \in (0,1)$.

It is obvious that, if one takes $h(\eta) = \eta$ and $h(\eta) = 1$, then the definition of h-convex function reduces to the definitions of classical convex function and classical P-function, respectively.

In [14], the authors introduced the definition of n-polynomial P-function and established related H–H inequality as follows:

Definition 2.4. Let $n \in \mathbb{N}$. A non-negative function $\varphi : \Im \subset \mathbb{R} \to \mathbb{R}$ is said to be n-polynomial P-function if

$$\varphi(\lambda\upsilon + (1-\lambda)\omega) \leq \frac{1}{n}\sum_{m=1}^{n}\left[2 - \lambda^m - (1-\lambda)^m\right]\left[\varphi(\upsilon) + \varphi(\omega)\right]$$

holds for all $\upsilon, \omega \in \Im$ and $\lambda \in [0,1]$.

Theorem 2.5. *Let $\varphi : [\upsilon, \omega] \to \mathbb{R}$ be a n-polynomial P-function. If $\upsilon < \omega$ and $\varphi \in L[\upsilon, \omega]$, then*

$$\frac{1}{4}\left(\frac{n}{n+2^{-n}-1}\right)\Phi\left(\frac{\upsilon+\omega}{2}\right) \leq \frac{1}{\omega-\upsilon}\int_{\upsilon}^{\omega}\varphi(\chi)d\chi \leq \left(\frac{\varphi(\upsilon)+\varphi(\omega)}{n}\right)\sum_{m=1}^{n}\frac{2m}{m+1}. \tag{2.3}$$

In [22], Numan introduced the class of n-fractional polynomial P-function and derived H–H inequality for this class of functions as follows:

Definition 2.6. Let $n \in \mathbb{N}$. A non-negative function $\varphi : \Im \subset \mathbb{R} \to \mathbb{R}$ is said to be n-fractional polynomial P-function if

$$\varphi(\lambda\upsilon + (1-\lambda)\omega) \leq \frac{1}{n}\sum_{m=1}^{n}\left[\lambda^{1/m} + (1-\lambda)^{1/m}\right]\left[\varphi(\upsilon) + \varphi(\omega)\right]$$

holds for all $\upsilon, \omega \in \Im$ and $\lambda \in [0,1]$.

Theorem 2.7. *Let $\varphi : [\upsilon, \omega] \to \mathbb{R}$ be an n-fractional polynomial P-function. If $\upsilon < \omega$ and $\varphi \in L[\upsilon, \omega]$, then*

$$\frac{n}{4\sum_{m=1}^{n}\left(\frac{1}{2}\right)^{1/m}}\varphi\left(\frac{\upsilon+\omega}{2}\right) \leq \frac{1}{\omega-\upsilon}\int_{\upsilon}^{\omega}\varphi(\chi)d\chi \leq \left(\frac{\varphi(\upsilon)+\varphi(\omega)}{n}\right)\sum_{m=1}^{n}\frac{2m}{m+1}. \tag{2.4}$$

Recently, in [18], Kadakal et al. construct a generalization of the definition of n-polynomial convex function called a generalized n-polynomial convex function as follows:

Definition 2.8. Let $n \in \mathbb{N}$ and $a_m \geq 0$ $\left(m = \overline{1,n}\right)$ such that $\sum_{m=1}^{n} a_m > 0$. A non-negative function $\varphi : \Im \subset \mathbb{R} \to \mathbb{R}$ is said to be generalized n-polynomial convex function if

$$\varphi(\lambda\upsilon + (1-\lambda)\omega) \leq \frac{\sum_{m=1}^{n} a_m\left[1-(1-\lambda)^m\right]}{\sum_{m=1}^{n} a_m}\varphi(\upsilon) + \frac{\sum_{m=1}^{n} a_m(1-\lambda^m)}{\sum_{m=1}^{n} a_m}\varphi(\omega)$$

holds for all $\upsilon, \omega \in \Im$ and $\lambda \in [0,1]$.

The main purpose of this paper is to introduce the concept of generalized n-fractional polynomial P-functions and establish some results connected with the right side of novel inequalities similar to the H–H inequality for this class of functions. Some applications to special means are also given.

3. The definition of generalized n-fractional polynomial P-functions

In this section, we construct a generalization of the definition of n-fractional polynomial P-function. We first give a new definition of a generalized n-fractional polynomial P-function. Then, we study some of its basic algebraic properties.

Definition 3.1. Let $n \in \mathbb{N}$ and $a_m \geq 0$ $\left(m = \overline{1,n}\right)$ such that $\sum_{m=1}^{n} a_m > 0$. A non-negative function $\varphi : \Im \subset \mathbb{R} \to \mathbb{R}$ is said to be generalized n-fractional polynomial P-function if

$$\varphi(\lambda \upsilon + (1-\lambda)\omega) \leq \frac{\sum_{m=1}^{n} a_m \left[\lambda^{1/m} + (1-\lambda)^{1/m}\right]}{\sum_{m=1}^{n} a_m} [\varphi(\upsilon) + \varphi(\omega)] \tag{3.1}$$

holds for all $\upsilon, \omega \in \Im$ and $\lambda \in [0,1]$.

We will denote by $GFPP(\Im)$ the class of generalized n-fractional polynomial P-functions on interval $\Im$.

Note that, every generalized n-fractional polynomial P-function is an h-convex function with $h(\lambda) = \frac{\sum_{m=1}^{n} a_m \left[\lambda^{1/m} + (1-\lambda)^{1/m}\right]}{\sum_{m=1}^{n} a_m}$. Therefore, if $\varphi, \psi \in GFPP(\Im)$, then

i) $\varphi + \psi \in GFPP(\Im)$ *and for* $c \in \mathbb{R}$ *(*$c \geq 0$*),* $c\varphi \in GFPP(\Im)$ *.(see* [27], *Proposition 10).*

ii) if φ and ψ be a similarly ordered functions on $\Im$, then $\varphi\psi \in GFPP(\Im)$*(see* [27], *Proposition 10).*

Also, if $\varphi : \Im \to K$ is a convex and $\psi \in GFPP(K)$ and nondecreasing, then $\psi \circ \varphi \in GFPP(\Im)$ *(see* [27], *Theorem 15).*

Remark 3.2. If one takes $n = 1$ in (3.1), then the class of generalized 1-fractional polynomial P-functions reduces to the class of classical P-functions.

Remark 3.3. If one takes $a_m = 1$ $\left(m = \overline{1,n}\right)$ in (3.1), then the class of generalized n-fractional polynomial P-functions reduces to the class of n-fractional polynomial P-functions.

Remark 3.4. Every non-negative convex function is a generalized n-fractional polynomial P-function. Indeed, since

$$\lambda \leq \lambda^{1/2} \leq \lambda^{1/3} \leq \ldots \leq \lambda^{1/n}$$

for every $\lambda \in [0,1]$ and $n \in \mathbb{N}$. One can write

$$\lambda \leq \frac{\sum_{m=1}^{n} a_m \lambda^{1/m}}{\sum_{m=1}^{n} a_m} \text{ and } 1-\lambda \leq \frac{\sum_{m=1}^{n} a_m (1-\lambda)^{1/m}}{\sum_{m=1}^{n} a_m}$$

for all $\lambda \in [0,1]$ and $n \in \mathbb{N}$. and thus, if φ is an non-negative convex function on an interval $\Im \subseteq \mathbb{R}$, then one has

$$\begin{aligned} \varphi(\lambda \upsilon + (1-\lambda)\omega) &\leq \lambda \varphi(\upsilon) + (1-\lambda)\varphi(\omega) \\ &\leq \frac{\sum_{m=1}^{n} a_m \left[\lambda^{1/m} + (1-\lambda)^{1/m}\right]}{\sum_{m=1}^{n} a_m} [\varphi(\upsilon) + \varphi(\omega)] \end{aligned}$$

for every $\upsilon, \omega \in \Im$, $\lambda \in [0,1]$ and $n \in \mathbb{N}$.

According to Remark (3.4), all nonnegative convex functions are given as an example of a generalized n-fractional polynomial P-function.

Theorem 3.5. *Let $\varphi : [\upsilon, \omega] \to \mathbb{R}$ be a generalized n-fractional polynomial P-function, then φ is bounded on $[\upsilon, \omega]$.*

Proof. Let $R = \max\{\varphi(\upsilon), \varphi(\omega)\}$ and $\chi \in [\upsilon, \omega]$ be an arbitrary point. There is a $\lambda \in [0,1]$ such that $\chi = \lambda\upsilon + (1-\lambda)\omega$. Since φ is a generalized n-fractional polynomial P-function on $[\upsilon, \omega]$,one has

$$\varphi(\chi) \leq \left[\frac{\sum_{m=1}^{n} a_m \left[\lambda^{1/m} + (1-\lambda)^{1/m}\right]}{\sum_{m=1}^{n} a_m}\right] [\varphi(\upsilon) + \varphi(\omega)] \leq 2R\left(\lambda^{1/m} + (1-\lambda)^{1/m}\right)$$

for every $\lambda \in [0,1]$. This shows that φ is bounded from above. It is also bounded from below as one can see by writing the arbitrary point $\chi \in [\upsilon, \omega]$ in the form $(\upsilon+\omega)/2 + \lambda$ or $(\upsilon+\omega)/2 - \lambda$, $\lambda \in [0, (\omega-\upsilon)/2]$ .One can accept $\chi = (\upsilon+\omega)/2 + \lambda$ since it will not loss the generality. Then

$$\begin{aligned}
\varphi\left(\frac{\upsilon+\omega}{2}\right) &\leq \varphi\left(\frac{1}{2}\left[\frac{\upsilon+\omega}{2} + \lambda\right] + \frac{1}{2}\left[\frac{\upsilon+\omega}{2} - \lambda\right]\right) \\
&\leq \frac{2\sum_{m=1}^{n} a_m \left(\frac{1}{2}\right)^{1/m}}{\sum_{m=1}^{n} a_m}\left[\varphi(\chi) + \varphi\left(\frac{\upsilon+\omega}{2} - \lambda\right)\right] \\
&\leq 2^{1-1/m}\left[\varphi(\chi) + \varphi\left(\frac{\upsilon+\omega}{2} - \lambda\right)\right]
\end{aligned}$$

or

$$\varphi(\chi) \geq 2^{1/m-1}\varphi\left(\frac{\upsilon+\omega}{2}\right) - \varphi\left(\frac{\upsilon+\omega}{2} - \lambda\right).$$

Using R as the upper bound, one has $-\varphi\left(\frac{\upsilon+\omega}{2} - \lambda\right) \geq -R$. So, one gets

$$\varphi(\chi) \geq 2^{1/m-1}\varphi\left(\frac{\upsilon+\omega}{2}\right) - R = r.$$

This completes the proof. □

Theorem 3.6. *Let $\varphi_\alpha : [\upsilon, \omega] \to \mathbb{R}$ be an arbitrary family of generalized n-fractional polynomial P-functions and let $\varphi(s) = \sup_\alpha \varphi_\alpha(s)$. If $I = \{s \in [\upsilon, \omega] : \varphi(s) < \infty\}$ is nonempty, then I is an interval and φ is a generalized n-fractional polynomial P-function on I.*

Proof. Let $\lambda \in [0,1]$ and $\upsilon, \omega \in I$ be fixed. Then

$$\begin{aligned}
&\varphi(\lambda\upsilon + (1-\lambda)\omega) \\
=& \sup_\alpha \varphi_\alpha(\lambda\upsilon + (1-\lambda)\omega) \\
\leq& \sup_\alpha \left[\frac{\sum_{m=1}^{n} a_m \left[\lambda^{1/m} + (1-\lambda)^{1/m}\right] [\varphi_\alpha(\upsilon) + \varphi_\alpha(\omega)]}{\sum_{m=1}^{n} a_m}\right]
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{\sum_{m=1}^{n} a_m \left[\lambda^{1/m} + (1-\lambda)^{1/m}\right]}{\sum_{m=1}^{n} a_m} \left[\sup_{\alpha} \varphi_\alpha(\upsilon) + \sup_{\alpha} \varphi_\alpha(\omega)\right] \\
&= \frac{\sum_{m=1}^{n} a_m \left[\lambda^{1/m} + (1-\lambda)^{1/m}\right] [\varphi(\upsilon) + \varphi(\omega)]}{\sum_{m=1}^{n} a_m} \\
&< \infty.
\end{aligned}
$$

This shows simultaneously that I is an interval and φ is a generalized n-fractional polynomial P-function.

Thus, the proof is completed. $\square$

4. H–H inequality for generalized n-fractional polynomial P-functions

The purpose of this section is to establish a novel version of H–H type inequalities for generalized n-fractional polynomial P-functions.

Theorem 4.1. *Let $\varphi : [\upsilon, \omega] \to \mathbb{R}$ be a generalized n-fractional polynomial P-function. If $\upsilon < \omega$ and $\varphi \in L[\upsilon, \omega]$, then*

$$
\begin{aligned}
\frac{1}{4}\left(\frac{\sum_{m=1}^{n} a_m}{\sum_{m=1}^{n} a_m \left(\frac{1}{2}\right)^{1/m}}\right) \varphi\left(\frac{\upsilon+\omega}{2}\right) &\le \frac{1}{\omega-\upsilon}\int_{\upsilon}^{\omega} \varphi(\chi)d\chi \\
&\le \left(\frac{\varphi(\upsilon)+\varphi(\omega)}{\sum_{m=1}^{n} a_m}\right) \sum_{m=1}^{n} a_m \left(\frac{2m}{m+1}\right).
\end{aligned} \tag{4.1}
$$

Proof. Since φ is a generalized n-fractional polynomial P-function, one gets

$$
\begin{aligned}
&\varphi\left(\frac{\upsilon+\omega}{2}\right) \\
&= \varphi\left(\frac{1}{2}[\lambda\upsilon + (1-\lambda)\omega] + \frac{1}{2}[(1-\lambda)\upsilon + \lambda\omega]\right) \\
&\le \frac{2}{\sum_{m=1}^{n} a_m} \sum_{m=1}^{n} a_m \left(\frac{1}{2}\right)^{1/m} [\varphi(\lambda\upsilon + (1-\lambda)\omega) + \varphi((1-\lambda)\upsilon + \lambda\omega)].
\end{aligned}
$$

Integrating the last inequality with respect to $\lambda \in [0,1]$ gives that

$$
\varphi\left(\frac{\upsilon+\omega}{2}\right) \le \frac{4}{\omega-\upsilon} \frac{\sum_{m=1}^{n} a_m \left(\frac{1}{2}\right)^{1/m}}{\sum_{m=1}^{n} a_m} \int_{\upsilon}^{\omega} \varphi(\chi)d\chi.
$$

From the property of the generalized n-fractional polynomial P-function φ, if one changes the variable as $\chi = \lambda\upsilon + (1-\lambda)\omega$, then one has

$$
\begin{aligned}
\frac{1}{\omega-\upsilon}\int_{\upsilon}^{\omega} \varphi(\chi)d\chi &= \int_0^1 \varphi(\lambda\upsilon + (1-\lambda)\omega)\, d\lambda \\
&\le \int_0^1 \left[\frac{\sum_{m=1}^{n} a_m \left[\lambda^{1/m} + (1-\lambda)^{1/m}\right] [\varphi(\upsilon)+\varphi(\omega)]}{\sum_{m=1}^{n} a_m}\right] d\lambda
\end{aligned}
$$

$$= \left(\frac{\varphi(\upsilon)+\varphi(\omega)}{\sum_{m=1}^{n} a_m}\right)\sum_{m=1}^{n} a_m \int_0^1 \left[\lambda^{1/m}+(1-\lambda)^{1/m}\right] d\lambda$$
$$= \left(\frac{\varphi(\upsilon)+\varphi(\omega)}{\sum_{m=1}^{n} a_m}\right)\sum_{m=1}^{n} a_m \left(\frac{2m}{m+1}\right),$$

where

$$\int_0^1 \left[\lambda^{1/m}+(1-\lambda)^{1/m}\right] d\lambda = \frac{2m}{m+1}.$$

This completes the proof. □

Remark 4.2. For $n=1$, the inequality (4.1) coincides with the inequality (2.2).

Remark 4.3. For $a_m = 1$ $\left(m=\overline{1,n}\right)$, the inequality (4.1) coincides with the inequality (2.4).

5. H–H type inequalities for generalized n-fractional polynomial P-functions

The aim of this section is to derive new estimates that refine H–H inequality for functions whose first derivative in absolute value, raised to a certain power is generalized n-fractional polynomial P-function.

In order to obtain some new results using our newly defined class of functions, we need the following crucial lemma:

Lemma 5.1 ([7]). *Let $\varphi : \Im \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on $\Im^\circ$, $\upsilon, \omega \in \Im^\circ$ with $\upsilon < \omega$. If $\varphi' \in L[\upsilon,\omega]$.The following identity holds:*

$$\frac{\varphi(\upsilon)+\varphi(\omega)}{2} - \frac{1}{\omega-\upsilon}\int_{\upsilon}^{\omega} \varphi(\chi)d\chi = \frac{\omega-\upsilon}{2}\int_0^1 (1-2\lambda)\varphi'(\lambda\upsilon+(1-\lambda)\omega)\, d\lambda.$$

Theorem 5.2. *Let $\varphi : \Im \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on $\Im^\circ$, $\upsilon, \omega \in \Im^\circ$ with $\upsilon < \omega$ and assume that $\varphi' \in L[\upsilon,\omega]$. If $|\varphi'|$ is a generalized n-fractional polynomial P-function on interval $[\upsilon,\omega]$, then the following inequality holds:*

$$\left|\frac{\varphi(\upsilon)+\varphi(\omega)}{2} - \frac{1}{\omega-\upsilon}\int_{\upsilon}^{\omega} \varphi(\chi)d\chi\right| \tag{5.1}$$
$$\leq \frac{\omega-\upsilon}{\sum_{m=1}^{n} a_m}\sum_{m=1}^{n} a_m \left[\frac{2m(1+2^{-1/m}m)}{2m^2+3m+1}\right] A\left(|\varphi'(\upsilon)|,|\varphi'(\omega)|\right),$$

where A is the arithmetic mean.

Proof. Applying Lemma 5.1 and the inequality

$$|\varphi'(\lambda\upsilon+(1-\lambda)\omega)| \leq \frac{\sum_{m=1}^{n} a_m \left[\lambda^{1/m}+(1-\lambda)^{1/m}\right]\left[|\varphi'(\upsilon)|+|\varphi'(\omega)|\right]}{\sum_{m=1}^{n} a_m},$$

one has

$$\left|\frac{\varphi(\upsilon)+\varphi(\omega)}{2} - \frac{1}{\omega-\upsilon}\int_{\upsilon}^{\omega} \varphi(\chi)d\chi\right|$$
$$\leq \left|\frac{\omega-\upsilon}{2}\int_0^1 (1-2\lambda)\varphi'(\lambda\upsilon+(1-\lambda)\omega)\, d\lambda\right|$$

$$\begin{aligned}
&\leq \frac{\omega-\upsilon}{2}\int_0^1 |1-2\lambda|\left(\frac{\sum_{m=1}^n a_m\left[\lambda^{1/m}+(1-\lambda)^{1/m}\right][|\varphi'(\upsilon)|+|\varphi'(\omega)|]}{\sum_{m=1}^n a_m}\right)d\lambda \\
&\leq \frac{\omega-\upsilon}{2\sum_{m=1}^n a_m}\left[|\varphi'(\upsilon)|+|\varphi'(\omega)|\right]\sum_{m=1}^n a_m\int_0^1 |1-2\lambda|\left[\lambda^{1/m}+(1-\lambda)^{1/m}\right]d\lambda \\
&= \frac{\omega-\upsilon}{2\sum_{m=1}^n a_m}\sum_{m=1}^n a_m\left[\frac{2m(1+2^{-1/m}m)}{2m^2+3m+1}\right]A\left(|\varphi'(\upsilon)|,|\varphi'(\omega)|\right),
\end{aligned}$$

where

$$\int_0^1 |1-2\lambda|\left[\lambda^{1/m}+(1-\lambda)^{1/m}\right]d\lambda=\frac{2m(1+2^{-1/m}m)}{2m^2+3m+1},$$

which completes the proof. □

Corollary 5.3. *If one takes $n=1$ in (5.1), then one has*

$$\left|\frac{\varphi(\upsilon)+\varphi(\omega)}{2}-\frac{1}{\omega-\upsilon}\int_\upsilon^\omega \varphi(\chi)d\chi\right| \leq \frac{\omega-\upsilon}{2}A\left(|\varphi'(\upsilon)|,|\varphi'(\omega)|\right). \tag{5.2}$$

Remark 5.4. For $a_m=1$ $\left(m=\overline{1,n}\right)$, the inequality (5.1) coincides with the inequality in [22, Teorem 4.2].

Theorem 5.5. *Let $\varphi:\Im\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable function on $\Im^\circ$, $\upsilon,\omega\in\Im^\circ$ with $\upsilon<\omega$, $q>1$, $1/p+1/q=1$ and assume that $\varphi'\in L[\upsilon,\omega]$. If $|\varphi'|^q$ is a generalized n-fractional polynomial P-function on interval $[\upsilon,\omega]$, then*

$$\begin{aligned}
&\left|\frac{\varphi(\upsilon)+\varphi(\omega)}{2}-\frac{1}{\omega-\upsilon}\int_\upsilon^\omega \varphi(\chi)d\chi\right| \\
&\leq \frac{\omega-\upsilon}{2}\left(\frac{1}{p+1}\right)^{1/p}\left(\frac{4}{\sum_{m=1}^n a_m}\sum_{m=1}^n a_m\frac{m}{m+1}\right)^{1/q}A^{1/q}\left(|\varphi'(\upsilon)|^q,|\varphi'(\omega)|^q\right),
\end{aligned} \tag{5.3}$$

where A is the arithmetic mean.

Proof. From Lemma 5.1, Hölder's integral inequality and the inequality

$$\left|\varphi'(\lambda\upsilon+(1-\lambda)\omega)\right|^q \leq \frac{\sum_{m=1}^n a_m\left[\lambda^{1/m}+(1-\lambda)^{1/m}\right][|\varphi'(\upsilon)|^q+|\varphi'(\omega)|^q]}{\sum_{m=1}^n a_m},$$

which is the generalized n-fractional polynomial P-function of $|\varphi'|^q$, one gets

$$\begin{aligned}
&\left|\frac{\varphi(\upsilon)+\varphi(\omega)}{2}-\frac{1}{\omega-\upsilon}\int_\upsilon^\omega \varphi(\chi)d\chi\right| \\
&\leq \frac{\omega-\upsilon}{2}\left(\int_0^1 |1-2\lambda|^p d\lambda\right)^{\frac{1}{p}}\left(\int_0^1 \left|\varphi'(\lambda\upsilon+(1-\lambda)\omega)\right|^q d\lambda\right)^{1/q} \\
&\leq \frac{\omega-\upsilon}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{|\varphi'(\upsilon)|^q+|\varphi'(\omega)|^q}{\sum_{m=1}^n a_m}\sum_{m=1}^n a_m\int_0^1\left[\lambda^{1/m}+(1-\lambda)^{1/m}\right]d\lambda\right)^{1/q}
\end{aligned}$$

19 Mar 2024 10:32:42 PDT
240214-Ozcan Version 2 - Submitted to Rocky Mountain J. Math.

$$= \frac{\omega-\upsilon}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{4}{\sum_{m=1}^{n}a_m}\sum_{m=1}^{n}a_m\frac{m}{m+1}\right)^{1/q}A^{1/q}\left(\left|\varphi'(\upsilon)\right|^q,\left|\varphi'(\omega)\right|^q\right),$$

where

$$\int_0^1 |1-2t|^p\,dt=\frac{1}{p+1}.$$

This completes the proof. □

Corollary 5.6. *If one takes $n=1$ in (5.3), then one has*

$$\left|\frac{\varphi(\upsilon)+\varphi(\omega)}{2}-\frac{1}{\omega-\upsilon}\int_\upsilon^\omega\varphi(\chi)d\chi\right|\le(\omega-\upsilon)\left(\frac{1}{2(p+1)}\right)^{1/p}A^{1/q}\left(\left|\varphi'(\upsilon)\right|^q,\left|\varphi'(\omega)\right|^q\right).$$

Remark 5.7. For $a_m=1$ $\left(m=\overline{1,n}\right)$, the inequality (5.3) coincides with the inequality in [22, Teorem 4.3].

Theorem 5.8. *Let $\varphi:\Im\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable function on $\Im^\circ$, $\upsilon,\omega\in\Im^\circ$ with $\upsilon<\omega$, $q\ge1$ and assume that $\varphi'\in L[\upsilon,\omega]$. If $|\varphi'|^q$ is a generalized n-fractional polynomial P-function on the interval $[\upsilon,\omega]$, then*

$$\left|\frac{\varphi(\upsilon)+\varphi(\omega)}{2}-\frac{1}{\omega-\upsilon}\int_\upsilon^\omega\varphi(\chi)d\chi\right| \le \frac{\omega-\upsilon}{2^{2-\frac{2}{q}}}\left(\frac{1}{\sum_{m=1}^{n}a_m}\sum_{m=1}^{n}a_m\frac{2m(1+2^{-1/m}m)}{2m^2+3m+1}\right)^{1/q}A^{1/q}\left(\left|\varphi'(\upsilon)\right|^q,\left|\varphi'(\omega)\right|^q\right), \tag{5.4}$$

where A is the arithmetic mean.

Proof. It follows from Lemma 5.1 and power mean integral inequality together with the property of generalized n-fractional polynomial P-function of $|\varphi'|^q$ that

$$\begin{aligned}
&\left|\frac{\varphi(\upsilon)+\varphi(\omega)}{2}-\frac{1}{\omega-\upsilon}\int_\upsilon^\omega\varphi(\chi)d\chi\right|\\
\le\ & \frac{\omega-\upsilon}{2}\left(\int_0^1|1-2\lambda|d\lambda\right)^{1-\frac{1}{q}}\left(\int_0^1|1-2\lambda|\left|\varphi'(\lambda\upsilon+(1-\lambda)\omega)\right|^q d\lambda\right)^{1/q}\\
\le\ & \frac{\omega-\upsilon}{2^{2-\frac{1}{q}}}\left(\frac{|\varphi'(\upsilon)|^q+|\varphi'(\omega)|^q}{\sum_{m=1}^{n}a_m}\sum_{m=1}^{n}a_m\int_0^1|1-2\lambda|\left[\lambda^{1/m}+(1-\lambda)^{1/m}\right]d\lambda\right)^{1/q}\\
=\ & \frac{\omega-\upsilon}{2^{2-\frac{2}{q}}}\left(\frac{1}{\sum_{m=1}^{n}a_m}\sum_{m=1}^{n}a_m\frac{2m(1+2^{-1/m}m)}{2m^2+3m+1}\right)^{1/q}A^{1/q}\left(\left|\varphi'(\upsilon)\right|^q,\left|\varphi'(\omega)\right|^q\right).
\end{aligned}$$

This completes the proof. □

Corollary 5.9. *Under the assumption of Theorem 5.8 with $q=1$, one gets the conclusion of Theorem 5.2.*

Corollary 5.10. *Under the assumption of Theorem 5.8, if one takes $a_m=1$ $\left(m=\overline{1,n}\right)$ in the inequality (5.4), then one gets the conclusion of* [22, Teorem 4.4].

Corollary 5.11. *Under the assumption of Theorem 5.8, if one takes $n=1$ in the inequality (5.4), then one gets the following inequality:*

$$\left|\frac{\varphi(\upsilon)+\varphi(\omega)}{2}-\frac{1}{\omega-\upsilon}\int_{\upsilon}^{\omega}\varphi(\chi)d\chi\right|\leq\frac{\omega-\upsilon}{2^{2-1/q}}A^{1/q}\left(\left|\varphi'(\upsilon)\right|^{q},\left|\varphi'(\omega)\right|^{q}\right).$$

Corollary 5.12. *Under the assumption of Theorem 5.8, if one takes $n=1$ and $q=1$ in the inequality (5.4),then one gets the following inequality:*

$$\left|\frac{\varphi(\upsilon)+\varphi(\omega)}{2}-\frac{1}{\omega-\upsilon}\int_{\upsilon}^{\omega}\varphi(\chi)d\chi\right|\leq\frac{\omega-\upsilon}{2}A\left(\left|\varphi'(\upsilon)\right|,\left|\varphi'(\omega)\right|\right),$$

which coincides with the inequality 5.2.

6. An Extension of Theorem 5.2

In this section we will denote by $\aleph$ an open and convex set of $\mathbb{R}^n$ $(n\geq 1)$.

We say that a function $\varphi:\aleph\to\mathbb{R}$ is a generalized n-fractional polynomial P-function on $\aleph$ if

$$\varphi(\lambda\upsilon+(1-\lambda)\omega)\leq\frac{\sum_{m=1}^{n}a_m\left[\lambda^{1/m}+(1-\lambda)^{1/m}\right][\varphi(\upsilon)+\varphi(\omega)]}{\sum_{m=1}^{n}a_m}$$

for all $\upsilon,\omega\in\aleph$ and $\lambda\in[0,1]$.

Lemma 6.1. *Let $\varphi:\aleph\to\mathbb{R}$ be a function. Then φ is a generalized n-fractional polynomial P-function on $\aleph$ if and only if for all $\upsilon,\omega\in\aleph$, the function $\Psi:[0,1]\to\mathbb{R}$, $\Psi(\lambda)=\varphi(\lambda\upsilon+(1-\lambda)\omega)$ is a generalized n-fractional polynomial P-function on $[0,1]$.*

Proof. "$\Longleftarrow$" Let $\upsilon,\omega\in\aleph$ be fixed. Assume that $\Psi:[0,1]\to\mathbb{R}$, $\Psi(\lambda)=\varphi(\lambda\upsilon+(1-\lambda)\omega)$ is a generalized n-fractional polynomial P-function on $[0,1]$.

Let $\lambda\in[0,1]$ be an arbitrary fixed. Clearly, $\lambda=(1-\lambda).0+\lambda.1$ and so,

$$\begin{aligned}\varphi(\lambda\upsilon+(1-\lambda)\omega) &= \Psi(\lambda)=\Psi((1-\lambda).0+\lambda.1)\\ &\leq \frac{\sum_{m=1}^{n}a_m\left[\lambda^{1/m}+(1-\lambda)^{1/m}\right][\Psi(0)+\Psi(1)]}{\sum_{m=1}^{n}a_m}\\ &= \frac{\sum_{m=1}^{n}a_m\left[\lambda^{1/m}+(1-\lambda)^{1/m}\right][\varphi(\upsilon)+\varphi(\omega)]}{\sum_{m=1}^{n}a_m}.\end{aligned}$$

This shows that φ is a generalized n-fractional polynomial P-function on $\aleph$.

"$\Longrightarrow$" Assume that φ is a generalized n-fractional polynomial P-function on $\aleph$. Let $\upsilon,\omega\in\aleph$ be fixed and define $\Psi:[0,1]\to\mathbb{R}$, $\Psi(\lambda)=\varphi(\lambda\upsilon+(1-\lambda)\omega)$. Let $\zeta_1,\zeta_2\in[0,1]$ and $\lambda\in[0,1]$.Then

$$\begin{aligned}&\Psi(\lambda\zeta_1+(1-\lambda)\zeta_2)\\ =\;&\varphi((\lambda\zeta_1+(1-\lambda)\zeta_2)\upsilon+(1-\lambda\zeta_1-(1-\lambda)\zeta_2)\omega)\\ =\;&\varphi(\lambda(\zeta_1\upsilon+(1-\zeta_1)\omega)+(1-\lambda)(\zeta_2\upsilon+(1-\zeta_2)\omega))\end{aligned}$$

$$\begin{aligned}
&\leq \frac{\sum_{m=1}^{n} a_m \left[\lambda^{1/m}+(1-\lambda)^{1/m}\right]\left[\varphi(\zeta_1\upsilon+(1-\zeta_1)\omega)+\varphi(\zeta_2\upsilon+(1-\zeta_2)\omega)\right]}{\sum_{m=1}^{n} a_m} \\
&= \frac{\sum_{m=1}^{n} a_m \left[\lambda^{1/m}+(1-\lambda)^{1/m}\right]\left[\Psi(\zeta_1)+\Psi(\zeta_2)\right]}{\sum_{m=1}^{n} a_m}.
\end{aligned}$$

We deduce that Ψ is a generalized n-fractional polynomial P-function on $[0,1]$.

So, the proof is completed. □

Using Lemma 6.1 we will prove an extension of Theorem 5.2 to functions of several variables.

Proposition 1. Assume $\varphi : \aleph \subseteq \mathbb{R}^n \to \mathbb{R}^+$ is a generalized n-fractional polynomial P-function on $\aleph$.Then for any $\upsilon,\omega \in \aleph$ and $\alpha,\beta \in (0,1)$ with $\alpha < \beta$, the following inequality holds:

$$\begin{aligned}
&\left| \frac{1}{2}\int_0^{\alpha} \varphi(\mu\upsilon+(1-\mu)\omega)\,d\mu + \frac{1}{2}\int_0^{\beta} \varphi(\mu\upsilon+(1-\mu)\omega)\,d\mu \right. \\
&\left. -\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\left(\int_0^{\theta} \varphi(\mu\upsilon+(1-\mu)\omega)\,d\mu\right)d\theta \right| \\
\leq\;& \frac{\beta-\alpha}{\sum_{m=1}^{n} a_m}\sum_{m=1}^{n} a_m \frac{2m(1+2^{-1/m}m)}{2m^2+3m+1} A\left(\varphi(\alpha\upsilon+(1-\alpha)\omega),\varphi(\beta\upsilon+(1-\beta)\omega)\right).
\end{aligned}$$

Proof. Let $\upsilon,\omega \in \aleph$ be fixed and $\alpha,\beta \in (0,1)$ with $\alpha < \beta$. Since φ is a generalized n-fractional polynomial P-function, by Lemma 6.1 it follows that the function

$$\Psi : [0,1] \to \mathbb{R}, \Psi(\lambda) = \varphi(\lambda\upsilon+(1-\lambda)\omega)$$

is a generalized n-fractional polynomial P-function on $[0,1]$.

Let define the function

$$\Upsilon : [0,1] \to \mathbb{R},\ \Upsilon(\lambda) = \int_0^{\lambda} \Psi(\mu)\,d\mu = \int_0^{\lambda} \varphi(\mu\upsilon+(1-\mu)\omega)\,d\mu.$$

Obviously, $\Upsilon'(\lambda) = \Psi(\lambda)$ for all $\lambda \in (0,1)$.

Since $\varphi(\aleph) \subseteq \mathbb{R}^+$ it shows that $\Psi \geq 0$ on $[0,1]$ and thus $\Upsilon' \geq 0$ on $(0,1)$.

Applying Theorem 5.2 to the function Υ we obtain

$$\begin{aligned}
&\left| \frac{\Upsilon(\alpha)+\Upsilon(\beta)}{2} - \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\Upsilon(\theta)d\theta \right| \\
\leq\;& \frac{\beta-\alpha}{\sum_{m=1}^{n} a_m}\sum_{m=1}^{n} a_m \frac{2m(1+2^{-1/m}m)}{2m^2+3m+1} A\left(\Upsilon'(\alpha),\Upsilon'(\beta)\right).
\end{aligned}$$

So, the proof is completed. □

7. Applications to the trapezoidal formula

Assume σ is a division of the interval $[\upsilon,\omega]$ such that

$$\sigma : \upsilon = \chi_0 < \chi_1 < \ldots < \chi_{n-1} < \chi_n = \omega.$$

For a function $\varphi:[\upsilon,\omega]\to\mathbb{R}$ we consider the trapezoidal formula

$$\tau(\varphi,\sigma)=\sum_{m=0}^{n-1}\frac{\varphi(\chi_m)+\varphi(\chi_{m+1})}{2}(\chi_{m+1}-\chi_m).$$

It is well known that if φ is twice differentiable on (υ,ω) and $\kappa=\sup_{\chi\in(\upsilon,\omega)}|\varphi''(\chi)|<\infty$ then

$$\int_\upsilon^\omega \varphi(\chi)d\chi=\tau(\varphi,\sigma)+\xi(\varphi,\sigma),$$

where $\xi(\varphi,\sigma)$ is the approximation error of the integral $\int_\upsilon^\omega\varphi(\chi)d\chi$ by the trapezoidal formula and satisfies

$$|\xi(\varphi,\sigma)|\le\frac{\kappa}{12}\sum_{m=0}^{n-1}(\chi_{m+1}-\chi_m)^3.$$

Proposition 2. Assume $\upsilon,\omega\in\mathbb{R}$ with $\upsilon<\omega$ and $\varphi:[\upsilon,\omega]\to\mathbb{R}$ is a differentiable mapping on (υ,ω). If $|\varphi'|$ is a generalized n-fractional polynomial P-function on $[\upsilon,\omega]$, then for each division σ of the interval $[\upsilon,\omega]$, one has

$$\begin{aligned}|\xi(\varphi,\sigma)|\le{}&\frac{2}{\sum_{m=1}^n a_m}\left(\sum_{m=1}^n a_m\frac{2m(1+2^{-1/m}m)}{2m^2+3m+1}\right)\left(\frac{\sum_{m=1}^n a_m\left(\frac{1}{2}\right)^{1/m}}{\sum_{m=1}^n a_m}\right)\\ &\times A\left(|\varphi'(\upsilon)|,|\varphi'(\omega)|\right)\sum_{m=0}^{n-1}(\chi_{m+1}-\chi_m)^2.\end{aligned}\tag{7.1}$$

Proof. By applying Theorem 5.2 on the sub-intervals $[\chi_m,\chi_{m+1}]$, $m=0,1,2,...,n-1$ given by the division σ and then adding from $m=0$ to $m=n-1$ one has

$$\begin{aligned}\left|\tau(\varphi,\sigma)-\int_\upsilon^\omega\varphi(\chi)d\chi\right|\le{}&\frac{1}{\sum_{m=1}^n a_m}\left(\sum_{m=1}^n a_m\frac{2m(1+2^{-1/m}m)}{2m^2+3m+1}\right)\\ &\times\sum_{m=0}^{n-1}(\chi_{m+1}-\chi_m)^2 A\left(|\varphi'(\chi_m)|,|\varphi'(\chi_{m+1})|\right).\end{aligned}\tag{7.2}$$

On the other hand, for each $\chi_m\in[\upsilon,\omega]$ there exists $\lambda_m\in[0,1]$ such that $\chi_m=\lambda_m\upsilon+(1-\lambda_m)\omega$. Since $|\varphi'|$ is a generalized n-fractional polynomial P-function and $\frac{\sum_{m=1}^n a_m\left[\lambda^{1/m}+(1-\lambda)^{1/m}\right]}{\sum_{m=1}^n a_m}\le\frac{2}{\sum_{m=1}^n a_m}\sum_{m=1}^n a_m\left(\frac{1}{2}\right)^{1/m}$ for all $\lambda\in[0,1]$, one gets

$$\begin{aligned}|\varphi'(\chi_m)|&\le\frac{\sum_{m=1}^n a_m\left[\lambda^{1/m}+(1-\lambda)^{1/m}\right][\varphi(\upsilon)+\varphi(\omega)]}{\sum_{m=1}^n a_m}\\ &\le\frac{4}{\sum_{m=1}^n a_m}\left(\sum_{m=1}^n a_m\left(\frac{1}{2}\right)^{1/m}\right)A\left(|\varphi'(\upsilon)|,|\varphi'(\omega)|\right)\end{aligned}\tag{7.3}$$

for each $m=0,1,2,...,n-1$. Relations (7.2) and (7.3) imply that the relation (7.1) holds true. Thus, Proposition 2 is proved. □

A comparable approach as that used in the proof of Proposition 6.1, utilizing Theorem 4.3 and Theorem 4.4 as foundations, demonstrates the validity of the following results.

Proposition 3. Assume $\upsilon,\omega\in\mathbb{R}$ with $\upsilon<\omega$ and $\varphi:[\upsilon,\omega]\to\mathbb{R}$ is a differentiable mapping on (υ,ω). If $|\varphi'|^q$, $q>1$, is a generalized n-fractional polynomial P-function on $[\upsilon,\omega]$, then for each division σ of the interval $[\upsilon,\omega]$, one has

$$\begin{aligned}|\xi(\varphi,\sigma)| &\le \frac{\sum_{m=1}^{n}a_m\left(\frac{1}{2}\right)^{1/m}}{\sum_{m=1}^{n}a_m}\left(\frac{1}{p+1}\right)^{1/p}\left(\frac{1}{\sum_{m=1}^{n}a_m}\sum_{m=1}^{n}a_m\frac{4m}{m+1}\right)^{1/q}\\ &\quad\times A^{1/q}\left(\left|\varphi'(\upsilon)\right|^q,\left|\varphi'(\omega)\right|^q\right)\sum_{m=0}^{n-1}(\chi_{m+1}-\chi_m)^2\end{aligned}$$

where $1/p+1/q=1$.

Proposition 4. Assume $\upsilon,\omega\in\mathbb{R}$ with $\upsilon<\omega$ and $\varphi:[\upsilon,\omega]\to\mathbb{R}$ is a differentiable mapping on (υ,ω). If $|\varphi'|^q$, $q\ge1$, is a generalized n-fractional polynomial P-function on $[\upsilon,\omega]$, then for each division σ of the interval $[\upsilon,\omega]$, one has

$$\begin{aligned}|\xi(\varphi,\sigma)| &\le \frac{\sum_{m=1}^{n}a_m\left(\frac{1}{2}\right)^{1/m}}{2^{1-\frac{2}{q}}\sum_{m=1}^{n}a_m}\left(\sum_{m=1}^{n}a_m\frac{2m(1+2^{-1/m}m)}{2m^2+3m+1}\right)^{1/q}\\ &\quad\times A^{1/q}\left(\left|\varphi'(\upsilon)\right|^q,\left|\varphi'(\omega)\right|^q\right)\sum_{m=0}^{n-1}(\chi_{m+1}-\chi_m)^2.\end{aligned}$$

8. Applications to special means

Consider the following special means of two non-negative numbers $\upsilon,\omega\in\mathbb{R}$ with $\omega>\upsilon$:

1. The arithmetic mean

$$A:=A(\upsilon,\omega)=\frac{\upsilon+\omega}{2},$$

2. The geometric mean

$$G:=G(\upsilon,\omega)=\sqrt{\upsilon\omega},\quad \upsilon,\omega\ge0$$

3. The harmonic mean

$$H:=H(\upsilon,\omega)=\frac{2\upsilon\omega}{\upsilon+\omega},\quad \upsilon,\omega>0,$$

4. The logarithmic mean

$$L:=L(\upsilon,\omega)=\begin{cases}\frac{\omega-\upsilon}{\ln\omega-\ln\upsilon}, & \upsilon\neq\omega\\ \upsilon, & \upsilon=\omega\end{cases}\;;\;\upsilon,\omega>0$$

5. The p-logaritmic mean

$$L_p:=L_p(u,v)=\begin{cases}\left(\frac{\omega^{p+1}-\upsilon^{p+1}}{(p+1)(\omega-\upsilon)}\right)^{\frac{1}{p}}, & \upsilon\neq\omega, p\in\mathbb{R}\backslash\{-1,0\}\\ \upsilon, & \upsilon=\omega\end{cases}\;;\;\upsilon,\omega>0.$$

19 Mar 2024 10:32:42 PDT
240214-Ozcan Version 2 - Submitted to Rocky Mountain J. Math.

6.The identric mean

$$I := I(\upsilon,\omega) = \frac{1}{e}\left(\frac{\omega^{\omega}}{\upsilon^{\upsilon}}\right)^{\frac{1}{v-u}}, \quad \upsilon,\omega > 0,$$

Proposition 5. Let $\upsilon,\omega \in [0,\infty)$ with $\upsilon < \omega$ and $r \in (-\infty,0)\cup[1,\infty)\setminus\{-1\}$. Then

$$\frac{\sum_{m=1}^{n} a_m}{2\sum_{m=1}^{n} a_m \left(\frac{1}{2}\right)^{1/m}} A^r(\upsilon,\omega) \le L_r^r(\upsilon,\omega) \le A(\upsilon^r,\omega^r)\frac{2}{\sum_{m=1}^{n} a_m}\sum_{m=1}^{n} a_m \frac{m}{m+1}.$$

Proof. The assertion follows from (4.1) for the function

$$\varphi(\chi) = \chi^r, \ \ \chi \in [0,\infty).$$

□

Proposition 6. Let $\upsilon,\omega \in (0,\infty)$ with $\upsilon < \omega$. Then

$$\frac{\sum_{m=1}^{n} a_m}{2\sum_{m=1}^{n} a_m \left(\frac{1}{2}\right)^{1/m}} A^{-1}(\upsilon,\omega) \le L^{-1}(\upsilon,\omega) \le \frac{2}{\sum_{m=1}^{n} a_m} H^{-1}(\upsilon,\omega)\sum_{m=1}^{n} a_m \frac{m}{m+1}.$$

Proof. The assertion follows from (4.1) for the function

$$\varphi(\chi) = \chi^{-1}, \ \ \chi \in (0,\infty).$$

□

Proposition 7. Let $\upsilon,\omega \in (0,1]$ with $\upsilon < \omega$. Then

$$\frac{2\ln G(\upsilon,\omega)}{\sum_{m=1}^{n} a_m}\sum_{m=1}^{n} a_m \frac{m}{m+1} \le \ln I \le \frac{\sum_{m=1}^{n} a_m}{2\sum_{m=1}^{n} a_m \left(\frac{1}{2}\right)^{1/m}} \ln A(\upsilon,\omega).$$

Proof. The assertion follows from (4.1) for the function

$$\varphi(\chi) = -\ln\chi, \ \ \chi \in (0,1].$$

□

9. Conclusions

In this article, the class of generalized n-fractional polynomial P-functions is introduced and related properties are given. Hermite-Hadamard inequality for the newly defined class of functions are establihed. Additionally, new refinements of the Hermite–Hadamard inequality for functions whose first derivatives in absolute value at certain power are generalized n-fractional polynomial P-function. New error estimates for the trapezoidal formula are also given. Finally, some applications to special means are given. The results obtained in this work are generalizations of previous results in the literature.

References

[1] J. Adams, Convex Functions in Economic Modeling. *J. Economic Literature*, **41** (2003), 1063–1078.
[2] A. O. Akdemir, S. I. Butt, M. Nadeem and M. A. Ragusa, Some new integral inequalities for a general variant of polynomial convex functions. *AIMS Math.* **7** (2022), 20461—20489.
[3] M. A. Ali, H. Budak and I. B. Sial, Generalized fractional integral inequalities for product of two convex functions. *Italian J. Pure Appl. Math.* **45** (2021), 689698.
[4] D. P. Bertsekas, *Nonlinear Programming*, Athena Scientific, 1999.
[5] S. Boyd and L. Vandenberghe, *Convex Optimization*, Cambridge University Press, 2004.
[6] S. I. Butt and J. Pečarić, Generalized Hermite-Hadamard's inequality. *Proc. A. Razmadze Math. Inst.* **163** (2013), 9–27.
[7] S. S. Dragomir and R. P. Agarwal, Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula. *Appl. Math. Lett.* **11** (1998), 91–95.
[8] S. S. Dragomir and C. E. M. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, Victoria University, Australia, 2000.
[9] S. S. Dragomir, J. Pecaric and L. E. Persson, Some inequalities of Hadamard Type. *Soochow J. Math.* **21** (2001), 335–341.
[10] K. Guan, Multiplicative convexity and its applications. *J. Math. Anal. Appl.* **362** (2010), 156–166.
[11] J. Hadamard, Étude sur les propriétés des fonctions entières en particulier d'une fonction considérée par Riemann. *J. Math. Pures Appl.* **58** (1893), 171–215.
[12] C. Hermite, Sur deux limites d'une integrale de finie. Mathesis, **3** (1883), 82.
[13] İ. İşcan, Construction of a new class of functions with their some properties and certain inequalities: n-fractional polynomial convex functions. *Miskolc Math. Notes*, **24** (2023), 1389-–1404.
[14] İ. İşcan and M. Kadakal, On n-Polynomial P-function and related inequalities. *Int. J. Math. Combin.*, **3** (2020), 16–25.
[15] İ. İşcan, S. Turhan and S. Maden, Hermite-Hadamard and Simpson-like type inequalities for differentiable p-quasi-convex functions. *Filomat*, **31** (2017), 5945–5953.
[16] H. Kadakal, Harmonic trigonometrically convexity. Filomat, **37** (2023), 8029–8038.
[17] H. Kadakal and M. Kadakal, Multiplicatively preinvex P-Functions. J. Sci. Arts, **23** (2023), 587–602.
[18] M. Kadakal, İ. İşcan and H. Kadakal, Construction of a new generalization for n-polynomial convexity with their certain inequalities. *Hacettepe J. Math. Stat.* (2023), DOI: 10.15672/hujms.1310861
[19] S. Maden, H. Kadakal, M. Kadakal and İ. İşcan, Some new integral inequalities for n-times differentiable convex and concave functions, *J. Nonlinear Sci. Appl.* **10** (2017), 6141–6148.
[20] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information,* Cambridge University Press, 2010.
[21] M. Nikolova, Pseudo-Convex Optimization Techniques in Image Processing. *IEEE Transactions on Image Processing*, **13** (2004), 13–20.
[22] S. Numan, On n-fractional polynomial P-functions. Filomat, **37** (2023), 7029—7039.
[23] S. Özcan, Hermite-Hadamard type inequalities for m-convex and (α,m)-convex functions. *J. Inequal. Appl.* **2020**:175 (2020), 1–10.
[24] S. Özcan, On refinements of some integral inequalities for differentiable prequasiinvex functions. *Filomat*, **33** (2019), 4377–4385.
[25] S. Özcan, S.: Hermite-Hadamard type inequalities for multiplicatively h-convex functions. *Konuralp J. Math.*, **8** (2020), 158–164.
[26] S. Özcan, Hermite–Hadamard type inequalities for multiplicatively P-functions. *Gümüşhane Üniv. Fen Bilim. Enst. Derg.* **10** (2020), 486-–491.
[27] S. Varošanec, On h-convexity. *J. Math. Anal. Appl.* **326** (2007), 303–311.
[28] M. M. Wilde, Quantum convexity theory: foundations and applications. *Quantum Information Processing*, **16** (2017), 72.
[29] J. Xie, M. A. Ali and T. Sitthiwirattham, Some new midpoint and trapezoidal type inequalities in multiplicative calculus with applications. *Filomat*, **37** (2023), 6665–6675.

[30] L. Zhang, Y. Peng and T. Du, On multiplicative Hermite–Hadamard- and Newton-type inequalities for multiplicatively (P,m)-convex functions. *J. Math. Anal. Appl.* **534** (2024), 128117

[31] D. Zhao, G. Gulshan, M. A. Ali and K. Nonlaopon, Some new midpoint and trapezoidal-type inequalities for general convex functions in q-calculus. *Mathematics*, **10** (2022), 444.

DEPARTMENT OF MATHEMATICS, KIRKLARELI UNIVERSITY, KIRKLARELI, TURKEY.
Email address: serapozcan87@gmail.com