Some supercongruences for q-trinomial coefficients

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Abstract. We study supercongruences for the q-trinomial coefficients $\tau_0(n,m,q)$, $T_0(n,m,q)$ and $T_1(n,m,q)$, which were first introduced by Andrews and Baxter. In particular, we completely determine $\tau_0(an,bn,q)$, $T_0(an,bn,q)$ and $T_1(an,bn,q)$ modulo the square of the cyclotomic polynomial $\Phi_n(q)$ for $(a,b)=(m,m-1)$.

Keywords: q-trinomial coefficients; q-congruences; cyclotomic polynomials

MR Subject Classifications: 11A07, 11B65, 13A05, 05A10

1 Introduction

In 1987, Andrews and Baxter [2] introduced six kinds of q-trinomial coefficients in the study of the solution of a model in statistical mechanics, which can be listed as follows:

$$\left(\binom{n}{m}\right)_q=\sum_{k=0}^{n}q^{k(k+m)}\begin{bmatrix}n\\k\end{bmatrix}\begin{bmatrix}n-k\\k+m\end{bmatrix},$$

$$\tau_0(n,m,q)=\sum_{k=0}^{n}(-1)^k q^{nk-\binom{k}{2}}\begin{bmatrix}n\\k\end{bmatrix}\begin{bmatrix}2n-2k\\n-m-k\end{bmatrix},$$

$$T_0(n,m,q)=\sum_{k=0}^{n}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}_{q^2}\begin{bmatrix}2n-2k\\n-m-k\end{bmatrix},$$

$$T_1(n,m,q)=\sum_{k=0}^{n}(-q)^k\begin{bmatrix}n\\k\end{bmatrix}_{q^2}\begin{bmatrix}2n-2k\\n-m-k\end{bmatrix},$$

$$t_0(n,m,q)=\sum_{k=0}^{n}(-1)^k q^{k^2}\begin{bmatrix}n\\k\end{bmatrix}_{q^2}\begin{bmatrix}2n-2k\\n-m-k\end{bmatrix},$$

$$t_1(n,m,q)=\sum_{k=0}^{n}(-1)^k q^{k(k-1)}\begin{bmatrix}n\\k\end{bmatrix}_{q^2}\begin{bmatrix}2n-2k\\n-m-k\end{bmatrix}.$$

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Here and in what follows, the q-binomial coefficients are defined as

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases} \dfrac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-k+1})}{(1-q)(1-q^2)\cdots(1-q^k)}, & \text{if } 0 \leqslant k \leqslant n, \\ 0, & \text{otherwise.} \end{cases}$$

Note that these q-trinomial coefficients are six apparently distinct q-analogues of the trinomial coefficients $\left(\binom{n}{m}\right)$, which are given by

$$(1+x+x^2)^n = \sum_{m=-n}^{n} \left(\binom{n}{m}\right) x^{m+n}.$$

It is well-known that the trinomial coefficients possess the following two simple formulas (see [15, page 43]):

$$\left(\binom{n}{m}\right) = \sum_{k=0}^{n} \binom{n}{k}\binom{n-k}{k+m},$$

and

$$\left(\binom{n}{m}\right) = \sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{2n-2k}{n-m-k}.$$

In the past two decades, q-analogues of congruences (q-congruences) were widely studied by many researchers. For recent developments on q-congruences, we refer the interested reader to [6–8, 11–14, 18–20].

It is remarkable that Andrews [1] showed that for any odd prime p,

$$\begin{bmatrix} 2p-1 \\ p-1 \end{bmatrix} \equiv q^{\frac{p(p-1)}{2}} \pmod{[p]_q^2}, \tag{1.1}$$

which gives a q-analogue of Babbage's congruence [3]. In order to understand (1.1), we recall some necessary notations. For polynomials $A_1(q), A_2(q), P(q) \in \mathbb{Z}[q]$, the q-congruence

$$A_1(q)/A_2(q) \equiv 0 \pmod{P(q)}$$

is understood as $A_1(q)$ is divisible by $P(q)$ and $A_2(q)$ is coprime with $P(q)$. In general, for rational functions $A(q), B(q) \in \mathbb{Z}(q)$,

$$A(q) \equiv B(q) \pmod{P(q)} \Longleftrightarrow A(q) - B(q) \equiv 0 \pmod{P(q)}.$$

The q-integers are defined as $[n]_q = (1-q^n)/(1-q)$ for $n \geq 1$, and the nth cyclotomic polynomial is given by

$$\Phi_n(q) = \prod_{\substack{1 \leq k \leq n \\ (n,k)=1}} (q - \zeta^k),$$

where ζ denotes an nth primitive root of unity.

It is worth mentioning that Straub [16, Theorem 2.2] extended (1.1) as follows (notice that ${2n-1 \brack n-1} = {2n \brack n}/(1+q^n)$):

$$ {an \brack bn} \equiv {a \brack b}_{q^{n^2}} - (a-b)b\binom{a}{b}\frac{n^2-1}{24}(q^n-1)^2 \pmod{\Phi_n(q)^3}, \tag{1.2} $$

which was further generalized by Zudilin [21].

The first author [10] investigated congruence properties for the q-trinomial coefficients $\left(\binom{an}{bn}\right)_q$ for $(a,b) \in \{(1,0),(2,1)\}$ and showed that for any positive integer n,

$$ \left(\binom{n}{0}\right)_q \equiv \mathcal{A}_n(q) \pmod{\Phi_n(q)^2}, \tag{1.3} $$

and

$$ \left(\binom{2n}{n}\right)_q \equiv 2\mathcal{A}_n(q) - n(1-q^n) \pmod{\Phi_n(q)^2}, \tag{1.4} $$

where $\mathcal{A}_n(q)$ is given by

$$ \mathcal{A}_n(q) = \begin{cases} (-1)^m(1+q^m)q^{m(3m-1)/2}, & \text{if } n = 3m, \\ (-1)^m q^{m(3m+1)/2}, & \text{if } n = 3m+1, \\ (-1)^{m+1} q^{(m+1)(3m+2)/2}, & \text{if } n = 3m+2. \end{cases} $$

It is remarkable that Chen, Xu and Wang [4] completely determined $\left(\binom{mn}{(m-1)n}\right)_q$ modulo $\Phi_n(q)^2$, which includes (1.3) and (1.4) as special cases.

In this paper, we aim to completely determine $\tau_0(an,bn,q)$, $T_0(an,bn,q)$ and $T_1(an,bn,q)$ modulo $\Phi_n(q)^2$ for $(a,b) = (m,m-1)$. The main results consist of the following three theorems.

Theorem 1.1 *If m and n are both positive integers, then the following holds modulo $\Phi_n(q)^2$:*

$$ \tau_0(mn,(m-1)n,q) \equiv 2m - nm(2m-1)(1-q^n) + (-1)^n q^{n(n+1)/2}\left(m(\mathcal{A}_n(q)+\mathcal{B}_n(q)-1) - \frac{3nm(m-1)}{2}(1-q^n)\right), $$

where $\mathcal{B}_n(q)$ is given by

$$ \mathcal{B}_n(q) = \begin{cases} (-1)^m(1+q^{2m})q^{m(3m-5)/2}, & \text{if } n = 3m, \\ (-1)^m q^{m(3m+1)/2}, & \text{if } n = 3m+1, \\ (-1)^{m+1} q^{(m-1)(3m+2)/2}, & \text{if } n = 3m+2. \end{cases} $$

Theorem 1.2 *If m and n are both positive integers, then the following holds modulo $\Phi_n(q)^2$:*

$$T_0(mn,(m-1)n,q) \equiv 2m - nm(2m-1)(1-q^n)$$
$$+(-1)^n\left((1+(-1)^n)(m-1)+1\right)\left(m - nm(m-1)(1-q^n) + 2m(\mathcal{A}_n(q)-1)\right).$$

Theorem 1.3 *If m and n are both positive integers, then the following holds modulo $\Phi_n(q)^2$:*

$$T_1(mn,(m-1)n,q) \equiv 2m - nm(2m-1)(1-q^n)$$
$$+(-1)^n\left((1+(-1)^n)(m-1)+1\right)\left(mq^n - nm(m-1)(1-q^n) + 2m(\mathcal{B}_n(q)-1)\right).$$

The rest of the paper is organized as follows. In Section 2, we first establish some preliminary results. The proofs of Theorems 1.1–1.3 will be given in Sections 3–5, respectively.

2 Preliminary results

In order to prove Theorems 1.1–1.3, we first require two q-binomial identities.

Lemma 2.1 *(See [9, Lemma 2.3].) For any non-negative integer n, we have*

$$(1-q^n)\sum_{k=0}^{\lfloor n/2\rfloor}\frac{(-1)^k q^{k(k-1)/2}}{1-q^{n-k}}\begin{bmatrix} n-k \\ k\end{bmatrix}$$
$$=\begin{cases}(-1)^m(1+q^m)q^{m(3m-1)/2}, & \text{if } n=3m,\\ (-1)^m q^{m(3m+1)/2}, & \text{if } n=3m+1,\\ (-1)^{m+1}q^{(m+1)(3m+2)/2}, & \text{if } n=3m+2.\end{cases} \tag{2.1}$$

Lemma 2.2 *For any non-negative integer n, we have*

$$(1-q^n)\sum_{k=0}^{\lfloor n/2\rfloor}\frac{(-1)^k q^{k(k-3)/2}}{1-q^{n-k}}\begin{bmatrix} n-k \\ k\end{bmatrix}$$
$$=\begin{cases}(-1)^m(1+q^{2m})q^{m(3m-5)/2}, & \text{if } n=3m,\\ (-1)^m q^{m(3m+1)/2}, & \text{if } n=3m+1,\\ (-1)^{m+1}q^{(m-1)(3m+2)/2}, & \text{if } n=3m+2.\end{cases} \tag{2.2}$$

We remark that Chu [5, Theorem 5] recently gave a common generalization of (2.1) and (2.2) through generating function technique. Note that $\mathcal{A}_n(q)$ and $\mathcal{B}_n(q)$ coincide with the right-hand sides of (2.1) and (2.2), respectively.

We also need the following congruence regarding q-binomial coefficients.

Lemma 2.3 *For positive integers m and n, we have*

$$\begin{bmatrix} mn \\ n \end{bmatrix}_{q^2} \equiv ((1+(-1)^n)(m-1)+1)\,(m-nm(m-1)(1-q^n)) \pmod{\Phi_n(q)^2}. \tag{2.3}$$

Proof. By [4, (2.7)], we have

$$\begin{bmatrix} mn \\ n \end{bmatrix} \equiv m - \frac{nm(m-1)}{2}(1-q^n) \pmod{\Phi_n(q)^2}. \tag{2.4}$$

It is clear that

$$q^n \equiv 1 \pmod{\Phi_n(q)}. \tag{2.5}$$

If n is odd, then $\Phi_n(q)|\Phi_n(q^2)$. It follows from (2.4) and (2.5) that

$$\begin{aligned} \begin{bmatrix} mn \\ n \end{bmatrix}_{q^2} &\equiv m - \frac{nm(m-1)}{2}(1+q^n)(1-q^n) \\ &\equiv m - nm(m-1)(1-q^n) \pmod{\Phi_n(q)^2}, \end{aligned}$$

which proves the case $n \equiv 1 \pmod 2$ of (2.3).

From (1.2) and the fact $\Phi_{2n}(q)|\Phi_n(q^2)$, we deduce that

$$\begin{bmatrix} 2mn \\ 2n \end{bmatrix}_{q^2} \equiv \begin{bmatrix} 2m \\ 2 \end{bmatrix}_{q^{2n^2}} \pmod{\Phi_{2n}(q)^2}. \tag{2.6}$$

Note that for any positive integer s,

$$\begin{aligned} q^{2sn^2} &= 1 - \left(1-(q^{2n})^{sn}\right) \\ &= 1-(1-q^{2n})(1+q^{2n}+q^{4n}+\cdots+q^{2n(sn-1)}) \\ &\equiv 1 - sn(1-q^{2n}) \pmod{\Phi_{2n}(q)^2}. \end{aligned}$$

Thus,

$$\begin{aligned} \begin{bmatrix} 2m \\ 2 \end{bmatrix}_{q^{2n^2}} &= \sum_{i=0}^{2m-2} q^{2in^2} \sum_{j=0}^{m-1} q^{4jn^2} \\ &\equiv \left(2m-1-n(2m-1)(m-1)(1-q^{2n})\right)\left(m-nm(m-1)(1-q^{2n})\right) \\ &\equiv (2m-1)\left(m-2nm(m-1)(1-q^{2n})\right) \pmod{\Phi_{2n}(q)^2}. \end{aligned} \tag{2.7}$$

It follows from (2.6) and (2.7) that for even positive integer n,

$$\begin{bmatrix} mn \\ n \end{bmatrix}_{q^2} \equiv (2m-1)\left(m - nm(m-1)(1-q^n)\right) \pmod{\Phi_n(q)^2},$$

which is the case $n \equiv 0 \pmod 2$ of (2.3). □

3 Proof of Theorem 1.1

Note that

$$\begin{aligned}
&\tau_0(mn,(m-1)n,q) \\
&= \sum_{k=0}^{n}(-1)^k q^{mnk-\binom{k}{2}} \begin{bmatrix} mn \\ k \end{bmatrix}\begin{bmatrix} 2mn-2k \\ n-k \end{bmatrix} \\
&= (-1)^n \sum_{k=0}^{n}(-1)^k q^{(n-k)((2m-1)n+k+1)/2} \begin{bmatrix} mn \\ n-k \end{bmatrix}\begin{bmatrix} 2n(m-1)+2k \\ k \end{bmatrix} \\
&= (-1)^n q^{n((2m-1)n+1)/2} \begin{bmatrix} mn \\ n \end{bmatrix} + \begin{bmatrix} 2mn \\ n \end{bmatrix} \\
&+ (-1)^n \sum_{k=1}^{n-1}(-1)^k q^{(n-k)((2m-1)n+k+1)/2} \begin{bmatrix} mn \\ n-k \end{bmatrix}\begin{bmatrix} 2n(m-1)+2k \\ k \end{bmatrix},
\end{aligned} \tag{3.1}$$

where we have performed the variable substitution $k \to n-k$ in the second step.

For $1 \le k \le n-1$, by (2.5) we have

$$\begin{aligned}
\begin{bmatrix} mn \\ n-k \end{bmatrix} &= \begin{bmatrix} mn \\ n \end{bmatrix} \frac{(1-q^{n-k+1})(1-q^{n-k+2})\dots(1-q^n)}{(1-q^{(m-1)n+1})(1-q^{(m-1)n+2})\cdots(1-q^{(m-1)n+k})} \\
&\equiv \begin{bmatrix} mn \\ n \end{bmatrix} \frac{(-1)^{k-1}q^{-k(k-1)/2}(1-q^n)}{1-q^k} \\
&\equiv \begin{bmatrix} mn \\ n \end{bmatrix} \frac{(-1)^{k}q^{-k(k+1)/2}(1-q^n)}{1-q^{n-k}} \pmod{\Phi_n(q)^2}.
\end{aligned} \tag{3.2}$$

Furthermore, by [17, Lemma 3.3] we have

$$\begin{bmatrix} 2k-1 \\ k \end{bmatrix} \equiv (-1)^k q^{k(3k-1)/2} \begin{bmatrix} n-k \\ k \end{bmatrix} \pmod{\Phi_n(q)}, \tag{3.3}$$

for $1 \le k \le n-1$. It follows from (2.5) and (3.3) that for $1 \le k \le n-1$,

$$\begin{aligned}
\begin{bmatrix} 2n(m-1)+2k \\ k \end{bmatrix} &= \frac{(1-q^{2n(m-1)+k+1})(1-q^{2n(m-1)+k+2})\dots(1-q^{2n(m-1)+2k})}{(1-q)(1-q^2)\cdots(1-q^k)} \\
&\equiv (1+q^k)\begin{bmatrix} 2k-1 \\ k \end{bmatrix} \\
&\equiv (-1)^k q^{k(3k-1)/2}(1+q^k)\begin{bmatrix} n-k \\ k \end{bmatrix} \pmod{\Phi_n(q)}. \qquad (3.4)
\end{aligned}$$

Combining (3.2) and (3.4) with the fact that

$$\frac{(n-k)((2m-1)n+k+1)}{2} = -\frac{k(k+1)}{2} - n(m-1)(k-n) + \frac{n(n+1)}{2},$$

we arrive at

$$\begin{aligned}
&\sum_{k=1}^{n-1}(-1)^k q^{(n-k)((2m-1)n+k+1)/2}\begin{bmatrix} mn \\ n-k \end{bmatrix}\begin{bmatrix} 2n(m-1)+2k \\ k \end{bmatrix} \\
&\equiv q^{n(n+1)/2}(1-q^n)\begin{bmatrix} mn \\ n \end{bmatrix}\sum_{k=1}^{n-1}\frac{(-1)^k q^{k(k-3)/2}(1+q^k)}{1-q^{n-k}}\begin{bmatrix} n-k \\ k \end{bmatrix} \\
&= q^{n(n+1)/2}\begin{bmatrix} mn \\ n \end{bmatrix}(\mathcal{A}_n(q)+\mathcal{B}_n(q)-2) \pmod{\Phi_n(q)^2}, \qquad (3.5)
\end{aligned}$$

where we have used (2.1) and (2.2) in the last step.

Noting that

$$q^{n((2m-1)n+1)/2} = q^{n(n+1)/2+(m-1)n^2},$$

and

$$\begin{aligned}
q^{(m-1)n^2} &= 1-(1-q^{(m-1)n})(1+q^{(m-1)n}+q^{2(m-1)n}+\cdots+q^{(m-1)n(n-1)}) \\
&\equiv 1-n(1-q^{(m-1)n}) \\
&= 1-n(1-q^n)(1+q^n+q^{2n}+\cdots+q^{(m-2)n}) \\
&\equiv 1-n(m-1)(1-q^n) \pmod{\Phi_n(q)^2},
\end{aligned}$$

we obtain

$$q^{n((2m-1)n+1)/2} \equiv q^{n(n+1)/2}(1-n(m-1)(1-q^n)) \pmod{\Phi_n(q)^2}. \qquad (3.6)$$

From (2.1) and (2.2), we deduce that

$$\mathcal{A}_n(q) - 1 \equiv 0 \pmod{\Phi_n(q)}, \tag{3.7}$$

and

$$\mathcal{B}_n(q) - 1 \equiv 0 \pmod{\Phi_n(q)}, \tag{3.8}$$

and so

$$\mathcal{A}_n(q) + \mathcal{B}_n(q) - 2 \equiv 0 \pmod{\Phi_n(q)}. \tag{3.9}$$

Finally, substituting (2.4), (3.5) and (3.6) into the right-hand side of (3.1) and using (3.9), we complete the proof of Theorem 1.1.

4 Proof of Theorem 1.2

Note that

$$\begin{aligned}
&T_0(mn,(m-1)n,q)\\
&=\sum_{k=0}^{n}(-1)^k\begin{bmatrix}mn\\k\end{bmatrix}_{q^2}\begin{bmatrix}2mn-2k\\n-k\end{bmatrix}\\
&=(-1)^n\sum_{k=0}^{n}(-1)^k\begin{bmatrix}mn\\n-k\end{bmatrix}_{q^2}\begin{bmatrix}2n(m-1)+2k\\k\end{bmatrix}\\
&=(-1)^n\begin{bmatrix}mn\\n\end{bmatrix}_{q^2}+\begin{bmatrix}2mn\\n\end{bmatrix}+(-1)^n\sum_{k=1}^{n-1}(-1)^k\begin{bmatrix}mn\\n-k\end{bmatrix}_{q^2}\begin{bmatrix}2n(m-1)+2k\\k\end{bmatrix}.
\end{aligned} \tag{4.1}$$

For $1 \le k \le n-1$, we have

$$\begin{aligned}
\begin{bmatrix}mn\\n-k\end{bmatrix}_{q^2}&=\begin{bmatrix}mn\\n\end{bmatrix}_{q^2}\frac{(1-q^{2(n-k+1)})(1-q^{2(n-k+2)})\dots(1-q^{2n})}{(1-q^{2(m-1)n+2})(1-q^{2(m-1)n+4})\cdots(1-q^{2(m-1)n+2k})}\\
&\equiv\begin{bmatrix}mn\\n\end{bmatrix}_{q^2}\frac{(-1)^{k-1}q^{-k(k-1)}(1-q^{2n})}{1-q^{2k}}\\
&\equiv\begin{bmatrix}mn\\n\end{bmatrix}_{q^2}\frac{2(-1)^kq^{-k^2}(1-q^n)}{(1+q^k)(1-q^{n-k})}\pmod{\Phi_n(q)^2}.
\end{aligned} \tag{4.2}$$

It follows from (2.1), (3.4) and (4.2) that

$$\sum_{k=1}^{n-1}(-1)^k \begin{bmatrix} mn \\ n-k \end{bmatrix}_{q^2} \begin{bmatrix} 2n(m-1)+2k \\ k \end{bmatrix}$$

$$\equiv 2(1-q^n) \begin{bmatrix} mn \\ n \end{bmatrix}_{q^2} \sum_{k=1}^{n-1} \frac{(-1)^k q^{k(k-1)/2}}{1-q^{n-k}} \begin{bmatrix} n-k \\ k \end{bmatrix}$$

$$= 2 \begin{bmatrix} mn \\ n \end{bmatrix}_{q^2} (\mathcal{A}_n(q)-1) \pmod{\Phi_n(q)^2}. \tag{4.3}$$

Finally, substituting (2.3), (2.4) and (4.3) into the right-hand side of (4.1) and using (3.7), we complete the proof of Theorem 1.2.

5 Proof of Theorem 1.3

Note that

$$T_1(mn, (m-1)n, q)$$

$$= \sum_{k=0}^{n}(-q)^k \begin{bmatrix} mn \\ k \end{bmatrix}_{q^2} \begin{bmatrix} 2mn-2k \\ n-k \end{bmatrix}$$

$$= \sum_{k=0}^{n}(-q)^{n-k} \begin{bmatrix} mn \\ n-k \end{bmatrix}_{q^2} \begin{bmatrix} 2n(m-1)+2k \\ k \end{bmatrix}$$

$$= (-q)^n \begin{bmatrix} mn \\ n \end{bmatrix}_{q^2} + \begin{bmatrix} 2mn \\ n \end{bmatrix} + \sum_{k=1}^{n-1}(-q)^{n-k} \begin{bmatrix} mn \\ n-k \end{bmatrix}_{q^2} \begin{bmatrix} 2n(m-1)+2k \\ k \end{bmatrix}. \tag{5.1}$$

Similarly to the proof of Theorem 1.2, by using (2.2), (4.2) and (3.4) we obtain

$$\sum_{k=1}^{n-1}(-q)^{n-k} \begin{bmatrix} mn \\ n-k \end{bmatrix}_{q^2} \begin{bmatrix} 2n(m-1)+2k \\ k \end{bmatrix}$$

$$\equiv 2(-1)^n \begin{bmatrix} mn \\ n \end{bmatrix}_{q^2} (\mathcal{B}_n(q)-1) \pmod{\Phi_n(q)^2}. \tag{5.2}$$

Substituting (2.3), (2.4) and (5.2) into the right-hand side of (5.1) and using (3.8), we complete the proof of Theorem 1.3.

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