Finite Time Stability and Relative Controllability of Fractional Multi-Delay Differential Systems☆

Xing Zhang[a], Mengmeng Li[a,*]

[a]*Department of Mathematics, Guizhou University, Guiyang 550025, Guizhou, China*

Abstract

This paper firstly studies finite time stability of the Riemann-Liouville fractional multi-delay differential equation via the multi-delayed perturbation of two parameter Mittag-Leffler type matrix function. Secondly, by using the multi-delayed Gramian matrix derived from the representation of the solution, the sufficient and necessary condition for the relative controllability of the linear multi-delay system is achieved. Further, the relative controllability result is extended to the semilinear system by means of the Krasnoselskii's fixed point theorem. Finally, three examples are presented to demonstrate the rationality of the key theoretical results.

Keywords: Multi-delayed perturbation of Mittag-Leffler functions, Finite time stability, Multi-delayed Gramian matrix, Relative controllability.

MSC: 34A08, 39B72, 93B05

1. Introduction

Unlike researching exponential stability, global asymptotic stability and Lyapunov stability in infinite time intervals, finite time stability (FTS) focuses on the properties of the system in finite time intervals. Up to now, there have been many results on this research. In [1, 2], using delayed Mittag-Leffler type matrix and fractional delayed cosine and sine type matrices, the authors investigated FTS of fractional delay differential equations. Thereafter, [3] researched FTS of Riemann-Liouville fractional delay differential equations by the delayed matrix function of Mittag-Leffler. Additionally, based on a generalized Gronwall inequality, the authors [4] studied the robust FTS problem of fractional-order systems with time-varying delay and nonlinear perturbation. For other methods, one can consult former works [5–11].

The theory and practice of system control are considered to be among the scientific fields that have had a significant impact on human productive activities and social life since the 20th century. Relative controllability of various control systems have been researched in [12–18]. In [19], the

☆This work is partially supported by the National Natural Science Foundation of China (12201148) and Science and Technology Program of Guizhou Province (No.QKHJC-ZK[2022]YB069).

*Corresponding author.

Email addresses: xzhangmath@126.com (Xing Zhang), mmli@gzu.edu.cn (Mengmeng Li)

authors studied relative controllability of impulsive multi-delay differential systems by the Gramian criteria and Krasnoselskii's fixed point theorem. Furthermore, by constructing solution of the multi-delay system without pairwise matrices permutation, the relative controllability of this system has been shown in [20].

Recently, Mahmudov has proposed delayed perturbation of Mittag-Leffler type matrix functions, which is an evolution of the classical Mittag-Leffler type matrix functions and delayed Mittag-Leffler type matrix functions in [21]. Then, he continued to promote the classical Mittag-Leffler type matrix function and obtained the multi-delayed perturbation of two parameter Mittag-Leffler type matrix function by introducing the concept of multivariate determining matrix equation in [22].

Motivated by [1, 19, 22], we firstly aim to study FTS of the linear fractional multi-delay differential system:

$$\begin{cases} ({}^{RL}D^{\alpha}_{0+}y)(\zeta) = By(\zeta) + \sum_{i=1}^{d} B_i y(\zeta-\vartheta_i) + g(\zeta),\ \zeta\in J,\ \vartheta_i>0, \\ D^{\alpha-1}_{0+}y(\zeta) = \varphi(0) = y_0, \\ y(\zeta) = \varphi(\zeta),\ \ -\vartheta\le\zeta<0, \end{cases} \tag{1}$$

where ${}^{RL}D^{\alpha}_{0+}(0<\alpha<1)$ denotes the Riemann-Liouville fractional derivative, $B, B_1, \cdots, B_d \in \mathbb{R}^{n\times n}$, $\vartheta := \max\{\vartheta_1,\vartheta_2,\cdots,\vartheta_d\}$, $\varphi(\cdot)\in C([-\vartheta,0],\mathbb{R}^n)$, $g\in C(J,\mathbb{R}^n)$, $J=(0,T]$ and $T>0$.

Additionally, we will investigate the relative controllability of multi-delay control system as below:

$$\begin{cases} ({}^{RL}D^{\alpha}_{0+}y)(\zeta) = By(\zeta) + \sum_{i=1}^{d} B_i y(\zeta-\vartheta_i) + g(\zeta,y(\zeta)) + Cu(\zeta),\ \zeta\in J, \\ D^{\alpha-1}_{0+}y(\zeta) = \varphi(0) = y_0, \\ y(\zeta) = \varphi(\zeta),\ \ -\vartheta\le\zeta<0, \end{cases} \tag{2}$$

where $C\in\mathbb{R}^{n\times n}$, $u\in L^2(J,\mathbb{R}^n)$ and $g: J\times\mathbb{R}^n\to\mathbb{R}^n$.

By [22, Theorem 4.2], the solution y of (1) can be formulated by

$$y(\zeta) = Y_{\alpha,\alpha}(\zeta)y_0 + \sum_{i=1}^{d}\int_{-\vartheta_i}^{\min(\zeta-\vartheta_i,0)} Y_{\alpha,\alpha}(\zeta-\vartheta_i-\rho)B_i\varphi(\rho)d\rho + \int_0^{\zeta} Y_{\alpha,\alpha}(\zeta-\rho)g(\rho)d\rho, \tag{3}$$

where $Y_{\alpha,\alpha}(\zeta)$ is defined in (4) with $\beta=\alpha$.

In viewing of (3), the solution of (2) can be represented as

$$\begin{aligned} y(\zeta) \ &= \ Y_{\alpha,\alpha}(\zeta)y_0 + \sum_{i=1}^{d}\int_{-\vartheta_i}^{\min(\zeta-\vartheta_i,0)} Y_{\alpha,\alpha}(\zeta-\vartheta_i-\rho)B_i\varphi(\rho)d\rho \\ &\quad + \int_0^{\zeta} Y_{\alpha,\alpha}(\zeta-\rho)(g(\rho,y(\rho))+Cu(\rho))d\rho. \end{aligned}$$

The innovations and difficulties of this paper contain the following aspects:

(i) With the complexity of the multivariate determining matrix equation, it is difficult to derive the norm estimation of the multi-delayed perturbation of two parameter Mittag-Leffler type matrix function.

(ii) Because the solution of the equation has a singularity, it is hard to estimate the norm of the solution in C_γ to analyze the FTS and relative controllability. From this perspective, our results enrich the qualitative theory in the area of fractional multi-delay differential equations.

The rest of the paper is organized as follows. In Section 2, we review some notations and definitions, and prove two vital Lemmas. In Section 3, we give some sufficient conditions to guarantee that (1) is FTS by researching the estimation of $Y_{\alpha,\beta}(\cdot)$. In Section 4, we consider relative controllability of the multi-delay control system for (2). In Section 5, we give some examples to illustrate the main theorems.

2. Preliminaries

Set $x \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$, the vector norm $\|x\| = \sum_{i=1}^n |x_i|$ and the matrix norm $\|A\| = \max_{1\le j\le n} \sum_{i=1}^n |a_{ij}|$. Let $C([a,b],\mathbb{R}^n)$ be the continuous function space with $\|y\| = \max_{a\le\zeta\le b}\|y(\zeta)\|$. For any $0<\gamma<1$, we denote $C_\gamma([a,b],\mathbb{R}^n) = \{y \in C((a,b],\mathbb{R}^n) : (\cdot - a)^\gamma y(\cdot) \in C([a,b],\mathbb{R}^n)\}$. Then $C_\gamma([a,b],\mathbb{R}^n)$ is a Banach space with $\|y\|_{C_\gamma} = \max_{a\le\zeta\le b}\|(\zeta-a)^\gamma y(\zeta)\|$. Denote $\lambda = \|B\| + \sum_{j=1}^{d}\|B_j\|$.

Definition 2.1. *(see [23]) For a function $y : [0,\infty) \to \mathbb{R}$, its Riemann-Liouville fractional derivative of order $0<\alpha<1$ can be defined as*

$$({}^{RL}D_{0+}^{\alpha} y)(\zeta) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{d\zeta}\int_0^\zeta (\zeta-\rho)^{-\alpha} y(\rho)d\rho,\ \zeta>0.$$

Definition 2.2. *(see [23]) For a function $y : [0,\infty) \to \mathbb{R}$, its fractional integral of order $0<\alpha<1$ can be defined as*

$$(I_{0+}^{\alpha} y)(\zeta) = \frac{1}{\Gamma(\alpha)}\int_0^\zeta (\zeta-\rho)^{\alpha-1} y(\rho)d\rho,\ \zeta>0.$$

Definition 2.3. *(see [22]) The coefficient matrices $Q_k(s_1,\cdots,s_d)$, $k=1,2,\cdots$, satisfy the following multivariate determinant equation*

$$\begin{aligned} Q_{k+1}(s_1,\cdots,s_d) &= BQ_k(s_1,\cdots,s_d) + \sum_{i=1}^{d} B_i Q_k(s_1,\cdots,s_i-\vartheta_i,\cdots,s_d), \\ Q_0(s_1,\cdots,s_d) &= Q_k(-\vartheta_1,\cdots,s_d) = \cdots = Q_k(s_1,\cdots,-\vartheta_d) = \Theta,\ Q_1(0,\cdots,0) = I, \\ k &= 0,1,2,\cdots,\ \text{and } s_i = 0,\vartheta_i,2\vartheta_i,\cdots, \end{aligned}$$

where I is an identity matrix, Θ is a zero matrix.

In [22], the authors introduced a shift operator $\mathcal{T}^\vartheta(\vartheta\in\mathbb{R})$ which causes a function f to its translation:

$$\mathcal{T}^\vartheta f(\zeta) = e^{\vartheta\frac{d}{d\zeta}} f := f(\zeta+\vartheta).$$

We now recall the multi-delayed perturbation of two parameter Mittag-Leffler type matrix function $Y_{\alpha,\beta}(\cdot)$ by using the shift operator $\mathcal{T}^\vartheta$.

Definition 2.4. *(see [22]) Let $0 < \alpha, \beta < 1$ and $B, B_1, \cdots, B_d \in \mathbb{R}^{n\times n}$. The multi-delayed perturbation of two parameter Mittag-Leffler type matrix function $Y_{\alpha,\beta}(\cdot) : \mathbb{R} \to \mathbb{R}^{n\times n}$ is written by*

$$Y_{\alpha,\beta}(\zeta) = \begin{cases} \Theta, \quad \zeta \in [-\vartheta, 0), \\ \sum\limits_{k=0}^{\infty} \sum\limits_{\substack{i_1+\cdots+i_d \le k \\ i_1,\cdots,i_d \ge 0}} Q_{k+1}(i_1\vartheta_1, i_2\vartheta_2, \cdots, i_d\vartheta_d) e^{-(i_1\vartheta_1+\cdots+i_d\vartheta_d)\frac{d}{d\zeta}} \frac{(\zeta)_+^{k\alpha+\beta-1}}{\Gamma(k\alpha+\beta)}, \ \zeta \in [0, \infty), \end{cases} \tag{4}$$

where $(\cdot)_+ := \max\{0, \cdot\}$. It follows from that $Q_{k+1}(i_1\vartheta_1, i_2\vartheta_2, \cdots, i_d\vartheta_d) = \Theta$, if $i_1 + \cdots + i_d \ge k+1$, $i_1, \cdots, i_d \ge 0$.

Definition 2.5. *(see [23]) Let $0 < \alpha, \beta < 1$, the Mittag-Leffler function $E_{\alpha,\beta}(\cdot^{\alpha})$ is written as*

$$E_{\alpha,\beta}(z^{\alpha}) = \sum_{k=0}^{\infty} \frac{z^{k\alpha}}{\Gamma(k\alpha+\beta)}, \ z \in \mathbb{C}.$$

Definition 2.6. *(see [1]) System (1) is FTS with regard to $\{0, J, \vartheta, \delta, \eta\}$ if and only if $\|\varphi\| < \delta$ and $\|y_0\| < \delta$ imply $\|y\|_{C_\gamma} < \eta$ for any $\zeta \in J$, where $\delta < \eta$.*

Definition 2.7. *(see [3]) System (2) is called relatively controllable, if for $\forall \varphi \in C([-\vartheta, 0], \mathbb{R}^n)$ and $y_1 \in \mathbb{R}^n$, $\exists u \in L^2(J, \mathbb{R}^n)$ such that (2) has a solution $y \in C([-\vartheta, T], \mathbb{R}^n)$ satisfying $y(\zeta_1) = y_1$.*

Lemma 2.8. *For any $\zeta \ge 0$, $0 < \alpha < 1$, $\beta > 0$ and $\alpha + \beta \ge 1$, one has*

$$\|Y_{\alpha,\beta}(\zeta)\| \le \zeta^{\beta-1} E_{\alpha,\beta}(\lambda\zeta^{\alpha}).$$

Proof. According to (4) and [22, Remark 3.5], we have

$$\|Y_{\alpha,\beta}(\zeta)\| \le \sum_{k=0}^{\infty} \sum_{\substack{i_0+i_1+\ldots+i_d=k \\ i_0,i_1,\ldots,i_d \ge 0}} \frac{k!}{i_0! i_1! \ldots i_d!} \|B\|^{i_0} \prod_{j=1}^{d} \|B_j^{i_j}\| \frac{\left(\zeta - \sum\limits_{j=1}^{d} i_j\vartheta_j\right)_+^{k\alpha+\beta-1}}{\Gamma(k\alpha+\beta)}.$$

Let $\vartheta_i > 0$ and $i_j \ge 0$, we obtain

$$\begin{aligned} \|Y_{\alpha,\beta}(\zeta)\| &\le \sum_{k=0}^{\infty} \sum_{\substack{i_0+i_1+\ldots+i_d=k \\ i_0,i_1,\ldots,i_d \ge 0}} \frac{k!}{i_0! i_1! \ldots i_d!} \|B\|^{i_0} \prod_{j=1}^{d} \|B_j\|^{i_j} \frac{(\zeta)_+^{k\alpha+\beta-1}}{\Gamma(k\alpha+\beta)} \\ &\le \sum_{k=0}^{\infty} (\|B\| + \sum_{j=1}^{d} \|B_j\|)^k \frac{\zeta^{k\alpha+\beta-1}}{\Gamma(k\alpha+\beta)} = \zeta^{\beta-1} E_{\alpha,\beta}(\lambda\zeta^{\alpha}). \end{aligned}$$

The proof is completed. □

Remark 2.9. *Let $\beta = 1$, one has $\|Y_{\alpha,1}(\zeta)\| \le E_{\alpha,1}(\lambda\zeta^{\alpha})$. Let $\frac{1}{2} \le \alpha = \beta < 1$, one has $\|Y_{\alpha,\alpha}(\zeta)\| \le \zeta^{\alpha-1} E_{\alpha,\alpha}(\lambda\zeta^{\alpha})$.*

Lemma 2.10. *For any $\zeta > 0$ and $\frac{1}{2} \le \alpha < 1$, one has*

$$\begin{aligned} \int_0^{\zeta} \|Y_{\alpha,\alpha}(\zeta-\rho)\| d\rho &\le \zeta^{\alpha} E_{\alpha,\alpha+1}(\lambda\zeta^{\alpha}), \\ \int_{-\vartheta_i}^{min(\zeta-\vartheta_i,0)} \|Y_{\alpha,\alpha}(\zeta-\vartheta_i-\rho)\| d\rho &\le \begin{cases} \zeta^{\alpha} E_{\alpha,\alpha+1}(\lambda\zeta^{\alpha}) - (\zeta-\vartheta_i)^{\alpha} E_{\alpha,\alpha+1}(\lambda(\zeta-\vartheta_i)^{\alpha}), \ \zeta \ge \vartheta_i, \\ \zeta^{\alpha} E_{\alpha,\alpha+1}(\lambda\zeta^{\alpha}), \ \zeta < \vartheta_i. \end{cases} \end{aligned}$$

Proof. Let $0 \le \rho \le \zeta$ and $0 \le \zeta - \rho < \zeta$, we have

$$\begin{aligned}\int_0^{\zeta} \|Y_{\alpha,\alpha}(\zeta-\rho)\| d\rho &\le \int_0^{\zeta} \sum_{k=0}^{\infty} \lambda^k \frac{(\zeta-\rho)_+^{k\alpha+\alpha-1}}{\Gamma(k\alpha+\alpha)} d\rho \le \int_0^{\zeta} \sum_{k=0}^{\infty} \lambda^k \frac{(\zeta-\rho)^{k\alpha+\alpha-1}}{\Gamma(\alpha k+\alpha)} d\rho \\ &\le \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(k\alpha+\alpha)} \int_0^{\zeta} (\zeta-\rho)^{k\alpha+\alpha-1} d\rho \le \sum_{k=0}^{\infty} \lambda^k \frac{\zeta^{k\alpha+\alpha}}{\Gamma(k\alpha+\alpha+1)} \\ &\le \zeta^{\alpha} E_{\alpha,\alpha+1}(\lambda\zeta^{\alpha}).\end{aligned}$$

Let $\zeta \ge \vartheta_i$, we have $\min(\zeta-\vartheta_i, 0) = 0$, which deduce $0 \le \zeta - \vartheta_i \le \zeta - \vartheta_i - \rho \le \zeta$. Thus

$$\begin{aligned}\int_{-\vartheta_i}^{min(\zeta-\vartheta_i,0)} \|Y_{\alpha,\alpha}(\zeta-\vartheta_i-\rho)\| d\rho &= \int_{-\vartheta_i}^{0} \|Y_{\alpha,\alpha}(\zeta-\vartheta_i-\rho)\| d\rho \\ &\le \int_{-\vartheta_i}^{0} \sum_{k=0}^{\infty} \lambda^k \frac{(\zeta-\vartheta_i-\rho)^{k\alpha+\alpha-1}}{\Gamma(k\alpha+\alpha)} d\rho \\ &\le \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(k\alpha+\alpha)} \int_{-\vartheta_i}^{0} (\zeta-\vartheta_i-\rho)^{k\alpha+\alpha-1} d\rho \\ &= \sum_{k=0}^{\infty} \frac{\lambda^k \zeta^{k\alpha+\alpha}}{\Gamma(k\alpha+\alpha+1)} - \sum_{k=0}^{\infty} \frac{\lambda^k (\zeta-\vartheta_i)^{k\alpha+\alpha}}{\Gamma(k\alpha+\alpha+1)} \\ &= \zeta^{\alpha} E_{\alpha,\alpha+1}(\lambda\zeta^{\alpha}) - (\zeta-\vartheta_i)^{\alpha} E_{\alpha,\alpha+1}(\lambda(\zeta-\vartheta_i)^{\alpha}).\end{aligned}$$

Similarly, when $\zeta < \vartheta_i$, we gain

$$\int_{-\vartheta_i}^{min(\zeta-\vartheta_i,0)} \|Y_{\alpha,\alpha}(\zeta-\vartheta_i-\rho)\| d\rho = \int_{-\vartheta_i}^{\zeta-\vartheta_i} \|Y_{\alpha,\alpha}(\zeta-\vartheta_i-\rho)\| d\rho \le \zeta^{\alpha} E_{\alpha,\alpha+1}(\lambda\zeta^{\alpha}).$$

The proof is completed. □

3. FTS results

In this part, by applying multi-delayed perturbation of two parameter Mittag-Leffler type matrix function, FTS results be presented. Firstly, we impose the following assumptions:

$[A_1]$ Suppose that $g(\cdot) \in C([0,T], \mathbb{R}^n)$ and $\|g\| = \max_{\zeta\in[0,T]} \|g(\zeta)\| < \infty$.

$[A_2]$ There exists $\mu(\cdot) \in L^q([0,T], \mathbb{R}^+)$, $\frac{1}{q} = 1 - \frac{1}{p}$, $p > 1$ such that $\|g(\zeta)\| \le \mu(\zeta)$ for any $\zeta \in [0,T]$ and $\phi(\zeta) = \left(\int_0^{\zeta} \mu^q(\rho) d\rho\right)^{\frac{1}{q}} < \infty$.

$[A_3]$ For any $0 < \gamma < 1$ and $\frac{1}{2} \le \alpha < 1$ such that $\alpha + \gamma - 1 \ge 0$.

Next, for every $\zeta \in [-\vartheta, T]$ and $p > 1$, we define

$$\begin{aligned}\Psi_1(\zeta) &= \frac{\zeta^{\alpha+\gamma+\frac{1}{p}-1}}{(p\alpha-p+1)^{\frac{1}{p}}} E_{\alpha,\alpha}(\lambda\zeta^{\alpha}), \\ \Psi_2(\zeta) &= \begin{cases} \sum_{i=1}^{d} \|B_i\| \zeta^{\alpha} E_{\alpha,\alpha+1}(\lambda\zeta^{\alpha}), \ \zeta < \vartheta_i, \\ \sum_{i=1}^{d} \|B_i\| \Big(\zeta^{\alpha} E_{\alpha,\alpha+1}(\lambda\zeta^{\alpha}) - (\zeta-\vartheta_i)^{\alpha} E_{\alpha,\alpha+1}(\lambda(\zeta-\vartheta_i)^{\alpha})\Big), \ \zeta \ge \vartheta_i, \end{cases} \\ \Psi_3(\zeta) &= \begin{cases} \sum_{i=1}^{d} \|B_i\| \zeta, \ \zeta < \vartheta_i, \\ \sum_{i=1}^{d} \|B_i\| \vartheta_i, \ \zeta \ge \vartheta_i. \end{cases}\end{aligned} \tag{5}$$

Theorem 3.1. *Suppose that* $[A_1]$ *and* $[A_3]$ *hold. System* (1) *is FTS with regard to* $\{0, J, \vartheta, \delta, \eta\}$ *if*

$$\zeta^{\alpha+\gamma-1}\delta E_{\alpha,\alpha}(\lambda\zeta^{\alpha}) + \zeta^{\gamma}\bigg(\delta\Psi_2(\zeta) + \zeta^{\alpha}E_{\alpha,\alpha+1}(\lambda\zeta^{\alpha})\|g\|\bigg) < \eta, \ \forall\zeta \in [0, T]. \tag{6}$$

Proof. The solution of (1) can be given by

$$y(\zeta) = Y_{\alpha,\alpha}(\zeta)y_0 + \sum_{i=1}^{d}\int_{-\vartheta_i}^{\min(\zeta-\vartheta_i,0)} Y_{\alpha,\alpha}(\zeta-\vartheta_i-\rho)B_i\varphi(\rho)d\rho + \int_0^{\zeta} Y_{\alpha,\alpha}(\zeta-\rho)g(\rho)d\rho,$$

By Lemma 2.10 via (6), we obtain

$$\begin{aligned}
&\|\zeta^{\gamma}y(\zeta)\| \\
\leq\ & \|\zeta^{\gamma}Y_{\alpha,\alpha}(\zeta)y_0\| + \zeta^{\gamma}\bigg(\sum_{i=1}^{d}\int_{-\vartheta_i}^{\min(\zeta-\vartheta_i,0)} \|Y_{\alpha,\alpha}(\zeta-\vartheta_i-\rho)B_i\varphi(\rho)\|d\rho + \int_0^{\zeta}\|Y_{\alpha,\alpha}(\zeta-\rho)g(\rho)\|d\rho\bigg) \\
\leq\ & \zeta^{\alpha+\gamma-1}\delta E_{\alpha,\alpha}(\lambda\zeta^{\alpha}) + \zeta^{\gamma}\bigg(\delta\sum_{i=1}^{d}\|B_i\|\int_{-\vartheta_i}^{\min(\zeta-\vartheta_i,0)}\|Y_{\alpha,\alpha}(\zeta-\vartheta_i-\rho)\|d\rho \\
& +\|g\|\int_0^{\zeta}\|Y_{\alpha,\alpha}(\zeta-\rho)\|d\rho\bigg) \\
\leq\ & \zeta^{\alpha+\gamma-1}\delta E_{\alpha,\alpha}(\lambda\zeta^{\alpha}) + \zeta^{\gamma}\bigg(\delta\Psi_2(\zeta) + \zeta^{\alpha}E_{\alpha,\alpha+1}(\lambda\zeta^{\alpha})\|g\|\bigg) < \eta.
\end{aligned}$$

This proof is completed. □

Theorem 3.2. *Suppose that* $[A_2]$ *and* $[A_3]$ *hold. System* (1) *is FTS with regard to* $\{0, J, \vartheta, \delta, \eta\}$ *if*

$$\zeta^{\alpha+\gamma-1}\delta E_{\alpha,\alpha}(\lambda\zeta^{\alpha}) + \zeta^{\gamma}\delta\Psi_2(\zeta) + \Psi_1(\zeta)\phi(\zeta) < \eta, \ \forall\zeta \in [0, T]. \tag{7}$$

Proof. By Lemma 2.10 via (7), one has

$$\begin{aligned}
&\|\zeta^{\gamma}y(\zeta)\| \\
\leq\ & \|\zeta^{\gamma}Y_{\alpha,\alpha}(\zeta)y_0\| + \sum_{i=1}^{d}\int_{-\vartheta_i}^{\min(\zeta-\vartheta_i,0)}\|\zeta^{\gamma}Y_{\alpha,\alpha}(\zeta-\vartheta_i-\rho)B_i\varphi(\rho)\|d\rho + \int_0^{\zeta}\|\zeta^{\gamma}Y_{\alpha,\alpha}(\zeta-\rho)g(\rho)\|d\rho \\
\leq\ & \zeta^{\alpha+\gamma-1}\delta E_{\alpha,\alpha}(\lambda\zeta^{\alpha}) + \zeta^{\gamma}\delta\sum_{i=1}^{d}\|B_i\|\int_{-\vartheta_i}^{\min(\zeta-\vartheta_i,0)}\|Y_{\alpha,\alpha}(\zeta-\vartheta_i-\rho)\|d\rho \\
& +\zeta^{\gamma}\int_0^{\zeta}\bigg(\sum_{k=0}^{\infty}\frac{\lambda^k(\zeta-\rho)^{k\alpha+\alpha-1}}{\Gamma(k\alpha+\alpha)}\bigg)\mu(\rho)d\rho \\
\leq\ & \zeta^{\alpha+\gamma-1}\delta E_{\alpha,\alpha}(\lambda\zeta^{\alpha}) + \zeta^{\gamma}\delta\Psi_2(\zeta) + \zeta^{\gamma}\sum_{k=0}^{\infty}\frac{\lambda^k}{\Gamma(k\alpha+\alpha)}\int_0^{\zeta}(\zeta-\rho)^{k\alpha+\alpha-1}\mu(\rho)d\rho \\
\leq\ & \zeta^{\alpha+\gamma-1}\delta E_{\alpha,\alpha}(\lambda\zeta^{\alpha}) + \zeta^{\gamma}\delta\Psi_2(\zeta) + \sum_{k=0}^{\infty}\frac{\lambda^k\zeta^{\gamma}}{\Gamma(k\alpha+\alpha)}\bigg(\int_0^{\zeta}(\zeta-\rho)^{p(k\alpha+\alpha-1)}d\rho\bigg)^{\frac{1}{p}}\bigg(\int_0^{\zeta}\mu^q(\rho)d\rho\bigg)^{\frac{1}{q}} \\
\leq\ & \zeta^{\alpha+\gamma-1}\delta E_{\alpha,\alpha}(\lambda\zeta^{\alpha}) + \zeta^{\gamma}\delta\Psi_2(\zeta) + \Psi_1(\zeta)\phi(\zeta) < \eta.
\end{aligned}$$

This proof is completed. □

12 Oct 2023 19:04:39 PDT
231012-LiMengmeng Version 1 - Submitted to Rocky Mountain J. Math.

Theorem 3.3. *Suppose that* $[A_1]$ *and* $[A_3]$ *hold. System* (1) *is FTS with regard to* $\{0, J, \vartheta, \delta, \eta\}$ *if*

$$\zeta^{\alpha+\gamma-1}\Big(\delta+\zeta\|g\|\Big)E_{\alpha,\alpha}(\lambda\zeta^{\alpha})+\zeta^{\gamma}\delta(\zeta+\Delta\vartheta)^{\alpha-1}E_{\alpha,\alpha}(\lambda(\zeta+\Delta\vartheta)^{\alpha})\Psi_3(\zeta)<\eta,\ \forall\zeta\in[0,T], \tag{8}$$

where $\Delta\vartheta=\vartheta-\underline{\vartheta}$ *and* $\underline{\vartheta}=\min\{\vartheta_1,\vartheta_2\cdots,\vartheta_d\}$.

Proof. For any $-\vartheta_i\le\rho\le\min(\zeta-\vartheta_i,0)$, one has $0\le\zeta-\vartheta_i-\rho\le\zeta$. By Lemma 2.8, one has

$$\begin{aligned}\|Y_{\alpha,\alpha}(\zeta-\vartheta_i-\rho)\| &\le \sum_{k=0}^{\infty}\frac{\lambda^k(\zeta-\vartheta_i-\rho)^{k\alpha+\alpha-1}}{\Gamma(k\alpha+\alpha)}\le\sum_{k=0}^{\infty}\frac{\lambda^k(\zeta-\underline{\vartheta}+\vartheta)^{k\alpha+\alpha-1}}{\Gamma(k\alpha+\alpha)}\\ &\le \sum_{k=0}^{\infty}\frac{\lambda^k(\zeta+\Delta\vartheta)^{k\alpha+\alpha-1}}{\Gamma(k\alpha+\alpha)}=(\zeta+\Delta\vartheta)^{\alpha-1}E_{\alpha,\alpha}(\lambda(\zeta+\Delta\vartheta)^{\alpha}).\end{aligned}$$

For $0\le\rho\le\zeta$, by Lemma 2.8, one has

$$\|Y_{\alpha,\alpha}(\zeta-\rho)\|\le\sum_{k=0}^{\infty}\frac{\lambda^k(\zeta-\rho)^{k\alpha+\alpha-1}}{\Gamma(k\alpha+\alpha)}\le\zeta^{\alpha-1}E_{\alpha,\alpha}(\lambda\zeta^{\alpha}).$$

Thus via (8), we have

$$\begin{aligned}&\|\zeta^{\gamma}y(\zeta)\|\\ \le\ &\zeta^{\alpha+\gamma-1}\delta E_{\alpha,\alpha}(\lambda\zeta^{\alpha})+\zeta^{\gamma}\delta\sum_{i=1}^{d}\|B_i\|\int_{-\vartheta_i}^{\min(\zeta-\vartheta_i,0)}\|Y_{\alpha,\alpha}(\zeta-\vartheta_i-\rho)\|d\rho\\ &+\zeta^{\gamma}\|g\|\int_0^{\zeta}\|Y_{\alpha,\alpha}(\zeta-\rho)\|d\rho\\ \le\ &\zeta^{\alpha+\gamma-1}\delta E_{\alpha,\alpha}(\lambda\zeta^{\alpha})+\zeta^{\gamma}\delta(\zeta+\Delta\vartheta)^{\alpha-1}E_{\alpha,\alpha}(\lambda(\zeta+\Delta\vartheta)^{\alpha})\sum_{i=1}^{d}\|B_i\|\int_{-\vartheta_i}^{\min(\zeta-\vartheta_i,0)}d\rho\\ &+\zeta^{\alpha+\gamma}\|g\|E_{\alpha,\alpha}(\lambda\zeta^{\alpha})\\ \le\ &\zeta^{\alpha+\gamma-1}\Big(\delta+\zeta\|g\|\Big)E_{\alpha,\alpha}(\lambda\zeta^{\alpha})+\zeta^{\gamma}\delta(\zeta+\Delta\vartheta)^{\alpha-1}E_{\alpha,\alpha}(\lambda(\zeta+\Delta\vartheta)^{\alpha})\Psi_3(\zeta)<\eta.\end{aligned}$$

This proof is completed. □

4. Relative controllability results

In this part, we discuss the relative controllability of (2).

4.1. Linear systems

Let $g(\zeta,y(\zeta))=\underbrace{(0,\cdots,0)^{\top}}_{n}:=\mathbf{0},\ \zeta\in J$. System (2) simplifies to the following multi-delay control system:

$$\begin{cases}({}^{RL}D_{0+}^{\alpha}y)(\zeta)=By(\zeta)+\sum_{i=1}^{d}B_iy(\zeta-\vartheta_i)+Cu(\zeta),\ \zeta\in J,\\ D_{0+}^{\alpha-1}y(\zeta)=\varphi(0)=y_0,\\ y(\zeta)=\varphi(\zeta),\ \ -\vartheta\le\zeta<0.\end{cases} \tag{9}$$

The solution has a form

$$y(\zeta)=Y_{\alpha,\alpha}(\zeta)y_0+\sum_{i=1}^{d}\int_{-\vartheta_i}^{\min(\zeta-\vartheta_i,0)}Y_{\alpha,\alpha}(\zeta-\vartheta_i-\rho)B_i\varphi(\rho)d\rho+\int_0^{\zeta}Y_{\alpha,\alpha}(\zeta-\rho)Cu(\rho)d\rho.$$

We introduce the multi-delayed Gramian matrix as follows:

$$W_{\vartheta_1,\cdots,\vartheta_d}[0,\zeta_1]=\int_0^{\zeta_1}Y_{\alpha,\alpha}(\zeta_1-\rho)CC^{\top}Y_{\alpha,\alpha}^{\top}(\zeta_1-\rho)d\rho, \tag{10}$$

where $\cdot^{\top}$ indicates the transpose of the matrix.

Theorem 4.1. *$W_{\vartheta_1,\cdots,\vartheta_d}[0,\zeta_1]$ is invertible matrix if and only if* (9) *is relatively controllable.*

Proof. "$\Rightarrow$" Since $W_{\vartheta_1,\cdots,\vartheta_d}[0,\zeta_1]$ is invertible matrix, then for arbitrary $y_1\in\mathbb{R}^n$, we can select $u\in L^2(J,\mathbb{R}^n)$ such that

$$u(\zeta)=C^{\top}Y_{\alpha,\alpha}^{\top}(\zeta_1-\zeta)W_{\vartheta_1,\cdots,\vartheta_d}^{-1}[0,\zeta_1]\xi, \tag{11}$$

where

$$\xi=y_1-Y_{\alpha,\alpha}(\zeta_1)y_0-\sum_{i=1}^{d}\int_{-\vartheta_i}^{\min(\zeta_1-\vartheta_i,0)}Y_{\alpha,\alpha}(\zeta_1-\vartheta_i-\rho)B_i\varphi(\rho)d\rho.$$

Then

$$\begin{aligned}y(\zeta_1)&=Y_{\alpha,\alpha}(\zeta_1)y_0+\sum_{i=1}^{d}\int_{-\vartheta_i}^{\min(\zeta_1-\vartheta_i,0)}Y_{\alpha,\alpha}(\zeta_1-\vartheta_i-\rho)B_i\varphi(\rho)d\rho+\int_0^{\zeta_1}Y_{\alpha,\alpha}(\zeta_1-\rho)Cu(\rho)d\rho\\&=Y_{\alpha,\alpha}(\zeta_1)y_0+\sum_{i=1}^{d}\int_{-\vartheta_i}^{\min(\zeta_1-\vartheta_i,0)}Y_{\alpha,\alpha}(\zeta_1-\vartheta_i-\rho)B_i\varphi(\rho)d\rho\\&\quad+\int_0^{\zeta_1}Y_{\alpha,\alpha}(\zeta_1-\rho)CC^{\top}Y_{\alpha,\alpha}^{\top}(\zeta_1-\rho)d\rho W_{\vartheta_1,\cdots,\vartheta_d}^{-1}[0,\zeta_1]\xi\\&=Y_{\alpha,\alpha}(\zeta_1)y_0+\sum_{i=1}^{d}\int_{-\vartheta_i}^{\min(\zeta_1-\vartheta_i,0)}Y_{\alpha,\alpha}(\zeta_1-\vartheta_i-\rho)B_i\varphi(\rho)d\rho+\xi=y_1.\end{aligned}$$

Thus, (9) is relatively controllable.

"$\Leftarrow$" Assume that $W_{\vartheta_1,\cdots,\vartheta_d}[0,\zeta_1]$ is irreversible, there exists a state $\widehat{y}\neq\mathbf{0}$ such that $\widehat{y}^{\top}W_{\vartheta_1,\cdots,\vartheta_d}[0,\zeta_1]\widehat{y}=0$. Thus, one has

$$\begin{aligned}0&=\widehat{y}^{\top}W_{\vartheta_1,\cdots,\vartheta_d}[0,\zeta_1]\widehat{y}\\&=\int_0^{\zeta_1}\widehat{y}^{\top}Y_{\alpha,\alpha}(\zeta_1-\rho)CC^{\top}Y_{\alpha,\alpha}^{\top}(\zeta_1-\rho)\widehat{y}d\rho\\&=\int_0^{\zeta_1}\|\widehat{y}^{\top}Y_{\alpha,\alpha}(\zeta_1-\rho)C\|^2d\rho,\end{aligned}$$

which implies that

$$\widehat{y}^{\top}Y_{\alpha,\alpha}(\zeta_1-\rho)C=\mathbf{0}^{\top},\ \forall\rho\in J. \tag{12}$$

12 Oct 2023 19:04:39 PDT
231012-LiMengmeng Version 1 - Submitted to Rocky Mountain J. Math.

Note that (9) is relatively controllable. Thus, there exist $u_1(\cdot)$ and $u_2(\cdot)$ that make the initial state to $\widetilde{y} = \mathbf{0}$ and another $\widehat{y} \neq \mathbf{0}$ at $\zeta = \zeta_1$, i.e.,

$$\begin{aligned} y(\zeta_1) &= Y_{\alpha,\alpha}(\zeta_1)y_0 + \sum_{i=1}^{d} \int_{-\vartheta_i}^{\min(\zeta_1-\vartheta_i,0)} Y_{\alpha,\alpha}(\zeta_1 - \vartheta_i - \rho)B_i\varphi(\rho)d\rho + \int_0^{\zeta_1} Y_{\alpha,\alpha}(\zeta_1 - \rho)Cu_1(\rho)d\rho \\ &= \mathbf{0}. \end{aligned} \tag{13}$$

and

$$\begin{aligned} y(\zeta_1) &= Y_{\alpha,\alpha}(\zeta_1)y_0 + \sum_{i=1}^{d} \int_{-\vartheta_i}^{\min(\zeta_1-\vartheta_i,0)} Y_{\alpha,\alpha}(\zeta_1 - \vartheta_i - \rho)B_i\varphi(\rho)d\rho + \int_0^{\zeta_1} Y_{\alpha,\alpha}(\zeta_1 - \rho)Cu_2(\rho)d\rho \\ &= \widehat{y}. \end{aligned} \tag{14}$$

By (13) and (14), we have

$$\widehat{y} = \int_0^{\zeta_1} Y_{\alpha,\alpha}(\zeta_1 - \rho)C[u_2(\rho) - u_1(\rho)]d\rho,$$

and by (12), one has

$$\widehat{y}^\top \widehat{y} = \int_0^{\zeta_1} \widehat{y}^\top Y_{\alpha,\alpha}(\zeta_1 - \rho)C[u_2(\rho) - u_1(\rho)]d\rho = 0.$$

Thus, one can obtain $\widehat{y}^\top \widehat{y} = 0$, which conflicts with $\widehat{y} \neq \mathbf{0}$. □

4.2. Semilinear systems

We need the following hypothesis:

$[A_4]$ Let $\mathcal{W} : L^2(J, \mathbb{R}^n) \to \mathbb{R}^n$ be given by

$$\mathcal{W}u = \int_0^{\zeta_1} Y_{\alpha,\alpha}(\zeta_1 - \rho)Cu(\rho)d\rho,$$

it has an inverse operator $\mathcal{W}^{-1}$, which takes values in $L^2(J, \mathbb{R}^n)\backslash \ker\mathcal{W}$. Then there is a constant $M > 0$ such that $M = \|\mathcal{W}^{-1}\|_{L_b(\mathbb{R}^n, L^2(J,\mathbb{R}^n)\backslash \ker\mathcal{W})}$. In viewing of [17], one has

$$M = \sqrt{\|W^{-1}_{\vartheta_1,\cdots,\vartheta_d}[0, \zeta_1]\|}\ . \tag{15}$$

$[A_5]$ For any $g \in C(J \times \mathbb{R}^n, \mathbb{R}^n)$ and there is $\omega \in C(J, \mathbb{R}^+)$ such that

$$\|g(\zeta, y) - g(\zeta, z)\| \le \omega(\zeta)\|y - z\|, \ y, z \in \mathbb{R}^n, \ \zeta \in J.$$

For any $y \in C_\gamma(J, \mathbb{R}^n)$, the function $u_y(\zeta)$ can be taken on

$$\begin{aligned} u_y(\zeta) &= \mathcal{W}^{-1}\Big(y_1 - Y_{\alpha,\alpha}(\zeta_1)y_0 - \sum_{i=1}^{d} \int_{-\vartheta_i}^{\min(\zeta_1-\vartheta_i,0)} Y_{\alpha,\alpha}(\zeta_1 - \vartheta_i - \rho)B_i\varphi(\rho)d\rho \\ &\quad - \int_0^{\zeta_1} Y_{\alpha,\alpha}(\zeta_1 - \rho)g(\rho, y(\rho))d\rho\Big)(\zeta), \ \zeta \in J. \end{aligned} \tag{16}$$

Using Krasnoselskii's fixed point theorem, we will show that (2) is relatively controllable. By (16), the operator $\mathcal{F}: C_\gamma(J,\mathbb{R}^n) \to C_\gamma(J,\mathbb{R}^n)$ is defined by

$$\begin{aligned}(\mathcal{F}y)(\zeta) &= Y_{\alpha,\alpha}(\zeta)y_0 + \sum_{i=1}^{d}\int_{-\vartheta_i}^{\min(\zeta-\vartheta_i,0)} Y_{\alpha,\alpha}(\zeta-\vartheta_i-\rho)B_i\varphi(\rho)d\rho \\ &\quad + \int_0^\zeta Y_{\alpha,\alpha}(\zeta-\rho)g(\rho,y(\rho))d\rho + \int_0^\zeta Y_{\alpha,\alpha}(\zeta-\rho)Cu_y(\rho)d\rho. \qquad (17)\end{aligned}$$

Hence, we need to prove that $\mathcal{F}$ exists a fixed point y, which exactly is a solution of (2). Besides, we verify $(\mathcal{F}y)(\zeta_1) = y_1$, which indicates that (2) is relatively controllable on $[-\vartheta,\zeta_1]$. Define

$$\begin{aligned} M_1 &= \zeta_1^{\alpha-1}E_{\alpha,\alpha+1}(\lambda\zeta_1^\alpha)\|y_0\| + \|\varphi\|\Psi_2(\zeta_1),\ M_2 = E_{\alpha,\alpha}(\lambda\zeta_1^\alpha)\mathbb{B}[1-\gamma,\alpha], \\ M_3 &= \zeta_1^\alpha E_{\alpha,\alpha+1}(\lambda\zeta_1^\alpha),\ \widetilde{M} = \sup_{\zeta\in J}\omega(\zeta),\ \|\widehat{g}\| = \sup_{\zeta\in J}\|g(\zeta,\mathbf{0})\|,\end{aligned}$$

where $\Psi_2(\cdot)$ defined in (5).

Theorem 4.2. *Let $0<\gamma<\alpha$, $\frac{1}{2}\le\alpha<1$ and $\alpha+\gamma>1$. Assume that $[A_4]$ and $[A_5]$ are held. Then (2) is relatively controllable provided that*

$$\zeta_1^\alpha \widetilde{M}M_2(1+MM_3\|C\|) < 1. \qquad (18)$$

Proof. Consider $\mathcal{F}$ defined in (17) on $\mathfrak{B}_r$, where $\mathfrak{B}_r = \{y\in C_\gamma(J,\mathbb{R}^n) : \|y\|_{C_\gamma}\le r\}$ and $r>0$. To make the following process apparent, we separate it into several steps as below:

Step 1. We prove that $\mathcal{F}(\mathfrak{B}_r)\subseteq\mathfrak{B}_r$. In terms of $[A_5]$ and Lemma 2.10, one has

$$\begin{aligned} &\int_0^\zeta \|Y_{\alpha,\alpha}(\zeta-\rho)\|\|g(\rho,y(\rho))\|d\rho \\ \le\ &\int_0^\zeta \|Y_{\alpha,\alpha}(\zeta-\rho)\|(\omega(\rho)\rho^{-\gamma}\|\rho^\gamma y(\rho)\| + \|g(\rho,\mathbf{0})\|)d\rho \\ \le\ &\widetilde{M}\|y\|_{C_\gamma}\int_0^\zeta(\zeta-\rho)^{\alpha-1}E_{\alpha,\alpha}(\lambda(\zeta-\rho)^\alpha)\rho^{-\gamma}d\rho + \|\widehat{g}\|\int_0^\zeta\|Y_{\alpha,\alpha}(\zeta-\rho)\|d\rho \\ \le\ &\widetilde{M}\|y\|_{C_\gamma}E_{\alpha,\alpha}(\lambda\zeta^\alpha)\int_0^\zeta(\zeta-\rho)^{\alpha-1}\rho^{-\gamma}d\rho + \|\widehat{g}\|\zeta^\alpha E_{\alpha,\alpha+1}(\lambda\zeta^\alpha) \\ \le\ &\widetilde{M}\zeta^{\alpha-\gamma}E_{\alpha,\alpha}(\lambda\zeta^\alpha)\mathbb{B}[1-\gamma,\alpha]\|y\|_{C_\gamma} + \|\widehat{g}\|\zeta^\alpha E_{\alpha,\alpha+1}(\lambda\zeta^\alpha) \\ \le\ &\zeta_1^{\alpha-\gamma}\widetilde{M}M_2\|y\|_{C_\gamma} + M_3\|\widehat{g}\|,\end{aligned}$$

where $\mathbb{B}[m,n] = \int_0^1 s^{m-1}(1-s)^{n-1}ds$ is a Beta function.

Taking into account of Lemmas 2.8 and 2.10, by $[A_4]$ and $[A_5]$, one has

$$\begin{aligned} &\|u_y(\zeta)\| \\ \le\ &\|\mathcal{W}^{-1}\|_{L_b(\mathbb{R}^n,L^2(J,\mathbb{R}^n)\setminus\ker\mathcal{W})}\Big(\|y_1\| + \|Y_{\alpha,\alpha}(\zeta_1)\|\|y_0\| \\ &+ \sum_{i=1}^d\|B_i\|\|\varphi\|\int_{-\vartheta_i}^{\min(\zeta_1-\vartheta_i,0)}\|Y_{\alpha,\alpha}(\zeta_1-\vartheta_i-\rho)\|d\rho + \int_0^{\zeta_1}\|Y_{\alpha,\alpha}(\zeta_1-\rho)\|\|g(\rho,y(\rho))\|d\rho\Big) \\ \le\ &M\Big(\|y_1\| + \zeta_1^{\alpha-1}E_{\alpha,\alpha}(\lambda\zeta_1^\alpha)\|y_0\| + \|\varphi\|\Psi_2(\zeta_1) + (\zeta_1^{\alpha-\gamma}\widetilde{M}M_2\|y\|_{C_\gamma} + M_3\|\widehat{g}\|)\Big) \\ \le\ &M(\|y_1\| + M_1 + M_3\|\widehat{g}\|) + \zeta_1^{\alpha-\gamma}M\widetilde{M}M_2\|y\|_{C_\gamma}.\end{aligned}$$

By Lemmas 2.8, 2.10 and (17), one has

$$\begin{aligned}
&\|\zeta^{\gamma}(\mathcal{F}y)(\zeta)\| \\
\leq\ & \|\zeta^{\gamma}Y_{\alpha,\alpha}(\zeta)y_0\| + \sum_{i=1}^{d}\int_{-\vartheta_i}^{\min(\zeta-\vartheta_i,0)} \|\zeta^{\gamma}Y_{\alpha,\alpha}(\zeta-\vartheta_i-\rho)B_i\varphi(\rho)\|d\rho \\
&+ \int_0^{\zeta}\|\zeta^{\gamma}Y_{\alpha,\alpha}(\zeta-\rho)g(\rho,y(\rho))\|d\rho + \int_0^{\zeta}\|\zeta^{\gamma}Y_{\alpha,\alpha}(\zeta-\rho)Cu_y(\rho)\|d\rho \\
\leq\ & \zeta^{\alpha+\gamma-1}E_{\alpha,\alpha}(\lambda\zeta^{\alpha})\|y_0\| + \zeta^{\gamma}\|\varphi\|\sum_{i=1}^{d}\|B_i\|\int_{-\vartheta_i}^{\min(\zeta-\vartheta_i,0)}\|Y_{\alpha,\alpha}(\zeta-\vartheta_i-\rho)\|d\rho \\
&+\zeta^{\gamma}\int_0^{\zeta}\|Y_{\alpha,\alpha}(\zeta-\rho)\|\|g(\rho,y(\rho))\|d\rho + \zeta^{\gamma}\|C\|\int_0^{\zeta}\|Y_{\alpha,\alpha}(\zeta-\rho)\|\|u_y(\rho)\|d\rho \\
\leq\ & \zeta_1^{\alpha+\gamma-1}E_{\alpha,\alpha}(\lambda\zeta_1^{\alpha})\|y_0\| + \zeta_1^{\gamma}\|\varphi\|\Psi_2(\zeta_1) + \zeta_1^{\gamma}(\zeta_1^{\alpha-\gamma}\widetilde{M}M_2\|y\|_{C_\gamma} + M_3\|\widehat{g}\|) \\
&+\zeta_1^{\gamma}\|C\|M_3\Big(M\big(\|y_1\| + M_1 + M_3\|\widehat{g}\|\big) + \zeta_1^{\alpha-\gamma}M\widetilde{M}M_2\|y\|_{C_\gamma}\Big) \\
\leq\ & \zeta_1^{\gamma}(M_1 + M_3\|\widehat{g}\|) + \zeta_1^{\gamma}MM_3\|C\|\big(\|y_1\| + M_1 + M_3\|\widehat{g}\|\big) + \zeta_1^{\alpha}\widetilde{M}M_2(1 + MM_3\|C\|)r \\
\leq\ & r,
\end{aligned}$$

for

$$r = \frac{\zeta_1^{\gamma}(M_1 + M_3\|\widehat{g}\|) + \zeta_1^{\gamma}MM_3\|C\|(\|y_1\| + M_1 + M_3\|\widehat{g}\|)}{1 - \zeta_1^{\alpha}\widetilde{M}M_2(1 + MM_3\|C\|)}.$$

So $\mathcal{F}(\mathfrak{B}_r) \subseteq \mathfrak{B}_r$, for $\zeta \in J$.

Next, we divide $\mathcal{F}$ into $\mathcal{F}_1$ and $\mathcal{F}_2$ on $\mathfrak{B}_r$ as below:

$$\begin{aligned}
(\mathcal{F}_1y)(\zeta) &= Y_{\alpha,\alpha}(\zeta)y_0 + \sum_{i=1}^{d}\int_{-\vartheta_i}^{\min(\zeta-\vartheta_i,0)} Y_{\alpha,\alpha}(\zeta-\vartheta_i-\rho)B_i\varphi(\rho)d\rho \\
&\quad + \int_0^{\zeta} Y_{\alpha,\alpha}(\zeta-\rho)Cu_y(\rho)d\rho,\ \zeta \in J, \\
(\mathcal{F}_2y)(\zeta) &= \int_0^{\zeta} Y_{\alpha,\alpha}(\zeta-\rho)g(\rho,y(\rho))d\rho,\ \zeta \in J.
\end{aligned}$$

Step 2. We prove that $\mathcal{F}_1$ is a contraction operator. For any $y, z \in \mathfrak{B}_r$ and $\zeta \in J$, we obtain

$$\begin{aligned}
&\|u_y(\zeta) - u_z(\zeta)\| \\
\leq\ & \|\mathcal{W}^{-1}\|_{L_b(\mathbb{R}^n, L^2(J,\mathbb{R}^n)\backslash\ker\mathcal{W})}\int_0^{\zeta_1}\|Y_{\alpha,\alpha}(\zeta_1-\rho)\|\|g(\rho,y(\rho)) - g(\rho,z(\rho))\|d\rho \\
\leq\ & M\int_0^{\zeta_1}\|Y_{\alpha,\alpha}(\zeta_1-\rho)\|\omega(\rho)\rho^{-\gamma}\|\rho^{\gamma}(y(\rho) - z(\rho))\|d\rho \\
\leq\ & M\widetilde{M}\|y - z\|_{C_\gamma}\int_0^{\zeta_1}\rho^{-\gamma}\|Y_{\alpha,\alpha}(\zeta_1-\rho)\|d\rho \\
\leq\ & \zeta_1^{\alpha-\gamma}M\widetilde{M}M_2\|y - z\|_{C_\gamma}.
\end{aligned}$$

and

$$\begin{aligned}
\|\zeta^{\gamma}((\mathcal{F}_1 y)(\zeta)-(\mathcal{F}_1 z)(\zeta))\| &\le \int_0^{\zeta} \zeta^{\gamma}\|Y_{\alpha,\alpha}(\zeta-\rho)\|\|C\|\|u_y(\rho)-u_z(\rho)\|d\rho \\
&\le \zeta_1^{\gamma}\int_0^{\zeta_1}\|Y_{\alpha,\alpha}(\zeta_1-\rho)\|d\rho\|C\|\zeta_1^{\alpha-\gamma}M\widetilde{M}M_2\|y-z\|_{C_\gamma} \\
&\le \zeta_1^{\alpha}M\widetilde{M}M_2M_3\|C\|\|y-z\|_{C_\gamma},
\end{aligned}$$

which gives that

$$\|\mathcal{F}_1 y-\mathcal{F}_1 z\|_{C_\gamma}\le L\|y-z\|_{C_\gamma},\ L:=\zeta_1^{\alpha}M\widetilde{M}M_2M_3\|C\|.$$

By (18), the operator $\mathcal{F}_1$ is a contraction.

Step 3. We proclaim that $\mathcal{F}_2$ is a continuous and compact mapping. Let $y_n\in\mathfrak{B}_r$ with $y_n\to y(n\to\infty)$ in $\mathfrak{B}_r$. By $[A_5]$, one has $g(\zeta,y_n(\zeta))\to g(\zeta,y(\zeta))$ in $C(J,\mathbb{R}^n)$ and hence by dominated convergence theorem, we have

$$\begin{aligned}
\|(\mathcal{F}_2 y_n)(\zeta)-(\mathcal{F}_2 y)(\zeta)\| &= \int_0^{\zeta}\|Y_{\alpha,\alpha}(\zeta-\rho)\|\|g(\rho,y_n(\rho))-g(\rho,y(\rho))\|d\rho \\
&\le \int_0^{\zeta}(\zeta-\rho)^{\alpha-1}E_{\alpha,\alpha}(\lambda(\zeta-\rho)^{\alpha})\|g(\rho,y_n(\rho))-g(\rho,y(\rho))\|d\rho \\
&\to 0 \text{ as } n\to\infty.
\end{aligned}$$

This indicates that $\mathcal{F}_2$ is continuous on $\mathfrak{B}_r$. In order to show that $\mathcal{F}_2$ is compact, the uniform boundedness and equicontinuous of $\mathcal{F}_2(\mathfrak{B}_r)$ need to be verified.

For arbitrary $y\in\mathfrak{B}_r$ and $0<\zeta\le\zeta+h\le\zeta_1$, one has

$$\begin{aligned}
&\|\zeta^{\gamma}((\mathcal{F}_2 y)(\zeta+h)-(\mathcal{F}_2 y)(\zeta))\| \\
=&\left\|\int_0^{\zeta+h}\zeta^{\gamma}Y_{\alpha,\alpha}(\zeta+h-\rho)g(\rho,y(\rho))d\rho-\int_0^{\zeta}\zeta^{\gamma}Y_{\alpha,\alpha}(\zeta-\rho)g(\rho,y(\rho))d\rho\right\| \\
\le& I_1+I_2,
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= \int_{\zeta}^{\zeta+h}\zeta^{\gamma}\|Y_{\alpha,\alpha}(\zeta+h-\rho)\|\|g(\rho,y(\rho)\|d\rho, \\
I_2 &= \int_0^{\zeta}\zeta^{\gamma}\|Y_{\alpha,\alpha}(\zeta+h-\rho)-Y_{\alpha,\alpha}(\zeta-\rho)\|\|g(\rho,y(\rho))\|d\rho.
\end{aligned}$$

For any $\zeta\in J$ and $0<\zeta\le\zeta+h\le\zeta_1$ as $h\to 0$, we have

$$\begin{aligned}
I_1 &\le \int_{\zeta}^{\zeta+h}\zeta^{\gamma}\|Y_{\alpha,\alpha}(\zeta+h-\rho)\|\|g(\rho,y(\rho)\|d\rho \\
&\le \zeta^{\gamma}\widetilde{M}\|y\|_{C_\gamma}\int_{\zeta}^{\zeta+h}\rho^{-\gamma}\|Y_{\alpha,\alpha}(\zeta+h-\rho)\|d\rho+\zeta^{\gamma}\|\widehat{g}\|\int_{\zeta}^{\zeta+h}\|Y_{\alpha,\alpha}(\zeta+h-\rho)\|d\rho \\
&\le \zeta^{\gamma}E_{\alpha,\alpha}(\lambda h^{\alpha})\widetilde{M}\|y\|_{C_\gamma}\int_{\zeta}^{\zeta+h}\rho^{-\gamma}(\zeta+h-\rho)^{\alpha-1}d\rho+h^{\alpha}\zeta^{\gamma}E_{\alpha,\alpha+1}(\lambda h^{\alpha})\|\widehat{g}\| \\
&\le \zeta^{\gamma}E_{\alpha,\alpha}(\lambda h^{\alpha})\widetilde{M}(\zeta+h)^{-\gamma}\|y\|_{C_\gamma}\int_{\zeta}^{\zeta+h}(\zeta+h-\rho)^{\alpha-1}d\rho+h^{\alpha}\zeta^{\gamma}E_{\alpha,\alpha+1}(\lambda h^{\alpha})\|\widehat{g}\| \\
&\le \frac{1}{\alpha}h^{\alpha}\zeta^{\gamma}(\zeta+h)^{-\gamma}E_{\alpha,\alpha}(\lambda h^{\alpha})\widetilde{M}\|y\|_{C_\gamma}+h^{\alpha}\zeta^{\gamma}E_{\alpha,\alpha+1}(\lambda h^{\alpha})\|\widehat{g}\|\to 0 \text{ as } h\to 0,
\end{aligned}$$

and

$$
\begin{aligned}
I_2 &\leq \int_0^\zeta \zeta^\gamma \|Y_{\alpha,\alpha}(\zeta+h-\rho)-Y_{\alpha,\alpha}(\zeta-\rho)\|\|g(\rho,y(\rho))\|d\rho \\
&\leq \zeta^\gamma \widetilde{M}\|y\|_{C_\gamma}\int_0^\zeta \rho^{-\gamma}\|Y_{\alpha,\alpha}(\zeta+h-\rho)-Y_{\alpha,\alpha}(\zeta-\rho)\|d\rho \\
&\quad +\zeta^\gamma\|\widehat{g}\|\int_0^\zeta \|Y_{\alpha,\alpha}(\zeta+h-\rho)-Y_{\alpha,\alpha}(\zeta-\rho)\|d\rho.
\end{aligned}
$$

Let $h\to 0$, one has

$$Y_{\alpha,\alpha}(\zeta+h-\rho)\to Y_{\alpha,\alpha}(\zeta-\rho) \text{ as } h\to 0,$$

which implies that $I_2\to 0$ as $h\to 0$.

According to the above, one has $\|\zeta^\gamma((\mathcal{F}_2y)(\zeta+h)-(\mathcal{F}_2y)(\zeta))\|\to 0$ as $h\to 0$ and $\mathcal{F}_2$ is equicontinuous. Owing to Step 1, it follows that $\mathcal{F}_2$ is uniformly bounded on $\mathfrak{B}_r$. By the Arzela-Ascoli theorem, $\mathcal{F}_2(\mathfrak{B}_r)$ is relatively compact in $C_\gamma(J,\mathbb{R}^n)$. Then $\mathcal{F}$ has a fixed point $y\in\mathfrak{B}_r$. Apparently, y satisfies $y(\zeta_1)=y_1$ which just is a solution of (2). This proof is completed. □

5. Examples

Example 5.1. *Set $\alpha=0.8$, $\vartheta_1=0.1$, $\vartheta_2=0.2$, $\vartheta=0.2$ and $T=0.4$. Consider*

$$
\begin{cases}
({}^{RL}D^\alpha_{0+}y)(\zeta)=By(\zeta)+\sum_{i=1}^{2}B_iy(\zeta-\vartheta_i)+g(\zeta),\ \zeta\in(0,0.4], \\
D^{\alpha-1}_{0+}y(\zeta)=\varphi(0)=(0.05,0.05)^\top, \\
y(\zeta)=(0.1\zeta+0.05,0.2\zeta+0.05)^\top,\ \ -0.2\leq\zeta<0,
\end{cases}
\tag{19}
$$

where

$$
y(\zeta)=\begin{pmatrix} y_1(\zeta)\\ y_2(\zeta)\end{pmatrix},\ g(\zeta)=\begin{pmatrix}\zeta\\ \zeta\end{pmatrix},\ B=\begin{pmatrix}0.1 & 0\\ 0 & 0.1\end{pmatrix},
$$

$$
B_1=\begin{pmatrix}0.15 & 0\\ 0.15 & 0.1\end{pmatrix},\ B_2=\begin{pmatrix}0.1 & 0\\ -0.15 & 0.15\end{pmatrix}.
$$

By (3), for any $\zeta\in(0,0.4]$, one has

$$
y(\zeta)=Y_{\alpha,\alpha}(\zeta)y_0+\sum_{i=1}^{2}\int_{-\vartheta_i}^{\min(\zeta-\vartheta_i,0)}Y_{\alpha,\alpha}(\zeta-\vartheta_i-\rho)B_i\varphi(\rho)d\rho+\int_0^\zeta Y_{\alpha,\alpha}(\zeta-\rho)g(\rho)d\rho,
$$

where

$$
\begin{aligned}
Y_{\alpha,\alpha}(\zeta) &= \sum_{k=0}^{\infty}B^k\frac{(\zeta)_+^{k\alpha+\alpha-1}}{\Gamma(k\alpha+\alpha)}+\sum_{k=1}^{\infty}\binom{k}{1}B^{k-1}B_1\frac{(\zeta-0.1)_+^{k\alpha+\alpha-1}}{\Gamma(k\alpha+\alpha)} \\
&\quad +\sum_{k=1}^{\infty}\binom{k}{1}B^{k-1}B_2\frac{(\zeta-0.2)_+^{k\alpha+\alpha-1}}{\Gamma(k\alpha+\alpha)}+\sum_{k=2}^{\infty}\binom{k}{1}\binom{k-1}{1}B^{k-2}B_1B_2\frac{(\zeta-0.3)_+^{k\alpha+\alpha-1}}{\Gamma(k\alpha+\alpha)} \\
&\quad +\sum_{k=2}^{\infty}\binom{k}{2}B^{k-2}B_1^2\frac{(\zeta-0.2)_+^{k\alpha+\alpha-1}}{\Gamma(k\alpha+\alpha)}+\sum_{k=3}^{\infty}\binom{k}{3}B^{k-3}B_1^3\frac{(\zeta-0.3)_+^{k\alpha+\alpha-1}}{\Gamma(k\alpha+\alpha)}.
\end{aligned}
$$

12 Oct 2023 19:04:39 PDT
231012-LiMengmeng Version 1 - Submitted to Rocky Mountain J. Math.

Set $p = q = 2$ and $\gamma = 0.4$. By computation, we have $\|B\| = 0.1$, $\|B_1\| = 0.3$, $\|B_2\| = 0.25$, $\lambda = \|B\| + \|B_1\| + \|B_2\| = 0.65$, $\|\varphi\| = 0.1$, $\|g\| = 0.8$, $\phi(0.4) = 0.2921$, $\Psi_1(0.4) = 0.8846$, $\Psi_2(0.4) = 0.1223$ and $\Psi_3(0.4) = 0.08$. We present FTS results of (19) in Table 1.

Theorem	$\|\varphi\|$	α	T	δ	$\|y\|_{C_\gamma}$	η	FTS
3.1	0.1	0.8	0.4	0.11	0.4826	0.49	Yes
3.2	0.1	0.8	0.4	0.11	0.3869	0.39(optimal)	Yes
3.3	0.1	0.8	0.4	0.11	0.4758	0.48	Yes

Table 1: FTS results of (19) with $T = 0.4$.

By Definition 2.6, we look for an applicable η making $\|y\|_{C_\gamma}$ of (1) does not over η on $(0, T]$. Firstly, we apply the explicit solution formula of (19) to obtain the corresponding $\eta = 0.2529$ when $T = 0.4$ (see Figure 1). Secondly, by reviewing Theorems 3.1, 3.2 and 3.3 for $[-0.2, 0.4]$, we contrast to the value η in Table 1 to pick out an optimal value $\eta = 0.39$.

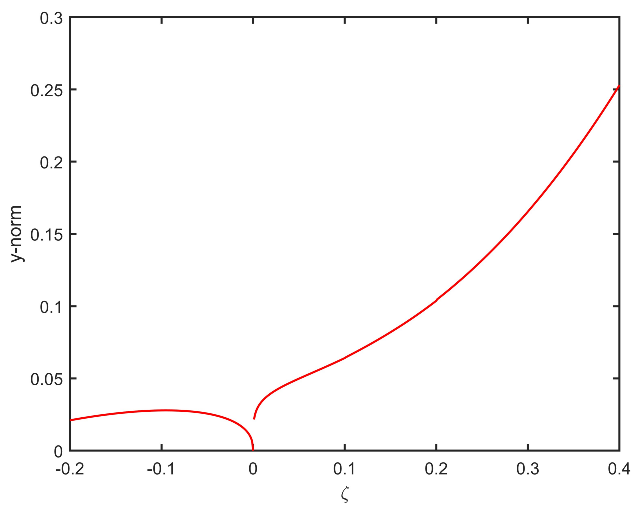

Figure 1: $\|\zeta^{0.4}y(\zeta)\|$ of (19) with $\zeta \in [-0.2, 0.4]$.

Example 5.2. *Set* $\alpha = 0.6$, $\vartheta_1 = 0.3$, $\vartheta_2 = 0.2$, $\vartheta = 0.3$ *and* $\zeta_1 = 0.6$. *Consider*

$$\begin{cases} ({}^{RL}D^{\alpha}_{0+}y)(\zeta) = By(\zeta) + \sum_{i=1}^{2} B_i y(\zeta - \vartheta_i) + Cu(\zeta), \ \zeta \in (0, 0.6], \\ D^{\alpha-1}_{0+}y(\zeta) = \varphi(0) = (0, 0)^\top, \\ y(\zeta) = (\zeta, \zeta^2)^\top, \ \ -0.3 \le \zeta < 0, \end{cases} \tag{20}$$

where $u \in L^2([0, 0.6], \mathbb{R}^2)$ *and*

$$y(\zeta) = \begin{pmatrix} y_1(\zeta) \\ y_2(\zeta) \end{pmatrix}, \quad B = \begin{pmatrix} 0.2 & 0.2 \\ 0 & 0.2 \end{pmatrix}, \ B_1 = \begin{pmatrix} 0.5 & 0.3 \\ 0 & 0.5 \end{pmatrix},$$

$$B_2 = \begin{pmatrix} 0.5 & -0.3 \\ 0 & 0.5 \end{pmatrix}, \ C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

By calculation, the multi-delayed Gramian matrix of (20) via (10) can be achieved as following:

$$W_{0.3,0.2}[0,0.6] = \int_0^{0.6} Y_{\alpha,\alpha}(0.6-\rho)CC^\top Y_{\alpha,\alpha}^\top(0.6-\rho)d\rho,$$

where

$$\begin{aligned} Y_{\alpha,\alpha}(0.6-\rho) &= \sum_{k=0}^{\infty} B^k \frac{(0.6-\rho)_+^{k\alpha+\alpha-1}}{\Gamma(k\alpha+\alpha)} + \sum_{k=1}^{\infty} \binom{k}{1} B^{k-1} B_2 \frac{(0.4-\rho)_+^{k\alpha+\alpha-1}}{\Gamma(k\alpha+\alpha)} \\ &\quad + \sum_{k=1}^{\infty} \binom{k}{1} B^{k-1} B_1 \frac{(0.3-\rho)_+^{k\alpha+\alpha-1}}{\Gamma(k\alpha+\alpha)} + \sum_{k=2}^{\infty} \binom{k}{2} B^{k-2} B_2^2 \frac{(0.2-\rho)_+^{k\alpha+\alpha-1}}{\Gamma(k\alpha+\alpha)} \\ &\quad + \sum_{k=2}^{\infty} \binom{k}{1}\binom{k-1}{1} B^{k-2} B_1 B_2 \frac{(0.1-\rho)_+^{k\alpha+\alpha-1}}{\Gamma(k\alpha+\alpha)}. \end{aligned}$$

By computation, one has

$$W_{0.3,0.2}[0,0.6] = \begin{pmatrix} 3.0433 & 0.2531 \\ 0.2531 & 3.0109 \end{pmatrix}, \quad W_{0.3,0.2}^{-1}[0,0.6] = \begin{pmatrix} 0.3309 & -0.0278 \\ -0.0278 & 0.3345 \end{pmatrix}.$$

Let $y(\zeta_1) = (y_1, y_2)$, by (11), one has $u \in L^2([0,0.6], \mathbb{R}^2)$ as

$$\begin{aligned} u(\zeta) &= C^\top Y_{\alpha,\alpha}^\top(0.6-\zeta) W_{0.3,0.2}^{-1}[0,0.6]\xi \\ &= C^\top \Bigg(\sum_{k=0}^{\infty} (B^\top)^k \frac{(0.6-\zeta)_+^{k\alpha+\alpha-1}}{\Gamma(k\alpha+\alpha)} + \sum_{k=1}^{\infty} \binom{k}{1} (B^\top)^{k-1} (B_2^\top) \frac{(0.4-\zeta)_+^{k\alpha+\alpha-1}}{\Gamma(k\alpha+\alpha)} \\ &\quad + \sum_{k=1}^{\infty} \binom{k}{1} (B^\top)^{k-1} (B_1^\top) \frac{(0.3-\zeta)_+^{k\alpha+\alpha-1}}{\Gamma(k\alpha+\alpha)} + \sum_{k=1}^{\infty} \binom{k}{2} (B^\top)^{k-2} (B_2^\top)^2 \frac{(0.2-\zeta)_+^{k\alpha+\alpha-1}}{\Gamma(k\alpha+\alpha)} \\ &\quad + \sum_{k=1}^{\infty} \binom{k}{1}\binom{k-1}{1} (B^\top)^{k-2} B_1^\top B_2^\top \frac{(0.1-\zeta)_+^{k\alpha+\alpha-1}}{\Gamma(k\alpha+\alpha)} \Bigg) W_{0.3,0.2}^{-1}[0,0.6]\xi, \end{aligned}$$

where

$$\begin{aligned} \xi &= y(\zeta_1) - Y_{\alpha,\alpha}(\zeta_1)y_0 - \sum_{i=1}^{2} \int_{-\vartheta_i}^{\min(\zeta_1-\vartheta_i,0)} Y_{\alpha,\alpha}(\zeta_1-\vartheta_i-\rho)B_i\varphi(\rho)d\rho \\ &= \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} - \int_{-0.3}^{0} Y_{\alpha,\alpha}(0.3-\rho)B_1\varphi(\rho)d\rho - \int_{-0.2}^{0} Y_{\alpha,\alpha}(0.4-\rho)B_2\varphi(\rho)d\rho \\ &= \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} - \begin{pmatrix} -0.0644 \\ 0.0128 \end{pmatrix} = \begin{pmatrix} y_1 + 0.0644 \\ y_2 - 0.0128 \end{pmatrix}. \end{aligned}$$

By Theorem 4.1, we know that (20) is relatively controllable. Figures 2 and 3 show the state $y(\zeta)$ of (20) when we set $y = (y_1, y_2)^\top = (0.8, 2)^\top$ and $y = (y_1, y_2)^\top = (1.2, 1.2)^\top$.

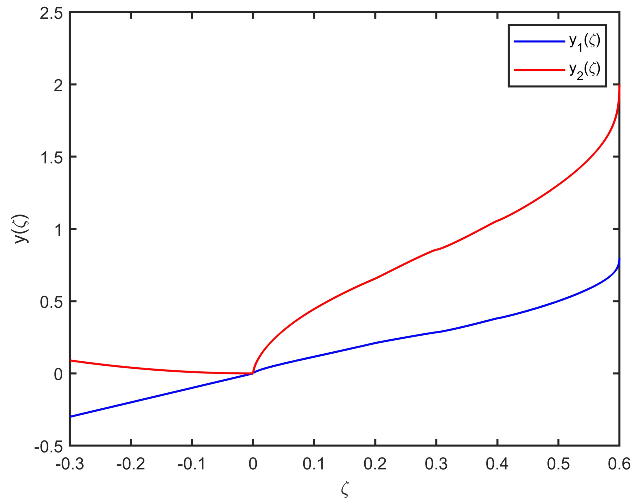

Figure 2: The state $y(\zeta)$ of (20) with $y = (y_1, y_2)^\top = (0.8, 2)^\top$.

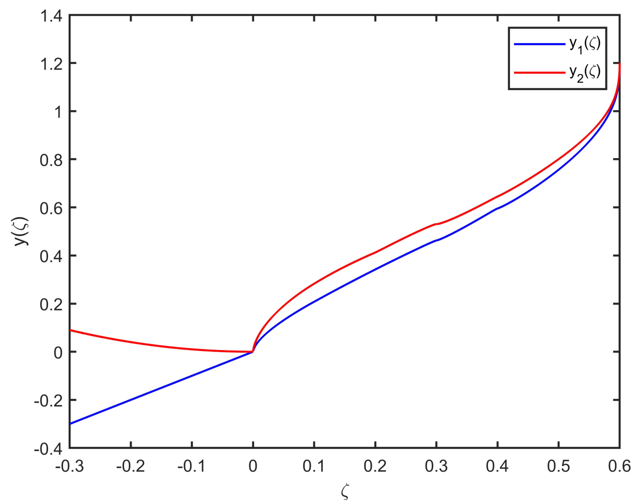

Figure 3: The state $y(\zeta)$ of (20) with $y = (y_1, y_2)^\top = (1.2, 1.2)^\top$.

Example 5.3. *Set* $\alpha=0.6$, $\vartheta_1=0.3$, $\vartheta_2=0.2$, $\vartheta=0.3$ *and* $\zeta_1=0.6$. *Consider*

$$\begin{cases} ({}^{RL}D_{0+}^{\alpha}y)(\zeta)=By(\zeta)+\sum_{i=1}^{2}B_iy(\zeta-\vartheta_i)+g(\zeta,y(\zeta))+Cu(\zeta),\ \zeta\in(0,0.6], \\ D_{0+}^{\alpha-1}y(\zeta)=\varphi(0)=(0,0)^{\top}, \\ y(\zeta)=(\zeta,\zeta^2)^{\top},\ \ -0.3\le\zeta<0, \end{cases} \tag{21}$$

where B, B_1, B_2 *and* C *are the same as in Example 5.2,* $u\in L^2([0,0.6],\mathbb{R}^2)$ *and*

$$g(\zeta,y(\zeta))=\begin{pmatrix} 10^{-2}\zeta y_1(\zeta) \\ 10^{-2}\zeta y_2(\zeta) \end{pmatrix}.$$

We now utilize (15) to calculate M. From Example 5.2, we obtain

$$W_{0.3,0.2}[0,0.6]=\begin{pmatrix} 3.0433 & 0.2531 \\ 0.2531 & 3.0109 \end{pmatrix},\ \ W_{0.3,0.2}^{-1}[0,0.6]=\begin{pmatrix} 0.3309 & -0.0278 \\ -0.0278 & 0.3345 \end{pmatrix}.$$

Therefore, one has $M=\sqrt{\|W_{0.3,0.2}^{-1}[0,0.6]\|}=\sqrt{0.3623}=0.6019$. Consequently, $\mathcal{W}$ satisfies assumption $[A_4]$. For any $\widetilde{y}(\zeta)$, $\widehat{y}(\zeta)\in\mathbb{R}^2$ and $\zeta\in[0,0.6]$, one has

$$\|g(\zeta,\widetilde{y}(\zeta))-g(\zeta,\widehat{y}(\zeta))\|\le 10^{-2}\zeta(|\widetilde{y}_1(\zeta)-\widehat{y}_1(\zeta)|+|\widetilde{y}_2(\zeta)-\widehat{y}_2(\zeta)|)\le 10^{-2}\zeta\|\widetilde{y}-\widehat{y}\|.$$

Hence, g satisfies the assumption $[A_5]$, where $\omega(\zeta)=10^{-2}\zeta\in C([0,0.6],\mathbb{R}^+)$.

Set $\gamma=0.5$, one has $\widetilde{M}=\sup_{\zeta\in[0,0.6]}\omega(\zeta)=0.006$, $\lambda=\|B\|+\|B_1\|+\|B_2\|=2$, $\|C\|=1$, $M_2=40.3693$ and $M_3=4.9883$. Therefore, one has

$$\zeta_1^{\alpha}\widetilde{M}M_2(1+MM_3\|C\|)=0.7135<1,$$

which guarantees that (18) holds. By Theorem 4.2, (21) is relatively controllable on $[0,0.6]$.

References

[1] M. Li, J. Wang, Finite time stability of fractional delay differential equations, Appl. Math. Lett. 64(2017), 170-176.

[2] C. Liang, J. Wang, D. O'Regan, Representation of a solution for a fractional linear system with pure delay, Appl. Math. Lett. 77(2018), 72-78.

[3] M. Li, J. Wang, Finite time stability and relative controllability of Riemann-Liouville fractional delay differential equations, Math. Methods Appl. Sci. 42(2019), 6607-6623.

[4] V. N. Phata, N. T. Thanh, New criteria for finite-time stability of nonlinear fractional-order delay systems: A Gronwall inequality approach, Appl. Math. Lett. 83(2018), 169-175.

[5] G. C. Wu, D. Baleanu, S. D. Zeng, Finite-time stability of discrete fractional delay systems: Gronwall inequality and stability criterion, Commun. Nonlinear Sci. Numer. Simul. 57(2018), 299-308.

[6] M. Li, J. Wang, Exploring delayed Mittag-Leffler type matrix functions to study finite time stability of fractional delay differential equations, Appl. Math. Comput. 324(2018), 254-265.

[7] M. Li, M. Fečkan, J. Wang, Representation and finite time stability of solution and relative controllability of conformable type oscillating systems, Math. Methods Appl. Sci. 46(2023), 3966-3982.

[8] X. Hei, R. Wu, Finite-time stability of impulsive fractional-order systems with time-delay, Appl. Math. Modelling, 40(2016), 4285-4290.

[9] X, Cao, J. Wang, Finite-time stability of a class of oscillating systems with two delays, Math. Methods Appl. Sci. 41(2018), 4943-4954.

[10] S. Abbas, M. Benchohra, M. Rivero, et al, Existence and stability results for nonlinear fractional order Riemann-Liouville Volterra-Stieltjes quadratic integral equations, Appl. Math. Comput. 247(2014), 319-328.

[11] M. Medved, M. Pospišil, Sufficient conditions for the asymptotic stability of nonlinear multidelay differential equations with linear parts defined by pairwise permutable matrices, Nonlinear Anal. 75(2012), 3348-3363.

[12] D. Y. Khusainov, G. V. Shuklin, Relative controllability in systems with pure delay, Internat. Appl. Mech. 41(2005), 210-221.

[13] J. Diblík, D. Y. Khusainov, M. Růžičková, Controllability of linear discrete systems with constant coefficients and pure delay, SIAM J. Control Optim. 47(2008), 1140-1149.

[14] J. Diblík, M. Fečkan, M. Pospíšil, On the new control functions for linear discrete delay systems, SIAM J. Control Optim. 52(2014), 1745-1760.

[15] Z. You, M. Fečkan, J. Wang, Relative controllability of fractional delay differential equations via delayed perturbation of Mittag-Leffler functions, J. Comput. Appl. Math. 378(2020), 112939.

[16] M. Pospíšil, Relative controllability of neutral differential equations with a delay, SIAM J. Control Optim. 55(2017), 835-855.

[17] J. Wang, Z. Luo, M. Fečkan, Relative controllability of semilinear delay differential systems with linear parts defined by permutable matrices, Eur. J. Control, 38(2017), 39-46.

[18] J. Wang, M. Fečkan, Y. Zhou, Controllability of Sobolev type fractional evolution systems, Dyn. Partial Differ. Equ. 11(2014), 71-87.

[19] Z. You, M. Fečkan, J. Wang, et al, Relative controllability of impulsive multi-delay differential systems, Nonlinear Anal. Model. Control, 27(2022), 70-90.

[20] Y. Si, J. Wang, Relative controllability of multiagent systems with pairwise different delays in states, Nonlinear Anal. Model. Control, 27(2022), 289-307.

[21] N. I. Mahmudov, Delayed perturbation of Mittag-Leffler functions and their applications to fractional linear delay differential equations, Math. Methods Appl. Sci. 42(2019), 5489-5497.

[22] N. I. Mahmudov, Multi-delayed perturbation of Mittag-Leffler type matrix function, J. Math. Anal. Appl. 505(2022), 125589.

[23] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier, 2006.