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STABLE CLOSE-TO-CONVEXITY AND RADIUS OF FULL CONVEXITY FOR SENSE-PRESERVING HARMONIC MAPPINGS

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ABSTRACT. Given a sense-preserving harmonic function defined in the open unit disk with its analytic part restricted to the class of starlike functions, several techniques are developed to construct stable close-to-convex harmonic mappings. Each technique is demonstrated through illustrations. Moreover, the radii of full convexity are computed for sense-preserving harmonic functions with the analytic part belonging to certain subclasses of univalent functions. The obtained bounds are sharp.

1. Introduction

A complex-valued harmonic function f defined in an open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ can be represented as $f = h + \bar{g}$, where the functions h and g are analytic in $\mathbb{D}$. Let $\mathscr{H}$ denotes the collection of such harmonic functions with the normalization $h(0) = h'(0) - 1 = g(0) = 0$ and $\mathscr{H}^0 \subseteq \mathscr{H}$ consists of harmonic functions which are further normalized by $g'(0) = 0$. We say that a harmonic function $f = h + \bar{g}$ is sense-preserving in $\mathbb{D}$ if the Jacobian $J_f(z) = |h'(z)|^2 - |g'(z)|^2 > 0$ for all $z \in \mathbb{D}$ which is equivalent to saying that the analytic function $w_f : \mathbb{D} \to \mathbb{C}$ defined as $w_f(z) = g'(z)/h'(z)$ satisfies $|w_f(z)| < 1$ for all $z \in \mathbb{D}$. The function w_f is known as the dilatation of f. The subclass of $\mathscr{H}^0$ consisting of sense-preserving univalent harmonic functions is denoted by $\mathscr{S}_H^0$ and it reduces to the classical family $\mathscr{S}$ of normalized univalent analytic functions in $\mathbb{D}$ if the co-analytic part of each harmonic function in $\mathscr{S}_H^0$ is zero. In 1984, Clunie and Sheil-Small [4] initiated the investigation of the class $\mathscr{S}_H^0$ and its geometric subclasses $\mathscr{C}_H^0$, $\mathscr{S}_H^{*0}$ and $\mathscr{K}_H^0$ consisting of functions mapping $\mathbb{D}$ onto a close-to-convex, starlike and convex domain respectively. The corresponding subclasses of $\mathscr{S}$ are denoted by $\mathscr{C}$, $\mathscr{S}^*$ and $\mathscr{K}$ respectively.

Several authors [2, 4, 9–13, 15–17, 30] have investigated the properties of sense-preserving harmonic functions by restricting their analytic part. If analytic part of a sense-preserving harmonic function f is convex, then f must be univalent in $\mathbb{D}$ by [4, Theorem 5.17]. However, if we restrict the analytic part of a sense-preserving harmonic function f to the family of starlike functions, then f need not be univalent in $\mathbb{D}$ (see [20, Example 1, p. 203]). Hotta and Michalski [9] studied the properties of a univalent harmonic function f with starlike analytic part and obtained the coefficient, distortion and growth estimates of the co-analytic part; and growth and Jacobian estimates of f. Klimek-Smęt and Michalski [12] carried out the similar analysis by considering the harmonic functions with convex analytic part. Zhu and Huang [30] extended these results by investigating the harmonic functions with analytic part as a univalent convex or starlike function of order $\beta \in [0, 1)$.

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Hernández and Martín [8] introduced the notion of stable harmonic mappings. A sense-preserving harmonic function $f = h + \bar{g}$ is said to be stable univalent (resp. stable close-to-convex and stable convex) if all the mappings $f_\varepsilon = h + \varepsilon\bar{g}$ with $|\varepsilon| = 1$ are univalent (resp. close-to-convex and convex) in $\mathbb{D}$. They proved that a sense-preserving harmonic mapping $f = h + \bar{g}$ is stable univalent (resp. stable close-to-convex and stable convex) if and only if the analytic functions $F_\varepsilon = h + \varepsilon g$ are univalent (resp. close-to-convex and convex) in $\mathbb{D}$ for each $|\varepsilon| = 1$. Let $\mathcal{S}\mathcal{S}^0_H$, $\mathcal{S}\mathcal{C}^0_H$ and $\mathcal{S}\mathcal{K}^0_H$ be the subclasses of $\mathcal{S}^0_H$ consisting of stable univalent, stable close-to-convex and stable convex harmonic mappings respectively. In Section 2, different techniques of constructing stable close-to-convex harmonic mappings are investigated from sense-preserving harmonic functions with their analytic part belonging to the class $\mathcal{S}^*$. In addition, concrete examples are provided to demonstrate the obtained results.

It is well-known that the hereditary property of convex analytic mappings does not generalize to harmonic functions. Chuaqui, Duren and Osgood [3] introduced the notion of fully convex functions that do inherit the property of convexity. A harmonic mapping of the unit disk is said to be fully convex if it maps every circle $|z| = r < 1$ in a one-to-one manner onto a convex curve. A fully convex harmonic function is necessarily univalent in $\mathbb{D}$ by Radó-Kneser-Choquet theorem [6, Section 3.1]. The radius of full convexity of the class $\mathcal{K}^0_H$ is $\sqrt{2}-1$ [25], while the radius of full convexity of the classes $\mathcal{S}^{*0}_H$ and $\mathcal{C}^0_H$ is $3-\sqrt{8}$ [22, 29]. However, the exact radius of full convexity of the class $\mathcal{S}^0_H$ is still unsettled. In the last section of the paper, the sharp radius of full convexity has been determined for sense-preserving harmonic functions with certain constraints on their analytic part.

2. Construction of stable close-to-convex harmonic mappings

Recall that an analytic function f in $\mathbb{D}$ with $f(0) = 0 = f'(0) - 1$ is close-to-convex in $\mathbb{D}$ if either of the following conditions are satisfied:

- $\operatorname{Re}(f'(z)/g'(z)) > 0$ for all $z \in \mathbb{D}$ for some $g \in \mathcal{K}$; or
- $\operatorname{Re}(zf'(z)/g(z)) > 0$ for all $z \in \mathbb{D}$ for some $g \in \mathcal{S}^*$.

These two conditions will be termed as *close-to-convexity criteria* and have been extensively used throughout this section.

Let $f = h + \bar{g} \in \mathcal{H}^0$ and $\phi_\varepsilon = h + \varepsilon g$ where $\varepsilon \in \mathbb{C}$ with $|\varepsilon| = 1$. It is easy to verify that if f is sense-preserving in $\mathbb{D}$, then

$$\left|\frac{\phi'_\varepsilon(z)}{h'(z)} - 1\right| < 1 \tag{2.1}$$

for all $z \in \mathbb{D}$. This, in turn, gives

$$\operatorname{Re}\frac{\phi'_\varepsilon(z)}{h'(z)} > 0 \quad \text{for all } z \in \mathbb{D}. \tag{2.2}$$

Thus if $f = h + \bar{g} \in \mathcal{H}^0$ is sense-preserving and the analytic function $h \in \mathcal{K}$, then by (2.2) and close-to-convexity criteria, ϕ_ε is close-to-convex in $\mathbb{D}$ for each $|\varepsilon| = 1$. Consequently, $f \in \mathcal{S}\mathcal{C}^0_H$. The following theorem gives another method of constructing stable close-to-convex harmonic mappings associated with the positive harmonic Alexander operator defined by Nagpal and Ravichandran [19, Definition 4.1, p. 582]. The following lemma due to Sakaguchi [27] will be used in our investigation.

Lemma 2.1. *[27, Lemma, p. 74] Let $D \in \mathcal{S}^*$ and N be analytic in $\mathbb{D}$ with $N(0) = N'(0) - 1 = 0$. If $\operatorname{Re}(N'(z)/D'(z)) > 0$ for $z \in \mathbb{D}$, then $\operatorname{Re}(N(z)/D(z)) > 0$ for $z \in \mathbb{D}$.*

Theorem 2.2. *Let $f = h + \bar{g} \in \mathcal{H}^0$ be sense-preserving harmonic function and $h \in \mathcal{S}^*$. If the analytic function $\psi \in \mathcal{K}$, then the harmonic function $F = H + \overline{G}$ is stable close-to-convex in $\mathbb{D}$, where H and G are given by*

$$H(z) = \int_0^z \frac{(\psi * h)(\xi)}{\xi} d\xi \quad \text{and} \quad G(z) = \int_0^z \frac{(\psi * g)(\xi)}{\xi} d\xi. \tag{2.3}$$

Here $*$ denotes the convolution or the Hadamard product of analytic functions.

Proof. Let $\phi_\varepsilon = h + \varepsilon g$ where $\varepsilon \in \mathbb{C}$ with $|\varepsilon| = 1$. Then (2.2) is satisfied so that Lemma 2.1 gives

$$\operatorname{Re} \frac{\phi_\varepsilon(z)}{h(z)} > 0 \tag{2.4}$$

for all $z \in \mathbb{D}$, since $h \in \mathcal{S}^*$. Also, as $\psi \in \mathcal{K}$, by invoking a result of Ruscheweyh [24, Theorem 2.4, p. 54], it follows that

$$\operatorname{Re} \frac{(\psi * h(\phi_\varepsilon/h))(z)}{(\psi * h)(z)} = \operatorname{Re} \frac{(\psi * \phi_\varepsilon)(z)}{(\psi * h)(z)} > 0$$

for all $z \in \mathbb{D}$. Moreover, it is known that the analytic function $\psi * h \in \mathcal{S}^*$ by [26]. Hence by close-to-convexity criteria, the analytic function

$$\int_0^z \frac{(\psi * \phi_\varepsilon)(\xi)}{\xi} d\xi = \int_0^z \frac{(\psi * h)(\xi)}{\xi} d\xi + \varepsilon \int_0^z \frac{(\psi * g)(\xi)}{\xi} d\xi$$

is close-to-convex for all $|\varepsilon| = 1$. This shows that $F \in \mathcal{SC}_H^0$. □

Remark 2.3. Under the hypothesis of Theorem 2.2, it can be shown that $|g(z)| < |h(z)|$ for all $z \in \mathbb{D} \setminus \{0\}$. To see this, let $0 \neq z \in \mathbb{D}$. As h is univalent and $h(0) = 0$, $h(z_0) \neq 0$. If $g(z_0) = 0$, then the inequality $|g(z_0)| < |h(z_0)|$ is automatically satisfied. If $g(z_0) \neq 0$, then (2.4) gives $\operatorname{Re}(1 + \varepsilon g(z_0)/h(z_0)) > 0$ and by choosing $\varepsilon = -e^{-i \arg(g(z_0)/h(z_0))}$, we conclude that the required inequality is satisfied.

Hotta and Michalski [9] proved a result which is a special case of Theorem 2.2. We present this result in the form of a corollary and it can be deduced by taking $\psi(z) = z/(1-z)$ in Theorem 2.2.

Corollary 2.4. *Let $f = h + \bar{g} \in \mathcal{H}^0$ be sense-preserving harmonic function and $h \in \mathcal{S}^*$. Then the positive harmonic Alexander operator $\Lambda_H^+ : \mathcal{H} \to \mathcal{H}$ defined by*

$$\Lambda_H^+[f] = \Lambda[h] + \overline{\Lambda[g]}, \quad f = h + \bar{g} \in \mathcal{H}$$

is stable close-to-convex in $\mathbb{D}$, where Λ is the Alexander integral operator defined as

$$\Lambda[p](z) = \int_0^z \frac{p(\xi)}{\xi} d\xi$$

for an analytic function p in $\mathbb{D}$ with $p(0) = p'(0) - 1 = 0$.

Let us illustrate Theorem 2.2 and Corollary 2.4 with an example.

Example 2.5. If $f_1 = h_1 + \bar{g}_1 \in \mathcal{H}^0$ where h_1 and g_1 are given by

$$h_1(z) = \frac{z}{(1-z)^2} \in \mathcal{S}^* \quad \text{and} \quad g_1(z) = \frac{-z+2z^2}{(1-z)^2} - \log(1-z),$$

then f_1 is sense-preserving in $\mathbb{D}$ with dilatation $w_{f_1}(z) = z$. By Corollary 2.4, the harmonic function $F_1 = H_1 + \overline{G}_1$ is in $\mathcal{SC}^0_H$, where H_1 and G_1 are given by

$$H_1(z) = \frac{z}{1-z} \quad \text{and} \quad G_1(z) = \frac{z}{1-z} + 2\log(1-z) + Li_2(z) = \sum_{n=2}^{\infty} \left(\frac{n-1}{n}\right)^2 z^n.$$

Here, Li_2 is the polylogarithm function of order 2 [14] defined by the power series

$$Li_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2} = z + \frac{z^2}{4} + \frac{z^3}{9} + \cdots.$$

If we take

$$\psi_0(z) = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right) \in \mathcal{K},$$

and make use of the fact that $(\psi_0 * \phi)(z) = (\Lambda[\phi](z) - \Lambda[\phi](-z))/2$ for an analytic function ϕ in $\mathbb{D}$ with $\phi(0) = \phi'(0) - 1 = 0$, where Λ is the Alexander integral operator, then

$$(\psi_0 * h_1)(z) = \frac{z}{1-z^2} \quad \text{and} \quad (\psi_0 * g_1)(z) = \sum_{n=1}^{\infty} \frac{4n^2}{(2n+1)^2} z^{2n+1}.$$

By Theorem 2.2, it follows that the harmonic function $F_2 = H_2 + \overline{G}_2$ is stable close-to-convex in $\mathbb{D}$, where $H_2 = \psi_0$ and

$$G_2(z) = \tanh^{-1}(z) - z - \frac{1}{2}z^3\Phi\left(z^2, 2, \frac{3}{2}\right) + \frac{1}{8}z^3\Phi\left(z^2, 3, \frac{3}{2}\right) = \sum_{n=1}^{\infty} \frac{4n^2}{(2n+1)^3} z^{2n+1}.$$

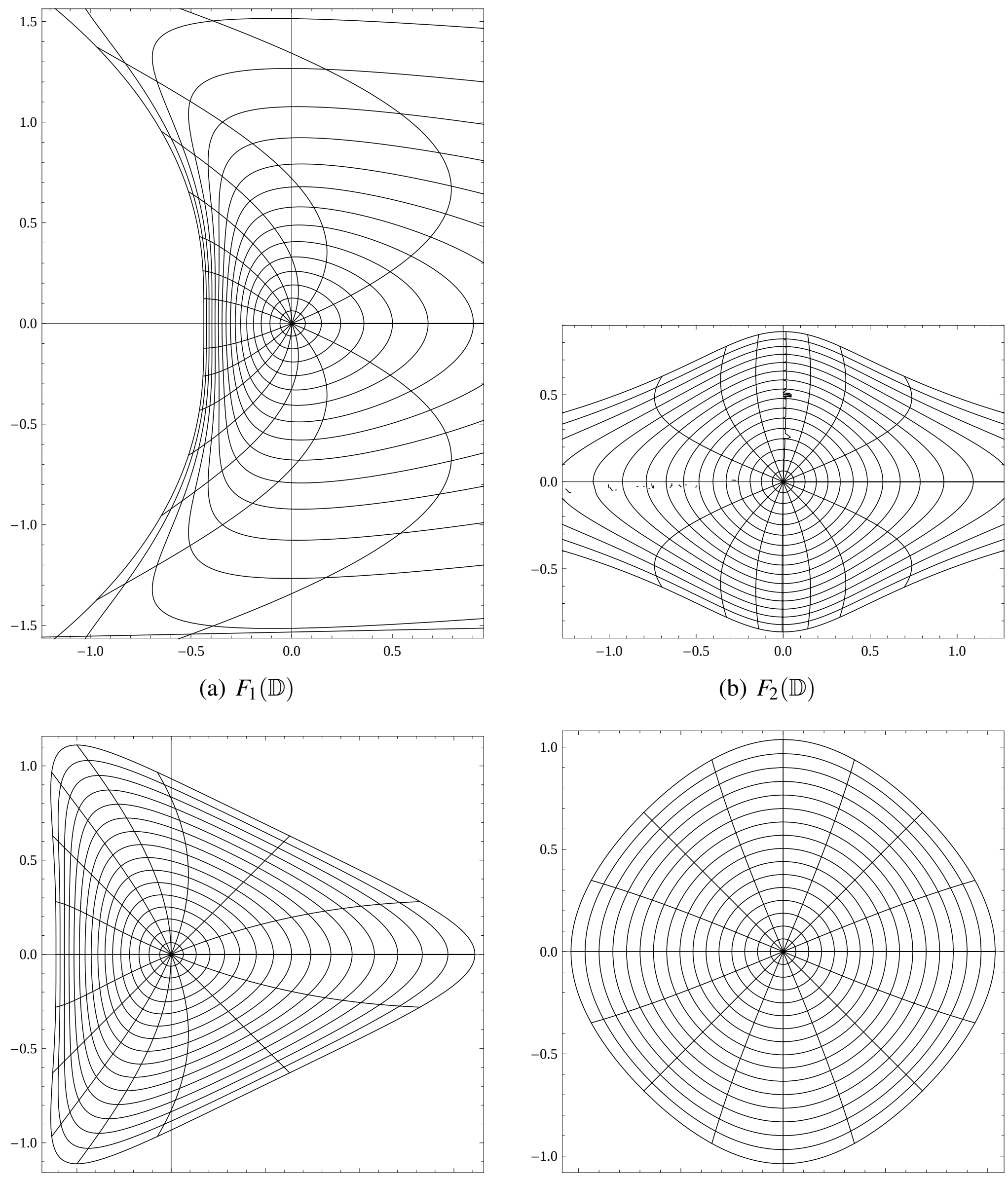
Note that $\Phi(z,s,a)$ is the Lerch transcendental function defined by the power series

$$\Phi(z,s,a) = \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s}$$

provided $a \neq 0, -1, -2, \ldots$. This series is convergent in $\mathbb{D}$ for all values of s (see [7]).

Similarly, if we take $f_2(z) = z + z^2/2 + \bar{z}^2/2 + \bar{z}^3/3$, then f_2 is sense-preserving in $\mathbb{D}$ with dilatation $w_{f_2}(z) = z$. Since the analytic part of f_2 is starlike in $\mathbb{D}$, therefore the harmonic function $F_3(z) = z + z^2/4 + \bar{z}^2/4 + \bar{z}^3/9$ is stable close-to-convex in $\mathbb{D}$ by Corollary 2.4. With $\psi_0 \in \mathcal{K}$ as defined above, Theorem 2.2 yields the harmonic function $F_4(z) = z + \bar{z}^3/27 \in \mathcal{SC}^0_H$. The image domains of the constructed functions F_i $(i = 1,2,3,4)$ are illustrated in Figure 1.

It is important to point out that Theorem 2.2 does not hold if we weaken the condition that $\psi \in \mathcal{K}$. To see this, consider the function $f(z) = z + z^2/2 - \bar{z}^2/2 - \bar{z}^3/3$. Its analytic part is starlike in $\mathbb{D}$ and the resulting harmonic function obtained from (2.3) by taking the non-convex function $\psi(z) = z/(1-z)^2$ is f itself which is not even univalent in $\mathbb{D}$ as $f(7e^{it}/8) = f(7e^{-it}/8)$ for $t = 2\tan^{-1}(3\sqrt{3})$. In the next theorem, stable close-to-convex harmonic functions are generated under the hypothesis of Theorem 2.2 with $\psi \in \mathcal{S}^*$.



(a) $F_1(\mathbb{D})$ (b) $F_2(\mathbb{D})$

(c) $F_3(\mathbb{D})$ (d) $F_4(\mathbb{D})$

FIGURE 1.

Theorem 2.6. *Let $f = h + \bar{g} \in \mathscr{H}^0$ be sense-preserving and $h \in \mathscr{S}^*$. If the analytic function $\psi \in \mathscr{S}^*$, then the harmonic function $F = H + \overline{G}$ is stable close-to-convex in $\mathbb{D}$, where H and G are given by*

$$H(z) = \int_0^z \frac{\psi(\xi)}{\xi} d\xi \quad \text{and} \quad G(z) = \int_0^z \frac{\psi(\xi) g(\xi)}{\xi h(\xi)} d\xi.$$

Proof. Let $\phi_\varepsilon = h+\varepsilon g$ for $|\varepsilon|=1$. Since $h\in\mathscr{S}^*$, therefore in view of (2.2) and Lemma 2.1, the condition (2.4) is satisfied, which may be rewritten as

$$\operatorname{Re}\frac{(\psi(z)\phi_\varepsilon(z)/h(z))}{\psi(z)} = \operatorname{Re}\frac{\phi_\varepsilon(z)}{h(z)} > 0 \tag{2.5}$$

for all $z\in\mathbb{D}$. As $\psi\in\mathscr{S}^*$, the close-to-convexity criteria implies that the analytic function

$$\int_0^z \frac{\psi(\xi)\phi_\varepsilon(\xi)}{\xi h(\xi)}d\xi = \int_0^z \frac{\psi(\xi)}{\xi}d\xi + \varepsilon\int_0^z \frac{\psi(\xi)g(\xi)}{\xi h(\xi)}d\xi$$

is close-to-convex in $\mathbb{D}$ for each $|\varepsilon|=1$. Hence F is stable close-to-convex in $\mathbb{D}$. □

Theorem 2.6 reduces to Corollary 2.4 by choosing $h=\psi$. Let us demonstrate Theorem 2.6 for the starlike functions $\psi(z)=z/(1-z)$ and $\psi(z)=z$ in the next example.

Example 2.7. Under the hypothesis of Theorem 2.6, if we take $\psi(z)=z/(1-z)\in\mathscr{S}^*$, then the harmonic function

$$F(z) = -\log(1-z) + \overline{\int_0^z \frac{g(\xi)}{(1-\xi)h(\xi)}d\xi}$$

is stable close-to-convex in $\mathbb{D}$. Therefore the harmonic functions f_1 and f_2 defined in Example 2.5 lead to the functions

$$P_1(z) = -\log(1-z) + \overline{Li_2(z) - 3z - 2\log(1-z) + z\log(1-z)}$$

and

$$P_2(z) = -\log(1-z) + \overline{\frac{1}{9}(-6z - 5\log(1-z) + 2\log(2+z) - 2\log 2)}$$

respectively in the class $\mathscr{SC}_H^0$.

Similarly, if we consider $\psi(z)=z$, then Theorem 2.6 generates the stable close-to-convex harmonic function

$$F(z) = z + \overline{\int_0^z \frac{g(\xi)}{h(\xi)}d\xi}.$$

As a consequence, the following harmonic functions

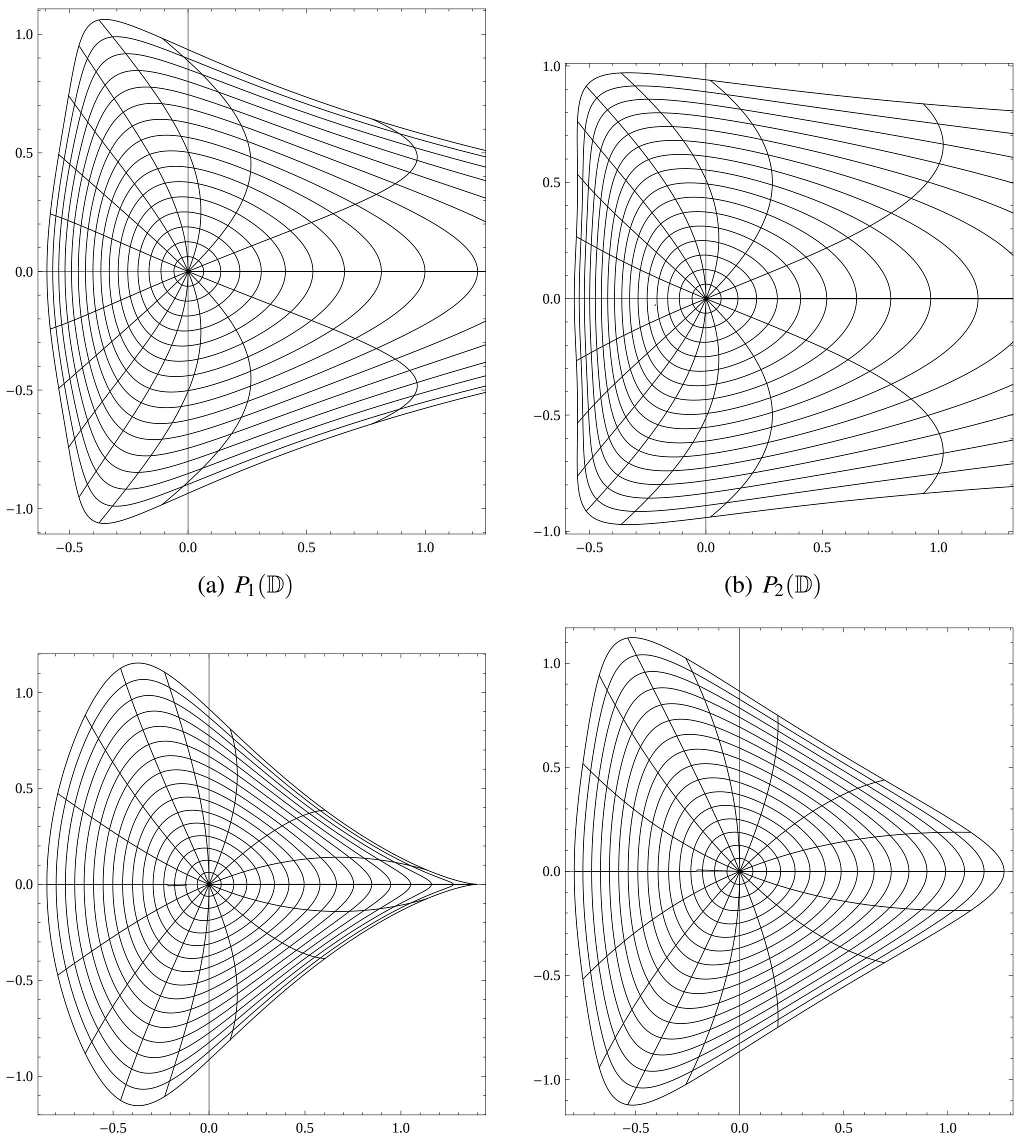
$$P_3(z) = z + \overline{Li_2(z) - \frac{5}{4}z(2-z) - \frac{1}{2}(3-z)(1-z)\log(1-z)}$$

and

$$P_4(z) = z + \overline{\frac{1}{3}(2\log(2+z) - (1-z)z - \log 4)}$$

lie in the class $\mathscr{SC}_H^0$ which are formed by considering the functions f_1 and f_2 respectively defined in Example 2.5. The image domains $P_i(\mathbb{D})$ are depicted in Figure 2 for $i=1,2,3,4$.

For particular choices of ψ, the conclusion of Theorem 2.6 can be further strengthened as seen by the following corollary. The proof makes use of the fact that an analytic function f in $\mathbb{D}$ with $f(0)=0$ and $f'(0)\neq 0$ maps the unit disk univalently onto a domain (i) convex in the direction of real axis if $\operatorname{Re}((1-z)^2 f'(z))>0$ for all $z\in\mathbb{D}$ [5]; and (ii) convex in the direction of imaginary axis if $\operatorname{Re}((1-z^2)f'(z))>0$ for all $z\in\mathbb{D}$ [23].



(a) $P_1(\mathbb{D})$

(b) $P_2(\mathbb{D})$

(c) $P_3(\mathbb{D})$

(d) $P_4(\mathbb{D})$

FIGURE 2.

Corollary 2.8. *Let $f = h + \bar{g} \in \mathscr{H}^0$ be sense-preserving in $\mathbb{D}$ and $h \in \mathscr{S}^*$.*

(i) *The harmonic function $F_1 = H_1 + \overline{G}_1$, where H_1 and G_1 are given by*

$$H_1(z) = \frac{z}{1-z} \quad \text{and} \quad G_1(z) = \int_0^z \frac{g(\xi)}{(1-\xi)^2 h(\xi)} d\xi$$

is stable close-to-convex in $\mathbb{D}$ and its range is convex in the direction of the real axis.

(ii) *The harmonic function* $F_2 = H_2 + \overline{G_2}$, *where* H_2 *and* G_2 *are given by*

$$H_2(z) = \frac{1}{2}\log\frac{1+z}{1-z} \quad \text{and} \quad G_2(z) = \int_0^z \frac{g(\xi)}{(1-\xi^2)h(\xi)}d\xi$$

is stable close-to-convex in $\mathbb{D}$ *and its range is convex in the direction of the imaginary axis.*

Proof. Clearly, $F_1, F_2 \in \mathscr{SC}_H^0$ by Theorem 2.6 with ψ as $z/(1-z)^2$ and $z/(1-z^2)$ respectively. The function $\phi_\varepsilon = h + \varepsilon g$ satisfies (2.4) so that

$$\mathrm{Re}\left((1-z)^2\frac{\phi_\varepsilon(z)}{(1-z)^2h(z)}\right) = \mathrm{Re}\left((1-z)^2\left(\frac{1}{(1-z)^2} + \frac{\varepsilon g(z)}{(1-z)^2h(z)}\right)\right) > 0$$

for all $\varepsilon \in \mathbb{C}$ with $|\varepsilon| = 1$ and for all $z \in \mathbb{D}$. This proves that the analytic functions

$$\int_0^z \frac{d\xi}{(1-\xi)^2} + \varepsilon\int_0^z \frac{g(\xi)}{(1-\xi)^2h(\xi)}d\xi$$

are univalent and their images are convex in the direction of the real axis for all $\varepsilon \in \mathbb{C}$ with $|\varepsilon| = 1$. By a theorem of Clunie and Sheil-Small [4, Theorem 5.3, p. 14], it follows that the harmonic function F_1 is stable close-to-convex and its range is convex in the direction of the real axis. This proves part (i). The proof of other part can be deduced by simply replacing the term $(1-z)^2$ by $(1-z^2)$. $\square$

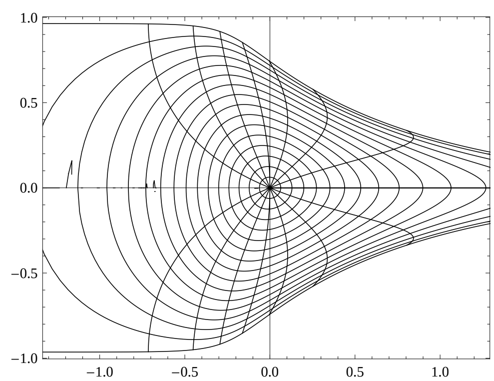
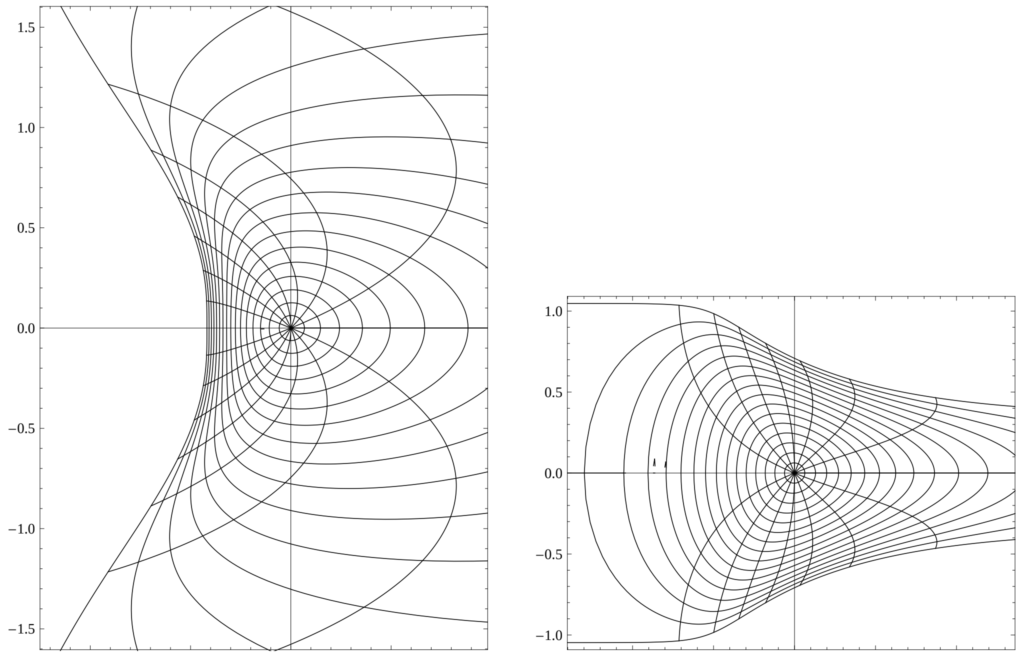


FIGURE 3. Image domain $F_2(\mathbb{D})$

With the function f_1 defined in Example 2.5, Corollary 2.8(i) yields the harmonic function F_1 which has the same expression as stated in Example 2.5. It is clearly evident from Figure 1(a) that image domain $F_1(\mathbb{D})$ is convex in the direction of the real axis. Also, Corollary 2.8(ii) provides a stable close-to-convex harmonic function $F_2(z) = (1/2)\log((1+z)/(1-z)) + \overline{G_2}(z)$ whose range is convex in the direction of the imaginary axis (see Figure 3), where G_2 is given by

$$G_2(z) = \frac{1}{6}\Big(\pi^2 - 6(\log 2)^2 - 3\log(1-z) + 3(\log(16)-3)\log(1+z) + 6Li_2(z) - 12Li_2\left(\frac{1+z}{2}\right)\Big).$$



(a) $W_1(\mathbb{D})$ (b) $W_2(\mathbb{D})$

FIGURE 4.

Another illustration is provided in the following example.

Example 2.9. If we consider the harmonic function f_2 as defined in Example 2.5, then Corollary 2.8 give rise to the stable close-to-convex harmonic functions

$$W_1(z) = \frac{z}{1-z} + \overline{\frac{1}{27}\left(\frac{15z}{1-z} - \log 4 + 16\log(1-z) + 2\log(2+z)\right)}$$

and

$$W_2(z) = \frac{1}{2}\log\frac{1+z}{1-z} + \overline{\frac{1}{18}\left(\log 16 - 5\log(1-z) - 3\log(1+z) - 4\log(2+z)\right)}$$

whose ranges are convex in the direction of the real axis and imaginary axis respectively. (see Figure 4).

The last result of this section generates harmonic functions in $\mathscr{SC}_H^0$ by dropping the condition on h as stated in Theorem 2.6.

Theorem 2.10. *Let $f = h + \bar{g} \in \mathscr{H}^0$ be sense-preserving in $\mathbb{D}$ with dilatation w_f. If $\psi \in \mathscr{S}^*$, then the harmonic function $F = H + \overline{G}$ is stable close-to-convex in $\mathbb{D}$, where H and G are given by*

$$H(z) = \int_0^z \frac{\psi(\xi)}{\xi}\,d\xi \quad \textit{and} \quad G(z) = \int_0^z \frac{w_f(\xi)\psi(\xi)}{\xi}\,d\xi.$$

In particular, the following harmonic functions belong to the class $\mathcal{SC}_H^0$:

$$Q_1(z) = \frac{z}{1-z} + \overline{\int_0^z \frac{w_f(\xi)}{(1-\xi)^2} d\xi}, \quad Q_2(z) = \frac{1}{2}\log\frac{1+z}{1-z} + \overline{\int_0^z \frac{w_f(\xi)}{1-\xi^2} d\xi}$$
$$Q_3(z) = -\log(1-z) + \overline{\int_0^z \frac{w_f(\xi)}{1-\xi} d\xi}, \quad Q_4(z) = z + \overline{\int_0^z w_f(\xi) d\xi}.$$

Moreover, the image domains $Q_1(\mathbb{D})$ and $Q_2(\mathbb{D})$ are convex in the direction of the real axis and imaginary axis respectively.

Proof. Since f is sense-preserving in $\mathbb{D}$, therefore the analytic functions $\phi_\varepsilon = h + \varepsilon g$ satisfy condition (2.2) which can be reformulated as

$$\operatorname{Re} \frac{(\psi(z)\phi_\varepsilon'(z)/h'(z))}{\psi(z)} = \operatorname{Re} \frac{\phi_\varepsilon'(z)}{h'(z)} > 0$$

for all $z \in \mathbb{D}$ and for all $\varepsilon \in \mathbb{C}$ with $|\varepsilon| = 1$. As $\psi \in \mathcal{S}^*$, therefore by close-to-convexity criteria, the analytic function

$$\int_0^z \frac{\psi(\xi)\phi_\varepsilon'(\xi)}{\xi h'(\xi)} d\xi = \int_0^z \frac{\psi(\xi)}{\xi} d\xi + \varepsilon \int_0^z \frac{\psi(\xi) g'(\xi)}{\xi h'(\xi)} d\xi = \int_0^z \frac{\psi(\xi)}{\xi} d\xi + \varepsilon \int_0^z \frac{w_f(\xi)\psi(\xi)}{\xi} d\xi$$

is close-to-convex in $\mathbb{D}$. Hence $F \in \mathcal{SC}_H^0$. By choosing ψ as $z/(1-z)^2$, $z/(1-z^2)$, $z/(1-z)$ and z, it is easy to deduce that the harmonic functions Q_i $(i = 1,2,3,4)$ are stable close-to-convex in $\mathbb{D}$ respectively. The convexity in one direction of the functions Q_1 and Q_2 is similar to the proof of Corollary 2.7 and therefore its details are omitted. □

The main key point of Theorem 2.10 relies on the observation that the analytic part of the sense-preserving harmonic function is not even required to be univalent in $\mathbb{D}$. Let us illustrate it with the help of following example.

Example 2.11. Let $f = h + \bar{g} \in \mathcal{H}^0$, where h and g are given by

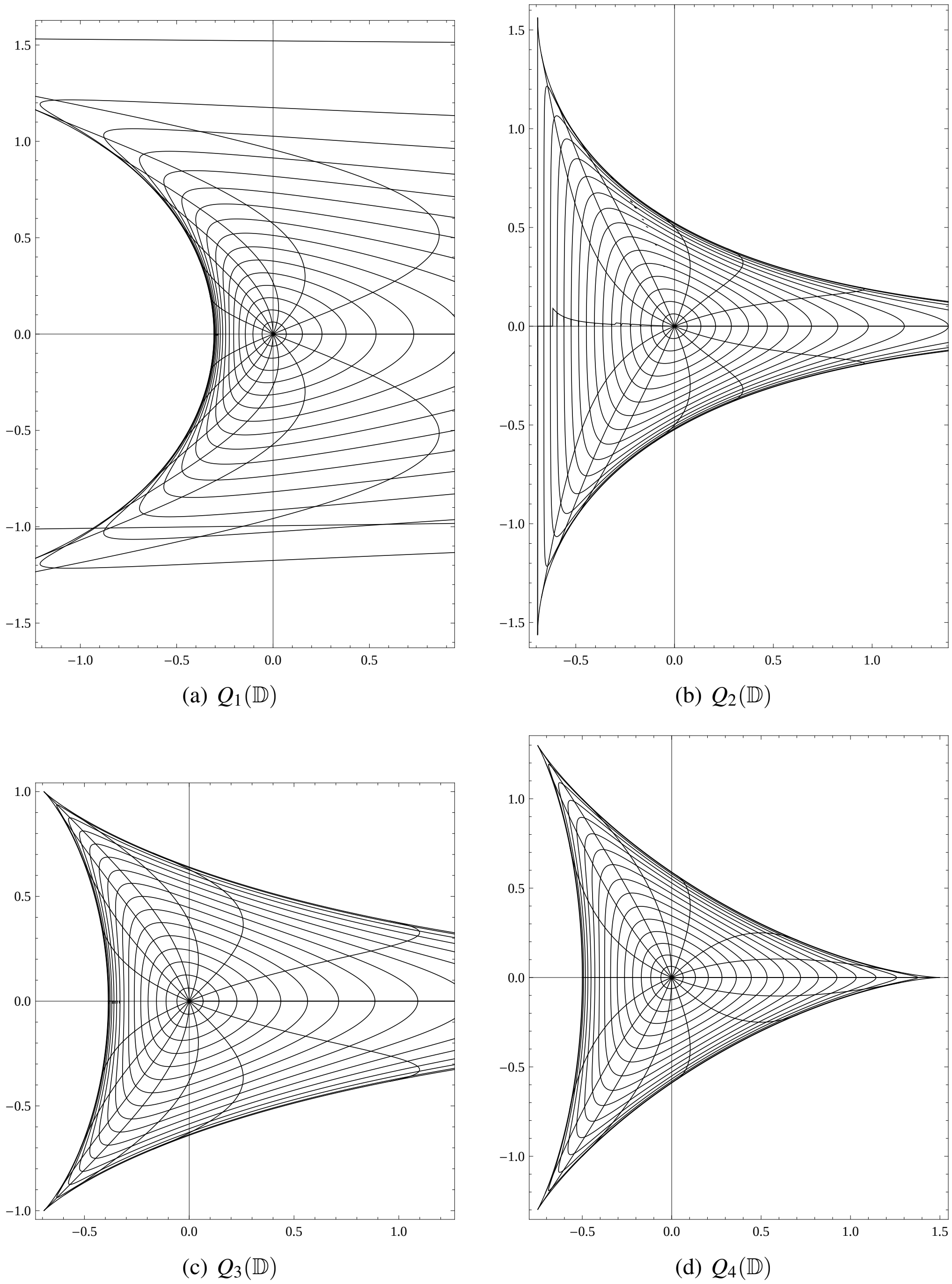
$$h(z) = z + z^2 \quad \text{and} \quad g(z) = \frac{1}{2}\bar{z}^2 + \frac{2}{3}\bar{z}^3.$$

Then f is sense-preserving in $\mathbb{D}$ with dilatation $w_f(z) = z$. Note that h is not univalent in $\mathbb{D}$. By Theorem 2.10, the following functions are in the class $\mathcal{SC}_H^0$:

$$Q_1(z) = \frac{z}{1-z} + \overline{\frac{z}{1-z} + \log(1-z)}, \quad Q_2(z) = \frac{1}{2}\log\frac{1+z}{1-z} - \overline{\frac{1}{2}\log(1-z^2)}$$

$$Q_3(z) = -\log(1-z) - \bar{z} - \overline{\log(1-z)}, \quad Q_4(z) = z + \frac{1}{2}\bar{z}^2.$$

The image domains $Q_i(\mathbb{D})$ are illustrated in Figure 5. Clearly, $Q_1(\mathbb{D})$ is convex in the direction of the real axis and $Q_2(\mathbb{D})$ is convex in the direction of imaginary axis.



(a) $Q_1(\mathbb{D})$ (b) $Q_2(\mathbb{D})$

(c) $Q_3(\mathbb{D})$ (d) $Q_4(\mathbb{D})$

FIGURE 5.

3. Radius of full convexity

In this section, we will determine the radius of fully convexity of the harmonic functions whose analytic part belong to various subclasses of univalent functions. Hernández and Martín [8, Corollary 4.1, p. 350] proved that a sense-preserving stable convex harmonic function is fully convex (see also [18, Theorem 2.3, p. 89]). The proof of the main result of this section makes use of this fact along with the results by Ratti [21] which evaluates the radius of convexity of the analytic functions f satisfying

$$\left|\frac{f'(z)}{g'(z)}-1\right|<1 \quad \text{for all } z\in\mathbb{D}$$

and g being a univalent, starlike, convex or a close-to-convex analytic function with the derivative having positive real part in $\mathbb{D}$.

Theorem 3.1. *Let $f=h+\bar{g}\in\mathcal{H}^0$ be a sense-preserving harmonic function.*

(i) *If $h\in\mathcal{S}$ or $\mathcal{S}^*$, then f is fully convex in $|z|<1/5$.*

(ii) *If h is convex of order α, that is, $\operatorname{Re}(1+zh''(z)/h'(z))>\alpha$ for all $z\in\mathbb{D}$, $0<\alpha\leq 1$, then f is fully convex in $|z|<r^*$, where $r^*=r^*(\alpha)$ is the smallest positive root of the equation $1-(3-2\alpha)r-2\alpha r^2=0$. In particular, if $h\in\mathcal{K}$, then f is fully convex in $|z|<1/3$.*

(iii) *If $\operatorname{Re}h'(z)>0$ for all $z\in\mathbb{D}$, then f is fully convex in $|z|<(\sqrt{17}-3)/4$.*

(iv) *If $\operatorname{Re}h'(z)>1/2$ for all $z\in\mathbb{D}$, then f is fully convex in $|z|<r_0$ where r_0 is the smallest positive root of the equation $r^4+2r^3+13r^2+4r-4=0$.*

All the bounds are sharp.

Proof. Let $\phi_\varepsilon=h+\varepsilon g$ where $\varepsilon\in\mathbb{C}$ with $|\varepsilon|=1$. Since f is sense-preserving in $\mathbb{D}$, therefore the inequality (2.1) is satisfied.

(i) If $h\in\mathcal{S}$ or $\mathcal{S}^*$, then by using [21, Theorems 1 and 2], ϕ_ε maps $|z|<1/5$ onto a convex domain for each $|\varepsilon|=1$. Therefore f is stable convex and hence fully convex in $|z|<1/5$. For sharpness, consider the harmonic function $f_1=h_1+\bar{g}_1$ as given in the Example 2.5. A calculation shows that

$$\begin{aligned}\frac{\partial}{\partial\theta}\left(\arg\left\{\frac{\partial}{\partial\theta}f_1(re^{i\theta})\right\}\right)\bigg|_{\theta=\pi}&=\frac{h_1'(-r)+g_1'(-r)-r(h_1''(-r)+g_1''(-r))}{h_1'(-r)-g_1'(-r)}\\&=\frac{1-5r}{(1+r)^2}\end{aligned}$$

which vanishes at $r=1/5$. Hence the obtained radius is sharp (see Figure 6(a)).

(ii) If h is convex of order α, then ϕ_ε satisfies

$$\operatorname{Re}\left(1+\frac{z\phi_\varepsilon''(z)}{\phi_\varepsilon'(z)}\right)>\operatorname{Re}\left(1+\frac{zh''(z)}{h'(z)}\right)-\frac{|z|}{1-|z|},\quad |z|=r \tag{3.1}$$

by using an estimate in [21, Equation (3.1), p. 485]. Also, $\operatorname{Re}(1+zh''(z)/h'(z))>(1+(2\alpha-1)r)/(1+r)$ (see [28, Lemma 3, p. 240]) so that (3.1) gives

$$\operatorname{Re}\left(1+\frac{z\phi_\varepsilon''(z)}{\phi_\varepsilon'(z)}\right)>\frac{1+(2\alpha-1)r}{1+r}-\frac{r}{1-r}=\frac{1-(3-2\alpha)r-2\alpha r^2}{1-r^2}>0$$

whenever $r < r^*$, where r^* is the smallest positive root of the equation $2\alpha r^2 + (3-2\alpha)r - 1 = 0$. For sharpness, consider the function $f_2 = h_2 + \bar{g}_2$, where h_2 and g_2 are given by

$$h_2(z) = \int_0^z \frac{d\xi}{(1+\xi)^{2-2\alpha}} \quad \text{and} \quad g_2(z) = -\int_0^z \frac{\xi}{(1+\xi)^{2-2\alpha}}\, d\xi.$$

It is easy to see that h_2 is convex of order α and dilatation of f_2 is $w_{f_2}(z) = -z$. A calculation shows that

$$\begin{aligned}\frac{\partial}{\partial\theta}\left(\arg\left\{\frac{\partial}{\partial\theta} f_2(re^{i\theta})\right\}\right)\bigg|_{\theta=0} &= \frac{h_2'(r) + g_2'(r) + r(h_2''(r) + g_2''(r))}{h_2'(r) - g_2'(r)} \\ &= \frac{1-(3-2\alpha)r - 2\alpha r^2}{(1+r)^2} = 0\end{aligned}$$

if $r = r^*$. In particular, for $\alpha = 0$, the function f_2 takes the form

$$f_2(z) = \frac{z}{1+z} + \overline{\frac{z}{1+z} - \log(1-z)}$$

which maps the sub-disk $|z| < 1/3$ onto a convex domain (see Figure 6(b)).

(iii) If $\operatorname{Re} h'(z) > 0$ for all $z \in \mathbb{D}$, then [21, Theorem 4, p. 486] shows that ϕ_ε maps $|z| < (\sqrt{17}-3)/4$ onto a convex domain. Consequently f is fully convex in $|z| < (\sqrt{17}-3)/4$. To check the sharpness of the bound, we consider the harmonic function $f_3 = h_3 + \bar{g}_3$ given by

$$h_3(z) = -z - 2\log(1-z) \quad \text{and} \quad g_3(z) = -2z - \frac{z^2}{2} - 2\log(1-z), \quad z \in \mathbb{D}.$$

Observe that f_3 is sense-preserving in $\mathbb{D}$ with dilatation $w_{f_3}(z) = z$ and

$$\begin{aligned}\frac{\partial}{\partial\theta}\left(\arg\left\{\frac{\partial}{\partial\theta} f_3(re^{i\theta})\right\}\right)\bigg|_{\theta=\pi} &= \frac{h_3'(-r) + g_3'(-r) - r(h_3''(-r) + g_3''(-r))}{h_3'(-r) - g_3'(-r)} \\ &= \frac{1-3r-2r^2}{(1+r)^2} = 0\end{aligned}$$

when $r = (\sqrt{17}-3)/4$ (see Figure 6(c)).

(iv) If $\operatorname{Re} h'(z) > 1/2$ for all $z \in \mathbb{D}$, then by [21, Theorem 5, p. 587], it is easy to deduce that ϕ_ε is convex and hence f is fully convex in $|z| < r_0$, where r_0 is the smallest positive root of the equation $r^4 + 2r^3 + 13r^2 + 4r - 4 = 0$. To check the sharpness of this result, consider the harmonic function $f_4 = h_4 + \bar{g}_4$, where h_4 and g_4 are defined as

$$h_4'(z) = \frac{1}{1+z\phi(z)} \quad \text{and} \quad g_4'(z) = -zh_4'(z)$$

with

$$\phi(z) = \frac{z+b}{1+bz}, \quad b = \frac{1}{2+r_0}.$$

Clearly $f_4 \in \mathscr{H}^0$ is sense-preserving in $\mathbb{D}$ with dilatation $w_{f_4}(z) = -z$ and $\operatorname{Re} h_4'(z) > 1/2$ for all $z \in \mathbb{D}$. Since

$$\begin{aligned} \frac{\partial}{\partial\theta}\left(\arg\left\{\frac{\partial}{\partial\theta} f_4(r_0 e^{i\theta})\right\}\right)\Bigg|_{\theta=0} &= \frac{h_4'(r_0) + g_4'(r_0) + r(h_4''(r_0) + g_4''(r_0))}{h_4'(r_0) - g_4'(r_0)} \\ &= \frac{4 - 4r_0 - 13r_0^2 - 2r_0^3 - r_0^4}{2(1+r_0)^2(2+r_0+r_0^2)} = 0, \end{aligned}$$

therefore the radius r_0 is best possible (see Figure 6(d)).

The shaded region in Figure 6 depicts the image of the sub-disks $|z| < 1/5$, $|z| < 1/3$, $|z| < (\sqrt{17}-3)/4$ and $|z| < r_0$ under the mappings f_1, f_2, f_3 and f_4 respectively. $\square$

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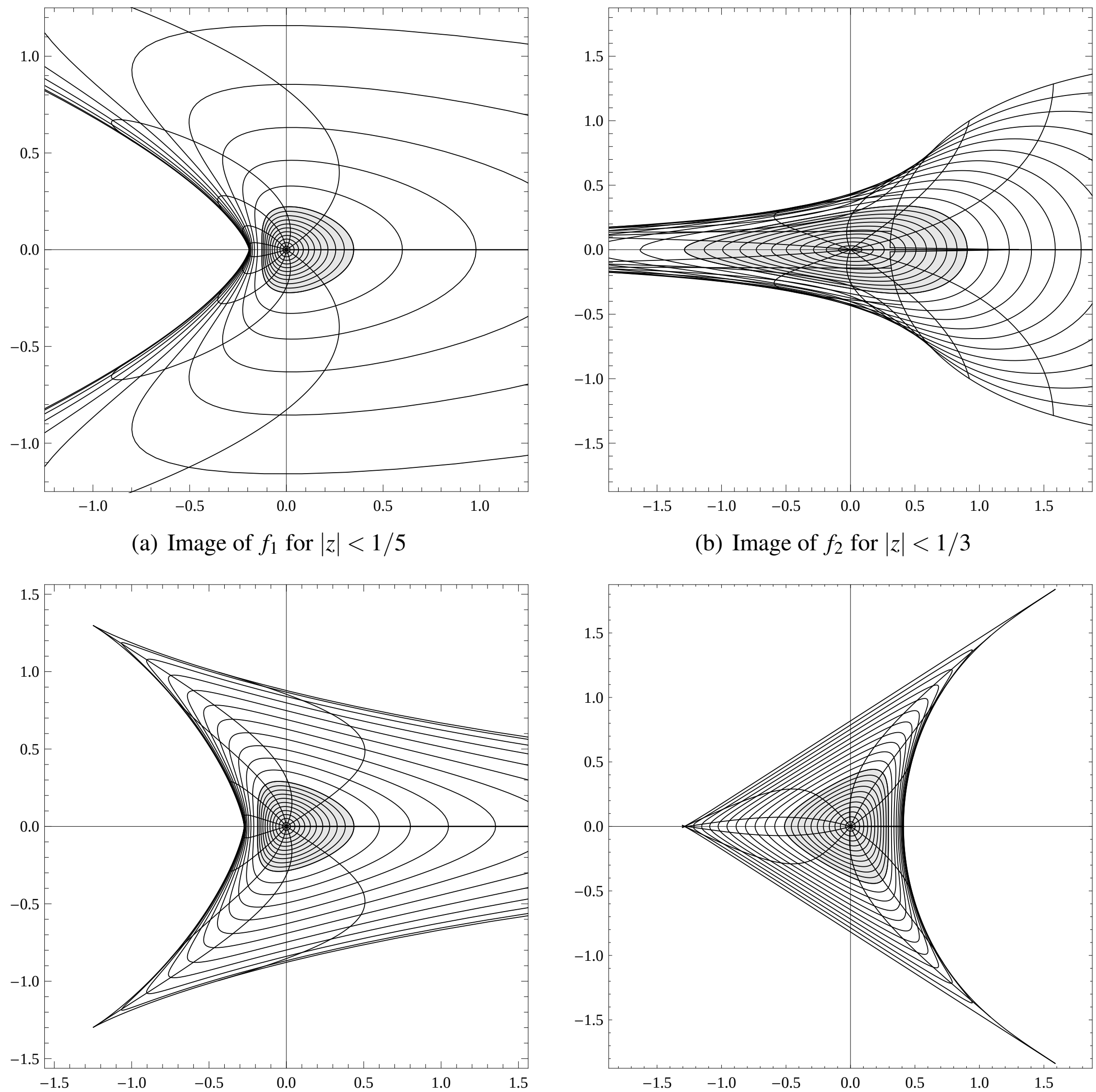

(a) Image of f_1 for $|z| < 1/5$

(b) Image of f_2 for $|z| < 1/3$

(c) Image of f_3 for $|z| < (\sqrt{17}-3)/4$

(d) Image of f_4 for $|z| < r_0$

FIGURE 6.

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