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Some inequalities for the Csiszár f-divergence operator mapping

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Abstract Let $\mathbf{A} = \{A_1, ..., A_n\}$ and $\mathbf{B} = \{B_1, ..., B_n\}$ be two finite sequences of strictly positive operators on a Hilbert space $\mathcal{H}$ and f, $h : \mathbb{I} \to \mathbb{R}$ continuous functions with $h > 0$. We consider the generalized Csiszár f-divergence operator mapping defined by

$$\mathbf{I}_{f\Delta h}(\mathbf{A}, \mathbf{B}) = \sum_{i=1}^{n} P_{f\Delta h}(A_i, B_i),$$

where

$$P_{f\Delta h}(A, B) := h(A)^{1/2} f(h(A)^{-1/2} B h(A)^{-1/2}) h(A)^{1/2}$$

is introduced for every strictly positive operator A and every self-adjoint operator B, where the spectra of the operators

$$A,\ A^{-1/2} B A^{-1/2} \text{ and } h(A)^{-1/2} B h(A)^{-1/2}$$

are contained in the closed interval $\mathbb{I}$.

In this paper we prove several inequalities for $\mathbf{I}_{f\Delta h}(\mathbf{A}, \mathbf{B})$ with applications to the relative operator (α, β)-entropy that contains as particular cases the usual and the generalized relative operator entropies.

Keywords Operator inequality · subadditivity · convexity · divergence · entropy · perspective.

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1 Introduction and Preliminaries

The classical perspective function associated to a continuous function $f : [0, \infty) \to \mathbb{R}$ is a function of two variables defined by $P_f(s, t) := sf(\frac{t}{s})$, cf. [18].

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For two discrete probability distributions $p = (p_1, ..., p_n)$ and $q = (q_1, ..., q_n)$ the f-divergence functional

$$I_f(p, q) = \sum_{i=1}^{n} P_f(p_i, q_i)$$

was introduced by Csiszár [5] as a distance function on the set of discrete probability distributions.

Let f and h be two real valued continuous functions defined on the closed interval $\mathbb{I}$ and $h > 0$. The value $f(A)$ is defined via the functional calculus as usual for a self–adjoint operator A whose spectrum is contained in $\mathbb{I}$. A fully noncommutative perspective of two variables (associated to f), by choosing an appropriate ordering, was introduced in [12] by setting

$$P_f(A, B) := A^{1/2} f(A^{-1/2} B A^{-1/2}) A^{1/2}$$

and the operator version of a fully noncommutative generalized perspective of two variables (associated to f and h) was also introduced by setting

$$P_{f\Delta h}(A, B) := h(A)^{1/2} f(h(A)^{-1/2} B h(A)^{-1/2}) h(A)^{1/2}$$

for every strictly positive operator A and every self-adjoint operator B on a Hilbert space $\mathcal{H}$, where the spectra of the operators

$$A, \ A^{-1/2} B A^{-1/2} \text{ and } h(A)^{-1/2} B h(A)^{-1/2}$$

are contained in the closed interval $\mathbb{I}$. Note that in this situation $P_{f\Delta h}(A, B) = P_f(h(A), B)$. Then, several striking matrix analogues of a classical result for operator convex functions were proved. More precisely, the necessary and sufficient conditions for the joint convexity of a fully noncommutative perspective and generalized perspective function were proved where restricting to the positive commuting matrices ensures Effros' approach announced in [13].

To provide some applications for some well-known noncommutative operator divergences, we recall the following definitions:

The relative entropy or Kullback-Leibler divergence [19] between two probability distributions $P = \{p_1, ..., p_n\}$, $Q = \{q_1, ..., q_n\}$ was defined as

$$D(P||Q) = \sum_{i=1}^{n} p_i \log \frac{p_i}{q_i}.$$

The χ^2-divergence was proposed by Pearson [31] via the formula

$$\chi^2(P, Q) = \sum_{i=1}^{n} \frac{(p_i - q_i)^2}{q_i}.$$

The Hellinger distance [3] was defined by

$$H(P, Q) = \frac{1}{2} \sum_{i=1}^{n} (\sqrt{p_i} - \sqrt{q_i})^2.$$

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Let $f(t) = -\log t$, $g(t) = \frac{(t-1)^2}{t}$, $h(t) = \frac{1}{2}(\sqrt{t}-1)^2$. Then,

$$I_f(P,Q) = D(P||Q),\ I_g(P,Q) = \chi^2(P,Q) \text{ and } I_h(P,Q) = H(P,Q).$$

Since $f(1) = g(1) = h(1) = 0$, one can observe that $D(P||Q)$, $\chi^2(P,Q)$, and $H(P,Q)$ are non-negative.

Now, we consider another useful divergence measure in information theory which is known as the Harmonic distance:

$$M(P,Q) = \sum_{i=1}^{n} \frac{2p_i q_i}{p_i + q_i}.$$

We have $M(P,Q) = I_f(P,Q)$ for $f(t) = 2(1+t^{-1})^{-1}$.

In noncommutative information theory, Fujii and Kamei [14] have introduced the relative operator entropy of two strictly positive operators A and B by the formula

$$S(A|B) = A^{1/2}(\log A^{-1/2}BA^{-1/2})A^{1/2}.$$

Later, this notion has been extended to by Furuta [16] for two strictly positive operators A and B and $\alpha \in \mathbb{R}$ by setting

$$S_\alpha(A|B) = A^{1/2}(A^{-1/2}BA^{-1/2})^\alpha(\log A^{-1/2}BA^{-1/2})A^{1/2}.$$

Some upper and lower bounds of the relative operator entropy and the generalized relative operator entropy have been determined in [10, 9, 23, 25, 32, 21, 2].

By using [27, Corollary 2.18(i)], we know that if $mA \le B \le MA$ for some $m, M \in [e^{\frac{2\alpha-1}{\alpha(1-\alpha)}}, \infty)$ with $m < M$ and $0 \le \alpha < 1$, then

$$\begin{aligned} 0 &\le S_\alpha(A|B) - \frac{m^\alpha \log m}{M-m}(MA-B) - \frac{M^\alpha \log M}{M-m}(B-mA) \\ &\le \frac{1}{4}(M-m)\Big(m^{\alpha-1}(1+\alpha\log m) - M^{\alpha-1}(1+\alpha\log M)\Big)A. \end{aligned} \tag{1}$$

The relative operator (α,β)-entropy was defined by the first author [22] as follows:

$$S_{\alpha,\beta}(A|B) = A^{\frac{\beta}{2}}(A^{-\frac{\beta}{2}}BA^{-\frac{\beta}{2}})^\alpha \log(A^{-\frac{\beta}{2}}BA^{-\frac{\beta}{2}})A^{\frac{\beta}{2}}.$$

In particular, one knows $S_{\alpha,1}(A|B) = S_\alpha(A|B)$ and $S_{0,1}(A|B) = S(A|B)$.

For some recent results concerning the relative operator entropy and some new estimates for Tsallis relative operator entropy see [15] and the references therein. A reverse inequality for Tsallis relative operator entropy involving a positive linear map was proved in [20]. In addition, a converse of Ando's inequality and an extension and reverse of the Löwner-Heinz inequality under certain conditions were obtained. Some results of [20] were also generalized in [30].

2 The Csiszár f-divergence operator mapping

Throughout this section we assume that f and h are continuous real valued functions defined on $[0,\infty)$ and $h>0$ unless we note otherwise. The following lemma was proved for the generalized perspective in [22,29].

Lemma 1 *Let r, s, and h be real valued and continuous functions on the closed interval $\mathbb{I}$. If $r(t)\leq s(t)$ for $t\in\mathbb{I}$, then*

$$P_{r\Delta h}(A,B)\leq P_{s\Delta h}(A,B)$$

for every strictly positive operator A and every self-adjoint operator B such that the spectrum of the operator A is in $\mathbb{I}$ and that of $h(A)^{-1/2}Bh(A)^{-1/2}$ is in $\mathbb{I}$.

Let $\mathbf{A}=\{A_1,...,A_n\}$ and $\mathbf{B}=\{B_1,...,B_n\}$ be two finite sequences of strictly positive operators and $f:\mathbb{I}\to\mathbb{R}$ a continuous function. We consider the Csiszár f-divergence operator mapping by setting

$$\mathbf{I}_f(\mathbf{A},\mathbf{B})=\sum_{i=1}^{n}P_f(A_i,B_i)$$

and the generalized Csiszár f-divergence operator mapping via

$$\mathbf{I}_{f\Delta h}(\mathbf{A},\mathbf{B})=\sum_{i=1}^{n}P_{f\Delta h}(A_i,B_i).$$

The joint convexity of the perspective and generalized perspective was proved in [12,24,28].

Theorem 1 *The following statements hold:*

(i) If f is operator convex, then P_f is jointly convex.
(ii) If f is operator convex with $f(0)\leq 0$ and h is operator concave, then $P_{f\Delta h}$ is jointly convex.
(iii) If f and h are operator concave with $f(0)\geq 0$, then $P_{f\Delta h}$ is jointly concave.

These results can be generalized to the Csiszár f-divergence operator mappings. The following corollary is a simple application of the joint convexity of the perspective.

Corollary 1 *The following statements hold:*

(i) If f is operator convex, then $\mathbf{I}_f$ is jointly convex.
(ii) If f is operator convex with $f(0)\leq 0$ and h is operator concave, then $\mathbf{I}_{f\Delta h}$ is jointly convex.
(iii) If f and h are operator concave with $f(0)\geq 0$, then $\mathbf{I}_{f\Delta h}$ is jointly concave.

For a continuous function $g:(0,\infty)\to\mathbb{R}$ the transpose function $\tilde{g}$ of g is defined by

$$\tilde{g}(x)=xg(x^{-1}),\ x>0.$$

Corollary 2 *Let $g : (0, \infty) \to \mathbb{R}$ be a continuous function. Then,*

$$\mathbf{I}_{\tilde{g}}(\mathbf{A}, \mathbf{B}) = \mathbf{I}_g(\mathbf{B}, \mathbf{A})$$

for two finite sequences of strictly positive operators $\mathbf{A}$ *and* $\mathbf{B}$.

Proof According to the relation between the perspective of g and $\tilde{g}$ [26] we have

$$P_{\tilde{g}}(A_i, B_i) = P_g(B_i, A_i).$$

By summing over i we deduce the result.

For a finite sequence of strictly positive operators $\mathbf{A} = \{A_1, ..., A_n\}$ on a Hilbert space $\mathcal{H}$ and a continuous function f, we set

$$S_{\mathbf{A}} := \sum_{i=1}^{n} A_i,$$

$$S_{f(\mathbf{A})} := \sum_{i=1}^{n} f(A_i),$$

where $f(\mathbf{A}) := \{f(A_1), ..., f(A_n)\}$ is a finite sequence of operators on $\mathcal{H}$.

Definition 1 We say that the continuous function f is subadditive if

$$f(S_{\mathbf{A}}) \leq S_{f(\mathbf{A})}$$

for a finite sequence of strictly positive operator $\mathbf{A}$. The function f is called superadditive if the reverse inequality holds, i.e., $f(S_{\mathbf{A}}) \geq S_{f(\mathbf{A})}$.

Remark 1 Let f and $h > 0$ be two continuous functions and $\mathbf{A}$, $\mathbf{B}$ two finite sequences of strictly positive operators. Then,

$$\mathbf{I}_{f \Delta h}(\mathbf{A}, \mathbf{B}) = \sum_{i=1}^{n} P_{f \Delta h}(A_i, B_i) = \sum_{i=1}^{n} P_f(h(A_i), B_i) = \mathbf{I}_f(h(\mathbf{A}), \mathbf{B}).$$

The proof of the following theorem is a direct application of *Hansen-Pedersen-Jensen inequality.*

Theorem 2 *If f is operator convex function, then*

$$P_f(S_{\mathbf{A}}, S_{\mathbf{B}}) \leq \mathbf{I}_f(\mathbf{A}, \mathbf{B}). \tag{2}$$

Proof Let $\mathbf{A} = \{A_1, A_2, ..., A_n\}$ and $\mathbf{B} = \{B_1, B_2, ..., B_n\}$ be two finite sequences of strictly positive operators on a Hilbert space $\mathcal{H}$. Since

$$\sum_{i=1}^{n} S_{\mathbf{A}}^{-1/2} A_i^{1/2} A_i^{1/2} S_{\mathbf{A}}^{-1/2} = I$$

and f is operator convex, it follows via [17, Theorem 2.1] that

$$\begin{aligned} f(S_{\mathbf{A}}^{-1/2}S_{\mathbf{B}}S_{\mathbf{A}}^{-1/2}) &= f\bigg(\sum_{i=1}^{n} S_{\mathbf{A}}^{-1/2}B_iS_{\mathbf{A}}^{-1/2}\bigg) \\ &= f\bigg(\sum_{i=1}^{n} S_{\mathbf{A}}^{-1/2}A_i^{1/2}(A_i^{-1/2}B_iA_i^{-1/2})A_i^{1/2}S_{\mathbf{A}}^{-1/2}\bigg) \\ &\le \sum_{i=1}^{n} S_{\mathbf{A}}^{-1/2}A_i^{1/2}f(A_i^{-1/2}B_iA_i^{-1/2})A_i^{1/2}S_{\mathbf{A}}^{-1/2}. \end{aligned}$$

This ensures that

$$S_{\mathbf{A}}^{1/2}f(S_{\mathbf{A}}^{-1/2}S_{\mathbf{B}}S_{\mathbf{A}}^{-1/2})S_{\mathbf{A}}^{1/2} \le \sum_{i=1}^{n} A_i^{1/2}f(A_i^{-1/2}B_iA_i^{-1/2})A_i^{1/2},$$

which implies the result.

In the dual case (when f is operator concave) the reverse inequality holds in (2).

Corollary 3 *If f is operator convex, then*

$$f(1)S_{\mathbf{A}} \le \mathbf{I}_f(\mathbf{A},\mathbf{B})$$

for two finite sequences of strictly positive operators $\mathbf{A},\mathbf{B}$ with $S_{\mathbf{A}} = S_{\mathbf{B}}$. Moreover, the reverse inequality holds for an operator concave function.

Proof Since $P_f(S_{\mathbf{A}}, S_{\mathbf{A}}) = f(1)S_{\mathbf{A}}$, the result follows from Theorem 2.

Theorem 3 *If f is operator convex with $f(0) \le 0$ and h is superadditive, then*

$$P_{f\Delta h}(S_{\mathbf{A}}, S_{\mathbf{B}}) \le \mathbf{I}_{f\Delta h}(\mathbf{A},\mathbf{B}). \tag{3}$$

Proof Let $\{A_1, A_2, ..., A_n\}$ and $\{B_1, B_2, ..., B_n\}$ be two sequences of strictly positive operators on a Hilbert space $\mathcal{H}$. Define $S_{\mathbf{B}} = \sum_{i=1}^{n} B_i$, $S_{h(\mathbf{A})} = \sum_{i=1}^{n} h(A_i)$, and $T_i := h(A_i)^{1/2}h(S_{\mathbf{A}})^{-1/2}$. It follows from superadditivity of h that

$$\begin{aligned} \sum_{i=1}^{n} T_i^*T_i &= \sum_{i=1}^{n} h(S_{\mathbf{A}})^{-1/2}h(A_i)^{1/2}h(A_i)^{1/2}h(S_{\mathbf{A}})^{-1/2} \\ &= h(S_{\mathbf{A}})^{-1/2}\sum_{i=1}^{n} h(A_i)h(S_{\mathbf{A}})^{-1/2} \\ &= h(S_{\mathbf{A}})^{-1/2}S_{h(\mathbf{A})}h(S_{\mathbf{A}})^{-1/2} \\ &\le h(S_{\mathbf{A}})^{-1/2}h(S_{\mathbf{A}})h(S_{\mathbf{A}})^{-1/2} = I. \end{aligned}$$

So, [17, Corollary 2.3] entails that

$$\begin{aligned}
&f(h(S_{\mathbf{A}})^{-1/2}S_{\mathbf{B}}h(S_{\mathbf{A}})^{-1/2})\\
&=f\left(\sum_{i=1}^{n}h(S_{\mathbf{A}})^{-1/2}B_ih(S_{\mathbf{A}})^{-1/2}\right)\\
&=f\left(\sum_{i=1}^{n}h(S_{\mathbf{A}})^{-1/2}h(A_i)^{1/2}(h(A_i)^{-1/2}B_ih(A_i)^{-1/2})h(A_i)^{1/2}h(S_{\mathbf{A}})^{-1/2}\right)\\
&=f\left(\sum_{i=1}^{n}T_i^*(h(A_i)^{-1/2}B_ih(A_i)^{-1/2})T_i\right)\\
&\le\sum_{i=1}^{n}T_i^*f(h(A_i)^{-1/2}B_ih(A_i)^{-1/2})T_i\\
&=\sum_{i=1}^{n}h(S_{\mathbf{A}})^{-1/2}h(A_i)^{1/2}f(h(A_i)^{-1/2}B_ih(A_i)^{-1/2})h(A_i)^{1/2}h(S_{\mathbf{A}})^{-1/2}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&h(S_{\mathbf{A}})^{1/2}f(h(S_{\mathbf{A}})^{-1/2}S_{\mathbf{B}}h(S_{\mathbf{A}})^{-1/2})h(S_{\mathbf{A}})^{1/2}\\
&\qquad\le\sum_{i=1}^{n}h(A_i)^{1/2}f(h(A_i)^{-1/2}B_ih(A_i)^{-1/2})h(A_i)^{1/2}.
\end{aligned}$$

From here we have

$$P_{f\Delta h}(S_{\mathbf{A}},S_{\mathbf{B}})\le\sum_{i=1}^{n}P_{f\Delta h}(A_i,B_i)=\mathbf{I}_{f\Delta h}(\mathbf{A},\mathbf{B}).$$

Corollary 4 *If f is operator concave with $f(0)\ge 0$ and h is superadditive, then the reverse inequality is valid in* (3).

We note that in Corollary 4 the condition $f(0)\ge 0$ can be removed for a positive operator concave function f.

Theorem 4 *Suppose that f, $h:(0,\infty)\to(0,\infty)$ are continuous functions. If f is operator monotone and h is superadditive, then the reverse inequality is valid in* (3).

Proof Let $\mathbf{A}=\{A_1,A_2,...,A_n\}$ and $\mathbf{B}=\{B_1,B_2,...,B_n\}$ be two sequences of strictly positive operators on a Hilbert space $\mathcal{H}$ and note that a positive operator monotone function f is operator concave ([4, Chapter V]). So,

$$\begin{aligned}
P_{f\Delta h}(S_{\mathbf{A}},S_{\mathbf{B}})&=P_f(h(S_{\mathbf{A}}),S_{\mathbf{B}})\\
&\ge P_f(S_{h(\mathbf{A})},S_{\mathbf{B}})\qquad\text{(by [26, Theorem 2.3])}\\
&\ge\mathbf{I}_f(h(\mathbf{A}),\mathbf{B})\qquad\text{(by Corollary 2)}\\
&=\mathbf{I}_{f\Delta h}(\mathbf{A},\mathbf{B})\qquad\text{(by Remark 1).}
\end{aligned}$$

We may consider a dual form of the above theorem when $f : (0, \infty) \to (0, \infty)$ is operator convex. In this case the operator monotonicity of f makes it to be an affine function. So, the inequality (3) holds when f is affine and h is subadditive without needing the condition $f(0) \leq 0$.

3 Bounds of the Csiszár *f*-divergence operator mapping

Throughout this section we assume that f, $h : [a, b] \subset \mathbb{R} \to \mathbb{R}$ are continuous real valued functions and $h > 0$. We verify the bounds of the Csiszár f-divergence operator mapping.

Theorem 5 *Let $\mathbf{A} = \{A_1, ..., A_n\}, \mathbf{B} = \{B_1, ..., B_n\}$ be two finite sequences of strictly positive operators. If $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$ is a concave and differentiable function and $h : [a, b] \to (0, \infty)$ is a continuous function such that $mh(A_i) \leq B_i \leq Mh(A_i)$ for some $m, M \in [a, b]$ with $0 < m < M$, then*

$$0 \leq \mathbf{I}_{f\Delta h}(\mathbf{A}, \mathbf{B}) - \frac{f(m)}{M-m}(MS_{h(\mathbf{A})} - S_{\mathbf{B}}) - \frac{f(M)}{M-m}(S_{\mathbf{B}} - mS_{h(\mathbf{A})}) \quad (4)$$
$$\leq \frac{1}{4}(M-m)\Big(f'(m) - f'(M)\Big)S_{h(\mathbf{A})}.$$

Proof Due to [11, Corollary 1] and for the concave and differentiable function f, we get

$$0 \leq f((1-c)x + cy) - (1-c)f(x) - cf(y) \quad (5)$$
$$\leq c(1-c)(y-x)(f'(x) - f'(y)),$$

where $c \in [0, 1]$ and x, $y \in [a, b]$ with $x < y$. Replacing $x = m$, $y = M$, and $c = \frac{u-m}{M-m} \in [0, 1]$ in (5), we find that

$$0 \leq f(u) - \frac{f(m)}{M-m}(M-u) - \frac{f(M)}{M-m}(u-m) \leq \frac{f'(m) - f'(M)}{M-m}\Psi(u), \quad (6)$$

where $\Psi(u) = (u-m)(M-u)$. The maximum value of $\Psi(u)$ is $\frac{1}{4}(M-m)^2$. So,

$$\frac{f'(m) - f'(M)}{M-m}\Psi(u) \leq \frac{1}{4}(M-m)(f'(m) - f'(M)). \quad (7)$$

Regarding (6) and (7) one can deduce

$$0 \leq f(u) - \frac{f(m)}{M-m}(M-u) - \frac{f(M)}{M-m}(u-m) \quad (8)$$
$$\leq \frac{1}{4}(M-m)(f'(m) - f'(M)).$$

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Using Lemma 1 and replacing $h(A_i)^{-1/2}B_ih(A_i)^{-1/2}$ with u and then multiplying both sides of the inequality (8) by $h(A_i)^{1/2}$, we get

$$\begin{aligned} 0 &\le P_f(h(A_i),B_i) - \frac{f(m)}{M-m}(Mh(A_i)-B_i) - \frac{f(M)}{M-m}(B_i - mh(A_i)) \\ &\le \frac{1}{4}(M-m)(f'(m)-f'(M))h(A_i). \end{aligned} \tag{9}$$

By summing over i in (9), we reach the desired results.

Remark 2 Under the hypotheses of Theorem 5, if $h : [a,b] \to (0,\infty)$ is the identity function, then

$$\begin{aligned} 0 &\le \mathbf{I}_f(\mathbf{A},\mathbf{B}) - \frac{f(m)}{M-m}(MS_{\mathbf{A}} - S_{\mathbf{B}}) - \frac{f(M)}{M-m}(S_{\mathbf{B}} - mS_{\mathbf{A}}) \\ &\le \frac{1}{4}(M-m)\Big(f'_+(m) - f'_-(M)\Big)S_{\mathbf{A}}, \end{aligned} \tag{10}$$

where f'_- and f'_+ are the left-hand and right-hand derivative of f, respectively. This inequality is a generalization of Theorem 7 from [7] and Theorem 2 from [25] in which only the case of a pair of operators was considered.

Corollary 5 *Let* $\mathbf{A} = \{A_1, ..., A_n\}$ *and* $\mathbf{B} = \{B_1, ..., B_n\}$ *be two finite sequences of strictly positive operators with* $\sum_{i=1}^n A_i = \sum_{i=1}^n B_i = I$ *and* $0 < mA_i \le B_i \le MA_i$. *Then,*

(i) $0 \le \sum_{i=1}^n (A_i^{-1} + B_i^{-1})^{-1} - \frac{mM+1}{(M+1)(m+1)} \le \frac{(m+M+2)(M-m)^2}{4(m+1)^2(M+1)^2}$,

(ii) $0 \le \sum_{i=1}^n S(A_i|B_i) + \frac{\log \frac{M^{m-1}}{m^{M-1}}}{M-m} \le K(\frac{M}{m})$,

where $K(h) = \frac{(h+1)^2}{4h}$, $h > 0$, *is the Kantorovich constant.*

Proof (i) Remark 2 indicates that the bounds of the Csiszár f-divergence operator mapping for the concave function $f(t) = 2(1+t^{-1})^{-1}$ are given by (i).

(ii) The bounds of the Csiszár f-divergence operator mapping for the concave function $f(t) = \log t$ are given by (ii) by using Remark 2.

Corollary 6 *Let* $\mathbf{A} = \{A_1, ..., A_n\}$ *and* $\mathbf{B} = \{B_1, ..., B_n\}$ *be two finite sequences of strictly positive operators with* $\sum_{i=1}^n A_i^\beta = \sum_{i=1}^n B_i = I$, $0 < mA_i^\beta \le B_i \le MA_i^\beta$, $0 \le \alpha < 1$, *and* $\beta \in \mathbb{R}$ *for some* $m, M \in [e^{\frac{2\alpha-1}{\alpha(1-\alpha)}}, \infty)$. *Then,*

$$\begin{aligned} 0 &\le \sum_{i=1}^n S_{\alpha,\beta}(A_i|B_i) + \frac{\log \frac{M^{(m-1)M^\alpha}}{m^{(M-1)m^\alpha}}}{M-m} \\ &\le \frac{1}{4}(M-m)\Big(m^{\alpha-1} - M^{\alpha-1} + \log \frac{m^{\alpha m^{\alpha-1}}}{M^{\alpha M^{\alpha-1}}}\Big). \end{aligned} \tag{11}$$

Proof According to Theorem 5, the bounds of the generalized Csiszár f-divergence operator mapping for the concave function $f(t) = t^\alpha \log t$ on $[e^{\frac{2\alpha-1}{\alpha(1-\alpha)}}, \infty)$ and $h(t) = t^\beta$ can be obtained as (11).

Theorem 6 *Let $\mathbf{A} = \{A_1, ..., A_n\}, \mathbf{B} = \{B_1, ..., B_n\}$ be two finite sequences of strictly positive operators. If $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ is a concave function and $h : [a,b] \to (0,\infty)$ is a continuous function such that $mh(A_i) \le B_i \le Mh(A_i)$ for some $m, M \in [a,b]$ with $0 < m < M$, then*

$$\begin{aligned} &2J_f(m,M)\mathbf{I}_{r\Delta h}(\mathbf{A},\mathbf{B}) \\ &\le \mathbf{I}_{f\Delta h}(\mathbf{A},\mathbf{B}) - \frac{f(m)}{M-m}(MS_{h(\mathbf{A})} - S_{\mathbf{B}}) - \frac{f(M)}{M-m}(S_{\mathbf{B}} - mS_{h(\mathbf{A})}) \\ &\le 2J_f(m,M)\mathbf{I}_{R\Delta h}(\mathbf{A},\mathbf{B}), \end{aligned} \tag{12}$$

where

$$J_f(m,M) := f\left(\frac{M+m}{2}\right) - \frac{f(m)+f(M)}{2},$$

$$r(u) := \min\left\{\frac{u-m}{M-m}, \frac{M-u}{M-m}\right\} = \frac{1}{2} - \left|\frac{u - \frac{M+m}{2}}{M-m}\right|,$$

$$R(u) := \max\left\{\frac{u-m}{M-m}, \frac{M-u}{M-m}\right\} = \frac{1}{2} + \left|\frac{u - \frac{M+m}{2}}{M-m}\right|$$

and $0 < m < M$.

Proof Regarding [6, Theorem 1], we have

$$2rJ_f(x,y) \le f((1-c)x + cy) - ((1-c)f(x) + cf(y)) \le 2RJ_f(x,y) \tag{13}$$

for all $x, y \in (a,b)$ and $c \in [0,1]$, where $r = \min\{c, 1-c\}$ and $R = \max\{c, 1-c\}$. Replacing $x = m$, $y = M$, and $c = \frac{u-m}{M-m}$ with $u \in [m,M]$ in (13), we observe

$$\begin{aligned} 2J_f(m,M)r(u) &\le f(u) - f(m)\frac{M-u}{M-m} - f(M)\frac{u-m}{M-m} \\ &\le 2J_f(m,M)R(u). \end{aligned} \tag{14}$$

Applying Lemma 1 and taking the generalized perspective, we get

$$\begin{aligned} &2J_f(m,M)P_r(h(A_i),B_i) \\ &\le P_f(h(A_i),B_i) - \frac{f(m)}{M-m}(Mh(A_i) - B_i) - \frac{f(M)}{M-m}(B_i - mh(A_i)) \\ &\le 2J_f(m,M)P_R(h(A_i),B_i). \end{aligned} \tag{15}$$

By summing the inequalities over i in (15), we conclude the result.

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Remark 3 Under the hypotheses of Theorem 6, if $h : [a,b] \to (0,\infty)$ is the identity function, then

$$
\begin{aligned}
&2J_f(m,M)\mathbf{I}_r(\mathbf{A},\mathbf{B}) \qquad (16)\\
&\leq \mathbf{I}_f(\mathbf{A},\mathbf{B}) - \frac{f(m)}{M-m}(MS_{\mathbf{A}} - S_{\mathbf{B}}) - \frac{f(M)}{M-m}(S_{\mathbf{B}} - mS_{\mathbf{A}})\\
&\leq 2J_f(m,M)\mathbf{I}_R(\mathbf{A},\mathbf{B}).
\end{aligned}
$$

This inequality is a generalization of Theorem 3 from [25] and Theorem 2 from [9] in which only the case of a pair of operators. Note that $S(B|A) = -S(A|B)$.

As a consequence of our main results, one realizes the bounds of the χ^2-divergence, Harmonic distance, and Kullback-Leibler divergence, respectively, as follows:

Corollary 7 *Let $P = \{p_1, ..., p_n\}$ and $Q = \{q_1, ..., q_n\}$ be two probability distributions with $\sum_{i=1}^n p_i = \sum_{i=1}^n q_i = 1$ and $0 < mp_i \leq q_i \leq Mp_i$. Then,*

(i) $-\frac{(M-m)(m+1)}{mM(m+M)} \leq \frac{(M-1)(1-m)}{mM} - \chi^2(P,Q) \leq \frac{(M-m)(M+1)}{mM(m+M)}$,

(ii) $0 \leq M(P,Q) - \frac{2(mM+1)}{(M+1)(m+1)} \leq \frac{(m+M+2)(M-m)^2}{2(m+1)^2(M+1)^2}$,

(iii) $-K(\frac{M}{m}) \leq D(P||Q) - \frac{\log \frac{M^{m-1}}{m^{M-1}}}{M-m} \leq 0.$

Proof (i) Consider $A_i = p_i I$, $B_i = q_i I$ in Remark 3, I is identity operator. So, the bounds of the Csiszár f-divergence operator mapping for the concave function $f(t) = -\frac{(t-1)^2}{t}$ can be obtained as follows:

$$
\begin{aligned}
\frac{(M-m)^2}{mM(m+M)} I_r(P,Q) &\leq -\chi^2(P,Q) + \frac{(M-1)(1-m)}{mM} \qquad (17)\\
&\leq \frac{(M-m)^2}{mM(m+M)} I_R(P,Q),
\end{aligned}
$$

where

$$
\begin{aligned}
I_r(P,Q) &= \frac{1}{2} - \frac{1}{2(M-m)} \sum_{i=1}^n |2q_i - (m+M)p_i|,\\
I_R(P,Q) &= \frac{1}{2} + \frac{1}{2(M-m)} \sum_{i=1}^n |2q_i - (m+M)p_i|.
\end{aligned}
$$

A simple verification and using the fact that the absolute value for real numbers satisfies the triangle inequality we reach

$$
\begin{aligned}
\frac{1}{2(M-m)} \sum_{i=1}^n |2q_i - (m+M)p_i| &\leq \frac{1}{2(M-m)} \left(2\sum_{i=1}^n q_i + (m+M)\sum_{i=1}^n p_i \right)\\
&= \frac{2+m+M}{2(M-m)}.
\end{aligned}
$$

This implies

$$I_R(P,Q) = \frac{1}{2} + \frac{1}{2(M-m)} \sum_{i=1}^{n} |2q_i - (m+M)p_i| \leq \frac{1}{2} + \frac{2+m+M}{2(M-m)},$$

$$\frac{1}{2} - \frac{2+m+M}{2(M-m)} \leq \frac{1}{2} - \frac{1}{2(M-m)} \sum_{i=1}^{n} |2q_i - (m+M)p_i| = I_r(P,Q).$$

Therefore, by replacing the lower and upper bounds of I_r and I_R in (17), respectively, we conclude the result.

(ii) Consider $A_i = p_i I$, $B_i = q_i I$ in Corollary 5(i). We realize the bounds of the Harmonic distance as required by (ii).

(iii) Consider $A_i = p_i I$, $B_i = q_i I$ in Corollary 5(ii). Then, we reach the bounds of the Kullback-Leibler divergence as required by (iii).

Note that from the part (iii) of the above corollary one may easily deduce that

$$\left| D(P||Q) - \frac{\log \frac{M^{m-1}}{m^{M-1}}}{M-m} \right| \leq K(\frac{M}{m}).$$

Theorem 7 *Let $\mathbf{A} = \{A_1, ..., A_n\}, \mathbf{B} = \{B_1, ..., B_n\}$ be two finite sequences of strictly positive operators. If $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ is a twice differentiable function and $h : [a,b] \to (0,\infty)$ is a continuous function such that $mh(A_i) \leq B_i \leq Mh(A_i)$ for some $m, M \in [a,b]$ with $0 < m < M$ and there exist the constants γ_1, γ_2 such that $\gamma_1 \leq f''(t) \leq \gamma_2$ for every $t \in (a,b)$, then*

$$\begin{aligned} &\frac{1}{2}\gamma_1 \mathbf{I}_{\Psi \Delta h}(\mathbf{A},\mathbf{B}) \\ &\leq \frac{f(m)}{M-m}(MS_{h(\mathbf{A})} - S_{\mathbf{B}}) + \frac{f(M)}{M-m}(S_{\mathbf{B}} - mS_{h(\mathbf{A})}) - \mathbf{I}_{f\Delta h}(\mathbf{A},\mathbf{B}) \\ &\leq \frac{1}{2}\gamma_2 \mathbf{I}_{\Psi \Delta h}(\mathbf{A},\mathbf{B}), \end{aligned} \tag{18}$$

where $\Psi(t) = (t-m)(M-t)$.

Proof In view of [1, Lemma 2.2], we get

$$\begin{aligned} \frac{1}{2}c(1-c)\gamma_1(y-x)^2 &\leq (1-c)f(x) + cf(y) - f((1-c)x + cy) \\ &\leq \frac{1}{2}c(1-c)\gamma_2(y-x)^2, \end{aligned} \tag{19}$$

where $c \in [0,1]$, x, $y \in [a,b]$. Substitute $x = m$, $y = M$, and $c = \frac{u-m}{M-m}$, in (19), to reach

$$\begin{aligned} \frac{1}{2}(u-m)(M-u)\gamma_1 &\leq \frac{M-u}{M-m}f(m) + \frac{u-m}{M-m}f(M) - f(u) \\ &\leq \frac{1}{2}(u-m)(M-u)\gamma_2. \end{aligned} \tag{20}$$

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Due to Lemma 1, we deduce

$$\begin{aligned}
&\frac{1}{2}\gamma_1 P_\Psi(h(A_i), B) \qquad (21)\\
&\le \frac{f(m)}{M-m}(Mh(A_i) - B) + \frac{f(M)}{M-m}(B - mh(A_i)) - P_f(h(A_i), B)\\
&\le \frac{1}{2}\gamma_2 P_\Psi(h(A_i), B).
\end{aligned}$$

Sum the obtained inequalities over i in (21) to obtain the results.

Remark 4 Under the hypotheses of Theorem 7, if $h : [a, b] \to (0, \infty)$ is the identity function, then

$$\begin{aligned}
\frac{1}{2}\gamma_1 \mathbf{I}_\Psi(\mathbf{A}, \mathbf{B}) &\le \frac{f(m)}{M-m}(MS_{\mathbf{A}} - S_{\mathbf{B}}) + \frac{f(M)}{M-m}(S_{\mathbf{B}} - mS_{\mathbf{A}}) - \mathbf{I}_f(\mathbf{A}, \mathbf{B}) \qquad (22)\\
&\le \frac{1}{2}\gamma_2 \mathbf{I}_\Psi(\mathbf{A}, \mathbf{B}).
\end{aligned}$$

This inequality is a generalization of Theorem 7 from [8], Theorem 4 from [9] and Theorem 4 from [25] in which only the case of a pair of operators. Note that $S(B|A) = -S(A|B)$.

Corollary 8 *Let $\mathbf{A} = \{A_1, ..., A_n\}$ and $\mathbf{B} = \{B_1, ..., B_n\}$ be two finite sequences of strictly positive operators with $\sum_{i=1}^n A_i = \sum_{i=1}^n B_i = I$ and $0 < mA_i \le B_i \le MA_i$. Then,*

$$\begin{aligned}
\frac{1}{8M\sqrt{M}}\mathbf{I}_\Psi(\mathbf{A}, \mathbf{B}) &\le 1 - \frac{(M-1)\sqrt{m} + (1-m)\sqrt{M}}{M-m} - \mathbf{I}_f(\mathbf{A}, \mathbf{B}) \qquad (23)\\
&\le \frac{1}{8m\sqrt{m}}\mathbf{I}_\Psi(\mathbf{A}, \mathbf{B}),
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{I}_\Psi(\mathbf{A}, \mathbf{B}) &= \sum_{i=1}^n (B_i - mA_i)A_i^{-1}(MA_i - B_i),\\
\mathbf{I}_f(\mathbf{A}, \mathbf{B}) &= \frac{1}{2}\sum_{i=1}^n A_i^{1/2}((A_i^{-1/2}B_iA_i^{-1/2})^{1/2} - I)^2 A_i^{1/2}.
\end{aligned}$$

Proof Consider $\Psi(t) = (t-m)(M-t)$, $f(t) = \frac{1}{2}(\sqrt{t} - 1)^2$ and note that

$$\frac{1}{4M\sqrt{M}} \le f''(t) \le \frac{1}{4m\sqrt{m}}$$

for every $t \in [m, M]$. Hence, by using Remark 4, one can get the bounds of the Csiszár f-divergence operator mapping as (23).

As another application of our results, we obtain the bounds of the Hellinger distance.

Corollary 9 *Let $P=\{p_1,...,p_n\}$ and $Q=\{q_1,...,q_n\}$ are two probability distributions with $\sum_{i=1}^n p_i=\sum_{i=1}^n q_i=1$ and $0<mp_i\le q_i\le Mp_i$. Then,*

$$\left|H(P,Q)+\frac{(M-1)\sqrt{m}+(1-m)\sqrt{M}}{M-m}-1\right|\le\frac{(M-m)^2}{32m\sqrt{m}}. \tag{24}$$

Proof The function $\Psi(t)=(t-m)(M-t)$ attains its maximum value at $t=\frac{M+m}{2}$ on the closed interval $[m,M]$ and the maximum value is $\frac{(M-m)^2}{4}$ and the minimum value is zero. Then, the inequality (23) can be rewritten as

$$0\le 1-\frac{(M-1)\sqrt{m}+(1-m)\sqrt{M}}{M-m}-\mathbf{I}_f(\mathbf{A},\mathbf{B})\le\frac{(M-m)^2}{32m\sqrt{m}}, \tag{25}$$

where $f(t)=\frac{1}{2}(\sqrt{t}-1)^2$. If one sets $A_i=p_iI$, $B_i=q_iI$ in (25), then

$$0\le 1-\frac{(M-1)\sqrt{m}+(1-m)\sqrt{M}}{M-m}-H(P,Q)\le\frac{(M-m)^2}{32m\sqrt{m}}$$

and so one reaches the desired result.

As a final result, consider $f(t)=-t^\alpha\log t$. Then,

$$I_f(P,Q)=\sum_{i=1}^n p_i^{(1-\alpha)}q_i^\alpha\log\frac{p_i}{q_i}.$$

We denote by $D_\alpha(P||Q)$ this new and generalized f-divergence functional and call it the relative α-entropy or Kullback-Leibler α-divergence.

Corollary 10 *Let $P=\{p_1,...,p_n\}$ and $Q=\{q_1,...,q_n\}$ are two probability distributions with $\sum_{i=1}^n p_i=\sum_{i=1}^n q_i=1$, $0<mp_i\le q_i\le Mp_i$ for some $m,M\in[e^{\frac{2\alpha-1}{\alpha(1-\alpha)}},\infty)$, and $0\le\alpha<1$. Then,*

$$\begin{aligned}0&\le\frac{\log\frac{M^{(m-1)M^\alpha}}{m^{(M-1)m^\alpha}}}{M-m}-D_\alpha(P||Q) \qquad (26)\\&\le\frac{1}{4}(M-m)\Big(m^{\alpha-1}-M^{\alpha-1}+\log\frac{m^{\alpha m^{\alpha-1}}}{M^{\alpha M^{\alpha-1}}}\Big).\end{aligned}$$

Proof Consider $A_i=p_iI$, $B_i=q_iI$, and $\beta=1$ in Corollary 6 and deduce the desired result.

We remark that the relative 0-entropy is the relative entropy or Kullback-Leibler divergence. Moreover, when $\alpha\to 0$ the inequalities (26) ensure

$$0\le\frac{\log\frac{M^{(m-1)}}{m^{(M-1)}}}{M-m}-D(P||Q)\le\frac{1}{4}(M-m)(m^{-1}-M^{-1})=K(\frac{M}{m}), \tag{27}$$

which confirm Corollary 7 (iii).

We also have the following upper and lower bounds for the difference

$$\frac{f(m)}{M-m}\left(MS_{h(\mathbf{A})}-S_{\mathbf{B}}\right)+\frac{f(M)}{M-m}\left(S_{\mathbf{B}}-mS_{h(\mathbf{A})}\right)-I_{f\Delta h}\left(\mathbf{A},\mathbf{B}\right)$$

under consideration.

Theorem 8 *With the assumptions of Theorem 7 and if there exists the constants* $\varphi_1 < \varphi_2$ *such that*

$$\varphi_1 \leq tf''(t) \leq \varphi_2 \text{ for all } t \in (m, M) \subset (0, \infty),$$

then

$$\begin{aligned} &\varphi_1 \mathbf{I}_{\Phi \Delta h}(\mathbf{A}, \mathbf{B}) \\ &\leq \frac{f(m)}{M-m}\left(MS_{h(\mathbf{A})} - S_{\mathbf{B}}\right) + \frac{f(M)}{M-m}\left(S_{\mathbf{B}} - mS_{h(\mathbf{A})}\right) - I_{f\Delta h}(\mathbf{A}, \mathbf{B}) \\ &\leq \varphi_2 \mathbf{I}_{\Phi \Delta h}(\mathbf{A}, \mathbf{B}), \end{aligned} \tag{28}$$

where

$$\Phi(t) := \frac{M-t}{M-m} m \ln m + \frac{t-m}{M-m} M \ln M - t \ln t.$$

Proof Consider the function $f_{\varphi_1}(t) := f(t) - \varphi_1 t \ln t$ for $t \in (m, M) \subset (0, \infty)$. Since f_{φ_1} is twice differentiable on (m, M) and

$$f''_{\varphi_1}(t) := f''(t) - \frac{\varphi_1}{t} = \frac{tf''(t) - \varphi_1}{t} \geq 0$$

then f_{φ_1} is convex on (m, M) and, as above, we have that

$$0 \leq \frac{M-u}{M-m} f_{\varphi_1}(m) + \frac{u-m}{M-m} f_{\varphi_1}(M) - f_{\varphi_1}(u) \tag{29}$$

for all $u \in [m, M]$.

Now, observe that by (29) we get

$$\begin{aligned} 0 \leq &\frac{M-u}{M-m} f(m) + \frac{u-m}{M-m} f(M) - f(u) \\ &- \varphi_1 \left(\frac{M-u}{M-m} m \ln m + \frac{u-m}{M-m} M \ln M\right), \end{aligned}$$

which gives that

$$\begin{aligned} &\varphi_1 \left(\frac{M-u}{M-m} m \ln m + \frac{u-m}{M-m} M \ln M - u \ln u\right) \\ &\leq \frac{M-u}{M-m} f(m) + \frac{u-m}{M-m} f(M) - f(u) \end{aligned} \tag{30}$$

for all $u \in [m, M]$.

We consider the function $f_{\varphi_2}(t) := \varphi_2 t \ln t - f(t)$ for $t \in (m, M) \subset (0, \infty)$. As above, we observe that f_{φ_2} is twice differentiable and convex and we also obtain the inequality

$$\begin{aligned} &\frac{M-u}{M-m} f(m) + \frac{u-m}{M-m} f(M) - f(u) \\ &\leq \varphi_2 \left(\frac{M-u}{M-m} m \ln m + \frac{u-m}{M-m} M \ln M - u \ln u\right) \end{aligned} \tag{31}$$

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for all $u \in [m, M]$.

By utilizing Lemma 1 , we deduce from (30) and (31) that

$$\begin{aligned} & \varphi_1 P_\Phi\left(h\left(A_i\right), B_i\right) \\ & \leq \frac{f(m)}{M-m}\left(M h\left(A_i\right)-B_i\right)+\frac{f(M)}{M-m}\left(B_i-m h\left(A_i\right)\right)-P_f\left(h\left(A_i\right), B_i\right) \\ & \leq \varphi_2 P_\Phi\left(h\left(A_i\right), B_i\right) \end{aligned} \tag{32}$$

for $i=1, \ldots, n$.

Sum the obtained inequalities over i from 1 to n in (32) to obtain the desired inequalities (28).

Theorem 9 *With the assumptions of Theorem 7 and if there exists the constants $\psi_1<\psi_2$ such that*

$$\psi_1 \leq t^2 f''(t) \leq \psi_2 \text{ for all } t \in(m, M) \subset(0, \infty),$$

then

$$\begin{aligned} & \psi_1 \mathbf{I}_{\Psi \Delta h}(\mathbf{A}, \mathbf{B}) \\ & \leq \frac{f(m)}{M-m}\left(M S_{h(\mathbf{A})}-S_{\mathbf{B}}\right)+\frac{f(M)}{M-m}\left(S_{\mathbf{B}}-m S_{h(\mathbf{A})}\right)-I_{f \Delta h}(\mathbf{A}, \mathbf{B}) \\ & \leq \psi_2 \mathbf{I}_{\Psi \Delta h}(\mathbf{A}, \mathbf{B}), \end{aligned} \tag{33}$$

where

$$\Psi(t):=\ln t-\frac{M-t}{M-m} \ln m-\frac{t-m}{M-m} \ln M.$$

Proof Consider the function $f_{\psi_1}(t):=f(t)+\psi_1 \ln t$ for $t \in(m, M) \subset(0, \infty)$. Since f_{ψ_1} is twice differentiable on (m, M) and

$$f_{\psi_1}''(t):=f''(t)-\frac{\psi_1}{t^2}=\frac{t^2 f''(t)-\psi_1}{t^2} \geq 0$$

then f_{ψ_1} is convex on (m, M) and, as above, we have that

$$0 \leq \frac{M-u}{M-m} f_{\psi_1}(m)+\frac{u-m}{M-m} f_{\psi_1}(M)-f_{\psi_1}(u) \tag{34}$$

for all $u \in[m, M]$.

Observe that by (34) we get

$$\begin{aligned} 0 \leq & \frac{M-u}{M-m} f(m)+\frac{u-m}{M-m} f(M) \\ & +\psi_1\left(\frac{M-u}{M-m} \ln m+\frac{u-m}{M-m} \ln M\right)-f(u)-\psi_1 \ln u, \end{aligned}$$

which gives that

$$\begin{aligned}&\psi_1\left(\ln t-\frac{M-u}{M-m}\ln m-\frac{u-m}{M-m}\ln M\right)\\&\leq\frac{M-u}{M-m}f(m)+\frac{u-m}{M-m}f(M)-f(t).\end{aligned}$$

In a similar way we derive

$$\begin{aligned}&\frac{M-u}{M-m}f(m)+\frac{u-m}{M-m}f(M)-f(t)\\&\leq\psi_2\left(\ln t-\frac{M-u}{M-m}\ln m-\frac{u-m}{M-m}\ln M\right).\end{aligned}$$

The proof now follows along the lines of the theorem above and the details are omitted.

Theorem 10 *With the assumptions of Theorem 7 and for* $p\in(-\infty,0)\cup(1,\infty)$, *if there exists the constants* $\delta_1<\delta_2$ *such that*

$$\delta_1\leq f''(t)\,t^{2-p}\leq\delta_2 \text{ for all } t\in(m,M)\subset(0,\infty),$$

then

$$\begin{aligned}&\frac{\delta_1}{p(p-1)}\mathbf{I}_{\Gamma_p\Delta h}(\mathbf{A},\mathbf{B}) &(35)\\&\leq\frac{f(m)}{M-m}\left(MS_{h(\mathbf{A})}-S_{\mathbf{B}}\right)+\frac{f(M)}{M-m}\left(S_{\mathbf{B}}-mS_{h(\mathbf{A})}\right)-I_{f\Delta h}(\mathbf{A},\mathbf{B})\\&\leq\frac{\delta_2}{p(p-1)}\mathbf{I}_{\Gamma_p\Delta h}(\mathbf{A},\mathbf{B}),\end{aligned}$$

where

$$\Gamma_p(t):=\frac{M-t}{M-m}m^p+\frac{t-m}{M-m}M^p-t^p.$$

Proof Consider the function $f_{\delta_1}(t):=f(t)-\frac{\delta_1}{p(p-1)}t^p$ for $t\in(m,M)\subset(0,\infty)$. Observe that

$$f''_{\delta_1}(t):=f''(t)-\delta_1t^{p-2}=\left(f''(t)\,t^{2-p}-\delta_1\right)t^{p-2}\geq0$$

for $t\in(m,M)$, which shows that f_{δ_1} is convex. Then

$$0\leq\frac{M-u}{M-m}f_{\delta_1}(m)+\frac{u-m}{M-m}f_{\delta_1}(M)-f_{\delta_1}(u) \tag{36}$$

for all $u\in[m,M]$.

Observe that by (36) we get

$$\begin{aligned}&\frac{\delta_1}{p(p-1)}\left(\frac{M-u}{M-m}m^p+\frac{u-m}{M-m}M^p-u^p\right)\\&\leq\frac{M-u}{M-m}f(m)+\frac{u-m}{M-m}f(M)-f(u)\end{aligned}$$

for all $u \in [m, M]$.

In a similar way we derive

$$\frac{M-u}{M-m} f(m) + \frac{u-m}{M-m} f(M) - f(u) \le \frac{\delta_2}{p(p-1)} \left(\frac{M-u}{M-m} m^p + \frac{u-m}{M-m} M^p - u^p \right)$$

for all $u \in [m, M]$.

By using a similar argument as above we obtain (35).

Example 1 For a given twice differentiable function f defined on $(0, \infty)$, if we take,

$$\delta_2 = \sup_{t \in (m,M)} f''(t) t^{2-p} \text{ and } \delta_1 = \inf_{t \in (m,M)} f''(t) t^{2-p}$$

assumed to be finite for some $p \in (-\infty, 0) \cup (1, \infty)$, then we get the corresponding lower and upper bounds for

$$\frac{f(m)}{M-m} \left(M S_{h(\mathbf{A})} - S_{\mathbf{B}} \right) + \frac{f(M)}{M-m} \left(S_{\mathbf{B}} - m S_{h(\mathbf{A})} \right) - I_{f \Delta h}(\mathbf{A}, \mathbf{B})$$

as provided by (35).

If we take $f(t) = t \ln t$ in (35), then

$$\delta_2 = \sup_{t \in (m,M)} f''(t) t^{2-p} = \sup_{t \in (m,M)} t^{1-p} = \begin{cases} m^{1-p} \text{ if } p > 1 \\ M^{1-p} \text{ if } p < 0 \end{cases}$$

and

$$\delta_1 = \inf_{t \in (m,M)} f''(t) t^{2-p} = \inf_{t \in (m,M)} t^{1-p} = \begin{cases} M^{1-p} \text{ if } p > 1 \\ m^{1-p} \text{ if } p < 0 \end{cases}$$

and we get from (35) that

$$\begin{aligned} &\frac{1}{p(p-1)} \mathbf{I}_{\Gamma_p \Delta h}(\mathbf{A}, \mathbf{B}) \times \begin{cases} M^{1-p} \text{ if } p > 1 \\ m^{1-p} \text{ if } p < 0 \end{cases} \\ &\le \frac{m \ln m}{M-m} \left(M S_{h(\mathbf{A})} - S_{\mathbf{B}} \right) + \frac{m \ln M}{M-m} \left(S_{\mathbf{B}} - m S_{h(\mathbf{A})} \right) - I_{(\cdot) \ln (\cdot) \Delta h}(\mathbf{A}, \mathbf{B}) \\ &\le \frac{1}{p(p-1)} \mathbf{I}_{\Gamma_p \Delta h}(\mathbf{A}, \mathbf{B}) \times \begin{cases} m^{1-p} \text{ if } p > 1 \\ M^{1-p} \text{ if } p < 0. \end{cases} \end{aligned} \tag{37}$$

Conflict of interest

The authors declare that they have no conflict of interest.

Declarations

We remark that the potential conflicts of interest and data sharing not applicable to this article and no data sets were generated during the current study. The authors declare that they have no fund for this work.

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