ROCKY MOUNTAIN JOURNAL OF MATHEMATICS
Vol. , No. , YEAR
https://doi.org/rmj.YEAR..PAGE

FREIMAN'S $(3k-4)$-LIKE RESULTS FOR SUBSET AND SUBSEQUENCE SUMS

MOHAN, JAGANNATH BHANJA, AND RAM KRISHNA PANDEY†

ABSTRACT. For a nonempty finite set A of integers, let $S(A) = \{\sum_{b \in B} b : \emptyset \neq B \subseteq A\}$ be the set of all nonempty subset sums of A. In 1995, Nathanson determined the minimum cardinality of $S(A)$ in terms of $|A|$ and described the structure of A for which $|S(A)|$ is the minimum. He asked to characterize the underlying set A if $|S(A)|$ is a small increment to its minimum size. Problems of such nature are inspired by the well-known Freiman's $3k-4$ theorem. In this paper, some results in the direction of Freiman's $3k-4$ theorem for the set of subset sums $S(A)$ are proved. Such results are also extended to the set of subsequence sums $S(\mathbb{A}) = \{\sum_{b \in \mathbb{B}} b : \emptyset \neq \mathbb{B} \subseteq \mathbb{A}\}$ of sequence $\mathbb{A}$, where the notation $\mathbb{B} \subseteq \mathbb{A}$, is used for $\mathbb{B}$ is a subsequence of $\mathbb{A}$. The results are further generalized to a generalization of subset and subsequence sums. The main idea of the proofs of the results is to write the set of subset sums $S(A)$ and the set of subsequence sums $S(\mathbb{A})$ in terms of the h-fold sumset hA and the h-fold restricted sumset $h^{\wedge}A$. Such representation also gives other proof of some of the results of Nathanson and Mistri *et al*.

1. Notation

Throughout the paper, we follow the following notations. We write $\mathbb{A} = \{a_1, a_2, \ldots, a_k\}_{\vec{r}}$ with $a_1 < a_2 < \cdots < a_k$ and $\vec{r} = (r_1, r_2, \ldots, r_k)$ to mean that $\mathbb{A}$ is a sequence consisting of k distinct integers $a_1, a_2, \ldots, a_k$ with a_i appearing r_i times in $\mathbb{A}$ for $i = 1, 2, \ldots, k$. By $|\mathbb{A}|$ we mean the number of terms (including multiplicities) in $\mathbb{A}$. By the size of sequence $\mathbb{A} = \{a_1, a_2, \ldots, a_k\}_{\vec{r}}$ we mean the number $\sum_{i=1}^{k} r_i$. For integers α and β, we define $\alpha * \mathbb{A} = \{\alpha a_1, \alpha a_2, \ldots, \alpha a_k\}_{\vec{r}}$, $\mathbb{A} + \beta = \{a + \beta : a \in \mathbb{A}\}$ and $[\alpha, \beta]_{\vec{r}} = \{\alpha, \alpha+1, \ldots, \beta\}_{\vec{r}}$ for $\alpha \leq \beta$. We use the usual set notation A to write the set $\{a_1, a_2, \ldots, a_k\}$ of distinct elements of sequence $\mathbb{A}$. If $A = \{a_1, a_2, \ldots, a_k\}$ is a nonempty finite set of integers, the notations defined above for sequences have the usual set theoretical meaning: $|A|$ denotes the number of elements in A, $\alpha * A = \{\alpha a_1, \alpha a_2, \ldots, \alpha a_k\}$, $A + \beta = \{a + \beta : a \in A\}$ and $[\alpha, \beta] = \{\alpha, \alpha+1, \ldots, \beta\}$ for $\alpha \leq \beta$. We denote the greatest common divisor of the integers $x_1, x_2, \ldots, x_k$ by $(x_1, x_2, \ldots, x_k)$ and write $d(A)$ in short, when $A = \{x_1, x_2, \ldots, x_k\}$ is a set. In addition, for a sum of the form $\sum_{x=a}^{b} f(x)$ with integers a, b such that $a > b$, we mean zero. Furthermore, we write $\mathbb{N}$ for the set of natural numbers and θ for the golden mean $\frac{1+\sqrt{5}}{2}$.

2. Introduction

Let A be a nonempty finite set of integers and h be a positive integer. The *h-fold sumset*, denoted by hA, is defined as the set of integers that can be written as a sum of h elements (not necessarily distinct) of

---

†The corresponding author.
2020 *Mathematics Subject Classification.* 11P70, 11B75, 11B13.
Key words and phrases. Subset sum; subsequence sum; sumset; restricted sumset; Freiman's $3k-4$ theorem.

A, and the *restricted h-fold sumset*, denoted by $h^{\wedge}A$, is defined as the set of integers that can be written as a sum of h distinct elements of A (see [1, 21]).

Two problems associated with sumsets that are studied extensively in the literature are direct and inverse problems. A direct problem is to determine the minimum cardinality and properties of the sumset and the inverse problem is to characterize the underlying set(s) when the cardinality of the sumset is known. The following are some of the classical results that give the minimum cardinality of h-fold sumset hA and $h^{\wedge}A$ and also describe the underlying set A when the cardinality of the sumset is minimum.

Theorem 2.1. [21, Theorem 1.4, Theorem 1.6] *Let A be a nonempty finite set of integers. Then, for $h \geq 1$, we have*

$$|hA| \geq h|A| - h + 1.$$

Moreover, if $h \geq 2$ and $|hA| = h|A| - h + 1$, then A is an arithmetic progression.

Theorem 2.2. [21, Theorem 1.9, Theorem 1.10] *Let A be a nonempty finite set of integers, and $1 \leq h \leq |A|$. Then*

$$|h^{\wedge}A| \geq h|A| - h^2 + 1.$$

Moreover, if $|h^{\wedge}A| = h|A| - h^2 + 1$ with $|A| \geq 5$ and $2 \leq h \leq |A| - 2$, then A is an arithmetic progression.

Freiman [10, 11] proved the following inverse theorem for the 2-fold sumset $2A$, which is well known as Freiman's $3k-4$ theorem.

Theorem 2.3. [11, Theorem 1.9] *Let $k \geq 3$. Let A be a set of k integers. If $|2A| = 2k - 1 + b \leq 3k - 4$, then A is a subset of an arithmetic progression of length at most $k + b$.*

This inverse theorem is a consequence of the following result.

Theorem 2.4. [11, Theorem 1.10] *Let $k \geq 3$. Let $A = \{a_0, a_1, \ldots, a_{k-1}\}$ be a set of integers such that $0 = a_0 < a_1 < \cdots < a_{k-1}$ and $d(A) = 1$. Then*

$$|2A| \geq \begin{cases} a_{k-1} + k, & \text{if } a_{k-1} \leq 2k-3; \\ 3k-3, & \text{if } a_{k-1} \geq 2k-2. \end{cases}$$

Lev [16] extended Theorem 2.4 to the sumsets hA for $h \geq 2$.

Theorem 2.5. [16, Theorem 1] *Let $k \geq 3$. Let $A = \{a_0, a_1, \ldots, a_{k-1}\}$ be a set of integers such that $0 = a_0 < a_1 < \cdots < a_{k-1}$ and $d(A) = 1$. Then, for $h \geq 2$, we have*

$$|hA| \geq |(h-1)A| + \min\{a_{k-1}, h(k-2) + 1\}.$$

For the restricted sumset $2^{\wedge}A$, the following was conjectured by Freiman and Lev [15], independently.

Conjecture 2.6. *Let $k > 7$. Let $A = \{a_0, a_1, \ldots, a_{k-1}\}$ be a set of integers such that $0 = a_0 < a_1 < \cdots < a_{k-1}$ and $d(A) = 1$. Then*

$$|2^{\wedge}A| \geq \begin{cases} a_{k-1} + k - 2, & \text{if } a_{k-1} \leq 2k-5; \\ 3k-7, & \text{if } a_{k-1} \geq 2k-4. \end{cases}$$

The lower bounds in Conjecture 2.6 are tight, as letting $A = \{0,1,\ldots,k-3\} \cup \{a_{k-1}-1, a_{k-1}\}$, we get $2^\wedge A = \{1,2,\ldots,2k-7\} \cup \{a_{k-1}-1,\ldots,a_{k-1}+k-3\} \cup \{2a_{k-1}-1\}$. Freiman *et al.* [12] made some progress on Conjecture 2.6 by proving the following result.

Theorem 2.7. [12, Theorem 1, Theorem 2] *Let $k \geq 3$. Let $A = \{a_0, a_1, \ldots, a_{k-1}\}$ be a set of integers such that $0 = a_0 < a_1 < \cdots < a_{k-1}$ and $d(A) = 1$. Then*

$$|2^\wedge A| \geq \begin{cases} 0.5(a_{k-1}+k)+k-3.5, & \text{if } a_{k-1} \leq 2k-3; \\ 2.5k-5, & \text{if } a_{k-1} \geq 2k-2. \end{cases}$$

A year later, Lev [15] improved Freiman *et al.* [12] results in the following theorem.

Theorem 2.8. [15, Theorem 1] *Let $k \geq 3$. Let $A = \{a_0, a_1, \ldots, a_{k-1}\}$ be a set of integers such that $0 = a_0 < a_1 < \cdots < a_{k-1}$ and $d(A) = 1$. Then*

$$|2^\wedge A| \geq \begin{cases} a_{k-1}+k-2, & \text{if } a_{k-1} \leq 2k-5; \\ (\theta+1)k-6, & \text{if } a_{k-1} \geq 2k-4. \end{cases}$$

In a recent paper, Daza et al. [7] have almost solved Conjecture 2.6, but we shall not make use of their result in this paper.

The purpose of this article is to prove results similar to Theorem 2.4 and Theorem 2.8 for the set of subset sums and the set of subsequence sums, which are defined below.

Let A be a nonempty finite set of k integers. For a nonempty subset B of A, the *subset sum* of B is defined as $s(B) = \sum_{b\in B} b$. The collection of all nonempty subset sums of A, denoted by $S(A)$, is defined as

$$S(A) := \Big\{ s(B) : \emptyset \neq B \subseteq A \Big\}.$$

Nathanson [22] initiated the study of direct and inverse problems for $S(A)$ over the group of integers. Such studies are done on other groups also (see [2, 4, 8, 13], and the references therein). However, in this article, we restrict ourselves to the group of integers only. Nathanson [22] determined the minimum cardinality of $S(A)$ in terms of $|A|$, and also gave a characterization of set A when $|S(A)|$ is the minimum (see Lemma 3.1). Lev [17] extended Nathanson's direct theorem to sequences of nonnegative integers (see Lemma 4.3). Mistri *et al.* [18] (also see [19]) extended Nathanson's inverse theorem to sequences of nonnegative integers (see Lemma 4.1 and Lemma 4.3) while giving a new proof of Lev's result. Jiang and Li [14] later proved the direct and inverse results for the subsequence sums when the sequence contains both positive and negative integers. For the sake of completeness, we define the subsequence sums below.

For a nonempty finite sequence $\mathbb{A}$ of integers, we denote by $S(\mathbb{A})$, the set of all subsequence sums of $\mathbb{A}$, i.e.,

$$S(\mathbb{A}) := \{ s(\mathbb{B}) : \emptyset \neq \mathbb{B} \subseteq \mathbb{A} \},$$

where $s(\mathbb{B}) = \sum_{b\in\mathbb{B}} b$.

The direct and inverse results for the usual subset and subsequence sums are further extended by Bhanja and Pandey [5, 6] considering the α-analog of subset and subsequence sums, which are defined below. For a given positive integer α, let

$$S_\alpha(A) := \{ s(B) : B \subseteq A, |B| \geq \alpha \},$$

and

$$S_\alpha(\mathbb{A}) := \{s(\mathbb{B}) : \mathbb{B} \subseteq \mathbb{A}, |\mathbb{B}| \geq \alpha\}.$$

Recently, Dwivedi and Mistri [9] reproved some results of Bhanja and Pandey using a generalization of h-fold sumset hA. The reader is also directed to see the article of Balandraud [3], where $S_\alpha(A)$ is introduced in this context, and also the minimum cardinality of $S_\alpha(A)$ is obtained over the finite cyclic groups of prime order.

In [22], Nathanson asked to prove Theorem 2.4-like result for $S(A)$. In this paper, we prove some results (see Theorems 3.2, 3.3, 4.2, 4.4, and 4.5) for $S(A)$ and $S(\mathbb{A})$ which are similar to Theorem 2.4 and Theorem 2.8. Our idea is to write $S(A)$ and $S(\mathbb{A})$ in terms of sumsets hA and $h^\wedge A$, and then use Theorem 2.4 and Theorem 2.8 to obtain Freiman like results for $S(A)$ and $S(\mathbb{A})$. Such representation will also lead us to give new proofs of some results of Nathanson [22] (see Lemma 3.1) and Mistri et al. [18] (see Lemma 4.1 and Lemma 4.3). Further, we prove analogous results for $S_\alpha(A)$ and $S_\alpha(\mathbb{A})$ in the last two sections of this paper. The proofs of the results of sections 5 and 6 are quite similar to the ones in sections 3 and 4, however, in sections 5 and 6 the proofs are more involved and depend heavily on α.

To prove the main results of sections 3 and 4 of this article, we first reprove the direct and inverse results for the usual subset and subsequence sums. In other results that we prove in sections 5 and 6, we directly use the already proven results for the α-analog of subset and subsequence sums. The following are the two results that we use to prove our results in sections 5 and 6.

Theorem 2.9. [5, Theorem 2.1, Theorem 2.2] *Let A be a set of k positive integers. Let $1 \leq \alpha \leq k$ be an integer. Then*

$$|S_\alpha(A)| \geq \frac{k(k+1)}{2} - \frac{\alpha(\alpha+1)}{2} + 1.$$

Moreover, if $k \geq 4$, $\alpha \leq k-2$, and $|S_\alpha(A)| = \frac{k(k+1)}{2} - \frac{\alpha(\alpha+1)}{2} + 1$, then $A = d * [1,k]$ for some positive integer d.

Theorem 2.10. [5, Theorem 3.1, Theorem 3.2] *Let $\mathbb{A} = \{a_1, a_2, \ldots, a_k\}_{\vec{r}}$ be a sequence of positive integers such that $a_1 < a_2 < \cdots < a_k$ and $\vec{r} = (r_1, r_2, \ldots, r_k)$ with $r_i \geq 1$ for all $i \in [1,k]$. Let $1 \leq \alpha \leq \sum_{i=1}^k r_i$ be an integer. Then there exists an integer $m \in [1,k]$ such that $\sum_{i=1}^{m-1} r_i \leq \alpha < \sum_{i=1}^m r_i$ and*

$$|S_\alpha(\mathbb{A})| \geq \sum_{i=1}^k i r_i - \sum_{i=1}^m i r_i + m\left(\sum_{i=1}^m r_i - \alpha\right) + 1.$$

Moreover, if $k \geq 4$, $\alpha \leq \sum_{i=1}^k r_i - 2$, and $|S_\alpha(\mathbb{A})| = \sum_{i=1}^k i r_i - \sum_{i=1}^m i r_i + m(\sum_{i=1}^m r_i - \alpha) + 1$, then $\mathbb{A} = d * [1,k]_{\vec{r}}$ for some positive integer d.

3. Freiman's theorem for subset sum

In the following lemma, we reprove the direct and inverse results of Nathanson for $S(A)$ when the set A contains positive integers. Then, in the next two theorems, we prove Freiman-like results for $S(A)$ in the cases in which A contains positive integers and A contains nonnegative integers with $0 \in A$.

Lemma 3.1. [22, Theorem 3, Theorem 5] *Let A be a set of k positive integers. Then*

$$|S(A)| \geq \frac{k(k+1)}{2}. \tag{3.1}$$

Moreover, if $k \geq 4$ and $|S(A)| = \dfrac{k(k+1)}{2}$, then $A = d * [1,k]$ for some positive integer d.

Proof. Let $A = \{a_1, a_2, \ldots, a_k\}$ with $0 < a_1 < a_2 < \cdots < a_k$. It is easy to see that the result holds for $k = 1, 2$. Assume that $k \geq 3$ and the result holds for all sets that have less than k elements. Let $B = A \setminus \{a_{k-1}, a_k\}$. Then $2^\wedge(A \cup \{0\})$ and $S(B) + a_{k-1} + a_k$ are two disjoint subsets of $S(A)$. By Theorem 2.2 and the induction hypothesis, we get

$$\begin{aligned} |S(A)| &\geq \left|2^\wedge(A \cup \{0\})\right| + |S(B) + a_{k-1} + a_k| \\ &\geq 2(k+1) - 3 + \frac{(k-2)(k-1)}{2} \\ &= \frac{k(k+1)}{2}. \end{aligned} \tag{3.2}$$

Now, suppose that $k \geq 4$ and $|S(A)| = \dfrac{k(k+1)}{2}$. Then by (3.2), we have $|2^\wedge(A \cup \{0\})| = 2(k+1) - 3$. Applying Theorem 2.2 on $A \cup \{0\}$, we get that $A \cup \{0\}$ is an arithmetic progression. Hence $A = a_1 * [1,k]$. □

Theorem 3.2. *Let $k \geq 3$. Let $A = \{a_1, a_2, \ldots, a_k\}$ be a set of k positive integers such that $a_1 < a_2 < \cdots < a_k$ and $d(A) = 1$. Then*

$$|S(A)| \geq \begin{cases} a_k + \dfrac{k(k-1)}{2}, & \text{if } a_k \leq 2k-3; \\ \theta(k+1) - 4 + \dfrac{k(k-1)}{2}, & \text{if } a_k \geq 2k-2. \end{cases}$$

Proof. From equation (3.2), we have the following inequality

$$|S(A)| \geq \left|2^\wedge(A \cup \{0\})\right| + |S(B) + a_{k-1} + a_k|,$$

where $B = A \setminus \{a_{k-1}, a_k\}$. Applying Theorem 2.8 on $A \cup \{0\}$ and Lemma 3.1 on B we obtain

$$\begin{aligned} |S(A)| &\geq \left|2^\wedge(A \cup \{0\})\right| + |S(B)| \\ &\geq \begin{cases} a_k + k - 1 + \dfrac{(k-1)(k-2)}{2}, & \text{if } a_k \leq 2(k+1) - 5; \\ (\theta+1)(k+1) - 6 + \dfrac{(k-1)(k-2)}{2}, & \text{if } a_k \geq 2(k+1) - 4, \end{cases} \\ &\geq \begin{cases} a_k + \dfrac{k(k-1)}{2}, & \text{if } a_k \leq 2k-3; \\ \theta(k+1) - 4 + \dfrac{k(k-1)}{2}, & \text{if } a_k \geq 2k-2. \end{cases} \end{aligned}$$

□

Theorem 3.3. *Let $k \geq 4$ and $A = \{a_0, a_1, \ldots, a_{k-1}\}$ be a set of k nonnegative integers such that $0 = a_0 < a_1 < \cdots < a_{k-1}$ and $d(A) = 1$. Then*

$$|S(A)| \geq \begin{cases} a_{k-1} + \frac{(k-1)(k-2)}{2} + 1, & \text{if } a_{k-1} \leq 2k-5; \\ \theta k - 3 + \frac{(k-1)(k-2)}{2}, & \text{if } a_{k-1} \geq 2k-4. \end{cases}$$

Proof. Set $B = \{a_1, a_2, \ldots, a_{k-1}\}$. Then B is a set of $k-1$ positive integers with $d(B) = 1$. Further, we have $S(A) = S(B) \cup \{0\}$. Thus, by Theorem 3.2, it follows that

$$|S(A)| \geq |S(B)| + 1 \geq \begin{cases} a_{k-1} + \frac{(k-1)(k-2)}{2} + 1, & \text{if } a_{k-1} \leq 2k-5; \\ \theta k - 3 + \frac{(k-1)(k-2)}{2}, & \text{if } a_{k-1} \geq 2k-4. \end{cases}$$

□

4. Freiman's theorem for subsequence sum

In this section, we start by giving a new proof of direct and inverse results of Mistri *et al.* [18] for $S(\mathbb{A})$ in Lemma 4.1. Then, using Lemma 4.1, we prove a Freiman-like result for $S(\mathbb{A})$ in Theorem 4.2 when the sequence $\mathbb{A}$ contains positive integers. In Theorem 4.4, we improve our previous bound assuming that every element of the sequence appears at least twice. To prove Theorem 4.4 we first prove Lemma 4.3. Further, in Theorem 4.5, we prove a similar Freiman's $3k-4$-like theorem for $S(\mathbb{A})$ when the sequence $\mathbb{A}$ contains nonnegative integers with $0 \in \mathbb{A}$.

Lemma 4.1. [18, Theorem 3.1, Theorem 3.2] *Let $\mathbb{A} = \{a_1, a_2, \ldots, a_k\}_{\vec{r}}$ be a sequence of positive integers such that $a_1 < a_2 < \cdots < a_k$, $\vec{r} = (r_1, r_2, \ldots, r_k)$ and $r_i \geq 1$ for all $i \in [1,k]$. Then*

$$|S(\mathbb{A})| \geq \sum_{i=1}^{k} i r_i. \tag{4.1}$$

Moreover, if $k \geq 4$ and $|S(\mathbb{A})| = \sum_{i=1}^{k} i r_i$, then $\mathbb{A} = d * [1,k]_{\vec{r}}$ for some positive integer d.

Proof. To prove (4.1), we use induction on k. For $k = 1$, we have $\mathbb{A} = (a_1)_{\vec{r_1}}$, and so $S(\mathbb{A}) = \{a_1, 2a_1, \ldots, r_1 a_1\}$. For $k = 2$, we have $\mathbb{A} = (a_1, a_2)_{\vec{r}}$ with $\vec{r} = (r_1, r_2)$. It is easy to see, in this case, that

$$S(\mathbb{A}) \supseteq \{ia_1 : i \in [1, r_1]\} \cup \{(r_1 - 1)a_1 + ia_2 : i \in [1, r_2]\} \cup \{r_1 a_1 + ia_2 : i \in [1, r_2]\},$$

where the three sets on the right hand side are pairwise disjoint. Therefore (4.1) holds for $k = 1, 2$. Assume that $k \geq 3$ and (4.1) holds for all sequences whose number of distinct terms is less than k. Set $\mathbb{B} = \{a_1, a_2, \ldots, a_{k-2}\}_{\vec{s}}$ with $\vec{s} = (r_1, r_2, \ldots, r_{k-2})$. Then $2^\wedge(A \cup \{0\})$ and $S(\mathbb{B}) + a_{k-1} + a_k$ are two disjoint subsets of $S(\mathbb{A})$. For $1 \leq i \leq r_{k-1} - 1$ and $1 \leq j \leq k-2$, define

$$s_{i,j} = \sum_{t=1, t \neq k-j-1}^{k-2} r_t a_t + (r_{k-j-1} - 1)a_{k-j-1} + (i+1)a_{k-1} + a_k,$$

and

$$s_{i,k-1} = \sum_{t=1}^{k-2} r_t a_t + (i+1)a_{k-1} + a_k.$$

Similarly, for $1 \le i \le r_k - 1$ and $1 \le j \le k-1$, define

$$u_{i,j} = \sum_{t=1, t\neq k-j}^{k-1} r_t a_t + (r_{k-j}-1)a_{k-j} + (i+1)a_k,$$

and

$$u_{i,k} = \sum_{t=1}^{k-1} r_t a_t + (i+1)a_k.$$

It is easy to see that

$$s_{i,1} < s_{i,2} < \cdots < s_{i,k-2} < s_{i,k-1} < s_{i+1,1},$$

$$s_{r_{k-1}-1,k-1} < u_{1,1},$$

and

$$u_{i,1} < u_{i,2} < \cdots < u_{i,k-1} < u_{i,k} < u_{i+1,1}.$$

Therefore, the elements $s_{i,j}$ and $u_{i,j}$ are all distinct, all are in the set $S(\mathbb{A})$, and bigger than the elements of $2^\wedge(A\cup\{0\})$ and $S(\mathbb{B})+a_{k-1}+a_k$. Note also that $s_{i,j}$ is not defined for $r_{k-1}=1$ and $u_{i,j}$ are not defined for $r_k = 1$. By Theorem 2.2 and the induction hypothesis we get

$$\begin{aligned}
|S(\mathbb{A})| &\ge \left|2^\wedge(A\cup\{0\})\right| + |S(\mathbb{B})+a_{k-1}+a_k| + \left|\bigcup_{i=1}^{r_{k-1}-1}\bigcup_{j=1}^{k-1} s_{i,j}\right| + \left|\bigcup_{i=1}^{r_k-1}\bigcup_{j=1}^{k} u_{i,j}\right| \\
&= \left|2^\wedge(A\cup\{0\})\right| + |S(\mathbb{B})| + \sum_{i=1}^{r_{k-1}-1}\sum_{j=1}^{k-1} 1 + \sum_{i=1}^{r_k-1}\sum_{j=1}^{k} 1 \\
&\ge 2(k+1) - 3 + \sum_{i=1}^{k-2} i r_i + (k-1)(r_{k-1}-1) + k(r_k-1) \\
&= \sum_{i=1}^{k} i r_i. \qquad (4.2)
\end{aligned}$$

Now suppose that $k\ge 4$ and $|S(\mathbb{A})| = \sum_{i=1}^{k} i r_i$. Then from (4.2) it follows that $|2^\wedge(A\cup\{0\})| = 2(k+1)-3$. Theorem 2.2 implies that $A\cup\{0\}$ is an arithmetic progression. Hence $\mathbb{A} = a_1 * [1,k]_{\vec{r}}$. □

Theorem 4.2. *Let $\mathbb{A} = \{a_1, a_2, \ldots, a_k\}_{\vec{r}}$ be a sequence of positive integers such that $a_1 < a_2 < \cdots < a_k$, $\vec{r} = (r_1, r_2, \ldots, r_k)$ and $r_i \ge 1$ for all $i \in [1,k]$. Let $d(A) = 1$. Then*

$$|S(\mathbb{A})| \ge \begin{cases} \sum_{i=1}^{k} i r_i + a_k - k, & \text{if } a_k \le 2k-3; \\ \sum_{i=1}^{k} i r_i + \theta(k+1) - k - 4, & \text{if } a_k \ge 2k-2. \end{cases}$$

Proof. Set $\mathbb{B} = \{a_1, a_2, \ldots, a_{k-2}\}_{\vec{s}}$ with $\vec{s} = (r_1, r_2, \ldots, r_{k-2})$. From (4.2) we have

$$|S(\mathbb{A})| \ge \left|2^\wedge(A\cup\{0\})\right| + |S(\mathbb{B})| + (k-1)(r_{k-1}-1) + k(r_k-1).$$

Applying Theorem 2.8 on $A\cup\{0\}$ and Lemma 4.1 on $\mathbb{B}$, we get

$$\begin{aligned}
&|S(\mathbb{A})| \\
&\geq \begin{cases} a_k+k-1+\sum_{i=1}^{k-2} ir_i+(k-1)(r_{k-1}-1)+k(r_k-1), & \text{if } a_k\leq 2(k+1)-5;\\ (\theta+1)(k+1)-6+\sum_{i=1}^{k-2} ir_i+(k-1)(r_{k-1}-1)+k(r_k-1), & \text{if } a_k\geq 2(k+1)-4, \end{cases}\\
&\geq \begin{cases} \sum_{i=1}^{k} ir_i+a_k-k, & \text{if } a_k\leq 2k-3;\\ \sum_{i=1}^{k} ir_i+\theta(k+1)-k-4, & \text{if } a_k\geq 2k-2. \end{cases}
\end{aligned}$$

This completes the proof of the theorem. □

In the next theorem, we prove an improved bound for $|S(\mathbb{A})|$ than that in Theorem 4.2, when every element of $\mathbb{A}$ appears at least twice in $\mathbb{A}$. Before that, we prove the following lemma, which is crucial for our next theorem.

Lemma 4.3. *Let $\mathbb{A}=\{a_1,a_2,\ldots,a_k\}_{\vec{r}}$ be a sequence of positive integers such that $k\geq 2$, $a_1<a_2<\cdots<a_k$, $\vec{r}=(r_1,r_2,\ldots,r_k)$, and $r_i\geq 2$ for all $i\in[1,k]$. Let $r:=\min\{r_1,r_2,\ldots,r_k\}$ and $\mathbb{B}'=\{a_1,a_2,\ldots,a_{k-1}\}_{\vec{t}}$ with $\vec{t}=(r_1,r_2,\ldots,r_{k-1})$. Then*

$$|S(\mathbb{A})|\geq |r(A\cup\{0\})\setminus\{0\}|+\left|S(\mathbb{B}')\right|+k(r_k-r). \tag{4.3}$$

Proof. If $r_k=r$, then $r(A\cup\{0\})\setminus\{0\}$ and $S(\mathbb{B}')+ra_k$ are two disjoint subsets of $S(\mathbb{A})$. Thus

$$|S(\mathbb{A})|\geq |r(A\cup\{0\})\setminus\{0\}|+\left|S(\mathbb{B}')+ra_k\right|=|r(A\cup\{0\})\setminus\{0\}|+\left|S(\mathbb{B}')\right|.$$

If $r_k>r$, for $1\leq i\leq r_k-r$ and $1\leq j\leq k-1$ we define

$$v_{i,j}=\sum_{t=1,t\neq k-j}^{k-1} r_ta_t+(r_{k-j}-1)a_{k-j}+(r+i)a_k,$$

and

$$v_{i,k}=\sum_{t=1}^{k-1} r_ta_t+(r+i)a_k.$$

Then

$$v_{i,1}<v_{i,2}<\cdots<v_{i,k-1}<v_{i,k}<v_{i+1,1}.$$

Therefore, the elements $v_{i,j}$ are all distinct, all are in the set $S(\mathbb{A})$, and bigger than the elements of $r(A\cup\{0\})$ and $S(\mathbb{B}')+ra_k$. Thus

$$\begin{aligned}
|S(\mathbb{A})| &\geq |r(A\cup\{0\})\setminus\{0\}|+\left|S(\mathbb{B}')+ra_k\right|+\left|\bigcup_{i=1}^{r_k-r}\bigcup_{j=1}^{k} v_{i,j}\right|\\
&=|r(A\cup\{0\})\setminus\{0\}|+\left|S(\mathbb{B}')\right|+k(r_k-r).
\end{aligned}$$

□

Theorem 4.4. *Let $\mathbb{A}=\{a_1,a_2,\ldots,a_k\}_{\vec{r}}$ be a sequence of positive integers such that $k\geq 2$, $a_1<a_2<\cdots<a_k$, $\vec{r}=(r_1,r_2,\ldots,r_k)$ and $r_i\geq 2$ for all $i\in[1,k]$. Let $d(A)=1$ and $\min\{r_1,r_2,\ldots,r_k\}=r$. Then*

$$|S(\mathbb{A})|\geq |(r-1)(A\cup\{0\})|+\min\{a_k,r(k-1)+1\}-1+\sum_{i=1}^{k-1} ir_i+k(r_k-r).$$

Proof. Set $\mathbb{B}' = \{a_1, a_2, \ldots, a_{k-1}\}_{\vec{t}}$ with $\vec{t} = (r_1, r_2, \ldots, r_{k-1})$. Then from (4.3) we have

$$|S(\mathbb{A})| \geq |r(A \cup \{0\}) \setminus \{0\}| + |S(\mathbb{B}')| + k(r_k - r).$$

Applying Theorem 2.5 on $A \cup \{0\}$ and Lemma 4.1 on $\mathbb{B}'$, we obtain

$$|S(\mathbb{A})| \geq |(r-1)(A \cup \{0\})| + \min\{a_k, r(k-1)+1\} - 1 + \sum_{i=1}^{k-1} i r_i + k(r_k - r).$$

This completes the proof of the theorem. □

In the following theorem we prove a result similar to Theorem 4.2 and Theorem 4.4, when the sequence $\mathbb{A}$ contains nonnegative integers with $0 \in \mathbb{A}$.

Theorem 4.5. *Let $\mathbb{A} = \{a_0, a_1, \ldots, a_{k-1}\}_{\vec{r}}$ be a sequence of nonnegative integers such that $0 = a_0 < a_1 < \cdots < a_{k-1}$, $\vec{r} = (r_0, r_1, \ldots, r_{k-1})$ and $r_i \geq 1$ for all $i \in [0, k-1]$. Let $d(A) = 1$ and $\min\{r_1, r_2, \ldots, r_{k-1}\} = r$.*

(1) If $r \geq 2$, then

$$|S(\mathbb{A})| \geq |(r-1)A| + \min\{a_{k-1}, r(k-2)+1\} + \sum_{i=1}^{k-1} (i-1) r_{i-1} + (k-1)(r_{k-1} - r).$$

(2) If $r = 1$, then

$$|S(\mathbb{A})| \geq \begin{cases} a_{k-1} - k + 2 + \sum_{i=1}^{k} (i-1) r_{i-1}, & \text{if } a_{k-1} \leq 2k-5; \\ \theta k - k - 2 + \sum_{i=1}^{k} (i-1) r_{i-1}, & \text{if } a_{k-1} \geq 2k-4. \end{cases}$$

Proof. Let $\mathbb{B}'' := \{a_1, a_2, \ldots, a_{k-1}\}_{\vec{v}}$ with $\vec{v} := (r_1, r_2, \ldots, r_{k-1})$. Then $d(B'') = 1$ and $S(\mathbb{A}) = S(\mathbb{B}'') \cup \{0\}$. If $r \geq 2$, then applying Theorem 4.4 on $\mathbb{B}''$, we obtain

$$\begin{aligned} |S(\mathbb{A})| &\geq |S(\mathbb{B}'')| + 1 \\ &\geq |(r-1)A| + \min\{a_{k-1}, r(k-2)+1\} - 1 + \sum_{i=1}^{k-2} i r_i + (k-1)(r_{k-1} - r) + 1 \\ &\geq |(r-1)A| + \min\{a_{k-1}, r(k-2)+1\} + \sum_{i=1}^{k-1} (i-1) r_{i-1} + (k-1)(r_{k-1} - r). \end{aligned}$$

If $r = 1$, then by Theorem 4.2, we have

$$\begin{aligned} |S(\mathbb{A})| &\geq |S(\mathbb{B}'')| + 1 \\ &\geq \begin{cases} a_{k-1} - k + 1 + \sum_{i=1}^{k-1} i r_i + 1, & \text{if } a_{k-1} \leq 2(k-1) - 3; \\ \theta k - k - 3 + \sum_{i=1}^{k-1} i r_i + 1, & \text{if } a_{k-1} \geq 2(k-1) - 2, \end{cases} \\ &= \begin{cases} a_{k-1} - k + 2 + \sum_{i=1}^{k} (i-1) r_{i-1}, & \text{if } a_{k-1} \leq 2k-5; \\ \theta k - k - 2 + \sum_{i=1}^{k} (i-1) r_{i-1}, & \text{if } a_{k-1} \geq 2k-4. \end{cases} \end{aligned}$$

This completes the proof of the theorem. □

5. Freiman's theorem for α-subset sum

In this section, we prove Freiman-like theorems for $S_\alpha(A)$, when the set A contains positive integers and when the set A contains nonnegative integers with $0 \in A$, in Theorem 5.1 and Theorem 5.2, respectively. To prove our results, we define

$$S_1^\alpha(A) := \{s(B) : B \subseteq A, 1 \leq |B| \leq |A| - \alpha\}.$$

Then $S_\alpha(A) = \sum_{a\in A} a - (S_1^\alpha(A) \cup \{0\})$. Therefore, $|S_\alpha(A)| = |S_1^\alpha(A)| + 1$ if $0 \notin A$ and $|S_\alpha(A)| = |S_1^\alpha(A)|$ if $0 \in A$.

Theorem 5.1. *Let $k \geq 3$. Let $A = \{a_1, a_2, \ldots, a_k\}$ be a set of k positive integers such that $a_1 < a_2 < \cdots < a_k$ and $d(A) = 1$. Let $\alpha \leq k-2$ be a positive integer. Then*

$$|S_\alpha(A)| \geq \begin{cases} a_k + \dfrac{k(k-1)}{2} - \dfrac{\alpha(\alpha+1)}{2} + 1, & \text{if } a_k \leq 2k-3; \\ \theta(k+1) - 4 + \dfrac{k(k-1)}{2} - \dfrac{\alpha(\alpha+1)}{2} + 1, & \text{if } a_k \geq 2k-2. \end{cases}$$

Proof. Set $B = A \setminus \{a_{k-1}, a_k\}$. Then

$$2^\wedge(A \cup \{0\}) \cup (S_1^\alpha(B) + a_{k-1} + a_k) \subset S_1^\alpha(A).$$

Here we are assuming that $S_1^\alpha(B) + a_{k-1} + a_k = \emptyset$ if $\alpha = |B|$. Observe that $2^\wedge(A \cup \{0\})$ and $S_1^\alpha(B) + a_{k-1} + a_k$ are disjoint. Thus

$$|S_\alpha(A)| = |S_1^\alpha(A)| + 1 \geq \left|2^\wedge(A \cup \{0\})\right| + |S_1^\alpha(B)| + 1 = \left|2^\wedge(A \cup \{0\})\right| + |S_\alpha(B)|.$$

If $a_k \leq 2k-3 = 2(k+1)-5$, then applying Theorem 2.8 on $A \cup \{0\}$ and Theorem 2.9 on B, we obtain

$$\begin{aligned} |S_\alpha(A)| &\geq a_k + k - 1 + \frac{(k-2)(k-1)}{2} - \frac{\alpha(\alpha+1)}{2} + 1 \\ &= a_k + \frac{k(k-1)}{2} - \frac{\alpha(\alpha+1)}{2} + 1. \end{aligned}$$

If $a_k \geq 2k-2 = 2(k+1)-4$, then again by Theorem 2.8 and Theorem 2.9, we get

$$\begin{aligned} |S_\alpha(A)| &\geq (\theta+1)(k+1) - 6 + \frac{(k-2)(k-1)}{2} - \frac{\alpha(\alpha+1)}{2} + 1 \\ &= \theta(k+1) - 4 + \frac{k(k-1)}{2} - \frac{\alpha(\alpha+1)}{2} + 1. \end{aligned}$$

This proves the theorem. □

We also have the following theorem when the set A has 0 as an element.

Theorem 5.2. *Let $k \geq 4$. Let $A = \{a_0, a_1, \ldots, a_{k-1}\}$ be a set of nonnegative integers such that $0 = a_0 < a_1 < \cdots < a_{k-1}$ and $d(A) = 1$. Let $\alpha \leq k-2$ be a positive integer. Then*

$$|S_\alpha(A)| \geq \begin{cases} a_{k-1} + \dfrac{(k-1)(k-2)}{2} - \dfrac{\alpha(\alpha-1)}{2} + 2, & \text{if } a_{k-1} \leq 2k-5; \\ \theta k - 4 + \dfrac{(k-1)(k-2)}{2} - \dfrac{\alpha(\alpha-1)}{2} + 2, & \text{if } a_{k-1} \geq 2k-4. \end{cases}$$

Proof. Set $B' = \{a_1, a_2, \ldots, a_{k-1}\}$. Then B' is a set of $k-1$ positive integers, $d(B') = 1$ and $S_1^{\alpha}(A) = S_1^{\alpha-1}(B') \cup \{0\}$. Then, from Theorem 5.1, it follows that

$$\begin{aligned}|S_\alpha(A)| &= |S_1^{\alpha}(A)| \\ &= \left|S_1^{\alpha-1}(B')\right| + 1 \\ &\geq \begin{cases} a_{k-1} + \dfrac{(k-1)(k-2)}{2} - \dfrac{\alpha(\alpha-1)}{2} + 2, & \text{if } a_{k-1} \leq 2k-5; \\ \theta k - 4 + \dfrac{(k-1)(k-2)}{2} - \dfrac{\alpha(\alpha-1)}{2} + 2, & \text{if } a_{k-1} \geq 2k-4. \end{cases}\end{aligned}$$

This proves the theorem. □

6. Freiman's theorem for α-subsequence sum

Let $\mathbb{A} = \{a_1, a_2, \ldots, a_k\}_{\vec{r}}$ be a sequence of positive integers, where $\vec{r} = (r_1, r_2, \ldots, r_k)$ with $r_i \geq 1$ for all $i \in [1,k]$. Let $\min\{r_1, r_2, \ldots, r_k\} = r$, and let $\alpha \leq \sum_{i=1}^{k} r_i - 2$ be a positive integer. In this section, we prove Freiman's $3k-4$-like results for $S_\alpha(\mathbb{A})$. The proofs are quite similar to the ones in Section 4, however, in this section, the proofs are more involved and depend heavily on α. In Theorem 6.1, we assume that $\alpha = \sum_{i=1}^{k} r_i - 2$. Then, in Theorems 6.2 and 6.3, we consider the case that $r \geq 2$ and $\alpha < \sum_{i=1}^{k} r_i - 2$. Further, in Theorems 6.4 and 6.5, we assume that $r = 1$ and $\alpha < \sum_{i=1}^{k} r_i - 2$. In Theorem 6.4, we consider all possible cases with $r = 1$, except the one that $r_{k-1} = 1$ and $r_k \neq 1$, with which we deal in Theorem 6.5. In all the above-mentioned theorems the sequence $\mathbb{A}$ contains positive integers. We prove similar results in Theorems 6.6, 6.7 and 6.8, when the sequence $\mathbb{A}$ contains nonnegative integers with $0 \in \mathbb{A}$.

Before proceeding to the results of this section, we first define

$$S_1^{\alpha}(\mathbb{A}) := \left\{ s(\mathbb{B}) : \mathbb{B} \subseteq \mathbb{A}, 1 \leq |\mathbb{B}| \leq \sum_{i=1}^{k} r_i - \alpha \right\}.$$

Then $S_\alpha(\mathbb{A}) = \sum_{a \in \mathbb{A}} a - (S_1^{\alpha}(\mathbb{A}) \cup \{0\})$. Therefore, $|S_\alpha(\mathbb{A})| = |S_1^{\alpha}(\mathbb{A})| + 1$ if $0 \notin \mathbb{A}$ and $|S_\alpha(\mathbb{A})| = |S_1^{\alpha}(\mathbb{A})|$ if $0 \in \mathbb{A}$.

Theorem 6.1. *Let $k \geq 3$. Let $\mathbb{A} = \{a_1, a_2, \ldots, a_k\}_{\vec{r}}$ be a sequence of positive integers such that $a_1 < a_2 < \cdots < a_k$, $\vec{r} = (r_1, r_2, \ldots, r_k)$ and $r_i \geq 1$ for all $i \in [1,k]$. Let $r = \min\{r_1, r_2, \ldots, r_k\}$. Let $\alpha = \sum_{i=1}^{k} r_i - 2$ and $d(A) = 1$. If $r = 1$, then*

$$|S_\alpha(\mathbb{A})| \geq \begin{cases} a_k + k, & \text{if } a_k \leq 2k-3; \\ (\theta+1)(k+1) - 4, & \text{if } a_k \geq 2k-2. \end{cases}$$

If $r \geq 2$, then

$$|S_\alpha(\mathbb{A})| \geq \begin{cases} a_k + k + 1, & \text{if } a_k \leq 2k-1; \\ 3k, & \text{if } a_k \geq 2k. \end{cases}$$

Proof. If $r = 1$, then

$$S_1^{\alpha}(\mathbb{A}) = 2^{\wedge}(A \cup \{0\}).$$

Therefore, by Theorem 2.8, we get

$$|S_\alpha(\mathbb{A})| = |S_1^\alpha(\mathbb{A})| + 1 \geq \begin{cases} a_k + k, & \text{if } a_k \leq 2k-3; \\ (\theta+1)(k+1)-4, & \text{if } a_k \geq 2k-2. \end{cases}$$

If $r \geq 2$, then

$$S_1^\alpha(\mathbb{A}) = 2(A \cup \{0\}) \setminus \{0\}.$$

Therefore, by Theorem 2.3, we get

$$|S_\alpha(\mathbb{A})| = |S_1^\alpha(\mathbb{A})| + 1 \geq \begin{cases} a_k + k + 1, & \text{if } a_k \leq 2k-1; \\ 3k, & \text{if } a_k \geq 2k. \end{cases}$$

□

Theorem 6.2. *Let $k \geq 3$. Let $\mathbb{A} = \{a_1, a_2, \ldots, a_k\}_{\vec{r}}$ be a sequence of positive integers such that $a_1 < a_2 < \cdots < a_k$, $\vec{r} = (r_1, r_2, \ldots, r_k)$ and $r_i \geq 2$ for all $i \in [1,k]$. Let $\min\{r_1, r_2, \ldots, r_k\} = r$ and $d(A) = 1$. Let $\alpha < \sum_{i=1}^{k} r_i - r$ be a positive integer. Then there exists an integer $m \in [1,k]$ such that $\sum_{i=1}^{m-1} r_i \leq \alpha < \sum_{i=1}^{m} r_i$ and*

$$|S_\alpha(\mathbb{A})| \geq |(r-1)(A \cup \{0\})| + \min\{a_k, r(k-1)+1\} + \sum_{i=1}^{k-1} i r_i - \sum_{i=1}^{m} i r_i + m\left(\sum_{i=1}^{m} r_i - \alpha\right) + k(r_k - r).$$

Proof. Set $\mathbb{B}_1 = \{a_1, a_2, \ldots, a_{k-1}, a_k\}_{\vec{s_1}}$ with $\vec{s_1} = (r_1, r_2, \ldots, r_{k-1}, r_k - r)$. Then

$$(r(A \cup \{0\}) \setminus \{0\}) \cup (S_1^\alpha(\mathbb{B}_1) + r a_k) \subset S_1^\alpha(\mathbb{A}),$$

where $(r(A \cup \{0\}) \setminus \{0\}) \cap (S_1^\alpha(\mathbb{B}_1) + r a_k) = \emptyset$. Therefore

$$|S_\alpha(\mathbb{A})| = |S_1^\alpha(\mathbb{A})| + 1 \geq |r(A \cup \{0\}) \setminus \{0\}| + |S_1^\alpha(\mathbb{B}_1)| + 1 = |r(A \cup \{0\})| + |S_\alpha(\mathbb{B}_1)| - 1.$$

If $m \leq k-1$, then applying Theorem 2.5 on $A \cup \{0\}$ and Theorem 2.10 on $\mathbb{B}_1$, we obtain

$$\begin{aligned} |S_\alpha(\mathbb{A})| &\geq |(r-1)(A \cup \{0\})| + \min\{a_k, r(k-1)+1\} - 1 + \sum_{i=1}^{k-1} i r_i + k(r_k - r) - \sum_{i=1}^{m} i r_i \\ &\quad + m\left(\sum_{i=1}^{m} r_i - \alpha\right) + 1 \\ &= |(r-1)(A \cup \{0\})| + \min\{a_k, r(k-1)+1\} + \sum_{i=1}^{k-1} i r_i - \sum_{i=1}^{m} i r_i + m\left(\sum_{i=1}^{m} r_i - \alpha\right) \\ &\quad + k(r_k - r). \end{aligned}$$

If $m = k$, then again applying Theorem 2.5 on $A \cup \{0\}$ and Theorem 2.10 on $\mathbb{B}_1$, we obtain

$$|S_\alpha(\mathbb{A})| \geq |(r-1)(A \cup \{0\})| + \min\{a_k, r(k-1)+1\} + k\left(\sum_{i=1}^{k-1} r_i + r_k - r - \alpha\right).$$

This proves the theorem. □

In the following theorem, we prove a similar result for the remaining values of α, i.e., $\sum_{i=1}^{k} r_i - r \leq \alpha < \sum_{i=1}^{k} r_i - 2$.

Theorem 6.3. *Let $k \geq 3$. Let $\mathbb{A} = \{a_1, a_2, \ldots, a_k\}_{\vec{r}}$ be a sequence of positive integers such that $a_1 < a_2 < \cdots < a_k$, $\vec{r} = (r_1, r_2, \ldots, r_k)$ and $r_i \geq 2$ for all $i \in [1,k]$. Let $\min\{r_1, r_2, \ldots, r_k\} = r$ and $d(A) = 1$. Let $\sum_{i=1}^{k} r_i - r \leq \alpha < \sum_{i=1}^{k} r_i - 2$ be a positive integer. Then*

$$|S_\alpha(\mathbb{A})| \geq \begin{cases} a_k - k + 2 + k\left(\sum_{i=1}^{k} r_i - \alpha\right), & \text{if } a_k \leq 2k-1; \\ k + 1 + k\left(\sum_{i=1}^{k} r_i - \alpha\right), & \text{if } a_k \geq 2k. \end{cases}$$

Proof. Set $\mathbb{B}_2 = \{a_1, a_2, \ldots, a_{k-1}, a_k\}_{\vec{s_2}}$ with $\vec{s_2} = (r_1, r_2, \ldots, r_{k-1}, r_k - 2)$. Then

$$(2(A \cup \{0\}) \setminus \{0\}) \cup (S_1^\alpha(\mathbb{B}_2) + 2a_k) \subset S_1^\alpha(\mathbb{A}),$$

where $(2(A \cup \{0\}) \setminus \{0\}) \cap (S_1^\alpha(\mathbb{B}_2) + 2a_k) = \emptyset$. Therefore,

$$|S_\alpha(\mathbb{A})| = |S_1^\alpha(\mathbb{A})| + 1 \geq |2(A \cup \{0\}) \setminus \{0\}| + |S_1^\alpha(\mathbb{B}_2)| + 1 = |2(A \cup \{0\})| + |S_\alpha(\mathbb{B}_2)| - 1.$$

Applying Theorem 2.4 on $A \cup \{0\}$ and Theorem 2.10 on $\mathbb{B}_2$ we obtain

$$\begin{aligned} |S_\alpha(\mathbb{A})| &\geq |2(A \cup \{0\})| - 1 + k\left(\sum_{i=1}^{k-1} r_i + r_k - 2 - \alpha\right) + 1 \\ &\geq \begin{cases} a_k + k + 1 + k\left(\sum_{i=1}^{k} r_i - \alpha\right) - 2k, & \text{if } a_k \leq 2(k+1) - 3; \\ 3(k+1) - 3 + k\left(\sum_{i=1}^{k} r_i - \alpha\right) - 2k, & \text{if } a_k \geq 2(k+1) - 2, \end{cases} \\ &= \begin{cases} a_k - k + 1 + k\left(\sum_{i=1}^{k} r_i - \alpha\right), & \text{if } a_k \leq 2k-1; \\ k + k\left(\sum_{i=1}^{k} r_i - \alpha\right), & \text{if } a_k \geq 2k. \end{cases} \end{aligned}$$

□

The case $r = 1$ is considered in the following two theorems.

Theorem 6.4. *Let $k \geq 3$. Let $\mathbb{A} = \{a_1, a_2, \ldots, a_k\}_{\vec{r}}$ be a sequence of positive integers such that $a_1 < a_2 < \cdots < a_k$, $\vec{r} = (r_1, r_2, \ldots, r_k)$ and $r_i \geq 1$ for all $i \in [1,k]$. Let $r_{k-1} \neq 1$ and $r_k \neq 1$ or $r_{k-1} = r_k = 1$ or $r_{k-1} \neq 1$ and $r_k = 1$. Let $d(A) = 1$. Let $\alpha < \sum_{i=1}^{k} r_i - 2$ be a positive integer. Then the following holds.*

(1) *If $\alpha < \sum_{i=1}^{k-1} r_i - 1$, then there exists an integer $m \in [1, k-1]$ such that $\sum_{i=1}^{m-1} r_i \leq \alpha < \sum_{i=1}^{m} r_i$ and*

$$|S_\alpha(\mathbb{A})| \geq \begin{cases} a_k - k + 1 + \sum_{i=1}^{k} i r_i - \sum_{i=1}^{m} i r_i + m\left(\sum_{i=1}^{m} r_i - \alpha\right), & \text{if } a_k \leq 2k-3; \\ \theta(k+1) - k - 3 + \sum_{i=1}^{k} i r_i - \sum_{i=1}^{m} i r_i + m\left(\sum_{i=1}^{m} r_i - \alpha\right), & \text{if } a_k \geq 2k-2. \end{cases}$$

(2) *If $\sum_{i=1}^{k-1} r_i - 1 \leq \alpha < \sum_{i=1}^{k} r_i - 2$, then*

$$|S_\alpha(\mathbb{A})| \geq \begin{cases} a_k - k + k\left(\sum_{i=1}^{k} r_i - \alpha\right), & \text{if } a_k \leq 2k-3; \\ \theta(k+1) - k - 4 + k\left(\sum_{i=1}^{k} r_i - \alpha\right), & \text{if } a_k \geq 2k-2. \end{cases}$$

Proof. Set $\mathbb{B}_3 = \{a_1, a_2, \ldots, a_{k-1}, a_k\}_{\vec{s_3}}$ with $\vec{s_3} = (r_1, r_2, \ldots, r_{k-2}, r_{k-1}-1, r_k-1)$. Then

$$2^\wedge(A \cup \{0\}) \cup (S_1^\alpha(\mathbb{B}_3) + a_{k-1} + a_k) \subset S_1^\alpha(\mathbb{A}),$$

where $2^\wedge(A \cup \{0\}) \cap (S_1^\alpha(\mathbb{B}_3) + a_{k-1} + a_k) = \emptyset$. Therefore,

$$|S_\alpha(\mathbb{A})| = |S_1^\alpha(\mathbb{A})| + 1 \geq \left|2^\wedge(A \cup \{0\})\right| + |S_1^\alpha(\mathbb{B}_3)| + 1 = \left|2^\wedge(A \cup \{0\})\right| + |S_\alpha(\mathbb{B}_3)|.$$

Case I ($r_{k-1} \geq 2$ and $r_k \geq 2$). If $\alpha < \sum_{i=1}^{k-2} r_i$, then $m \leq k-2$ for both $\mathbb{A}$ and $\mathbb{B}_3$. Applying Theorem 2.8 on $A \cup \{0\}$ and Theorem 2.10 on $\mathbb{B}_3$, we get

$$\begin{aligned}
&|S_\alpha(\mathbb{A})| \\
&\geq \left|2^\wedge(A \cup \{0\})\right| + |S_\alpha(\mathbb{B}_3)| \\
&\geq \begin{cases} a_k + k - 1 + \sum_{i=1}^k ir_i - (k-1) - k - \sum_{i=1}^m ir_i + m\left(\sum_{i=1}^m r_i - \alpha\right) + 1, \\ \text{ if } a_k \leq 2k-3; \\ (\theta+1)(k+1) - 6 + \sum_{i=1}^k ir_i - (k-1) - k - \sum_{i=1}^m ir_i + m\left(\sum_{i=1}^m r_i - \alpha\right) + 1, \\ \text{ if } a_k \geq 2k-2, \end{cases} \\
&= \begin{cases} a_k - k + 1 + \sum_{i=1}^k ir_i - \sum_{i=1}^m ir_i + m\left(\sum_{i=1}^m r_i - \alpha\right), & \text{if } a_k \leq 2k-3; \\ \theta(k+1) - k - 3 + \sum_{i=1}^k ir_i - \sum_{i=1}^m ir_i + m\left(\sum_{i=1}^m r_i - \alpha\right), & \text{if } a_k \geq 2k-2. \end{cases}
\end{aligned}$$

If $\sum_{i=1}^{k-2} r_i \leq \alpha < \sum_{i=1}^{k-1} r_i - 1$, then $m = k-1$ for both $\mathbb{A}$ and $\mathbb{B}_3$. Applying Theorem 2.8 on $A \cup \{0\}$ and Theorem 2.10 on $\mathbb{B}_3$, we get

$$\begin{aligned}
&|S_\alpha(\mathbb{A})| \\
&\geq \left|2^\wedge(A \cup \{0\})\right| + |S_\alpha(\mathbb{B}_3)| \\
&\geq \begin{cases} a_k + k - 1 + \sum_{i=1}^k ir_i - (2k-1) - \left(\sum_{i=1}^{k-1} ir_i - (k-1)\right) + (k-1)\left(\sum_{i=1}^{k-1} r_i - 1 - \alpha\right) + 1, \\ \text{ if } a_k \leq 2k-3; \\ (\theta+1)(k+1) - 6 + \sum_{i=1}^k ir_i - (2k-1) - \left(\sum_{i=1}^{k-1} ir_i - (k-1)\right) + (k-1)\left(\sum_{i=1}^{k-1} r_i - 1 - \alpha\right) + 1, \\ \text{ if } a_k \geq 2k-2; \end{cases} \\
&= \begin{cases} a_k - k + 1 + \sum_{i=1}^k ir_i - \sum_{i=1}^{k-1} ir_i + (k-1)\left(\sum_{i=1}^{k-1} r_i - \alpha\right), & \text{if } a_k \leq 2k-3; \\ \theta(k+1) - k - 3 + \sum_{i=1}^k ir_i - \sum_{i=1}^{k-1} ir_i + (k-1)\left(\sum_{i=1}^{k-1} r_i - \alpha\right), & \text{if } a_k \geq 2k-2. \end{cases}
\end{aligned}$$

If $\sum_{i=1}^{k-1} r_i - 1 \leq \alpha < \sum_{i=1}^{k} r_i - 2$, then applying Theorem 2.8 on $A \cup \{0\}$ and Theorem 2.10 on $\mathbb{B}_3$, we get

$$
\begin{aligned}
&|S_\alpha(\mathbb{A})| \\
&\geq \left|2^\wedge(A\cup\{0\})\right| + |S_\alpha(\mathbb{B}_3)| \\
&\geq \begin{cases} a_k + k - 1 + k\left(\sum_{i=1}^{k} r_i - 2 - \alpha\right) + 1, & \text{if } a_k \leq 2k-3; \\ (\theta+1)(k+1) - 6 + k\left(\sum_{i=1}^{k} r_i - 2 - \alpha\right) + 1, & \text{if } a_k \geq 2k-2, \end{cases} \\
&= \begin{cases} a_k - k + k\left(\sum_{i=1}^{k} r_i - \alpha\right), & \text{if } a_k \leq 2k-3; \\ \theta(k+1) - k - 4 + k\left(\sum_{i=1}^{k} r_i - \alpha\right), & \text{if } a_k \geq 2k-2. \end{cases}
\end{aligned}
$$

Case II ($r_{k-1} \geq 2$ and $r_k = 1$). In this case, the sequence $\mathbb{B}_3$ has $k-1$ distinct elements and the vector $\vec{s_3}$ has $k-1$ terms. If $\alpha < \sum_{i=1}^{k-2} r_i$, then $m \leq k-2$ for both $\mathbb{A}$ and $\mathbb{B}_3$. Applying Theorem 2.8 on $A \cup \{0\}$ and Theorem 2.10 on $\mathbb{B}_3$, we get

$$
\begin{aligned}
&|S_\alpha(\mathbb{A})| \\
&\geq \left|2^\wedge(A\cup\{0\})\right| + |S_\alpha(\mathbb{B}_3)| \\
&\geq \begin{cases} a_k + k - 1 + \sum_{i=1}^{k-2} i r_i + (k-1)(r_{k-1}-1) - \sum_{i=1}^{m} i r_i + m\left(\sum_{i=1}^{m} r_i - \alpha\right) + 1, \\ \text{ if } a_k \leq 2k-3; \\ (\theta+1)(k+1) - 6 + \sum_{i=1}^{k-2} i r_i + (k-1)(r_{k-1}-1) - \sum_{i=1}^{m} i r_i + m\left(\sum_{i=1}^{m} r_i - \alpha\right) + 1, \\ \text{ if } a_k \geq 2k-2, \end{cases} \\
&= \begin{cases} a_k - k + 1 + \sum_{i=1}^{k} i r_i - \sum_{i=1}^{m} i r_i + m\left(\sum_{i=1}^{m} r_i - \alpha\right), & \text{if } a_k \leq 2k-3; \\ \theta(k+1) - k - 3 + \sum_{i=1}^{k} i r_i - \sum_{i=1}^{m} i r_i + m\left(\sum_{i=1}^{m} r_i - \alpha\right), & \text{if } a_k \geq 2k-2. \end{cases}
\end{aligned}
$$

If $\sum_{i=1}^{k-2} r_i \leq \alpha < \sum_{i=1}^{k-1} r_i - 1 = \sum_{i=1}^{k} r_i - 2$, then $m = k-1$ for both $\mathbb{A}$ and $\mathbb{B}_3$. Applying Theorem 2.8 on $A \cup \{0\}$ and Theorem 2.10 on $\mathbb{B}_3$, we get

$$
\begin{aligned}
&|S_\alpha(\mathbb{A})| \\
&\geq \left|2^\wedge(A\cup\{0\})\right| + |S_\alpha(\mathbb{B}_3)| \\
&\geq \begin{cases} a_k + k - 1 + (k-1)\left(\sum_{i=1}^{k-2} r_i + r_{k-1} - 1 - \alpha\right) + 1, & \text{if } a_k \leq 2k-3; \\ (\theta+1)(k+1) - 6 + (k-1)\left(\sum_{i=1}^{k-2} r_i + r_{k-1} - 1 - \alpha\right) + 1, & \text{if } a_k \geq 2k-2, \end{cases} \\
&= \begin{cases} a_k - k + 1 + \sum_{i=1}^{k} i r_i - \sum_{i=1}^{k-1} i r_i + (k-1)\left(\sum_{i=1}^{k-1} r_i - \alpha\right), & \text{if } a_k \leq 2k-3; \\ \theta(k+1) - k - 3 + \sum_{i=1}^{k} i r_i - \sum_{i=1}^{k-1} i r_i + (k-1)\left(\sum_{i=1}^{k-1} r_i - \alpha\right), & \text{if } a_k \geq 2k-2. \end{cases}
\end{aligned}
$$

Case III ($r_{k-1} = r_k = 1$). In this case, the sequence $\mathbb{B}_3$ has $k-2$ distinct elements and the vector $\vec{s_3}$ has $k-2$ terms. Thus, $m \leq k-2$ for both $\mathbb{A}$ and $\mathbb{B}_3$. Applying Theorem 2.8 on $A \cup \{0\}$ and Theorem

2.10 on $\mathbb{B}_3$, we get

$$\begin{aligned}
&|S_\alpha(\mathbb{A})|\\
&\geq \left|2^\wedge(A\cup\{0\})\right| + |S_\alpha(\mathbb{B}_3)|\\
&\geq \begin{cases} a_k+k-1+\sum_{i=1}^{k-2} ir_i - \sum_{i=1}^{m} ir_i + m\left(\sum_{i=1}^{m} r_i-\alpha\right)+1, & \text{if } a_k\leq 2k-3;\\ (\theta+1)(k+1)-6+\sum_{i=1}^{k-2} ir_i - \sum_{i=1}^{m} ir_i + m\left(\sum_{i=1}^{m} r_i-\alpha\right)+1, & \text{if } a_k\geq 2k-2,\end{cases}\\
&= \begin{cases} a_k-k+1+\sum_{i=1}^{k} ir_i - \sum_{i=1}^{m} ir_i + m\left(\sum_{i=1}^{m} r_i-\alpha\right), & \text{if } a_k\leq 2k-3;\\ \theta(k+1)-k-3+\sum_{i=1}^{k} ir_i - \sum_{i=1}^{m} ir_i + m\left(\sum_{i=1}^{m} r_i-\alpha\right), & \text{if } a_k\geq 2k-2.\end{cases}
\end{aligned}$$

□

Theorem 6.5. *Let $k\geq 3$. Let $\mathbb{A}=\{a_1,a_2,\ldots,a_k\}_{\vec{r}}$ be a sequence of positive integers such that $a_1<a_2<\cdots<a_k$, $\vec{r}=(r_1,r_2,\ldots,r_k)$, $r_i\geq 1$ for all $i\in[1,k-2]$, $r_{k-1}=1$ and $r_k\geq 2$. Let $d(A)=1$. Let $\alpha<\sum_{i=1}^{k} r_i-2$ be a positive integer. Then the following holds.*

(1) *If $\alpha<\sum_{i=1}^{k-2} r_i$, then there exists an integer $m\in[1,k-2]$ such that $\sum_{i=1}^{m-1} r_i\leq\alpha<\sum_{i=1}^{m} r_i$ and*

$$|S_\alpha(\mathbb{A})|\geq\begin{cases} a_k+k+\sum_{i=1}^{k-2} ir_i-\sum_{i=1}^{m} ir_i+m\left(\sum_{i=1}^{m} r_i-\alpha\right)+(k-m+1)(r_k-1),\\ \quad \text{if } a_k\leq 2k-3;\\ \theta(k+1)+k-4+\sum_{i=1}^{k-2} ir_i-\sum_{i=1}^{m} ir_i+m\left(\sum_{i=1}^{m} r_i-\alpha\right)+(k-m+1)(r_k-1),\\ \quad \text{if } a_k\geq 2k-2.\end{cases}$$

(2) *If $\sum_{i=1}^{k-2} r_i\leq\alpha<\sum_{i=1}^{k} r_i-2$, then*

$$|S_\alpha(\mathbb{A})|\geq\begin{cases} a_k-k+2+(k-1)\left(\sum_{i=1}^{k} r_i-\alpha\right), & \text{if } a_k\leq 2k-3;\\ \theta(k+1)-k-2+(k-1)\left(\sum_{i=1}^{k} r_i-\alpha\right), & \text{if } a_k\geq 2k-2.\end{cases}$$

Proof. Set $\mathbb{B}_4=\{a_1,a_2,\ldots,a_{k-2}\}_{\vec{s_4}}$ with $\vec{s_4}=(r_1,r_2,\ldots,r_{k-2})$. If $\alpha<\sum_{i=1}^{k-2} r_i$, then $2^\wedge(A\cup\{0\})$ and $(S_1^\alpha(\mathbb{B}_4)+a_{k-1}+a_k)$ are two disjoint subsets of $S_1^\alpha(\mathbb{A})$. For $1\leq i\leq r_k-1$ and $1\leq j\leq k-m-1$, define

$$v_{i,j}=\left(\sum_{\ell=1}^{m} r_\ell-\alpha\right)a_m+\sum_{t=m+1,t\neq k-j}^{k-1} r_t a_t+(r_{k-j}-1)a_{k-j}+(i+1)a_k,$$

$$v_{i,k-m}=\left(\sum_{\ell=1}^{m} r_\ell-\alpha-1\right)a_m+\sum_{t=m+1}^{k-1} r_t a_t+(i+1)a_k,$$

and

$$v_{i,k-m+1}=\left(\sum_{\ell=1}^{m} r_\ell-\alpha\right)a_m+\sum_{t=m+1}^{k-1} r_t a_t+(i+1)a_k.$$

It is easy to see that

$$v_{i,1}<v_{i,2}<\cdots<v_{i,k-m-1}<v_{i,k-m}<v_{i,k-m+1}<v_{i+1,1}.$$

Therefore, the elements $v_{i,j}$ are all distinct, all are in the set $S_1^{\alpha}(\mathbb{A})$ and bigger than the elements of $2^{\wedge}(A\cup\{0\})$ and $S_1^{\alpha}(\mathbb{B}_4)+a_{k-1}+a_k$. Therefore, we have

$$\begin{aligned}|S_{\alpha}(\mathbb{A})| &= |S_1^{\alpha}(\mathbb{A})|+1\\ &\geq \left|2^{\wedge}(A\cup\{0\})\right|+|S_1^{\alpha}(\mathbb{B}_4)|+\left|\bigcup_{i=1}^{r_k-1}\bigcup_{j=1}^{k-m+1} v_{i,j}\right|+1\\ &= \left|2^{\wedge}(A\cup\{0\})\right|+|S_{\alpha}(\mathbb{B}_4)|+\sum_{i=1}^{r_k-1}\sum_{j=1}^{k-m+1} 1.\end{aligned}$$

By Theorem 2.8 and Theorem 2.10, we have

$$\begin{aligned}&|S_{\alpha}(\mathbb{A})|\\ &\geq \begin{cases} a_k+k-1+\sum_{i=1}^{k-2} ir_i-\sum_{i=1}^{m} ir_i+m\left(\sum_{i=1}^{m} r_i-\alpha\right)+1+(k-m+1)(r_k-1), \\ \text{ if } a_k\leq 2(k+1)-5; \\ (\theta+1)(k+1)-6+\sum_{i=1}^{k-2} ir_i-\sum_{i=1}^{m} ir_i+m\left(\sum_{i=1}^{m} r_i-\alpha\right)+1+(k-m+1)(r_k-1), \\ \text{ if } a_k\geq 2(k+1)-4, \end{cases}\\ &= \begin{cases} a_k+k+\sum_{i=1}^{k-2} ir_i-\sum_{i=1}^{m} ir_i+m\left(\sum_{i=1}^{m} r_i-\alpha\right)+(k-m+1)(r_k-1), \\ \text{ if } a_k\leq 2k-3; \\ \theta(k+1)+k-4+\sum_{i=1}^{k-2} ir_i-\sum_{i=1}^{m} ir_i+m\left(\sum_{i=1}^{m} r_i-\alpha\right)+(k-m+1)(r_k-1), \\ \text{ if } a_k\geq 2k-2. \end{cases}\end{aligned}$$

Now, let $\sum_{i=1}^{k-2} r_i\leq\alpha<\sum_{i=1}^{k} r_i-2=\sum_{i=1}^{k-2} r_i+r_k-1$. Set $a'_{k-1}=a_k$, $r'_{k-1}=r_k-1$, and $\mathbb{B}_5=\{a_1,a_2,\ldots,a_{k-2},a'_{k-1}\}_{\vec{s}_5}$ with $\vec{s}_5=(r_1,r_2,\ldots,r_{k-2},r'_{k-1})$. Then

$$2^{\wedge}(A\cup\{0\})\cup(S_1^{\alpha}(\mathbb{B}_5)+a_{k-1}+a_k)\subset S_1^{\alpha}(\mathbb{A}),$$

where $2^{\wedge}(A\cup\{0\})\cap(S_1^{\alpha}(\mathbb{B}_5)+a_{k-1}+a_k)=\emptyset$. Thus,

$$|S_{\alpha}(\mathbb{A})|=|S_1^{\alpha}(\mathbb{A})|+1\geq\left|2^{\wedge}(A\cup\{0\})\right|+|S_1^{\alpha}(\mathbb{B}_5)|+1=\left|2^{\wedge}(A\cup\{0\})\right|+|S_{\alpha}(\mathbb{B}_5)|.$$

As $\sum_{i=1}^{k-2} r_i\leq\alpha<\sum_{i=1}^{k-2} r_i+r'_{k-1}$, by Theorem 2.10, we have

$$|S_{\alpha}(\mathbb{B}_5)|\geq(k-1)\left(\sum_{i=1}^{k-2} r_i+r'_{k-1}-\alpha\right)+1=(k-1)\left(\sum_{i=1}^{k-2} r_i+r_k-1-\alpha\right)+1.$$

This equation together with Theorem 2.8 applying on $A \cup \{0\}$, give

$$\begin{aligned}|S_\alpha(\mathbb{A})| &= \left|2^\wedge(A\cup\{0\})\right| + |S_\alpha(\mathbb{B}_5)| \\ &\geq \begin{cases} a_k + k - 1 + (k-1)\left(\sum_{i=1}^{k-2} r_i + r_k - 1 - \alpha\right) + 1, & \text{if } a_k \leq 2k-3; \\ (\theta+1)(k+1) - 6 + (k-1)\left(\sum_{i=1}^{k-2} r_i + r_k - 1 - \alpha\right) + 1, & \text{if } a_k \geq 2k-2, \end{cases} \\ &= \begin{cases} a_k - k + 2 + (k-1)\left(\sum_{i=1}^{k} r_i - \alpha\right), & \text{if } a_k \leq 2k-3; \\ \theta(k+1) - k - 2 + (k-1)\left(\sum_{i=1}^{k} r_i - \alpha\right), & \text{if } a_k \geq 2k-2. \end{cases}\end{aligned}$$

This proves the theorem. □

In the following three theorems, the sequence $\mathbb{A}$ contains nonnegative integers with $0 \in \mathbb{A}$.

Theorem 6.6. *Let $k \geq 4$. Let $\mathbb{A} = \{a_0, a_1, \ldots, a_{k-1}\}_{\vec{r}}$ be a sequence of nonnegative integers such that $0 = a_0 < a_1 < \cdots < a_{k-1}$, $\vec{r} = (r_0, r_1, \ldots, r_{k-1})$ and $r_i \geq 1$ for all $i \in [0, k-1]$. Let $d(A) = 1$ and $r = \min\{r_1, r_2, \ldots, r_{k-1}\}$. Let $\alpha = \sum_{i=0}^{k-1} r_i - 2$. If $r = 1$, then*

$$|S_\alpha(\mathbb{A})| \geq \begin{cases} a_{k-1} + k - 1, & a_{k-1} \leq 2k-5; \\ (\theta+1)k - 5, & a_{k-1} \geq 2k-4. \end{cases}$$

If $r \geq 2$, then

$$|S_\alpha(\mathbb{A})| \geq \begin{cases} a_{k-1} + k, & a_{k-1} \leq 2k-3; \\ 3k-3, & a_{k-1} \geq 2k-2. \end{cases}$$

Proof. If $r = 1$, then

$$S_\alpha(\mathbb{A}) = 2^\wedge A \cup \{0\}.$$

Therefore, by Theorem 2.8, we get

$$|S_\alpha(\mathbb{A})| = \left|2^\wedge A\right| + 1 \geq \begin{cases} a_{k-1} + k - 1, & a_{k-1} \leq 2k-5; \\ (\theta+1)k - 5, & a_{k-1} \geq 2k-4. \end{cases}$$

If $r \geq 2$, then

$$S_\alpha(\mathbb{A}) = 2A.$$

Therefore, by Theorem 2.3, we get

$$|S_\alpha(\mathbb{A})| = \left|2^\wedge A\right| + 1 \geq \begin{cases} a_{k-1} + k, & a_{k-1} \leq 2k-3; \\ 3k-3, & a_{k-1} \geq 2k-2. \end{cases}$$

□

Theorem 6.7. *Let $k \geq 4$. Let $\mathbb{A} = \{a_0, a_1, \ldots, a_{k-1}\}_{\vec{r}}$ be a sequence of nonnegative integers such that $0 = a_0 < a_1 < \cdots < a_{k-1}$, $\vec{r} = (r_0, r_1, \ldots, r_{k-1})$ and $r_i \geq 1$ for all $i \in [0, k-1]$. Let $d(A) = 1$ and $\min\{r_1, r_2, \ldots, r_{k-1}\} = r \geq 2$. Let $\alpha < \sum_{i=0}^{k-1} r_i - 2$ be a positive integer. Then the following holds.*

(1) *If* $0 < \alpha < \sum_{i=0}^{k-1} r_i - r$, *then there exists an integer* $m \in [1,k]$ *such that* $\sum_{i=0}^{m-2} r_i \leq \alpha < \sum_{i=0}^{m-1} r_i$ *and*

$$|S_\alpha(\mathbb{A})| \geq |(r-1)A| + \min\{a_{k-1}, r(k-2)+1\} + \sum_{i=1}^{k-1}(i-1)r_{i-1} - \sum_{i=1}^{m}(i-1)r_{i-1} + (m-1)\left(\sum_{i=1}^{m} r_{i-1} - \alpha\right) + (k-1)(r_{k-1}-r).$$

(2) *If* $\sum_{i=0}^{k-1} r_i - r \leq \alpha < \sum_{i=0}^{k-1} r_i - 2$, *then*

$$|S_\alpha(\mathbb{A})| \geq \begin{cases} a_{k-1} - k + 2 + (k-1)\left(\sum_{i=1}^{k} r_{i-1} - \alpha\right), & \text{if } a_{k-1} \leq 2k-3; \\ k-1+(k-1)\left(\sum_{i=1}^{k} r_{i-1} - \alpha\right), & \text{if } a_{k-1} \geq 2k-2. \end{cases}$$

Proof. Set $\mathbb{B}_6 := \{a_1, a_2, \ldots, a_{k-1}\}_{\vec{s_6}}$ with $\vec{s_6} := (r_1, r_2, \ldots, r_{k-1})$. Then $d(B_6) = 1$. Observe that $S_1^\alpha(\mathbb{A}) = S(\mathbb{B}_6) \cup \{0\}$ if $0 < \alpha \leq r_0$ and $S_1^\alpha(\mathbb{A}) = S_1^{\alpha - r_0}(\mathbb{B}_6) \cup \{0\}$ if $r_0 < \alpha \leq \sum_{i=0}^{k-1} r_i - 2$. Note also that, if $\alpha > r_0$ and $\sum_{i=0}^{m-2} r_i \leq \alpha < \sum_{i=0}^{m-1} r_i$ for some $m \in [2,k]$, then $\sum_{i=1}^{m-2} r_i \leq \alpha - r_0 < \sum_{i=1}^{m-1} r_i$. Therefore the integer m for $\mathbb{A}$ will work as $m-1$ for $\mathbb{B}_6$.

If $0 < \alpha \leq r_0$, we have from Theorem 4.4 that

$$\begin{aligned} |S_\alpha(\mathbb{A})| &= |S_1^\alpha(\mathbb{A})| \\ &= |S(\mathbb{B}_6)| + 1 \\ &\geq |(r-1)A| + \min\{a_{k-1}, r(k-2)+1\} - 1 + \sum_{i=1}^{k-2} i r_i + (k-1)(r_{k-1} - r) + 1 \\ &= |(r-1)A| + \min\{a_{k-1}, r(k-2)+1\} + \sum_{i=1}^{k-1}(i-1)r_{i-1} + (k-1)(r_{k-1}-r). \end{aligned}$$

If $r_0 < \alpha < \sum_{i=0}^{k-1} r_i - 2$, then we have two possibilities: either $r_0 < \alpha < \sum_{i=0}^{k-1} r_i - r$ or $\sum_{i=0}^{k-1} r_i - r \leq \alpha < \sum_{i=0}^{k-1} r_i - 2$. In the first case, we have $0 < \alpha - r_0 < \sum_{i=1}^{k-1} r_i - r$, so by Theorem 6.2, we obtain

$$\begin{aligned}
&|S_\alpha(\mathbb{A})| \\
&= |S_1^\alpha(\mathbb{A})| \\
&= \left|S_1^{\alpha-r_0}(\mathbb{B}_6)\right| + 1 \\
&= \left|S_{\alpha-r_0}(\mathbb{B}_6)\right| \\
&\geq |(r-1)A| + \min\{a_{k-1}, r(k-2)+1\} + \sum_{i=1}^{k-2} ir_i - \sum_{i=1}^{m-1} ir_i + (m-1)\left(\sum_{i=1}^{m-1} r_i - (\alpha - r_0)\right) \\
&\quad + (k-1)(r_{k-1} - r) \\
&= |(r-1)A| + \min\{a_{k-1}, r(k-2)+1\} + \sum_{i=1}^{k-1}(i-1)r_{i-1} - \sum_{i=1}^{m}(i-1)r_{i-1} \\
&\quad + (m-1)\left(\sum_{i=1}^{m} r_{i-1} - \alpha\right) + (k-1)(r_{k-1} - r).
\end{aligned}$$

In the second case, we have $\sum_{i=1}^{k-1} r_i - r \leq \alpha - r_0 < \sum_{i=1}^{k-1} r_i - 2$. By Theorem 6.3, we get

$$\begin{aligned}
|S_\alpha(\mathbb{A})| &= \left|S_{\alpha-r_0}(\mathbb{B}_6)\right| \\
&\geq \begin{cases} a_{k-1} - k + 2 + (k-1)\left(\sum_{i=1}^{k-1} r_i - (\alpha - r_0)\right), & \text{if } a_{k-1}; \leq 2k-3 \\ k-1 + (k-1)\left(\sum_{i=1}^{k-1} r_i - (\alpha - r_0)\right), & \text{if } a_{k-1} \geq 2k-2, \end{cases} \\
&= \begin{cases} a_{k-1} - k + 2 + (k-1)\left(\sum_{i=1}^{k} r_{i-1} - \alpha\right), & \text{if } a_{k-1} \leq 2k-3; \\ k-1 + (k-1)\left(\sum_{i=1}^{k} r_{i-1} - \alpha\right), & \text{if } a_{k-1} \geq 2k-2. \end{cases}
\end{aligned}$$

This completes the proof of the theorem. □

The following theorem is for $r = 1$.

Theorem 6.8. *Let $k \geq 4$. Let $\mathbb{A} = \{a_0, a_1, \ldots, a_{k-1}\}_{\vec{r}}$ be a sequence of nonnegative integers such that $0 = a_0 < a_1 < \cdots < a_{k-1}$, $\vec{r} = (r_0, r_1, \ldots, r_{k-1})$ and $r_i \geq 1$ for all $i \in [0, k-1]$. Let $d(A) = 1$ and $\min\{r_1, r_2, \ldots, r_{k-1}\} = r = 1$. Let $\alpha < \sum_{i=0}^{k-1} r_i - 2$ be a positive integer. Then the following holds.*

(1) *If* $0<\alpha<\sum_{i=0}^{k-2} r_i-1$ *with* $r_{k-1}\neq 1$ *and* $r_k\neq 1$ *or* $r_{k-1}=r_k=1$ *or* $r_{k-1}\neq 1$ *and* $r_k=1$*, then there exists an integer* $m\in[1,k]$ *such that* $\sum_{i=0}^{m-2} r_i\leq\alpha<\sum_{i=0}^{m-1} r_i$ *and*

$$|S(\mathbb{A})| \geq \begin{cases} a_{k-1}-k+2+\sum_{i=1}^{k}(i-1)r_{i-1}-\sum_{i=1}^{m}(i-1)r_{i-1}+(m-1)\left(\sum_{i=1}^{m} r_{i-1}-\alpha\right), \\ \text{if } a_{k-1}\leq 2k-5; \\ \theta k-k-2+\sum_{i=1}^{k}(i-1)r_{i-1}-\sum_{i=1}^{m}(i-1)r_{i-1}+(m-1)\left(\sum_{i=1}^{m} r_{i-1}-\alpha\right), \\ \text{if } a_{k-1}\geq 2k-4. \end{cases}$$

(2) *If* $\sum_{i=0}^{k-2} r_i-1\leq\alpha<\sum_{i=0}^{k-1} r_i-2$ *with* $r_{k-1}\neq 1$ *and* $r_k\neq 1$ *or* $r_{k-1}=r_k=1$ *or* $r_{k-1}\neq 1$ *and* $r_k=1$*, then*

$$|S(\mathbb{A})| \geq \begin{cases} a_{k-1}-k+1+(k-1)\left(\sum_{i=1}^{k} r_{i-1}-\alpha\right), & \text{if } a_{k-1}\leq 2k-5; \\ \theta k-k-3+(k-1)\left(\sum_{i=1}^{k} r_{i-1}-\alpha\right), & \text{if } a_{k-1}\geq 2k-4. \end{cases}$$

(3) *If* $0<\alpha\leq r_0$ *with* $r_{k-2}=1$ *and* $r_{k-1}\neq 1$*, then*

$$|S_\alpha(\mathbb{A})| \geq \begin{cases} a_{k-1}-k+2+\sum_{i=1}^{k}(i-1)r_{i-1}, & \text{if } a_{k-1}\leq 2k-5; \\ \theta k-k-2+\sum_{i=1}^{k}(i-1)r_{i-1}, & \text{if } a_{k-1}\geq 2k-4. \end{cases}$$

(4) *If* $r_0<\alpha<\sum_{i=0}^{k-3} r_i$ *with* $r_{k-2}=1$ *and* $r_{k-1}\neq 1$*, then there exists an integer* $m\in[1,k]$ *such that* $\sum_{i=0}^{m-2} r_i\leq\alpha<\sum_{i=0}^{m-1} r_i$ *and*

$$|S_\alpha(\mathbb{A})| \geq \begin{cases} a_{k-1}+1+\sum_{i=1}^{k-1}(i-1)r_{i-1}-\sum_{i=1}^{m}(i-1)r_{i-1}+(m-1)\left(\sum_{i=1}^{m} r_{i-1}-\alpha\right)+(k-m)(r_{k-1}-1), \\ \text{if } a_{k-1}\leq 2k-5; \\ \theta k-3+\sum_{i=1}^{k-1}(i-1)r_{i-1}-\sum_{i=1}^{m}(i-1)r_{i-1}+(m-1)\left(\sum_{i=1}^{m} r_{i-1}-\alpha\right)+(k-m)(r_{k-1}-1), \\ \text{if } a_k\geq 2k-4. \end{cases}$$

(5) *If* $\sum_{i=0}^{k-3} r_i\leq\alpha<\sum_{i=0}^{k-1} r_i-2$ *with* $r=r_{k-2}=1$ *and* $r_{k-1}\neq 1$*, then*

$$|S_\alpha(\mathbb{A})| \geq \begin{cases} a_{k-1}-k+3+(k-2)\left(\sum_{i=1}^{k} r_{i-1}-\alpha\right), & \text{if } a_{k-1}\leq 2k-5; \\ \theta k-k-1+(k-2)\left(\sum_{i=1}^{k} r_{i-1}-\alpha\right), & \text{if } a_{k-1}\geq 2k-4. \end{cases}$$

Proof. Set $\mathbb{B}_7:=\{a_1,a_2,\ldots,a_{k-1}\}_{\vec{s_7}}$ with $\vec{s_7}:=(r_1,r_2,\ldots,r_{k-1})$. Then $d(B_7)=1$. Observe that $S_1^{\alpha}(\mathbb{A})=S(\mathbb{B}_7)\cup\{0\}$ if $0<\alpha\leq r_0$ and $S_1^{\alpha}(\mathbb{A})=S_1^{\alpha-r_0}(\mathbb{B}_7)\cup\{0\}$ if $r_0<\alpha\leq\sum_{i=0}^{k-1} r_i-2$. Note also that, if $\alpha>r_0$ and $\sum_{i=0}^{m-2} r_i\leq\alpha<\sum_{i=0}^{m-1} r_i$ for some $m\in[2,k]$, then $\sum_{i=1}^{m-2} r_i\leq\alpha-r_0<\sum_{i=1}^{m-1} r_i$. Therefore the integer m for $\mathbb{A}$ will work as $m-1$ for $\mathbb{B}_7$.

Case I ($r_{k-1} \neq 1$ and $r_k \neq 1$ or $r_{k-1} = r_k = 1$ or $r_{k-1} \neq 1$ and $r_k = 1$). If $0 < \alpha \leq r_0$, we have from Theorem 4.2 that

$$\begin{aligned}|S_\alpha(\mathbb{A})| &= |S_1^\alpha(\mathbb{A})|\\ &= |S(\mathbb{B}_7)| + 1\\ &\geq \begin{cases} a_{k-1} - (k-1) + \sum_{i=1}^{k-1} ir_i + 1, & \text{if } a_{k-1} \leq 2(k-1) - 3;\\ \theta k - (k-1) - 4 + \sum_{i=1}^{k-1} ir_i + 1, & \text{if } a_{k-1} \geq 2(k-1) - 2.\end{cases}\\ &= \begin{cases} a_{k-1} - k + 2 + \sum_{i=1}^{k} (i-1) r_{i-1}, & \text{if } a_{k-1} \leq 2k-5;\\ \theta k - k - 2 + \sum_{i=1}^{k} (i-1) r_{i-1}, & \text{if } a_{k-1} \geq 2k-4.\end{cases}\end{aligned}$$

If $r_0 < \alpha < \sum_{i=0}^{k-1} r_i - 2$, then we have two possibilities: either $0 < \alpha - r_0 < \sum_{i=1}^{k-2} r_i - 1$ or $\sum_{i=1}^{k-2} r_i - 1 \leq \alpha - r_0 < \sum_{i=1}^{k-1} r_i - 2$. If $\alpha - r_0 < \sum_{i=1}^{k-2} r_i - 1$, then by Theorem 6.4, we get

$$\begin{aligned}&|S_\alpha(\mathbb{A})|\\ &= |S_1^\alpha(\mathbb{A})|\\ &= \left|S_1^{\alpha - r_0}(\mathbb{B}_7)\right| + 1\\ &= \left|S_{\alpha - r_0}(\mathbb{B}_7)\right|\\ &\geq \begin{cases} a_{k-1} - k + 2 + \sum_{i=1}^{k-1} ir_i - \sum_{i=1}^{m-1} ir_i + (m-1)\left(\sum_{i=1}^{m-1} r_i - (\alpha - r_0)\right),\\ \text{ if } a_{k-1} \leq 2(k-1) - 3;\\ \theta k - k - 2 + \sum_{i=1}^{k-1} ir_i - \sum_{i=1}^{m-1} ir_i + (m-1)\left(\sum_{i=1}^{m-1} r_i - (\alpha - r_0)\right),\\ \text{ if } a_{k-1} \geq 2(k-1) - 2,\end{cases}\\ &= \begin{cases} a_{k-1} - k + 2 + \sum_{i=1}^{k} (i-1) r_{i-1} - \sum_{i=1}^{m} (i-1) r_{i-1} + (m-1)\left(\sum_{i=1}^{m} r_{i-1} - \alpha\right),\\ \text{ if } a_{k-1} \leq 2k-5;\\ \theta k - k - 2 + \sum_{i=1}^{k} (i-1) r_{i-1} - \sum_{i=1}^{m} (i-1) r_{i-1} + (m-1)\left(\sum_{i=1}^{m} r_{i-1} - \alpha\right),\\ \text{ if } a_{k-1} \geq 2k-4.\end{cases}\end{aligned}$$

If $\sum_{i=1}^{k-2} r_i - 1 \leq \alpha - r_0 < \sum_{i=1}^{k-1} r_i - 2$, then again by Theorem 6.4, we get

$$\begin{aligned}&|S_\alpha(\mathbb{A})|\\ &= \left|S_{\alpha - r_0}(\mathbb{B}_7)\right|\\ &\geq \begin{cases} a_{k-1} - k + 1 + (k-1)\left(\sum_{i=1}^{k-1} r_i - (\alpha - r_0)\right), & \text{if } a_{k-1} \leq 2(k-1) - 3;\\ \theta k - k - 3 + (k-1)\left(\sum_{i=1}^{k-1} r_i - (\alpha - r_0)\right), & \text{if } a_{k-1} \geq 2(k-1) - 2,\end{cases}\\ &= \begin{cases} a_{k-1} - k + 1 + (k-1)\left(\sum_{i=1}^{k} r_{i-1} - \alpha\right), & \text{if } a_{k-1} \leq 2k-5;\\ \theta k - k - 3 + (k-1)\left(\sum_{i=1}^{k} r_{i-1} - \alpha\right), & \text{if } a_{k-1} \geq 2k-4.\end{cases}\end{aligned}$$

Case II ($r_{k-2} = 1$ and $r_{k-1} \neq 1$). If $0 < \alpha \leq r_0$, we have from Theorem 4.2 that

$$\begin{aligned}|S_\alpha(\mathbb{A})| &= |S(\mathbb{B}_7)| + 1\\ &\geq \begin{cases} a_{k-1} - (k-1) + \sum_{i=1}^{k-1} ir_i + 1, & \text{if } a_{k-1} \leq 2(k-1) - 3;\\ \theta k - (k-1) - 4 + \sum_{i=1}^{k-1} ir_i + 1, & \text{if } a_{k-1} \geq 2(k-1) - 2.\end{cases}\\ &= \begin{cases} a_{k-1} - k + 2 + \sum_{i=1}^{k} (i-1) r_{i-1}, & \text{if } a_{k-1} \leq 2k-5;\\ \theta k - k - 2 + \sum_{i=1}^{k} (i-1) r_{i-1}, & \text{if } a_{k-1} \geq 2k-4.\end{cases}\end{aligned}$$

If $r_0 < \alpha < \sum_{i=0}^{k-3} r_i$, then $0 < \alpha - r_0 < \sum_{i=1}^{k-3} r_i$. Thus, it follows from the proof of Theorem 6.5 that

$$\begin{aligned}&|S(\mathbb{A})|\\ &= \left|S_{\alpha - r_0}(\mathbb{B}_7)\right|\\ &\geq \begin{cases} a_{k-1} + k - 1 + \sum_{i=1}^{k-3} ir_i - \sum_{i=1}^{m-1} ir_i + (m-1)\left(\sum_{i=1}^{m-1} r_i - (\alpha - r_0)\right) + (k-m)(r_{k-1} - 1),\\ \quad \text{if } a_{k-1} \leq 2(k-1) - 3;\\ \theta k + k - 5 + \sum_{i=1}^{k-3} ir_i - \sum_{i=1}^{m-1} ir_i + (m-1)\left(\sum_{i=1}^{m-1} r_i - (\alpha - r_0)\right) + (k-m)(r_{k-1} - 1),\\ \quad \text{if } a_k \geq 2(k-1) - 2,\end{cases}\\ &= \begin{cases} a_{k-1} + 1 + \sum_{i=1}^{k-1} (i-1) r_{i-1} - \sum_{i=1}^{m} (i-1) r_{i-1} + (m-1)\left(\sum_{i=1}^{m} r_{i-1} - \alpha\right) + (k-m)(r_{k-1} - 1),\\ \quad \text{if } a_{k-1} \leq 2k - 5;\\ \theta k - 3 + \sum_{i=1}^{k-1} (i-1) r_{i-1} - \sum_{i=1}^{m} (i-1) r_{i-1} + (m-1)\left(\sum_{i=1}^{m} r_{i-1} - \alpha\right) + (k-m)(r_{k-1} - 1),\\ \quad \text{if } a_k \geq 2k - 4.\end{cases}\end{aligned}$$

If $\sum_{i=0}^{k-3} r_i \leq \alpha < \sum_{i=0}^{k-1} r_i - 2$, then we have again from Theorem 6.5 that

$$\begin{aligned}|S(\mathbb{A})| &= \left|S_{\alpha - r_0}(\mathbb{B}_7)\right|\\ &\geq \begin{cases} a_{k-1} - k + 3 + (k-2)\left(\sum_{i=1}^{k-1} r_i - (\alpha - r_0)\right), & \text{if } a_{k-1} \leq 2(k-1) - 3;\\ \theta k - k - 1 + (k-2)\left(\sum_{i=1}^{k-1} r_i - (\alpha - r_0)\right), & \text{if } a_{k-1} \geq 2(k-1) - 2,\end{cases}\\ &= \begin{cases} a_{k-1} - k + 3 + (k-2)\left(\sum_{i=1}^{k} r_{i-1} - \alpha\right), & \text{if } a_{k-1} \leq 2k - 5;\\ \theta k - k - 1 + (k-2)\left(\sum_{i=1}^{k} r_{i-1} - \alpha\right), & \text{if } a_{k-1} \geq 2k - 4.\end{cases}\end{aligned}$$

This completes the proof of the theorem. □

Acknowledgment

We are grateful to the anonymous referee for his/her useful comments that improved the presentation of the article. The first author would like to thank the Council of Scientific and Industrial Research (CSIR), India, for providing the grant to carry out the research with Grant No. 09/143(0925)/2018-EMR-I. A part of this work was done when the second author was at the Institute of Mathematical Sciences in Chennai, India. He thanks IMSc Chennai for the postdoctoral fellowship he received during that period.

References

1. B. Bajnok, Additive Combinatorics: A Menu of Research Problems, CRC Press, 2018.
2. É. Balandraud, An addition theorem and maximal zero-sum free sets in $\mathbb{Z}/p\mathbb{Z}$, *Israel J. Math.* **188** (2012), 405–429.
3. É. Balandraud, Addition theorems in $\mathbb{F}_p$ via the polynomial method, arXiv: 1702.06419 (2017).
4. É. Balandraud, B. Girard, S. Griffiths, and Y. O. Hamidoune, Subset sums in abelian groups, *European J. Comb.* **34** (2013), 1269–1286.
5. J. Bhanja and R. K. Pandey, Inverse problems for certain subsequence sums in integers, *Discrete Math.* **343**(12) (2020), article no. 112148 (11 pages).
6. J. Bhanja and R. K. Pandey, On the minimum size of subset and subsequence sums in integers, *Comptes Rendus Math.* **360** (2022), 1099–1111.
7. D. Daza, M. Huicochea, C. Martos, and C. Trujillo, A Freiman-type theorem for restricted sumsets, *Int. J. Number Theory* **19**(10) (2023), 2309–2332.
8. M. DeVos, L. Goddyn, B. Mohar, and R. Šámal, A quadratic lower bound for subset sums, *Acta Arith.* **129** (2007), 187–195.
9. H. K. Dwivedi and R. K. Mistri, Direct and inverse problems for subset sums with certain restrictions, *Integers* **22** (2022), Paper No. A112, 13 pp.
10. G. A. Frieman, On the addition of finite sets I, *Izv. Vysh. Uchebn. Zaved. Mat.* **13**(6) (1959), 202–213.
11. G. A. Freiman, *Foundations of structural theory of set addition*, vol. **37**, Translations of Mathematical Monographs, American Mathematical Society, Providence, R.I., 1973.
12. G. A. Freiman, L. Low, and J. Pitman, Sumsets with distinct summands and the conjecture of Erdős-Heilbronn on sums of residues, *Astérisque* **258** (1999), 163–172.
13. S. Griffiths, Asymptotically tight bounds on subset sums, *Acta Arith.* **138** (2009), 53–72.
14. X.-W. Jiang and Y.-L. Li, On the cardinality of subsequence sums, *Int. J. Number Theory* **14**(3) (2018), 661–668.
15. V. F. Lev, Restricted set addition in groups I: the classical setting, *J. London Math. Soc.* **62**(1) (2000), 27–40.
16. V. F. Lev, Structure theorem for multiple addition and the Frobenius problem, *J. Number Theory* **58**(1) (1996), 79–88.
17. V. F. Lev, The structure of multisets with a small number of subset sums, In Structure theory of set addition, *Astérisque* **258** (1999), 179–186.
18. R. K. Mistri, R. K. Pandey, and O. Prakash, Subsequence sums: direct and inverse problems, *J. Number Theory* **148** (2015), 235–256.
19. R. K. Mistri, R. K. Pandey, and O. Prakash, Subset and subsequence sums in integers, *J. Comb. Number Theory* **8** (2016), 207–223.
20. F. Monopoli, A generalization of sumsets modulo a prime, *J. Number Theory* **157** (2015), 271–279.
21. M. B. Nathanson, *Additive Number Theory: Inverse Problems and the Geometry of Sumsets*, Springer, 1996.
22. M. B. Nathanson, Inverse theorems for subset sums, *Trans. Am. Math. Soc.* **347**(4) (1995), 1409–1418.

DEPARTMENT OF MATHEMATICS, INDIAN INSTITUTE OF TECHNOLOGY ROORKEE, UTTARAKHAND, 247667, INDIA
Email address: mohan98math@gmail.com

DEPARTMENT OF MATHEMATICS, INDIAN INSTITUTE OF INFORMATION TECHNOLOGY, DESIGN AND MANUFACTURING, KANCHEEPURAM, CHENNAI-600127, INDIA
Email address: jagannath@iiitdm.ac.in

DEPARTMENT OF MATHEMATICS, INDIAN INSTITUTE OF TECHNOLOGY ROORKEE, UTTARAKHAND, 247667, INDIA
Email address: ram.pandey@ma.iitr.ac.in