# LOCAL UNIQUE FACTORIZATION DOMAINS WITH INFINITELY MANY NONCATENARY POSETS

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ABSTRACT. We demonstrate a class of local (Noetherian) unique factorization domains (UFDs) that are noncatenary at infinitely many places. In particular, if $A$ is in our class of UFDs, then the prime spectrum of $A$, when viewed as a partially ordered set, has infinitely many disjoint (except at the maximal ideal) copies of a noncatenary saturated finite poset. As a consequence of our result, there are infinitely many height one prime ideals $P$ of $A$ such that $A/P$ is noncatenary. We also construct a countable local UFD $A$ satisfying the property that for *every* height one prime ideal $P$ of $A$, $A/P$ is noncatenary.

## 1. INTRODUCTION

An overarching goal of commutative algebra is to understand the set of prime ideals (the prime spectrum) of a commutative ring. One way of understanding this set is to examine its structure as a partially ordered set (poset) under containment. One open question is, given a partially ordered set $X$, when is $X$ isomorphic to the prime spectrum of a Noetherian ring? This question has not been answered fully, although much progress has been made (see, for example, [11] for a nice survey). Relatedly, we can focus on a subset of the prime spectrum of a Noetherian ring and ask the question: given a partially ordered set $X$, can it be embedded into the prime spectrum of a Noetherian ring in a way that preserves saturated chains? Particularly, we are interested in whether or not Noetherian rings with desirable algebraic properties can have prime spectra that behave strangely. Recall that a ring $R$ is said to be noncatenary if there exist prime ideals $P$ and $Q$ of $R$ such that $P \subset Q$ and there are saturated chains of prime ideals of different lengths that start at $P$ and end at $Q$. This is an example of a strange property a ring can have that will be investigated in more detail in this paper. It was previously conjectured that noncatenary Noetherian rings do



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not exist. However, in 1956, this conjecture was disproved by Nagata in [10] when he constructed a noncatenary Noetherian domain. Then, in 1979, Heitmann proved in [3] that, given any finite partially ordered set $X$, there exists a Noetherian domain $R$ such that $X$ can be embedded into the prime spectrum of $R$ in a way that preserves saturated chains. Since posets can be arbitrarily noncatenary, this shows that there is no limit to "how noncatenary" a Noetherian domain can be.

These results regarding noncatenary Noetherian rings raise the question of whether or not a Noetherian ring with nice algebraic properties can have unusual posets embedded in its prime spectrum. In particular, one might ask whether or not a Noetherian unique factorization domain (UFD) can be noncatenary. The answer to this question was not known until Heitmann constructed a noncatenary UFD in [4] in 1993. Later, more examples of families of noncatenary UFDs were constructed (see, for example, [1] and [8]). Finally, in [2] it was shown, that, similar to Heitmann's result in [3] for Noetherian rings, given a finite partially ordered set $X$, there exists a Noetherian UFD $R$ such that $X$ can be embedded into the prime spectrum of $R$ in a way that preserves saturated chains. It is natural to ask whether or not this result holds if $X$ is countably infinite.

Particularly, in this article, we are interested in whether a Noetherian UFD can be "infinitely noncatenary." More specifically, we ask if there exists a Noetherian UFD whose prime spectrum contains infinitely many disjoint noncatenary posets. We use ideas from [8], [4], and [1] to show in Theorem 3.2 that such a Noetherian UFD does indeed exist. A consequence of Theorem 3.2 is that there exist Noetherian UFDs $A$ satisfying the property that for infinitely many height one prime ideals $P$ of $A$, the ring $A/P$ is noncatenary. Hence, whereas in previous literature it is shown that the prime spectrum of a Noetherian UFD can contain finite noncatenary posets, Theorem 3.2 indicates the previously unknown result that UFDs can be much more noncatenary; in fact, infinitely noncatenary. This is surprising given that Noetherian UFDs are usually considered to be very well-behaved rings. Moreover, in Theorem 4.7, we construct a four-dimensional local (Noetherian) UFD such that the quotient ring at *every* height one prime ideal is noncatenary. Thus, we have shown that there exists a Noetherian UFD that is not only noncatenary at infinitely many places, but is, in some sense, noncatenary everywhere.



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We now describe Theorem 3.2 in more detail. Let $T$ be a complete local ring with depth at least two such that no integer of $T$ is a zerodivisor. Letting $\{P_{0,1}, \ldots, P_{0,s}\}$ be the minimal prime ideals of $T$, assume that $\dim(T/P_{0,i}) = n_i \geq 3$ for all $i = 1, 2, \ldots, s$. Then Theorem 3.2 states that there exists a subring $A$ of $T$ such that $A$ is a local UFD whose completion is $T$. Moreover, $A$ has an infinite set of height one prime ideals $\{J_n\}_{n \in \mathbb{N}}$ such that, for each $n \in \mathbb{N}$, there are saturated chains of prime ideals of lengths $n_1 - 1, n_2 - 1, \ldots, n_{s-1} - 1$, and $n_s - 1$ that start at $J_n$, end at the maximal ideal of $A$, and are disjoint except at $J_n$ and at the maximal ideal of $A$. It then follows that the prime spectrum of $A$ contains infinitely many disjoint copies (except at the maximal ideal) of a noncatenary saturated finite poset. It also follows that $A$ is noncatenary and $A/J_n$ is noncatenary for every $n \in \mathbb{N}$.

To illustrate the ideas behind the proof of Theorem 3.2, we first explain the strategy of the proof using the complete local ring $T = k[[x, y, z, w, t]]/((x) \cap (y, z))$ where k is a field and $x, y, z, w, t$ are indeterminates. Note that this is the ring Heitmann used in [4] to exhibit the first example of a noncatenary local (Noetherian) UFD. Let $x, y, z, w, t$ now denote their images in $T$. Then the minimal prime ideals of $T$ are $P_{0,1} = (x)$ and $P_{0,2} = (y, z)$, and we have $\dim(T/P_{0,1}) = 4$ and $\dim(T/P_{0,2}) = 3$ . Therefore, $T$ is not equidimensional and $\dim(T) = 4$. By Theorem 3.8 in [1], $T$ is the completion of at least one noncatenary local (Noetherian) UFD. We build off of the methods in [1], [4], and [8] to find a particular subring $A$ of $T$ that is a local UFD whose completion is $T$. We construct $A$ in such a way that there are infinitely many pairs of height one prime ideals $\{P_n^{(1)}, P_n^{(2)}\}_{n \in \mathbb{N}}$ of $T$ (the first containing $P_{0,1}$ and the second containing $P_{0,2}$) such that each pair is “glued together" in $A$ to different height one prime ideals, i.e. $P_n^{(1)} \cap A = P_n^{(2)} \cap A \neq (0)$ for all $n \in \mathbb{N}$. We also adjoin generators of carefully chosen coheight one prime ideals of $T$ to $A$ (infinitely many containing $P_{0,1}$ and infinitely many containing $P_{0,2}$) to ensure that $A$ has infinitely many noncatenary posets of prime ideals. Specifically, the prime spectrum of $A$ contains the poset in Figure 1, which is the poset in Figure 2 repeated infinitely many times. See Example 3.3 for more details. Note that since $A$ is a local UFD whose completion is $T$ and since all local UFDs with dimension less than four are catenary, it is necessary that the ring $T$ we chose has dimension

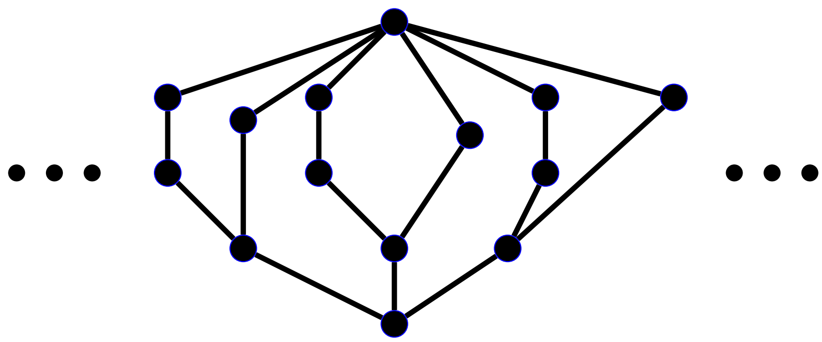

FIGURE 1

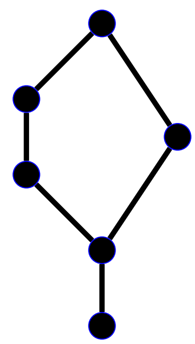

FIGURE 2

at least four. It is also necessary that $T$ is not equidimensional since if the completion of a local ring is equidimensional, then, by Theorem 31.6 in [9], that ring must be universally catenary.

In Theorem 4.7 we use the complete local ring $T = \mathbb{Q}[[x, y, z, w, t]]/((x) \cap (y, z))$ to construct a local UFD that is noncatenary at every height one prime ideal. The proof of this theorem requires a more in-depth analysis of the properties of chains of prime ideals of $T$. We use an inductive process to construct the desired subring $A$, which is a local (Noetherian) UFD such that the completion of $A$ at its maximal ideal is $T$ and such that every height one prime ideal is contained in a maximal saturated chain of prime ideals of length three and a different such chain of length four. In particular, every height one prime ideal of A is contained in a subset of the prime spectrum that is isomorphic to the poset in Figure 2.

This paper is organized in the following way. In Section 2, we recall the notion of an N-subring of a complete local ring, first introduced in [4]. The bulk of Section 2 is devoted to stating and proving results about N-subrings that are essential for our constructions. We state and prove our main result, Theorem 3.2, in Section 3. In particular, in Section 3.1, we demonstrate a class of local (Noetherian) UFDs such that, if $A$ is in the class, then the prime spectrum of the local UFD $A$ produced by Theorem 3.2 has infinitely many disjoint (except at the maximal ideal) copies of a noncatenary saturated finite poset, which is determined by a given complete local ring $T$. In this setting, we are viewing the prime spectrum of $A$ as a poset with respect to inclusion and we are using the term "saturated" describing the finite poset to mean that if $P \subset Q$ is saturated in the finite poset, then $P \subset Q$ is saturated in the prime spectrum of $A$. (For the definition of a noncatenary poset see Section 2 of [8]). Finally, in Section 4, we construct a four-dimensional countable local (Noetherian) UFD $A$ such that, for every height one prime ideal $P$ of $A$, the ring $A/P$ is noncatenary.

## 2. Preliminaries

We first establish some terminology. In this paper, all rings are assumed to be commutative with unity. When $R$ is a Noetherian ring with exactly one maximal ideal, we say $R$ is *local*. If $R$ has exactly one maximal ideal but is not necessarily Noetherian, we call it *quasi-local*. We use the notation $(R, M)$ when $R$ is a quasi-local ring and $M$ is the maximal ideal of $R$. If $(R, M)$ is a local ring, we use $\widehat{R}$ to denote the $M$-adic completion of $R$. We use the standard notation Spec$(R)$ to denote the set of prime ideals of the ring $R$. If $R$ is a local ring, we informally refer to two prime ideals $P, Q \in$ Spec $(\widehat{R})$ as being "glued together" in $R$ if $P \cap R = Q \cap R$.

Before we begin, we recall what it means for a ring to be noncatenary.

**Definition 2.1.** *A ring $R$ is called noncatenary if there exists $P, Q \in$ Spec $(R)$ with $P \subset Q$, such that there are (at least) two saturated chains of prime ideals from $P$ to $Q$ with different lengths. If no such pair of prime ideals exists, then $R$ is said to be catenary.*

The following prime avoidance lemma will be used multiple times in our construction.

**Lemma 2.2** ([4], Lemma 2)**.** *Let $T$ be a complete local ring with maximal ideal $M$, let $C$ be a countable set of prime ideals in Spec $(T)$ such that $M \notin C$, and let $D$ be a countable set of elements of $T$. If $I$ is an ideal of $T$ which is contained in no single $P$ in $C$, then*

$$I \not\subset \bigcup\{(P + r) : P \in C, r \in D\}.$$

To construct our noncatenary UFDs, we use ideas and results from [4] which rely heavily on a specific type of subring of a complete local ring called an N-subring.

**Definition 2.3** ([4])**.** *Let $(T, M)$ be a complete local ring. We say a quasi-local subring $(R, M \cap R)$ of $T$ is an N-subring of $T$ if $R$ is a UFD and*

(1) *$|R| \leq max\{\aleph_0, |T/M|\}$, with equality only when $T/M$ is countable,*

(2) *$Q \cap R = (0)$ for all $Q \in Ass(T)$, and*

(3) *If $t \in T$ is regular and $P \in Ass(T/tT)$, then $ht(P \cap R) \leq 1$.*

The remainder of this section will be devoted to stating and proving results about N-subrings that will be crucial for our constructions.

The following lemma establishes machinery to adjoin an element of $T$ to an N-subring of $T$ and ensure that the result is still an N-subring of $T$. Note that, if $R$ is a subring of a ring $T$ and $P$ is a prime ideal of $T$, then there is an injective map from $R/(P \cap R)$ to $T/P$. Thus, it makes sense to say that an element $x + P \in T/P$ is either algebraic or transcendental over the ring $R/(P \cap R)$.

**Lemma 2.4** ([6], Lemma 11)**.** *Let $(T, M)$ be a complete local ring and let $\mathfrak{p} \in \text{Spec}(T)$. Let $R$ be an N-subring of $T$ with $\mathfrak{p} \cap R = (0)$. Suppose $C \subset \text{Spec}(T)$ satisfies the following conditions:*

(1) *$M \notin C$,*

(2) *$\mathfrak{p} \in C$,*

(3) $\{P \in \text{Spec}(T) \mid P \in \text{Ass}(T/rT)$ with $0 \neq r \in R\} \subset C$, *and*

(4) $\text{Ass}(T) \subset C$.

*Let $x \in T$ be such that $x \notin P$ and $x + P$ is transcendental over $R/(P \cap R)$ as an element of $T/P$ for every $P \in C$. Then, $S = R[x]_{(M \cap R[x])}$ is an N-subring of $T$ properly containing $R$, $|S| = \max\{\aleph_0, |R|\}$, and $\mathfrak{p} \cap S = (0)$.*

In the proof of Theorem 3.2, we ensure that our final UFD contains generating sets of carefully chosen prime ideals of $T$. Lemma 2.5 shows how to extend a given N-subring to another N-subring that contains a generating set for an ideal $Q \in \text{Spec}(T)$ satisfying certain properties. Later, in Lemma 2.6, we will show that this can actually be applied infinitely many times so that our N-subring contains generating sets for infinitely many $Q$ satisfying certain properties.

**Lemma 2.5.** *Let $(T, M)$ be a complete local ring with depth$(T) \geq 2$, let $R$ be a countable N-subring of $T$, and let $\mathfrak{p}$ be a nonmaximal prime ideal of $T$ such that $\mathfrak{p} \cap R = (0)$. Let $Q$ be a prime ideal of $T$ such that $Q \not\subseteq \mathfrak{p}$ and $Q \not\subseteq P$ for any $P \in Ass(T)$ and any $P \in Ass(T/zT)$ with $z$ a nonzero regular element of $T$. Then there exists a countable N-subring $S$ of $T$ such that $R \subseteq S$, prime elements in $R$ are prime in $S$, $S$ contains a generating set for $Q$, and $\mathfrak{p} \cap S = (0)$.*

*Proof.* Let $Q = (a_1, a_2, \ldots, a_m)$ for $a_i \in T$. Define

$$C = \text{Ass}(T) \cup \{P \in \text{Spec}\,(T) \mid P \in \text{Ass}(T/rT) \text{ for some } 0 \neq r \in R\} \cup \{\mathfrak{p}\}.$$

Note that, since depth$(T) \geq 2$, $M \notin C$ and, since $R$ is countable, so is $C$. In addition, our hypotheses imply that $Q \not\subseteq P$ for all $P \in C$. Lemma 2.2 with $D = \{0\}$ gives that $Q \not\subseteq \bigcup_{P \in C} P$. So, let $q_1 \in Q$ such that $q_1 \notin P$ for any $P \in C$.

Fix some $P \in C$. If $a_1 + tq_1 + P = a_1 + t'q_1 + P$ for $t, t' \in T$ then $q_1(t - t') \in P$. Since $q_1 \notin P$, we have $t - t' \in P$ and so $t + P = t' + P$. It follows that if $t + P \neq t' + P$ then $a_1 + tq_1 + P \neq a_1 + t'q_1 + P$. Now let $D_{(P)}$ be a full set of coset representatives for the cosets $t + P \in T/P$ that make $(a_1 + q_1 t) + P$ algebraic over $R/(P \cap R)$. Since the algebraic closure of $R/(P \cap R)$ in $T/P$ is countable, we have that $D_{(P)}$ is countable. Let $D = \bigcup_{P \in C} D_{(P)}$, and note that $D$ is countable. Use Lemma 2.2 to find $m_1 \in M$ such that $m_1 \notin \bigcup\{(P + r) \mid P \in C, r \in D\}$. It follows that $(a_1 + q_1 m_1) + P$ is transcendental over $R/(P \cap R)$ for all $P \in C$. By Lemma 2.4, if $\tilde{a}_1 = a_1 + q_1 m_1$

then $R_1 = R[\tilde{a}_1]_{(M \cap R[\tilde{a}_1])}$ is a countable N-subring of $T$ with $\mathfrak{p} \cap R_1 = (0)$. Let $P \in \text{Ass}(T)$. Then $\tilde{a}_1 + P$ is transcendental over $R/(P \cap R)$ and $P \cap R = (0)$. It follows that $\tilde{a}_1$ is transcendental over $R$ and so prime elements in $R$ are prime in $R_1$. Note that $(\tilde{a}_1, a_2, \ldots, a_m) + MQ = Q$ and so by Nakayama's Lemma, $Q = (a_1, a_2, \ldots, a_m) = (\tilde{a}_1, a_2, \ldots, a_m)$.

Repeat the above process with $R$ replaced by $R_1$ to find $q_2 \in Q$ and $m_2 \in M$ so that, if $\tilde{a}_2 = a_2 + q_2 m_2$ then $R_2 = R_1[\tilde{a}_2]_{(M \cap R_1[\tilde{a}_2])}$ is a countable N-subring of $T$, $\mathfrak{p} \cap R_2 = (0)$, prime elements in $R_1$ are prime in $R_2$, and $Q = (a_1, a_2, \ldots, a_m) = (\tilde{a}_1, a_2, \ldots, a_m) = (\tilde{a}_1, \tilde{a}_2, \ldots, a_m)$.

Continue the process to find a countable N-subring $R_m$ of $T$ such that $R \subseteq R_m$, $\mathfrak{p} \cap R_m = (0)$, prime elements in $R$ are prime in $R_m$ and $R_m$ contains a generating set for $Q$. Then $S = R_m$ is the desired N-subring of $T$. □

The following lemma shows that we can repeat Lemma 2.5 infinitely many times. In other words, given a countable set of prime ideals of $T$ satisfying certain conditions, the lemma allows us to adjoin a generating set for each of these prime ideals to an N-subring of $T$, with the resulting ring being an N-subring of $T$.

**Lemma 2.6.** *Let $(T, M)$ be a complete local ring with depth$(T) \geq 2$, let $R$ be a countable N-subring of $T$, and let $\mathfrak{p}$ be a nonmaximal prime ideal of $T$ with $\mathfrak{p} \cap R = (0)$. Let $\{Q_j\}_{j \in \mathbb{N}}$ be a countable set of prime ideals of $T$ such that for every $j \in \mathbb{N}$, $Q_j \not\subseteq \mathfrak{p}$ and $Q_j \not\subseteq P$ for any $P \in$ Ass$(T)$ and any $P \in$ Ass$(T/zT)$ with $z$ a nonzero regular element of $T$. Then there exists a countable N-subring $S$ of $T$ such that $R \subseteq S$, $\mathfrak{p} \cap S = (0)$, prime elements in $R$ are prime in $S$, and, for every $j \in \mathbb{N}$, $S$ contains a generating set for $Q_j$.*

*Proof.* Let $R_1$ be the countable N-subring obtained by applying Lemma 2.5 with $Q = Q_1$. We inductively define $R_k$ for every $k > 1$. Assume that $k > 1$ and $R_{k-1}$ has been defined so that for $\ell \leq k-1$, $R_\ell$ is a countable N-subring of $T$ containing generating sets for $Q_1, Q_2, \ldots, Q_\ell$, $\mathfrak{p} \cap R_\ell = (0)$, and prime elements of $R_{\ell-1}$ are prime in $R_\ell$. Define $R_k$ to be the countable N-subring obtained from Lemma 2.5 so that $R_k$ contains a generating set for $Q_k$, $\mathfrak{p} \cap R_k = (0)$, and prime elements of $R_{k-1}$ are prime in $R_k$.



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Let $S = \bigcup_{k\in\mathbb{N}} R_k$. We claim that this is the desired N-subring of $T$. For all $k \in \mathbb{N}$, $R_k$ is countable and prime elements of $R_{k-1}$ are prime in $R_k$. By Lemma 6 in [4], $S$ is a countable N-subring of $T$ such that prime elements in $R$ are prime in $S$. Furthermore, a generating set for $Q_k$ is contained in $R_k$, so, for every $j \in \mathbb{N}$, $S$ contains a generating set for $Q_j$. Finally, since $\mathfrak{p} \cap R_j = (0)$ for every $j \in \mathbb{N}$, we have $\mathfrak{p} \cap S = (0)$. □

We use the next result in the proof of Theorem 3.2 to identify height one prime ideals of $T$ that will be glued together in our final UFD.

**Lemma 2.7.** *Let $(T, M)$ be a complete local ring and let $R$ be a countable N-subring of $T$. Let $Q$ be a prime ideal of $T$ such that $Q \not\subseteq P$ for any $P \in$ Ass$(T)$ and any $P \in$ Ass$(T/rT)$ with $0 \neq r \in R$. Let $X = \{Q_1, Q_2, \ldots, Q_n\}$ be a (possibly empty) set of prime ideals of $T$ such that $Q \not\subseteq Q_j$ for all $j = 1, 2, \ldots, n$. Then there exists a height one prime ideal $P'$ of $T$ such that $P' \subseteq Q$ and $P' \not\subseteq P$ for any $P \in$ Ass$(T)$ and any $P \in$ Ass$(T/rT)$ with $0 \neq r \in R$. Moreover, if $X$ is not empty then $P' \not\subseteq Q_j$ for all $j = 1, 2, \ldots, n$.*

*Proof.* Use Lemma 2.2 with

$$C = \text{Ass}(T) \cup \{P \in \text{Spec}\,(T) \mid P \in \text{Ass}(T/rT) \text{ for some } 0 \neq r \in R\} \cup X,$$

$D = \{0\}$ and $I = Q$ to find $q \in Q$ such that $q$ is not an element in any prime ideal in $C$. Then $q$ is a nonzero regular element of $T$. Let $P'$ be a minimal prime ideal of $qT$ contained in $Q$. By the principal ideal theorem, $\text{ht}(P') = 1$. By the way $q$ was chosen and since $q \in P'$, we have that $P' \not\subseteq P$ for any $P \in C$, as desired. □

Note that, in the above lemma, if $Q$ contains only one minimal prime ideal of $T$ then $P'$ also contains only one minimal prime ideal of $T$.

In the next lemma, we show that, given an N-subring $R$ of $T$ and given certain height one prime ideals $P_1, \ldots, P_s$ of $T$, we can adjoin a special element $\widetilde{x}$ of $T$ to $R$ and obtain another N-subring of $T$. Specifically, $\widetilde{x}$ will satisfy the property that it is in $P_i$ for all $i = 1, 2, \ldots, s$. Our final UFD



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$A$ in the proof of Theorem 3.2 will satisfy the property that $P_i \cap A = \widetilde{x}A$ for all $i = 1, 2, \ldots, s$. In other words, the prime ideals $P_1, \ldots, P_s$ will be glued together in $A$.

**Lemma 2.8.** *Let $(T, M)$ be a complete local ring with depth$(T) \geq 2$ and suppose $R$ is a countable N-subring of $T$. Let $P_1, \ldots, P_s$ be height one prime ideals of $T$ such that, for every $i = 1, 2, \ldots, s$ we have that $P_i \not\subseteq P$ for any $P \in$ Ass$(T)$ and any $P \in$ Ass$(T/rT)$ with $0 \neq r \in R$. Let $X$ be a (possibly empty) finite set of prime ideals of $T$ such that $P_i \not\subseteq Q$ for every $Q \in X$ and for every $i = 1, 2, \ldots, s$. Then, there exists $\widetilde{x} \in \bigcap_{i=1}^{s} P_i$ with $\widetilde{x} \notin \bigcup_{Q \in X} Q$ such that $S = R[\widetilde{x}]_{(R[\widetilde{x}] \cap M)}$ is an N-subring of $T$ with $P_i \cap S = \widetilde{x}S$ for every $i = 1, 2, \ldots, s$. Moreover, prime elements in $R$ are prime in $S$.*

*Proof.* Define

$$C = \text{Ass}(T) \cup \{P \in \text{Spec}\,(T) \mid P \in \text{Ass}(T/rT) \text{ for some } 0 \neq r \in R\} \cup X$$

and note that $C$ is countable and $M \notin C$. Now, if $i \in \{1, 2, \ldots, s\}$, we have $P_i \not\subseteq P$ for all $P \in C$. For every $i = 1, 2, \ldots, s$, apply Lemma 2.2 to find $x_i \in P_i$ such that $x_i \notin P$ for all $P \in C$. Define $x = \prod_{i=1}^{s} x_i$ and note that $x \in \bigcap_{i=1}^{s} P_i$ and $x \notin P$ for every $P \in C$. Now, fix $P \in C$ and let $t, t' \in T$. If $x(1+t) + P = x(1+t') + P$ as elements of $T/P$ then $x(t - t') \in P$. Since $x \notin P$, we have $t - t' \in P$ and so $t + P = t' + P$. It follows that if $t + P \neq t' + P$ then $x(1+t) + P \neq x(1+t') + P$. Let $D_{(P)}$ be a full set of coset representatives for the cosets $t + P$ that make $x(1+t) + P$ algebraic over $R/(P \cap R)$, and note that $D_{(P)}$ is countable. Let $D = \bigcup_{P \in C} D_{(P)}$. Use Lemma 2.2 to find $\alpha \in M$ so that $x(1+\alpha) + P \in T/P$ is transcendental over $R/(P \cap R)$ for every $P \in C$. Define $\widetilde{x} = x(1+\alpha)$. Then $\widetilde{x} \notin \bigcup_{Q \in X} Q$. By Lemma 2.4, $S = R[\widetilde{x}]_{(M \cap R[\widetilde{x}])}$ is an N-subring of $T$. Since $\widetilde{x}$ is transcendental over $R$, prime elements of $R$ are prime in $S$. Fix $i \in \{1, 2, \ldots, s\}$. Since $\widetilde{x} \in S$ and $\widetilde{x} \in P_i$, we have $\widetilde{x}S \subseteq P_i \cap S$. Since $R$ is a domain, $\widetilde{x}S$ is a prime ideal of $S$. Now $\widetilde{x}$ is a regular element of $T$ and $P_i$ is a height one prime ideal of $T$. It follows that $P_i \in \text{Ass}(T/\widetilde{x}T)$. As $S$ is an N-subring of $T$ we have ht$(P_i \cap S) = 1$ and so $P_i \cap S = \widetilde{x}S$. □

## 3. The Main Result

In this section, we state and prove our main result, Theorem 3.2, which can be used to show that there exist local UFDs $A$ that are “infinitely" noncatenary. More specifically, the result shows that there exist local UFDs $A$ such that $A$ has infinitely many height one prime ideals $\{J_n\}_{n=1}^{\infty}$ satisfying the property that, for every $n \in \mathbb{N}$, there are saturated chains of prime ideals of different lengths starting at $J_n$ and ending at the maximal ideal of $A$. It follows that $A$ is noncatenary and that $A/J_n$ is noncatenary for every $n \in \mathbb{N}$. After proving Theorem 3.2, we provide an illustrative example. Finally, in Section 3.1, we show that Theorem 3.2 can be used to find an infinite family of local UFDs that satisfy the above conditions.

The first step in the proof of Theorem 3.2 is to start with a complete local ring $T$ with $s$ minimal prime ideals and then, for each minimal prime ideal $P$ of $T$, we identify infinitely many coheight one prime ideals of $T$ that contain $P$ and satisfy the hypotheses of Lemma 2.6. We then use Lemma 2.6 to find an N-subring $R_0$ of $T$ that contains generating sets for all of our chosen coheight one prime ideals. We strategically adjoin elements of $T$ to $R_0$ to construct our final local UFD $A$, and so $A$ will contain $R_0$. It follows that $A$ will contain generators of all of our chosen coheight one prime ideals of $T$. This property will be useful in proving that $A$ contains our desired saturated chains of prime ideals.

We use Lemma 3.1 to find our infinite sets of coheight one prime ideals of $T$.

**Lemma 3.1.** *Let $(T, M)$ be a local catenary ring with depth$(T) \geq 2$ and let $P_0$ be a minimal prime ideal of $T$ with dim$(T/P_0) = n \geq 3$. Then there are infinitely many prime ideals $Q$ of $T$ satisfying the conditions that $P_0$ is the only minimal prime ideal contained in $Q$, dim$(T/Q) = 1$, and $Q \not\subseteq P$ for any $P \in$ Ass$(T)$ and any $P \in$ Ass$(T/zT)$ with $z$ a nonzero regular element of $T$.*

*Proof.* Let $P_0 \subsetneq Q_1 \subsetneq \cdots \subsetneq Q_{n-1} \subsetneq M$ be a saturated chain of prime ideals of $T$ obtained from Lemma 2.8 in [1] such that, for each $i = 1, 2, \ldots, n-1$, $Q_i \notin \mathrm{Ass}(T)$ and $P_0$ is the only minimal prime ideal contained in $Q_i$. Suppose that $Q_{n-2} \subseteq P$ for some $P \in \mathrm{Ass}(T)$. As depth$(T) \geq 2$, $M \notin \mathrm{Ass}(T)$. Since $T$ is catenary and $Q_{n-2} \notin \mathrm{Ass}(T)$, we have $\dim(T/P) = 1$. By Theorem

17.2 in [9], $2 \leq \operatorname{depth}(T) \leq \dim(T/P) = 1$, a contradiction. It follows that $Q_{n-2} \not\subseteq P$ for all $P \in \operatorname{Ass}(T)$. Thus, there exists a regular element $x \in T$ with $x \in Q_{n-2}$.

Let

$$X = \{Q \in \text{Spec}\,(T) \,|\, Q_{n-2} \subsetneq Q \subsetneq M \text{ is saturated}\}.$$

Since $T$ is Noetherian, $X$ has infinitely many elements. Suppose $Q \in X$ such that $Q$ contains $P_1$ where $P_1$ is a minimal prime ideal of $T$ satisfying $P_1 \neq P_0$. Then $Q$ is a minimal prime ideal of $Q_{n-2} + P_1$, of which there are only finitely many. The two sets $\operatorname{Ass}(T)$ and $\operatorname{Ass}(T/xT)$ are finite. Thus, the set

$$Y = \{Q \in X \,|\, Q \notin \operatorname{Ass}(T), Q \notin \operatorname{Ass}(T/xT) \text{ and}$$
$$P_0 \text{ is the only minimal prime ideal of } T \text{ contained in } Q\}$$

contains infinitely many elements. Let $Q \in Y$. Then $\dim(T/Q) = 1$ and $Q \not\subseteq P$ for any $P \in \operatorname{Ass}(T)$.

Since $T_Q$ is a flat extension of $T$ and $x$ is a regular element of $T$, we have that $x$ is a regular element of $T_Q$. As $Q \notin \operatorname{Ass}(T/xT)$, the corollary to Theorem 6.2 in [9] gives that $QT_Q \notin \operatorname{Ass}(T_Q/xT_Q)$. It follows that $\operatorname{depth}(T_Q) \geq 2$. Now suppose that $Q \subseteq P$ for some $P \in \operatorname{Ass}(T/zT)$ where $z$ is a nonzero regular element of $T$. Then $P = M$ or $P = Q$. As $\operatorname{depth}(T) \geq 2$, $M \notin \operatorname{Ass}(T/zT)$ and so $P = Q$. Therefore, $Q \in \operatorname{Ass}(T/zT)$. It follows that $QT_Q \in \operatorname{Ass}(T_Q/zT_Q)$, contradicting that $\operatorname{depth}(T_Q) \geq 2$. Hence all elements of $Y$ satisfy the desired conditions. □

We are now ready to prove the main result of this paper. Note that if $T$ in Theorem 3.2 is chosen so that $n_i \neq n_j$ for at least one pair of $i, j \in \{1, \ldots, s\}$ then the resulting local UFD $A$ is necessarily noncatenary.

**Theorem 3.2.** *Let $(T, M)$ be a complete local ring such that no integer of $T$ is a zerodivisor of $T$ and such that depth$(T) \geq 2$. Let $\{P_{0,1}, \ldots, P_{0,s}\}$ be the minimal prime ideals of $T$ and suppose that for $i = 1, 2, \ldots, s$, we have $\dim(T/P_{0,i}) = n_i \geq 3$. Then there exists a local UFD $(A, M \cap A)$ such that $\widehat{A} = T$ and such that, for all $n \in \mathbb{N}$ and for all $i = 1, 2, \ldots, s$, there exist saturated* *chains of prime ideals* $(0) \subsetneq J_n \subsetneq J_{2,n}^{(i)} \subsetneq \cdots \subsetneq J_{n_i-1,n}^{(i)} \subsetneq M \cap A$ *of* $A$ *satisfying* $J_{a,b}^{(i)} = J_{c,d}^{(j)}$ *if and only if* $i = j$*,* $a = c$*, and* $b = d$*, and* $J_n = J_m$ *if and only if* $n = m$*.*

In the statement of Theorem 3.2 and in its proof, the notation $P^{(i)}$ for a prime ideal $P$ has no relation to the $i$th symbolic power of $P$ but is just a notational label.

*Proof.* The strategy for our proof is to first use Lemma 3.1 to find, for every $i = 1, 2, \ldots s$, infinitely many prime ideals $\{Q_n^{(i)}\}_{n \in \mathbb{N}}$ of $T$ that satisfy the assumptions of Lemma 2.6, as well as the properties that, for every $n \in \mathbb{N}$, $Q_n^{(i)}$ contains $P_{0,i}$ and $Q_n^{(i)}$ has coheight equal to one (i.e. $\dim(T/Q_n^{(i)}) = 1$). We then note that a localization of the prime subring of $T$ is an N-subring, and we use Lemma 2.6 to obtain an N-subring $R_0$ of $T$ that contains generators of $Q_n^{(i)}$ for all $i = 1, 2, \ldots, s$ and for all $n \in \mathbb{N}$. Our final UFD $A$ will contain $R_0$ and hence $A$ will contain generators of all of the $Q_n^{(i)}$'s. Next, we alternate using Lemma 2.7 and Lemma 2.8 infinitely many times to find appropriate height one prime ideals of $T$ that will be glued together in $A$, along with elements that will generate the prime ideals of $A$ that the height one prime ideals will glue to. In particular, for each $j = 1, 2, \ldots s$, we find infinitely many height one prime ideals $\{P_m^{(j)}\}_{m \in \mathbb{N}}$ of $T$ such that the prime ideal $P_m^{(j)}$ contains $P_{0,j}$ and satisfies $P_m^{(j)} \subseteq Q_n^{(i)}$ if and only if $i = j$ and $n = m$. In this way, each identified height one prime ideal is paired with an identified coheight one prime ideal. We rely on results from [4] to finish the construction our UFD $A$. We then carefully construct saturated chains of prime ideals of $T$. Each chain will contain one of our identified height one prime ideals $P_n^{(i)}$, along with its corresponding coheight one prime ideal $Q_n^{(i)}$. When we intersect these chains with $A$, the nonmaximal prime ideals of height at least two will remain distinct, while $P_n^{(i)} \cap A = P_m^{(j)} \cap A$ (i.e. $P_n^{(i)}$ and $P_m^{(j)}$ are glued together in $A$) if and only if $n = m$. Moreover, $P_n^{(i)} \cap A \neq (0)$ for all $i = 1, 2, \ldots s$ and for all $n \in \mathbb{N}$. These properties will imply that our desired saturated chains of prime ideals of $A$ are obtained by intersecting our chains from $T$ with $A$. We now begin implementing this strategy.



**19 Aug 2024 10:52:10 PDT**
**240304-Loepp Version 3 - Submitted to Rocky Mountain J. Math.**

For each minimal prime ideal $P_{0,i}$, apply Lemma 3.1 to find infinitely many coheight one prime ideals $\{Q_n^{(i)}\}_{n\in\mathbb{N}}$ of $T$ above each $P_{0,i}$ such that $Q_n^{(i)} \neq Q_m^{(i)}$ if $n \neq m$ and each $Q_n^{(i)} \not\subseteq P$ for any $P \in \operatorname{Ass}(T)$ and any $P \in \operatorname{Ass}(T/zT)$ with $z$ a nonzero regular element of $T$ .

Let $\Pi$ be the prime subring of $T$ and let $R$ be $\Pi$ localized at $M \cap \Pi$. Then $R$ is a countable N-subring of $T$. By Lemma 2.6, there is a countable N-subring $R_0$ of $T$ that contains a generating set for $Q_n^{(i)}$ for every $i = 1, 2, \ldots, s$ and for every $n \in \mathbb{N}$.

Now that we have infinitely many coheight one prime ideals of $T$ above each minimal prime ideal, all of which have a generating set contained in $R_0$, we next focus on finding $s$ height one prime ideals $P_1^{(i)}$ of $T$ such that $P_{0,i} \subseteq P_1^{(i)} \subseteq Q_1^{(i)}$. These $s$ prime ideals will then be glued together in an N-subring that is an extension of $R_0$ to a height one prime ideal. To do this, use Lemma 2.7 letting $X$ be the empty set to find height one prime ideals $P_1^{(i)}$ of $T$ for each $i = 1, 2, \ldots, s$ satisfying $P_1^{(i)} \subseteq Q_1^{(i)}$ and $P_1^{(i)} \not\subseteq P$ for any $P \in \operatorname{Ass}(T)$ and for any $P \in \operatorname{Ass}(T/rT)$ with $0 \neq r \in R_0$. By Lemma 2.8 there exists $\widetilde{x}_1 \in \bigcap_{i=1}^s P_1^{(i)}$ such that $R_1 = R_0[\widetilde{x}_1]_{(M\cap R_0[\widetilde{x}_1])}$ is a countable N-subring of $T$ with $P_1^{(i)} \cap R_1 = \widetilde{x}_1 R_1$ for every $i = 1, 2, \ldots, s$. Moreover, prime elements of $R_0$ are prime in $R_1$.

Our goal now is to find $s$ height one prime ideals $P_2^{(i)}$ of $T$ such that $P_{0,i} \subseteq P_2^{(i)} \subseteq Q_2^{(i)}$ and such that each $P_2^{(i)}$ is not contained in any $Q_1^{(j)}$. These $s$ prime ideals will then be glued together in an N-subring that is an extension of $R_1$ to a height one prime ideal. To do this, we essentially repeat the process in the previous paragraph. Specifically, for each $i = 1, 2, \ldots s$, use Lemma 2.7 with $X = \{Q_1^{(i)}\}$ to find height one prime ideals $P_2^{(i)}$ of $T$ for each $i = 1, 2, \ldots, s$ satisfying $P_2^{(i)} \subseteq Q_2^{(i)}$ and $P_2^{(i)} \not\subseteq P$ for any $P \in \operatorname{Ass}(T)$ and for any $P \in \operatorname{Ass}(T/rT)$ with $0 \neq r \in R_1$. Moreover, $P_2^{(i)} \not\subseteq Q_1^{(i)}$ for all $i = 1, 2, \ldots, s$. Suppose $P_2^{(i)} \subseteq Q_1^{(k)}$ for some $k \neq i$. Then $Q_1^{(k)}$ contains $P_{0,i}$ and $P_{0,k}$, a contradiction. It follows that $P_2^{(i)} \not\subseteq Q_1^{(k)}$ for all $i = 1, 2, \ldots, s$ and for all $k = 1, 2, \ldots, s$.

By Lemma 2.8 using $X = \{Q_1^{(1)}, Q_1^{(2)}, \ldots, Q_1^{(s)}\}$ there exists $\widetilde{x}_2 \in \bigcap_{i=1}^s P_2^{(i)}$ with $\widetilde{x}_2 \notin Q_1^{(i)}$ for all $i = 1, 2, \ldots, s$ such that $R_2 = R_1[\widetilde{x}_2]_{(M\cap R_1[\widetilde{x}_2])}$ is a countable N-subring of $T$ with $P_2^{(i)} \cap R_2 =$

$\widetilde{x}_2 R_2$ for every $i = 1, 2, \ldots, s$. Moreover, prime elements of $R_1$ are prime in $R_2$. In particular, $\widetilde{x}_1$ is a prime element of $R_2$.

Repeat this process using Lemma 2.7 with $X = \{Q_1^{(i)}, Q_2^{(i)}\}$ to find height one prime ideals $P_3^{(i)}$ and then Lemma 2.8 to find an element $\widetilde{x}_3$ of $T$ and an N-subring $R_3$ of $T$. Continue inductively so that, if $P_{n-1}^{(i)}$ for every $i = 1, 2, \ldots, s$, $\widetilde{x}_{n-1}$, and $R_{n-1}$ have been defined, use Lemma 2.7 with $X = \{Q_1^{(i)}, Q_2^{(i)}, \ldots, Q_{n-1}^{(i)}\}$ and Lemma 2.8 with $X = \bigcup_{i=1}^{s}\{Q_1^{(i)}, Q_2^{(i)}, \ldots, Q_{n-1}^{(i)}\}$ to define the following:

- height one prime ideals $P_n^{(i)}$ of $T$ for each $i = 1, 2, \ldots, s$ satisfying $P_n^{(i)} \subseteq Q_n^{(i)}$ and $P_n^{(i)} \not\subseteq P$ for any $P \in \mathrm{Ass}(T)$ and for any $P \in \mathrm{Ass}(T/rT)$ with $0 \neq r \in R_{n-1}$. Moreover, for all $i = 1, 2, \ldots, s$ and for all $k = 1, 2, \ldots, s$, $P_n^{(i)} \not\subseteq Q_j^{(k)}$ whenever $j < n$.
- an element $\widetilde{x}_n$ of $T$ such that $\widetilde{x}_n \in \bigcap_{i=1}^{s} P_n^{(i)}$ and, for all $i = 1, 2, \ldots, s$ we have $\widetilde{x}_n \notin Q_j^{(i)}$ whenever $j < n$.
- a countable N-subring $R_n$ of $T$ containing $\widetilde{x}_1, \widetilde{x}_2, \ldots, \widetilde{x}_n$ such that $P_n^{(i)} \cap R_n = \widetilde{x}_n R_n$ for every $i = 1, 2, \ldots, s$. Moreover, prime elements of $R_{n-1}$ are prime in $R_n$. In particular, $\widetilde{x}_1, \widetilde{x}_2, \ldots, \widetilde{x}_n$ are prime elements of $R_n$.

By Lemma 6 in [4], which gives sufficient conditions for a union of an ascending chain of N-subrings to be an N-subring and for prime elements in one of the N-subrings to remain prime in the union, $S = \bigcup_{n=1}^{\infty} R_n$ is a countable N-subring of $T$ and $\widetilde{x}_n$ is a prime element of $S$ for every $n \in \mathbb{N}$.

At this point, we use the construction in [4] to build our UFD $A$. In particular, in the proof of Theorem 8 in [4], one starts with a complete local ring $\tilde{T}$ and a localization of the prime subring of $\tilde{T}$ and then constructs a UFD whose completion is $\tilde{T}$. For our proof, we replace the localization of the prime subring of $T$ in the proof of Theorem 8 in [4] with the N-subring $S$ above. We then follow the proof of Theorem 8 in [4] to construct a UFD $(A, M \cap A)$. In particular, $A$ contains $S$, $\widehat{A} \cong T$, and prime elements of $S$ are prime in $A$. Since $A$ contains $S$, it contains a generating set for each $Q_n^{(i)}$ and so $(Q_n^{(i)} \cap A)T = Q_n^{(i)}$. Also note that $\widetilde{x}_n$ is a prime element of $A$ for every $n \in \mathbb{N}$ and, for every $i = 1, 2, \ldots, s$ and for every $n \in \mathbb{N}$, we have $P_n^{(i)} \cap A = \widetilde{x}_n A$. Because $T/Q_n^{(i)} =$

$T/(Q_n^{(i)} \cap A)T$ is the completion of $A/(Q_n^{(i)} \cap A)$, we have $1 = \dim(T/Q_n^{(i)}) = \dim(A/(Q_n^{(i)} \cap A)$ for every $i = 1, 2, \ldots, s$ and for every $n \in \mathbb{N}$.

By construction, for each $n \in \mathbb{N}$, the height one prime ideals $P_n^{(i)}$ for $i = 1, \ldots, s$ get glued together in $A$ to the height one prime ideal $\tilde{x}_n A$. Define $J_n = \tilde{x}_n A = P_n^{(i)} \cap A$. Note that $J_n = J_m$ if and only if $n = m$. Now for each $n \in \mathbb{N}$, we will show that there are $s$ disjoint saturated chains of prime ideals of $A$ all starting at $J_n$ and ending at $M \cap A$ where the $i$th chain contains $Q_n^{(i)} \cap A$ and has length $n_i - 1$. To find these chains, we first define a prime ideal $J_{2,n}^{(i)}$ of $A$ for all $i = 1, 2, \ldots, s$ and for all $n \in \mathbb{N}$. For a fixed $i \in \{1, 2, \ldots, s\}$ and a fixed $n \in \mathbb{N}$, the prime ideal $J_{2,n}^{(i)}$ will contain $J_n$ and have height two. To define $J_{2,n}^{(i)}$, we identify a height two prime ideal $Q_{2,n}^{(i)}$ of $T$ that is contained in $Q_n^{(i)}$ and strictly contains $P_n^{(i)}$. The ideal $J_{2,n}^{(i)}$ will be the intersection of $Q_{2,n}^{(i)}$ with $A$.

Observe that $\widetilde{x}_n A \subseteq Q_n^{(i)} \cap A$. Suppose we have $\widetilde{x}_n A = Q_n^{(i)} \cap A$. Then there is a $y_n \in M \cap A$ with $y_n \notin Q_n^{(i)} \cap A = \widetilde{x}_n A$. In this case, $M \cap A$ is a minimal prime ideal of $(\widetilde{x}_n, y_n)A$ and so, by the generalized principal ideal theorem, $\text{ht}(M \cap A) \leq 2$. It follows that $\dim(T) = \dim(A) \leq 2$, a contradiction. Hence $\widetilde{x}_n A \subsetneq Q_n^{(i)} \cap A$. If $Q_n^{(i)} \cap A = Q_n^{(k)} \cap A$ where $i \neq k$, then $Q_n^{(i)} = Q_n^{(k)}$, a contradiction. Thus, by prime avoidance, there exists $p_n^{(i)} \in A$ such that $p_n^{(i)} \in Q_n^{(i)}$, $p_n^{(i)} \notin \widetilde{x}_n A$ and $p_n^{(i)} \notin Q_n^{(k)}$ for $k \neq i$. Let $Q_{2,n}^{(i)}$ be a minimal prime ideal of $P_n^{(i)} + p_n^{(i)} T$ that is contained in $Q_n^{(i)}$ and note that $P_n^{(i)} \subsetneq Q_{2,n}^{(i)}$ is saturated. In particular, $\text{ht}(Q_{2,n}^{(i)}) = 2$. Define $J_{2,n}^{(i)} = Q_{2,n}^{(i)} \cap A$ and note that, since $T$ is a faithfully flat extension of $A$, $\text{ht}(J_{2,n}^{(i)}) \leq 2$. It follows that $(0) \subsetneq J_n \subsetneq J_{2,n}^{(i)}$ is saturated. Also observe that $J_{2,n}^{(i)} = J_{2,n}^{(k)}$ if and only if $i = k$.

Note that $J_{2,n}^{(i)} = Q_n^{(i)} \cap A$ if and only if $\dim(T/P_{0,i}) = n_i = 3$. If we are in this case, we have completed defining our chain. So suppose $J_{2,n}^{(i)} \subsetneq Q_n^{(i)} \cap A$. We define $Q_{t,n}^{(i)}$ and $J_{t,n}^{(i)}$ inductively for $t \geq 3$. Assume $Q_{t-1,n}^{(i)}$ and $J_{t-1,n}^{(i)}$ have been defined and suppose $J_{t-1,n}^{(i)} \subsetneq Q_n^{(i)} \cap A$. Let $q_{t,n}^{(i)} \in Q_n^{(i)} \cap A$ with $q_{t,n}^{(i)} \notin J_{t-1,n}^{(i)}$, and let $Q_{t,n}^{(i)}$ be a minimal prime ideal of $Q_{t-1,n}^{(i)} + q_{t,n}^{(i)} T$ that is contained in $Q_n^{(i)}$. Define $J_{t,n}^{(i)} = Q_{t,n}^{(i)} \cap A$. As we continue inductively defining $Q_{t,n}^{(i)}$ and $J_{t,n}^{(i)}$, eventually there will be an $\ell \in \mathbb{N}$ such that $J_{\ell-1,n}^{(i)} \subsetneq Q_n^{(i)} \cap A$ and $J_{\ell,n}^{(i)} = Q_n^{(i)} \cap A$. At this point, we stop, and so we only define $Q_{t,n}^{(i)}$ and $J_{t,n}^{(i)}$ until $t = \ell$. Note that $Q_{\ell,n}^{(i)} = Q_n^{(i)}$ and if $\mu < \rho \leq \ell$, then $J_{\mu,n}^{(i)} \subsetneq J_{\rho,n}^{(i)}$. Observe that $P_{0,i} \subsetneq P_n^{(i)} \subsetneq Q_{2,n}^{(i)} \subsetneq \cdots \subsetneq Q_{\ell,n}^{(i)} \subsetneq M$ is saturated. Since $T$ is catenary,

$\ell = n_i - 1$. By the going down property, the chain $(0) \subsetneq J_n \subsetneq J^{(i)}_{2,n} \subsetneq \cdots \subsetneq J^{(i)}_{n_i-1,n} \subsetneq M \cap A$ is saturated.

Finally, we show that if $J^{(i)}_{a,b} = J^{(k)}_{c,d}$ then $i = k$, $a = c$, and $b = d$. Suppose $J^{(i)}_{a,b} = J^{(k)}_{c,d}$ for some $b \neq d$. Without loss of generality, assume that $b < d$. Then $\widetilde{x}_d \in J^{(i)}_{a,b} \subseteq Q^{(i)}_b$, a contradiction since $\tilde{x}_d \notin Q^{(i)}_j$ for $j < d$, and so we have $b = d$ and $J^{(i)}_{a,b} = J^{(k)}_{c,b}$. Then $p^{(i)}_b \in Q^{(k)}_b$. If $i \neq k$ then this contradicts the way $p^{(i)}_b$ was chosen. It follows that $i = k$, and thus, $a = c$ as well. □

Recall that all local Noetherian UFDs of dimension at most three are catenary. In addition, if $A$ is a local ring and $\widehat{A}$ is equidimensional, then $A$ is universally catenary. Since we are interested in the case where $A$ is noncatenary, Theorem 3.2 is most interesting in our setting when $T$ is a complete local ring of dimension at least four that is not equidimensional (i.e. $n_i \neq n_j$ for some $i, j \in \{1, 2, \ldots, s\}$). In this case, the local UFD $A$ given by the theorem has infinitely many height one prime ideals $J_1, J_2, \ldots$ such that $A/J_n$ is noncatenary for every $n \in \mathbb{N}$. More specifically, for every $n \in \mathbb{N}$, if $i \in \{1, 2, \ldots, s\}$, then $A/J_n$ has a saturated chain of prime ideals of length $n_i - 1$ that starts at the zero ideal of $A/J_n$ and ends at the maximal ideal of $A/J_n$.

We now turn to the example introduced in Section 1.

**Example 3.3.** *Let $T = \mathbb{C}[[x, y, z, w, t]]/((x) \cap (y, z))$. Then $T$ satisfies the conditions of Theorem 3.2. Let $x, y, z, w, t$ denote their images in $T$. The minimal prime ideals of $T$ are $(x)$ and $(y, z)$, and we have dim$(T/(x)) = 4$ and dim$(T/(y, z)) = 3$. By Theorem 3.2, $T$ is the completion of a UFD $A$ such that $A$ has infinitely many height one prime ideals $\{J_n\}_{n\in\mathbb{N}}$ satisfying the condition that, for every $n \in \mathbb{N}$, there is a saturated chain of prime ideals of length 3 starting at $J_n$ and ending at the maximal ideal of $A$, and there is a saturated chain of prime ideals of length 2 starting at $J_n$ and ending at the maximal ideal of $A$. Moreover, all of these chains are disjoint (except at the maximal ideal of $A$). As a consequence, $A/J_n$ is noncatenary for every $n \in \mathbb{N}$. We note that one has some choice for the elements of the chains having coheight one. In the proof of Theorem 3.2, one can choose the prime ideals $Q^{(i)}_n$ to satisfy the required conditions. The elements of the chains in $A$ of coheight one will be the prime ideals $Q^{(i)}_n \cap A$ of $A$. For example, for our* *given $T$, let $\{\alpha_n\}_{n\in\mathbb{N}}$ be distinct elements of $\mathbb{C}$. One could choose $Q_n^{(1)}$ to be $(x,y,w,t+\alpha_n z)$ and $Q_n^{(2)}$ to be $(y,z,w,t+\alpha_n x)$. In this case, the coheight one ideals in the chains of length 3 will be $(x,y,w,t+\alpha_n z)\cap A$ and the coheight one ideals in the chains of length 2 will be $(y,z,w,t+\alpha_n x)\cap A$. Furthermore, the generator $\widetilde{x}_n$ of the height one prime ideal $J_n$ of $A$ will be a regular element of $(x,y,w,t+\alpha_n z)\cap(y,z,w,t+\alpha_n x)$.*

Example 3.3 can be generalized to produce a class of local UFDs that are noncatenary at infinitely many places.

3.1. **A class of local UFDs that are noncatenary at infinitely many places.** Let $s\geq 2$ and let $n=1+2+3+\cdots+s=s(s+1)/2$. Let $T_s'=\mathbb{C}[[x_1,x_2,\ldots,x_n,x_{n+1},x_{n+2}]]$, and define the ideal $I_s$ of $T_s'$ by

$$I_s=(x_1)\cap(x_2,x_3)\cap(x_4,x_5,x_6)\cap\cdots\cap(x_{n-s+1},\ldots,x_n).$$

Let $T_s=T_s'/I_s$ and let $x_1,x_2,\ldots,x_n$ denote their images in $T_s$. The minimal prime ideals of $T_s$ are $\{(x_1),(x_2,x_3),(x_4,x_5,x_6),\ldots,(x_{n-s+1},\ldots,x_n)\}$ and we have $\dim(T_s/((x_{n-s+1},\ldots,x_n)))=2+(s-1)s/2\geq 3$. It follows that the coheight of every minimal prime ideal of $T_s$ is at least 3. Note that, if $P$ and $P'$ are distinct minimal prime ideals of $T_s$, then $\dim(T_s/P)\neq\dim(T_s/P')$. Since $T_s$ satisfies the conditions of Theorem 3.2, it is the completion of a UFD $A_s$ such that $A_s$ has infinitely many height one prime ideals $\{J_n\}_{n\in\mathbb{N}}$ satisfying the condition that, for every $n\in\mathbb{N}$, there are saturated chains of prime ideals of $s$ different lengths that start at $J_n$ and end at the maximal ideal of $A_s$. Moreover, all of these chains are disjoint (except at the maximal ideal of $A_s$). As a consequence, $A_s/J_n$ is noncatenary for every $n\in\mathbb{N}$, and so, for every $s\in\mathbb{N}$, $A_s$ is noncatenary at infinitely many places.

## 4. A Very Noncatenary UFD

In this section, we construct a dimension four countable local UFD $A$ such that, for *every* height one prime ideal $P$ of $A$, the ring $A/P$ is noncatenary. To do this, we start with the complete local ring

$$T = \mathbb{Q}[[x,y,z,w,t]]/((x)\cap(y,z)).$$

The ring $A$ will be a subring of $T$ with $\widehat{A} \cong T$. Therefore, $T$ is a faithfully flat extension of $A$, and this fact will help us show that $A$ satisfies our desired property. To show that $\widehat{A} \cong T$, we use the following result.

**Proposition 4.1** ([5], Proposition 1)**.** *Let $(R, M\cap R)$ be a quasi-local subring of a complete local ring $(T,M)$ such that the map $R \longrightarrow T/M^2$ is onto and $IT \cap R = I$ for every finitely generated ideal $I$ of $R$. Then $R$ is Noetherian and the natural homomorphism $\widehat{R} \longrightarrow T$ is an isomorphism.*

To show that $\widehat{A} \cong T$ using Proposition 4.1, we guarantee that the map $A \longrightarrow T/M^2$ is onto and $IT \cap A = I$ for every finitely generated ideal $I$ of $A$. We use the next lemma when constructing $A$ to ensure that it satisfies these two properties.

**Lemma 4.2** ([7], Lemma 3.7)**.** *Let $(T,M)$ be a complete local ring with depth$(T) \geq 2$ and let $\mathfrak{p}$ be a nonmaximal prime ideal of $T$. Let $(R, M\cap R)$ be an infinite N-subring of $T$ with $\mathfrak{p}\cap R = (0)$ and let $u \in T$. Then there exists an N-subring $(S, M\cap S)$ of $T$ such that*

(1) $R \subseteq S \subseteq T$,

(2) $u + M^2$ *is in the image of the map* $S \longrightarrow T/M^2$,

(3) $|S| = |R|$,

(4) $\mathfrak{p}\cap S = (0)$,

(5) *prime elements of $R$ are prime in $S$, and*

(6) *for every finitely generated ideal $I$ of $S$, we have $IT \cap S = I$.*

In the next lemma, we show that, if $T$ is a complete local ring satisfying certain conditions and $R$ is a countable N-subring of $T$, then we can enlarge $R$ to another countable N-subring $A$ of $T$ whose completion is $T$. Moreover, for a given nonmaximal ideal $\mathfrak{p}$ of $T$, if $\mathfrak{p}\cap R = (0)$, then $\mathfrak{p}\cap A = (0)$ as well.

**Lemma 4.3.** *Let $(T, M)$ be a complete local ring containing the rationals with $T/M$ countable and depth$(T) \geq 2$. Let $\mathfrak{p}$ be a nonmaximal prime ideal of $T$. Let $(R, M \cap R)$ be a countable N-subring of $T$ with $\mathfrak{p} \cap R = (0)$. Then there exists a countably infinite N-subring $(A, M \cap A)$ of $T$ such that $R \subseteq A$, prime elements of $R$ are prime in $A$, $A$ is Noetherian, $\widehat{A} \cong T$, and $\mathfrak{p} \cap A = (0)$.*

*Proof.* Since $T/M$ is countable, $T/M^2$ is countable. Enumerate the elements of $T/M^2$ as $u_0 + M^2, u_1 + M^2, u_2 + M^2, \ldots,$. Let $R_0 = R$, a countably infinite N-subring of $T$ with $\mathfrak{p} \cap R = (0)$. We will extend $R_0 = R$ by using Lemma 4.2 infinitely many times, once for each $u_n \in T$. Let $R_1$ be the countable N-subring of T obtained from Lemma 4.2 that extends $R_0 = R$ using $u = u_0$. Then for $n \geq 2$, let $R_n$ be the countable N-subring of T obtained from Lemma 4.2 that extends $R_{n-1}$ using $u = u_{n-1}$.

Let $A = \bigcup_{j=0}^{\infty} R_j$ and note that $R \subseteq A$. By Lemma 6 in [4], $A$ is a countable N-subring of $T$ and prime elements of $R$ are prime in $A$. By construction, $\mathfrak{p} \cap A = (0)$ and the map $A \longrightarrow T/M^2$ is onto. Now let $I$ be a finitely generated ideal of $A$ and let $c \in IT \cap A$. We have $I = (a_1, a_2, \ldots, a_m)$ for some $a_i \in A$. Choose $N$ so that $c, a_1, a_2, \ldots, a_m \in R_N$. Then $c \in (a_1, \ldots, a_m)T \cap R_N = (a_1, \ldots, a_m)R_N \subseteq I$. It follows that $IT \cap A = I$. By Proposition 4.1, we have that $A$ is Noetherian and $\widehat{A} \cong T$. □

As mentioned previously, we begin our construction with the complete local ring

$$T = \mathbb{Q}[[x, y, z, w, t]]/((x) \cap (y, z)).$$

The next three results, Lemma 4.4, Lemma 4.5, and Lemma 4.6, demonstrate facts about this ring that we use in Theorem 4.7 to construct the UFD $A$ that is noncatenary at every height one prime ideal. Before we state and prove these lemmas, we give a short outline of the proof of Theorem 4.7 and explain how Lemma 4.4, Lemma 4.5, and Lemma 4.6 are used in that proof.

In the rest of this section, let $x, y, z, w, t$ denote their corresponding images in $T$. To prove Theorem 4.7, we first use Lemma 4.3 to find an N-subring $A_0$ of $T$ such that the completion of $A_0$ is $T$ and such that $(x, y, z) \cap A_0 = (0)$. For each height one prime ideal $a_j A_0$ of $A_0$, we let $P_{1,a_j}$

be a minimal prime ideal (in $T$) of $(a_j, x)$ and we let $P_{2,a_j}$ be a minimal prime ideal of $(a_j, y, z)$. Lemma 4.5 allows us to conclude that $P_{1,a_j}$ does not contain $(y, z)$ and $P_{2,a_j}$ does not contain $(x)$. Lemma 4.4 then produces two coheight one prime ideals of $T$, $Q_{1,a_j}$ and $Q_{2,a_j}$, each containing only one minimal prime ideal of $T$ and satisfying $P_{1,a_j} \subseteq Q_{1,a_j}$ and $P_{2,a_j} \subseteq Q_{2,a_j}$. We then use Lemma 4.6 to show that these coheight one prime ideals satisfy the conditions of Lemma 2.6, and so we can find an N-subring $R_1$ that contains a generating set for each $Q_{i,a_j}$. We apply Lemma 4.3 to $R_1$ to obtain an N-subring $A_1$ of $T$ with $\widehat{A_1} \cong T$, $R_1 \subseteq A_1$ and $(x, y, z) \cap A_1 = (0)$. We continue this process to define an infinite chain of N-subrings of $T$, $A_0 \subseteq R_1 \subseteq A_1 \subseteq R_2 \subseteq A_2 \cdots$. The union of this chain produces a local UFD $A$ such that $A/P$ is noncatenary for every height one prime ideal $P$.

**Lemma 4.4.** *Let $a$ be a nonzero regular element of the complete local ring*

$$T = \mathbb{Q}[[x, y, z, w, t]]/((x) \cap (y, z)).$$

*Suppose that the ideal $(a, x)$ has a minimal prime ideal $P_1$ that does not contain $(y, z)$ and that the ideal $(a, y, z)$ has a minimal prime ideal $P_2$ that does not contain $(x)$. Then there exist prime ideals $Q_1$ and $Q_2$ of $T$ such that, for $i = 1, 2$, we have $P_i \subsetneq Q_i$, $Q_i$ only contains one minimal prime ideal of $T$ and dim$(T/Q_i) = 1$.*

*Proof.* Let $M = (x, y, z, w, t)$. Note that $P_1$ and $P_2$ contain only one minimal prime ideal of $T$ and that the chains $(x) \subsetneq P_1$ and $(y, z) \subsetneq P_2$ are saturated. Since $T$ is Noetherian and catenary, there are infinitely many prime ideals strictly between $P_2$ and $M$. If $Q$ is such a prime ideal containing $(x)$, then $Q$ is a minimal prime ideal of $P_2 + (x)$, of which there are only finitely many. Therefore, we can choose $Q_2$ to satisfy the conditions that $P_2 \subsetneq Q_2 \subsetneq M$ and $Q_2$ does not contain $(x)$.

There exists a prime ideal $J$ of $T$ such that $P_1 \subsetneq J \subsetneq M$ and $\dim(T/J) = 1$. Since $T$ is catenary, $P_1 \subsetneq J$ is not saturated. There are infinitely many prime ideals strictly between $P_1$ and $J$. If $I$ is such a prime ideal containing $(y, z)$, then $I$ is a minimal prime ideal of $P_1 + (y, z)$ of which there are only finitely many. Thus, there is a prime ideal $I$ of $T$ satisfying the conditions that $P_1 \subsetneq I \subsetneq J \subsetneq M$ and $I$ does not contain $(y, z)$. By a similar argument replacing $P_1$ by $I$, there exists a prime ideal $Q_1$ of $T$ satisfying the conditions that $P_1 \subsetneq I \subsetneq Q_1 \subsetneq M$ and $Q_1$ does not contain $(y, z)$. $\square$

**Lemma 4.5.** *Let $a$ be an element of the complete local ring $T = \mathbb{Q}[[x, y, z, w, t]]/((x) \cap (y, z))$ satisfying the condition that $a \notin (x, y, z)$. Then every minimal prime ideal of the ideal $(a, x)$ does not contain $(y, z)$ and every minimal prime ideal of the ideal $(a, y, z)$ does not contain $(x)$.*

*Proof.* Let $P_1$ be a minimal prime ideal of $(a, x)$. Then in the ring $T/(x)$, the principal ideal theorem gives us that $\text{ht}(P_1/(x)) = 1$. Suppose $(y, z) \subseteq P_1$. Then $(x, y, z) \subseteq P_1$ and so in the ring $T/(x)$, $(x, y, z)/(x) \subseteq P_1/(x)$, and it follows that $\text{ht}((x, y, z)/(x)) \leq 1$. But $(x, y)/(x)$ is a prime ideal of $T/(x)$ and we have $(x)/(x) \subsetneq (x, y)/(x) \subsetneq (x, y, z)/(x)$, a contradiction.

Now let $P_2$ be a minimal prime ideal of $(a, y, z)$ and suppose that $(x) \subseteq P_2$. Then $(x, y, z) \subseteq P_2$. In the ring $T/(y, z)$, both of the prime ideals $P_2/(y, z)$ and $(x, y, z)/(y, z)$ have height one, and so $P_2/(y, z) = (x, y, z)/(y, z)$. Thus, $a + (y, z) \in (x, y, z)/(y, z)$ and we have that $a \in (x, y, z)$, a contradiction. $\square$

**Lemma 4.6.** *Let $Q$ be a prime ideal of the complete local ring $T = \mathbb{Q}[[x, y, z, w, t]]/((x) \cap (y, z))$ satisfying dim$(T/Q) = 1$. Then, $Q \nsubseteq P$ for any $P \in$ Ass$(T/vT)$ with $v$ a nonzero regular element of $T$.*

*Proof.* Note that $\text{depth}(T) = 3$ and so if $v$ is a nonzero regular element of $T$ then $(x, y, z, w, t) \notin \text{Ass}(T/vT)$. So suppose that $Q \in \text{Ass}(T/vT)$ for some nonzero regular element $v$ of $T$. Then Theorem 17.2 in [9] gives us that the dimension of the ring $(T/vT)/(Q/vT) \cong T/Q$ is greater than or equal to the depth of the ring $T/vT$. This implies that the depth of $T/vT$ is at most one, contradicting that the depth of $T$ is 3. $\square$

We are now equipped with the tools needed to prove the main result of this section.

**Theorem 4.7.** *Let $T = \mathbb{Q}[[x, y, z, w, t]]/((x) \cap (y, z))$. There exists a local UFD $A$ such that $\widehat{A} \cong T$ and such that, for every height one prime ideal $P$ of $A$, $A/P$ is noncatenary.*

*Proof.* Let $R_0 = \mathbb{Q}$, let $\mathfrak{p} = (x, y, z)$, and let $M = (x, y, z, w, t)$. Observe that $R_0$ is an infinite N-subring of $T$ and $\mathfrak{p} \cap R_0 = (0)$. Use Lemma 4.3 to find a countable N-subring $A_0$ of $T$ such that $A_0$ is Noetherian, $\widehat{A_0} \cong T$, and $\mathfrak{p} \cap A_0 = (0)$. Since $\widehat{A_0} \cong T$, the map $A_0 \longrightarrow T/M^2$ is onto and $IT \cap A_0 = I$ for every ideal $I$ of $A_0$.

Because $A_0$ is a Noetherian UFD, all of its height one prime ideals are principal. Enumerate the height one prime ideals of $A_0$ by $a_1A_0, a_2A_0, \ldots, a_nA_0, \ldots$. Note that, for all $j \geq 1$, we have $a_j \notin \mathfrak{p}$. For $j \geq 1$, let $P_{1,a_j}$ be a minimal prime ideal (in $T$) of $(a_j, x)$ and let $P_{2,a_j}$ be a minimal prime ideal of $(a_j, y, z)$. By Lemma 4.5, $P_{1,a_j}$ does not contain $(y, z)$ and $P_{2,a_j}$ does not contain $(x)$. By Lemma 4.4, there are prime ideals $Q_{1,a_j}$ and $Q_{2,a_j}$ of $T$ such that $P_{1,a_j} \subseteq Q_{1,a_j}$, $P_{2,a_j} \subseteq Q_{2,a_j}$, $Q_{1,a_j}$ and $Q_{2,a_j}$ only contain one minimal prime ideal of $T$, $\dim(T/Q_{1,a_j}) = 1$, and $\dim(T/Q_{2,a_j}) = 1$. Since $\dim(T/\mathfrak{p}) > 1$, we have $Q_{1,a_j} \not\subseteq \mathfrak{p}$ and $Q_{2,a_j} \not\subseteq \mathfrak{p}$ for all $j \geq 1$. Note that if $P \in \mathrm{Ass}(T) = \{(x), (y, z)\}$, then $Q_{1,a_j} \not\subseteq P$ and $Q_{2,a_j} \not\subseteq P$ for all $j \geq 1$. By Lemma 4.6, for $j \geq 1$ we have $Q_{1,a_j} \not\subseteq P$ and $Q_{2,a_j} \not\subseteq P$ for any $P \in \mathrm{Ass}(T/vT)$ with $v$ a nonzero regular element of $T$. Use Lemma 2.6 to find a countable N-subring $R_1$ of $T$ such that $A_0 \subseteq R_1$, $\mathfrak{p} \cap R_1 = (0)$, prime elements in $A_0$ are prime in $R_1$, and for every $j \geq 1$, $R_1$ contains a generating set for $Q_{1,a_j}$ and for $Q_{2,a_j}$.

By Lemma 4.3 there exists a countable N-subring $A_1$ of $T$ such that $R_1 \subseteq A_1$, prime elements in $R_1$ are prime in $A_1$, $A_1$ is Noetherian, $\widehat{A_1} \cong T$, and $\mathfrak{p} \cap A_1 = (0)$. Note that $IT \cap A_1 = I$ for every ideal $I$ of $A_1$. Also note that prime elements in $A_0$ are prime in $A_1$. Thus, if $a_jA_0$ is a height one prime ideal of $A_0$, then $a_jA_1$ is a height one prime ideal of $A_1$. However, $A_1$ might have additional height one prime ideals, and so we need to repeat the procedure replacing $A_0$ with $A_1$. That is, enumerate the height one prime ideals of $A_1$, find the appropriate height one and coheight one prime ideals of $T$, and adjoin the generators of the coheight one prime ideals to find a countable N-subring $R_2$. Continue to form a countably infinite chain of N-subrings $R_0 \subseteq A_0 \subseteq R_1 \subseteq A_1, \subseteq \cdots$ of $T$.

Define $A = \bigcup_{n=0}^{\infty} A_n$. By Lemma 6 in [4], $A$ is a countable N-subring of $T$ and elements that are prime in $A_j$ for some $j \geq 0$ are also prime in $A$. Since $A$ is an N-subring, it is a UFD. By construction, the map $A \longrightarrow T/M^2$ is onto and $IT \cap A = I$ for every finitely generated ideal $I$ of $A$. By Proposition 4.1, $A$ is Noetherian and $\widehat{A} \cong T$.

Let $P$ be any height one prime ideal of $A$. Since $A$ is a UFD, $P$ is principal, and so $P = aA$ for some $a \in A$. Choose $N$ so that $a \in A_N$. Suppose that $a = p_1 p_2 \cdots p_m$ is the prime factorization of $a$ in $A_N$. Then $p_1, p_2, \ldots, p_m$ are all prime elements in $A$, and so $a = p_1 p_2 \ldots p_m$ is also the prime factorization of $a$ in $A$. As $a$ is prime in $A$, we have that $m = 1$ and $a = p_1$. It follows that $a$ is prime in $A_N$. Thus $A_{N+1}$, and hence $A$, contains a generating set for prime ideals $Q_1$ and $Q_2$ of $T$ where $Q_1$ contains a minimal prime ideal $P_1$ of $(a, x)$, $Q_2$ contains a minimal prime ideal $P_2$ of $(a, y, z)$, $Q_1$ and $Q_2$ contain only one minimal prime ideal of $T$, $\dim(T/Q_1) = 1$, and $\dim(T/Q_2) = 1$. Note that $P_1$ and $P_2$ are both height one prime ideals of $T$. Thus, $\text{ht}(P_1 \cap A) = 1$ and $\text{ht}(P_2 \cap A) = 1$. Therefore, $P_1 \cap A = aA$ and $P_2 \cap A = aA$. The rest of the argument is similar to the argument in the proof of Theorem 3.2, and so we omit some of the detailed explanations. The completion of $A/(Q_1 \cap A)$ is $T/Q_1$ and it follows that $\dim(A/(Q_1 \cap A)) = 1$. Similarly, $\dim(A/(Q_2 \cap A)) = 1$. Therefore, $aA \subsetneq Q_1 \cap A$ and $aA \subsetneq Q_2 \cap A$. Let $b \in Q_1 \cap A$ with $b \notin aA$ and let $c \in Q_2 \cap A$ with $c \notin aA$. Let $Q_1'$ be a minimal prime ideal of $P_1 + bT$ contained in $Q_1$ and let $Q_2'$ be a minimal prime ideal of $P_2 + cT$ contained in $Q_2$. Then $Q_2' = Q_2$, $(0) \subsetneq aA \subsetneq Q_1' \cap A$ is saturated and $(0) \subsetneq aA \subsetneq Q_2' \cap A = Q_2 \cap A \subsetneq M \cap A$ is saturated. We now have $Q_1' \subsetneq Q_1$ and so $Q_1' \cap A \subsetneq Q_1 \cap A$. Let $d \in Q_1 \cap A$ with $d \notin Q_1'$, and let $I$ be a minimal prime ideal of $Q_1' + dT$ contained in $Q_1$. Then $I = Q_1$ and we have that $(0) \subsetneq aA \subsetneq Q_1' \cap A \subsetneq I \cap A = Q_1 \cap A \subsetneq M \cap A$ is saturated. We have shown that there is a saturated noncatenary poset of prime ideals in the prime spectrum of $A$ that is isomorphic to Figure 2 where the minimal node is $(0)$, the node above it is $P = aA$, and the maximal node is $M \cap A$. It follows that $A/aA = A/P$ is noncatenary. $\square$

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**19 Aug 2024 10:52:10 PDT**
**240304-Loepp Version 3 - Submitted to Rocky Mountain J. Math.**