BOUNDARY BEHAVIOUR OF WEIGHTED BERGMAN KERNELS

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Abstract. For a planar domain $D \subset \mathbb{C}$ and an admissible weight function μ on it, some aspects of the boundary behaviour of the corresponding weighted Bergman kernel $K_{D,\mu}$ are studied. First, under the assumption that μ extends continuously to a smooth boundary point p of D and is non-vanishing there, we obtain a precise relation between $K_{D,\mu}$ and the classical Bergman kernel K_D near p. Second, when viewed as functions of such weights, the weighted Bergman kernel is shown to have a suitable additive and multiplicative property near such boundary points.

1. Introduction

Let $D \subset \mathbb{C}^n$ be a domain and μ a positive measurable function on it. Let $L^2_\mu(D)$ denote the space of all functions on D that are square integrable with respect to μdV, where dV denotes standard Lebesgue measure, and set $\mathcal{O}_\mu(D) = L^2_\mu(D) \cap \mathcal{O}(D)$. The class of weights μ for which $\mathcal{O}_\mu(D) \subset L^2_\mu(D)$ is closed, and for any $z \in D$, the point evaluations $z \mapsto f(z) \in \mathbb{C}$ are bounded on $\mathcal{O}_\mu(D)$ were considered by Pasternak–Winiarski in [11], [12] and termed *admissible* therein. In this situation, there is a reproducing kernel $K_{D,\mu}(z,w)$ which is the weighted Bergman kernel with weight μ. Let $K_{D,\mu}(z) = K_{D,\mu}(z,z)$ for $z \in D$. A sufficient condition for μ to be admissible is that μ^{-a} be locally integrable in D with respect to dV for some $a > 0$ – see [12]. In particular, this holds if $1/\mu \in L^\infty_{\text{loc}}(D)$. As usual, the classical Bergman kernel and its restriction to the diagonal will be denoted by $K_D(z,w)$ and $K_D(z)$ respectively, and this corresponds to the case $\mu \equiv 1$.

The purpose of this note is to quantitatively compare $K_{D,\mu}(z)$ with $K_D(z)$. Let us fix the notations first. For quantities A, B, the notation $A \sim \lambda B$ will mean that A/B approaches λ when suitable limits are taken, while $A \approx B$ will mean that A/B is bounded above and below by positive constants. As is customary, for a domain $D \subset \mathbb{C}^n$, $\delta(z)$ denotes the Euclidean distance of z to the boundary ∂D. Finally, for a domain $D \subset \mathbb{C}^n$ and a fixed base point $p \in \partial D$, let $\mathcal{A}(D,p)$ be the collection of all admissible weights $\mu \in C^0(D) \cap L^\infty(D)$ that extend continuously to p with $\mu(p) > 0$.

Theorem 1.1. *Let $D \subset \mathbb{C}$ be a bounded domain and suppose that $p \in \partial D$ is a C^2-smooth boundary point. Let $\mu \in \mathcal{A}(D,p)$. Then*

$$K_{D,\mu}(z) \sim \frac{1}{\mu(p)} K_D(z) \approx \frac{1}{\mu(p)} \left(\delta(z)\right)^{-2}$$

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as $z \to p$. Further, if D is simply connected, then for integers $\alpha, \beta \geq 0$,

$$\frac{\partial^{\alpha+\beta}}{\partial z^\alpha \partial \overline{z}^\beta} K_{D,\mu}(z) \sim \frac{1}{\mu(p)} \frac{\partial^{\alpha+\beta}}{\partial z^\alpha \partial \overline{z}^\beta} K_D(z) \approx \frac{1}{\mu(p)} \left(\delta(z)\right)^{-(\alpha+\beta+2)}$$

as $z \to p$.

Various estimates have been obtained for weighted Bergman kernels in different settings with varying assumptions on the weights. From this extensive basket of results, we mention [2, 3, 4, 5, 6] as prototypes that are perhaps closest in spirit to Theorem 1.1 in the sense that they deal with estimates for these kernels on a given domain with suitable assumptions on the weights such as log–plurisuperharmonicity ([2, 3]) or admitting a power series representation involving the defining function and its logarithm ([4]). On the other hand, Theorem 1.1 rests only on the continuity of μ at a boundary point and provides precise boundary estimates for the weighted kernel; for simply connected domains, it provides information on all derivatives of the weighted kernel near such a point. The scaling principle is used to reduce the proof to the paradigm of a Ramadanov type theorem for a family of varying domains, that are not necessarily monotone and equipped with varying weights – see [10] for a related Ramadanov theorem of this type. The presence of varying weights is due to the fact that the weighted kernels transform as

$$K_{D_1,\mu}(z,w) = Jf(z) K_{D_2,\mu\circ f^{-1}}(f(z), f(w)) \overline{Jf(w)} \tag{1.1}$$

under a biholomorphism $f : D_1 \to D_2$; here, Jf is the Jacobian determinant of f. Another ingredient in the proof of Theorem 1.1 that we have had to rely on is the Riemann mapping theorem, and it is for this reason that we do not know whether it holds for domains in $\mathbb{C}^n$.

The weights we have considered do not vanish on the boundary. At the other extreme, take $D \subset \mathbb{C}^n$, a smoothly bounded domain and consider the class of weights given by $\mu_d(z) = K_D^{-d}(z)$ for some integer $d \geq 0$. These weights vanish on the boundary ∂D and are admissible since $1/\mu_d = K_D^d$ is locally bounded on D. Denote the corresponding weighted Bergman kernel by $K_{D,d}$. The intrinsic advantage of μ_d as a weight is that $K_{D,d}$ transforms much like the unweighted kernel under biholomorphisms. It can be checked that

$$K_{D_1,d}(z,w) = (Jf(z))^{d+1} K_{D_2,d}(f(z), f(w)) (\overline{Jf(w)})^{d+1}$$

for a biholomorphism $f : D_1 \to D_2$ – this holds for domains in $\mathbb{C}^n$ also. Adapting the techniques from [1], shows that

$$K_{D,d}(z) \sim (2d+1)(K_D(z))^{d+1}$$

as $z \to p \in \partial D$.

Theorem 1.1 has a useful consequence. It clarifies the relation between the Bergman kernel associated to a sum or product of suitable weights and the sum or product of such kernels arising from individual weights.

Corollary 1.2. *Let $D \subset \mathbb{C}$ and $p \in \partial D$ be as in Theorem 1.1. For $1 \leq i \leq n$, let μ_i be a finite collection of weights in $\mathcal{A}(D,p)$. Let $\alpha_i > 0$ be a given n-tuple of positive reals. Then, as $z \to p$,*

(i) $K^n_{D,\mu_1\cdot\mu_2\cdots\mu_n}(z) \sim \left(\mu_1(p)\cdot\mu_2(p)\cdots\mu_n(p)\right)^{n-1} \left(K_{D,\mu_1}\cdot K_{D,\mu_2}\cdots K_{D,\mu_n}\right)(z)$, *and*

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(ii) $K_{D,\alpha_1\mu_1+\alpha_2\mu_2+\ldots+\alpha_n\mu_n}(z) \sim \lambda\left(\alpha_1 K_{D,\mu_1} + \alpha_2 K_{D,\mu_2} + \ldots + \alpha_n K_{D,\mu_n}\right)(z)$, *where*

$$\frac{1}{\lambda} = \sum_{i,j=1}^{n} \alpha_i \alpha_j \frac{\mu_i(p)}{\mu_j(p)}.$$

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2. Weights that do not vanish on the boundary

In this section, we will prove Theorem 1.1 and Corollary 1.2.

Proof of Theorem 1.1. A quick recap of the scaling principle is as follows: let $D \subset \mathbb{C}$ be a domain and $p \in \partial D$ a C^2-smooth boundary point of D. Let ψ be a C^2-smooth defining function for D in a neighbourhood U of p. Choose a sequence p_j in D converging to p. For $j \geq 1$, let

$$T_j(z) = \frac{z - p_j}{-\psi(p_j)}, \qquad z \in \mathbb{C}$$

and set $D_j = T_j(D)$. Note that $0 \in D_j$ for every j as $T_j(p_j) = 0$. Let $K \subset \mathbb{C}$ be a compact set. Since $\psi(p_j) \to 0$ as $j \to \infty$, the $T_j(U)$'s eventually contain every compact set and hence, K in particular. By writing the Taylor series expansion of ψ near $z = p_j$, we see that

$$\psi \circ T_j^{-1}(\zeta) = \psi(p_j + \zeta(-\psi(p_j))) = \psi(p_j) + 2\Re\left(\frac{\partial \psi}{\partial z}(p_j)\zeta\right)(-\psi(p_j)) + (\psi(p_j))^2\, O(1)$$

where $O(1)$ is a term that is uniformly bounded on K as j varies. Therefore, the functions $\psi \circ T_j^{-1}$ are well defined on K for all large j.

Now note that the defining functions of D_j near $T_j(p) \in \partial D_j$ are given by

$$\psi_j(z) = \frac{1}{(-\psi(p_j))}\psi \circ T_j^{-1}(z) = -1 + 2\Re\left(\frac{\partial \psi}{\partial z}(p_j)z\right) + (-\psi(p_j))\, O(1).$$

We now claim that the ψ_j's converge to

$$\psi_\infty(z) = -1 + 2\Re\left(\frac{\partial \psi}{\partial z}(p)z\right)$$

on all compact subsets of $\mathbb{C}$. For simplicity, assume that $\frac{\partial \psi}{\partial z}(p) = 1$ by a suitable normalization. Therefore, $\psi_\infty(z) = -1 + 2\Re z$. Let $\mathcal{H}$ denote the half space defined by ψ_∞, i.e.,

$$\mathcal{H} = \{z : -1 + 2\Re z < 0\}.$$

To see the convergence of ψ_j's to ψ_∞ on compact sets of $\mathbb{C}$, note that ψ_j is a defining function of D_j on $T_j(U)$, and $T_j(U)$ eventually contains $K \cap D_j$ for any compact subset $K \subset \mathbb{C}$. Hence

$$K \cap D_j = \{z \in K : \psi_j(z) < 0\}$$

for sufficiently large j. Since $\psi_j \to \psi_\infty$ uniformly on compacts, $K \cap D_j$ converges in the Hausdorff sense to $\{z \in K : \psi_\infty(z) < 0\} = K \cap \mathcal{H}$. Thus, the domains D_j converge to $\mathcal{H}$ in the Hausdorff sense on every compact subset of $\mathbb{C}$. The same reasoning shows that the domains $T_j(D \cap U)$ also converge to the same limiting half-space $\mathcal{H}$ in the Hausdorff sense on compact sets. In particular, every compact subset of $\mathcal{H}$ is eventually contained in D_j

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(as well as $T_j(D \cap U)$), and every compact subset that is disjoint from $\overline{\mathcal{H}}$ is eventually disjoint from $\overline{D}_j$ (as well as $\overline{T_j(D \cap U)}$).

Moving ahead, we want to study the behavior of $K_D(p_j, p_j)$ as $j \to \infty$. The transformation formula for the weighted Bergman kernels under biholomorphisms $T_j : D \to D_j$ gives

$$K_{D,\mu}(z, w) = T_j{}'(z)\, K_{D_j, \mu \circ T_j^{-1}}(T_j(z), T_j(w))\, \overline{T_j{}'(w)}, \qquad z, w \in D. \tag{2.1}$$

Let $\mu_j = \mu \circ T_j^{-1}$. Since $T_j' \equiv -1/\psi(p_j)$, we have

$$K_{D,\mu}(z, w) = \frac{1}{(\psi(p_j))^2}\, K_{D_j, \mu_j}(T_j(z), T_j(w)), \qquad z, w \in D. \tag{2.2}$$

Since $T_j(p_j) = 0$ for all $j \geq 1$, we obtain

$$K_{D,\mu}(p_j, p_j) = \frac{1}{(\psi(p_j))^2}\, K_{D_j, \mu_j}(0, 0).$$

Therefore, it is enough to study the behavior of $K_{D_j, \mu_j}(0, 0)$ as $j \to \infty$.

Case 1. Since D is simply connected and T_j's are affine maps, the domains D_j's are also simply connected. Let $F_j : D_j \to \mathcal{H}$ be the Riemann maps such that $F_j(0) = 0$ and $F_j{}'(0) > 0$. Let $\nu_j = \mu_j \circ F_j^{-1} = \mu \circ T_j^{-1} \circ F_j^{-1}$. The transformation formula for the weighted Bergman kernels under F_j's gives

$$K_{D_j, \mu_j}(z, w) = F_j'(z)\, K_{\mathcal{H}, \mu_j \circ F_j^{-1}}(F_j(z), F_j(w))\, \overline{F_j'(w)}, \qquad z, w \in D_j. \tag{2.3}$$

Let $\nu_j = \mu_j \circ F_j^{-1} = \mu \circ T_j^{-1} \circ F_j^{-1}$. Note that $\nu_j(z) = \mu\left(p_j - \psi(p_j) F_j^{-1}(z)\right)$. By Proposition 2.1 below, $F_j^{-1}(K)$ is uniformly bounded on compact sets $K \subset \mathbb{C}$. Since μ extends continuously to p, it follows that

$$\nu_j \to \nu_\infty \equiv \mu(p) > 0$$

locally uniformly on $\mathcal{H}$. The problem is therefore reduced to studying the Riemann maps F_j and the kernels $K_{\mathcal{H}, \nu_j}$. Note that the domain is now fixed (i.e., $\mathcal{H}$) and only the weights vary.

Proposition 2.1. *The Riemann maps F_j converge to the identity map $i_{\mathcal{H}}$ locally uniformly on $\mathcal{H}$.*

Proof. Since every compact $K \subset \mathbb{C} \setminus \overline{\mathcal{H}}$ is eventually contained in $\mathbb{C} \setminus \overline{D_j}$, both families $\{F_j\}$ and $\{F_j^{-1}\}$ omit at least two values in $\mathbb{C}$ and fix the origin. Therefore, $\{F_j\}$ and $\{F_j^{-1}\}$ are normal families.

Pick a subsequence $\{j_k\}$ such that both $\{F_{j_k}\}$ and $\{F_{j_k}^{-1}\}$ converge locally uniformly on $\mathcal{H}$ to some holomorphic functions F, $G : \mathcal{H} \to \overline{\mathcal{H}}$ respectively and $F(0) = G(0) = 0$. Since $\mathcal{H}$ is biholomorphic to the unit disc $\mathbb{D}$, we may assume that $F, G : \mathbb{D} \to \overline{\mathbb{D}}$ and there is an interior point $z_0 \in \mathbb{D}$ that is fixed by both F, G. Now neither F nor G can map an interior point of $\mathbb{D}$ to a point on $\partial\mathbb{D}$, for in this case, either would reduce to a constant function by a standard argument using peak functions that exist at each point on $\partial\mathbb{D}$. This would contradict the fact that F, G both fix $z_0 \in \mathbb{D}$. Going back to $\mathcal{H}$, this means that $F, G : \mathcal{H} \to \mathcal{H}$.

Now, $F_{j_k} \circ F_{j_k}^{-1} \equiv id_{\mathcal{H}}$ and $F_{j_k}^{-1} \circ F_{j_k} \equiv id_{D_j}$. Since $F_{j_k} \circ F_{j_k}^{-1}$ converges to $F \circ G$ locally uniformly on $\mathcal{H}$, we have $F \circ G \equiv id_{\mathcal{H}}$. Since any compact $K \subset \mathcal{H}$ is eventually contained in D_j, the sequence $F_{j_k}^{-1} \circ F_{j_k}$ converges to $G \circ F$ uniformly on K. So, $(G \circ F)|_K \equiv id_K$ and therefore $G \circ F \equiv id_{\mathcal{H}}$. Thus, F is an automorphism of $\mathcal{H}$ such that $F(0) = 0$ and $F'(0) > 0$.

Let $\varphi : \mathcal{H} \to \mathbb{D}$ be the Riemann map such that $\varphi(0) = 0$ and $\varphi'(0) > 0$. Then, $\phi = \varphi \circ F \circ \varphi^{-1}$ is an automorphism of $\mathbb{D}$ such that $\phi(0) = 0$ and $\phi'(0) > 0$. So, $\phi \equiv id_{\mathbb{D}}$ and therefore $F \equiv Id_{\mathcal{H}}$.

Hence, the Riemann maps F_j converge to the identity map $id_{\mathcal{H}}$ locally uniformly on $\mathcal{H}$. □

The following characterization of weighted Bergman kernels from [10] will be used.

Lemma 2.2. *Let Ω be a domain in $\mathbb{C}^n$ and ν be an admissible weight on Ω. For $w \in \Omega$, let $S_{\nu,w}(\Omega) \subset \mathcal{O}_\nu(\Omega)$ denote the set of all functions f such that $f(w) \geq 0$ and $\|f\|_{L^2_\nu(\Omega)} \leq \sqrt{f(w)}$. Then the weighted Bergman function $K_{\Omega,\nu}(\cdot, w)$ is uniquely characterized by the properties:*

(i) *$K_{\Omega,\nu}(\cdot, w) \in S_{\nu,w}(\Omega)$;*
(ii) *if $f \in S_{\nu,w}(\Omega)$ and $f(w) \geq K_{\Omega,\nu}(w, w)$, then $f(\cdot) \equiv K_{\Omega,\nu}(\cdot, w)$.*

Proposition 2.3. *We have that*

$$\lim_{j\to\infty} K_{\mathcal{H},\nu_j} = K_{\mathcal{H},\nu_\infty}$$

locally uniformly on $\mathcal{H} \times \mathcal{H}$.

Proof. Let $W \subset \mathcal{H}$ be a compact. Choose a neighborhood $\widetilde{W} \subset\subset \mathcal{H}$ of W. Since $\{F_j^{-1}\}$ is a normal family on $\mathcal{H}$, there exists a compact set $W_0 \subset \mathbb{C}$ such that $F_j^{-1}(\widetilde{W}) \subset W_0$ for all $j \geq 1$. Choose a neighborhood $V \subset \mathbb{C}$ of p such that $\mu \geq c$ a.e. on $D \cap V$ for some constant $c > 0$. Since the domains $T_j(V)$ eventually contain every compact subset, $W_0 \subset T_j(V)$ for all large j. Thus, $\nu_j \geq c$ a.e. on $\widetilde{W}$ for all large j.

For a domain $\Omega \subset \mathbb{C}$, an admissible weight ρ on Ω and $z \in \Omega$, it is straightforward to check that (proof is similar to the classical case; see [7])

$$K_{\Omega,\rho}(z,z)^{-1/2} = \min\{\|g\|_{L^2_\rho(\Omega)} : g \in \mathcal{O}_\rho(\Omega),\, g(z) = 1\}, \tag{2.4}$$

where we assume that $K_{\Omega,\rho}(z, z) > 0$. Therefore, if $\tilde{\Omega} \subset \Omega$ is a domain with an admissible weight $\tilde{\rho}$ on $\tilde{\Omega}$ such that $\tilde{\rho} \leq \rho$ a.e. on $\tilde{\Omega}$, then $K_{\tilde{\Omega},\tilde{\rho}}(z, z) \geq K_{\Omega,\rho}(z, z)$ for every $z \in \tilde{\Omega}$. This is the monotonicity property. It holds even when $K_{\Omega,\rho}(z, z) = 0$ because $K_{\tilde{\Omega},\tilde{\rho}}(z, z)$ is always non-negative.

Since D is bounded and $\mu \in L^\infty(D)$, the constant function 1 belongs to $\mathcal{O}_\mu(D)$. This implies that $K_{D,\mu}(z, z) > 0$ for every $z \in D$. Now (2.1) and (2.3) show that $K_{\mathcal{H},\nu_j}(z, z) > 0$ for all $z \in \mathcal{H}$ and $j \geq 1$.

Now for $z, w \in W$ and $j \geq j_0$, where j_0 is sufficiently large,

$$\begin{aligned}
|K_{\mathcal{H},\nu_j}(z,w)| &= |\langle K_{\mathcal{H},\nu_j}(\cdot,w), K_{\mathcal{H},\nu_j}(\cdot,z)\rangle_{L^2_{\nu_j}(\mathcal{H})}| \leq \|K_{\mathcal{H},\nu_j}(\cdot,w)\|_{L^2_{\nu_j}(\mathcal{H})}\|K_{\mathcal{H},\nu_j}(\cdot,z)\|_{L^2_{\nu_j}(\mathcal{H})} \\
&= \sqrt{K_{\mathcal{H},\nu_j}(w,w)}\sqrt{K_{\mathcal{H},\nu_j}(z,z)} \leq \sqrt{K_{\widetilde{W},c}(w,w)}\sqrt{K_{\widetilde{W},c}(z,z)} \\
&= \frac{1}{c}\sqrt{K_{\widetilde{W}}(w,w)}\sqrt{K_{\widetilde{W}}(z,z)} \leq \frac{1}{c}\sup_{\xi\in W} K_{\widetilde{W}}(\xi,\xi) =: C < \infty,
\end{aligned}$$

by the monotonicity property. Hence, $\{K_{\mathcal{H},\nu_j}\}$ is a normal family. It suffices to show that every subsequence converges to the kernel $K_{\mathcal{H},\nu_\infty}$. By taking a subsequence and retaining notion, assume that

$$\lim_{j\to\infty} K_{\mathcal{H},\nu_j} = k$$

locally uniformly on $\mathcal{H} \times \mathcal{H}$ for some function k on $\mathcal{H} \times \mathcal{H}$ which is holomorphic in the first variable and anti-holomorphic in the second.

Fix $w \in \mathcal{H}$. For a compact set $W \subset \mathcal{H}$,

$$\begin{aligned}
\int_W |k(z,w)|^2 \nu_\infty(z)dz &\leq \liminf_{j\to\infty} \int_W |K_{\mathcal{H},\nu_j}(z,w)|^2 \nu_j(z)dz \\
&\leq \liminf_{j\to\infty} \int_{\mathcal{H}} |K_{\mathcal{H},\nu_j}(z,w)|^2 \nu_j(z)dz = \liminf_{j\to\infty} K_{\mathcal{H},\nu_j}(w,w) \\
&= k(w,w)
\end{aligned}$$

and therefore

$$\int_{\mathcal{H}} |k(z,w)|^2 \nu_\infty(z)dz \leq k(w,w).$$

Therefore, $k(\cdot,w) \in \mathcal{O}_{\nu_\infty}(\mathcal{H})$. Also, $k(w,w) \geq 0$ and $\|k(\cdot,w)\|_{L^2_{\nu_\infty}(\mathcal{H})} \leq \sqrt{k(w,w)}$. Thus, the function $k(\cdot,w) \in S_{\nu_\infty,w}(\mathcal{H})$. Since $\mu \in L^\infty(D)$, there exists a constant $M > 0$ such that $\nu_j \leq M$ for all j and hence

$$\int_{\mathcal{H}} \left|\frac{K_{\mathcal{H},\nu_\infty}(z,w)}{K_{\mathcal{H},\nu_\infty}(w,w)}\right|^2 \nu_j(z)dz \leq \frac{M}{\mu(p)} \int_{\mathcal{H}} \left|\frac{K_{\mathcal{H},\nu_\infty}(z,w)}{K_{\mathcal{H},\nu_\infty}(w,w)}\right|^2 \nu_\infty(z)dz < \infty.$$

for all j. This shows that $K_{\mathcal{H},\nu_\infty}(\cdot,w)/K_{\mathcal{H},\nu_\infty}(w,w) \in \mathcal{O}_{\nu_j}(\mathcal{H})$ for every j. Furthermore, note that $K_{\mathcal{H},\nu_j}(\cdot,w)/K_{\mathcal{H},\nu_j}(w,w)$ is the unique solution of the extremal problem

$$\min\{\|f\|_{L^2_{\nu_j}(\mathcal{H})} : f \in \mathcal{O}_{\nu_j}(\mathcal{H}),\ f(w) = 1\},$$

which follows from (2.4). So,

$$\frac{1}{K_{\mathcal{H},\nu_j}(w,w)} = \int_{\mathcal{H}} \left|\frac{K_{\mathcal{H},\nu_j}(z,w)}{K_{\mathcal{H},\nu_j}(w,w)}\right|^2 \nu_j(z)\,dz \leq \int_{\mathcal{H}} \left|\frac{K_{\mathcal{H},\nu_\infty}(z,w)}{K_{\mathcal{H},\nu_\infty}(w,w)}\right|^2 \nu_j(z)\,dz.$$

Taking limits as $j \to \infty$ on both sides and using the dominated convergence theorem, we see that

$$\frac{1}{k(w,w)} \leq \int_{\mathcal{H}} \left|\frac{K_{\mathcal{H},\nu_\infty}(z,w)}{K_{\mathcal{H},\nu_\infty}(w,w)}\right|^2 \nu_\infty(z)\,dz = \frac{1}{K_{\mathcal{H},\nu_\infty}(w,w)},$$

i.e., $k(w,w) \geq K_{\mathcal{H},\nu_\infty}(w,w)$. Therefore, $k(\cdot,w) \equiv K_{\mathcal{H},\nu_\infty}(\cdot,w)$ by Lemma 2.2. Since $w \in \mathcal{H}$ was arbitrary, $k = K_{\mathcal{H},\nu_\infty}$. □

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Hence, (2.3), Proposition 2.1 and Proposition 2.3 show that

$$\lim_{j\to\infty} K_{D_j,\mu_j} = K_{\mathcal{H},\nu_\infty}$$

locally uniformly on $\mathcal{H}\times\mathcal{H}$. The same holds for all the derivatives of the weighted Bergman kernel, where derivative estimates follow from the Cauchy integral formula.

For integers α, $\beta \geq 0$, let us denote $\frac{\partial^{\alpha+\beta}}{\partial\xi^\alpha\partial\bar{\zeta}^\beta}K_{\Omega,\rho}(\xi,\zeta)$ by $K^{\alpha,\beta}_{\Omega,\rho}$. Upon differentiating (2.2), we obtain

$$K^{\alpha,\beta}_{D,\mu}(z,w) = \frac{1}{(-\psi(p_j))^{\alpha+\beta+2}}\, K^{\alpha,\beta}_{D_j,\mu_j}(T_j(z),T_j(w)), \qquad z,w\in D.$$

Thus,

$$K^{\alpha,\beta}_{D,\mu}(p_j,p_j) = \frac{1}{(-\psi(p_j))^{\alpha+\beta+2}}\, K^{\alpha,\beta}_{D_j,\mu_j}(0,0) \sim \frac{1}{(-\psi(p_j))^{\alpha+\beta+2}} K^{\alpha,\beta}_{\mathcal{H},\nu_\infty}(0,0) \tag{2.5}$$

as $j\to\infty$. Note that for $\mu\equiv 1$, we have $\mu_j\equiv 1$. Therefore,

$$\lim_{j\to\infty}\frac{K^{\alpha,\beta}_{D,\mu}(p_j,p_j)}{K^{\alpha,\beta}_{D}(p_j,p_j)} = \lim_{j\to\infty}\frac{K^{\alpha,\beta}_{D_j,\mu_j}(0,0)}{K^{\alpha,\beta}_{D_j}(0,0)} = \frac{K^{\alpha,\beta}_{\mathcal{H},\nu_\infty}(0,0)}{K^{\alpha,\beta}_{\mathcal{H}}(0,0)} = \frac{1}{\mu(p)}\frac{K^{\alpha,\beta}_{\mathcal{H}}(0,0)}{K^{\alpha,\beta}_{\mathcal{H}}(0,0)} = \frac{1}{\mu(p)}.$$

We shall now compute $K^{\alpha,\beta}_{\mathcal{H}}$. Applying the transformation formula for the Bergman kernels under the biholomorphism $f:\mathcal{H}\longrightarrow\mathbb{D}$ defined by

$$f(z) = \frac{2z+1}{-2z+3}$$

gives that for $z,w\in\mathcal{H}$ (see [7])

$$\begin{aligned} K_{\mathcal{H}}(z,w) &= f'(z)\,K_{\mathbb{D}}(f(z),f(w))\,\overline{f'(w)} \\ &= \frac{8}{(-2z+3)^2}\frac{(-2z+3)^2(-2\bar{w}+3)^2}{64\pi(1-z-\bar{w})^2}\frac{8}{(-2\bar{w}+3)^2} = \frac{1}{\pi(1-z-\bar{w})^2}. \end{aligned}$$

Therefore, for integers $\alpha,\beta\geq 0$

$$\frac{\partial^{\alpha+\beta}}{\partial z^\alpha\partial\bar{w}^\beta}K_{\mathcal{H}}(z,w) = \frac{\alpha!\beta!}{\pi(1-z-\bar{w})^{\alpha+\beta+2}}, \qquad z,w\in\mathcal{H}.$$

So, $K^{\alpha,\beta}_{\mathcal{H}}(0,0) = \alpha!\beta!/\pi$. Since $|\psi(z)|\sim 2\delta(z)$, it follows that

$$K^{\alpha,\beta}_{D,\mu}(p_j) = \frac{1}{|\psi(p_j)|^{\alpha+\beta+2}}K^{\alpha,\beta}_{D_j,\mu_j}(0) \approx \frac{1}{\delta(p_j)^{\alpha+\beta+2}}K^{\alpha,\beta}_{D_j,\mu_j}(0),$$

and $K^{\alpha,\beta}_{D_j,\mu_j}(0)$ converges to $K^{\alpha,\beta}_{\mathcal{H},\nu_\infty}(0) = (1/\mu(p))K^{\alpha,\beta}_{\mathcal{H}}(0)$ which is a finite constant. Therefore, we have proved

$$\frac{\partial^{\alpha+\beta}}{\partial z^\alpha\partial\bar{z}^\beta}K_{D,\mu}(z) \sim \frac{1}{\mu(p)}\frac{\partial^{\alpha+\beta}}{\partial z^\alpha\partial\bar{z}^\beta}K_D(z) \approx \frac{1}{\mu(p)}(\delta(z))^{-(\alpha+\beta+2)} \tag{2.6}$$

as $z\to p$. This completes the simply connected case.

□

Case 2. When D is not simply connected, $K_{D,\mu}(z)$ can be localized near p as in Proposition 2.4 below. This is possible because p, being a C^2-smooth boundary point, is a local holomorphic peak point (see [7] for details).

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Proposition 2.4. *[Localization] Let $D \subset \mathbb{C}$ be a bounded domain and suppose that $p \in \partial D$ is a local holomorphic peak point. Let $\mu \in \mathcal{A}(D, p)$. Then there exists a neighbourhood U of p such that*

$$\lim_{\zeta \to p} \frac{K_{D,\mu}(\zeta, \zeta)}{K_{D\cap U,\mu}(\zeta, \zeta)} = 1$$

where $K_{D\cap U,\mu}$ denotes the Bergman kernel for $D \cap U$ with respect to the weight $\mu|_{D\cap U}$.

Proof. The standard proof using L^2-methods works verbatim for the weighted case because of the lower and upper bounds on the weight μ as $\mu \in \mathcal{A}(D, p)$. The assumptions on μ imply that $K_{D\cap U,\mu}$ is well-defined and positive. Further, since $L^2(D) \subset L^2_\mu(D)$, several auxiliary functions that appear in the standard proof and belong to pertinent L^2-spaces are actually in $L^2_\mu(D)$. The details are omitted. See [7] for details. □

Choose a neighborhood U of p in $\mathbb{C}$ as in Proposition 2.4 such that $D \cap U$ is simply connected. Note that $\mu|_{D\cap U} \in \mathcal{A}(D \cap U, p)$. For a sequence p_j in $D \cap U$ converging to the boundary point p, it therefore follows from Case 1 that

$$\lim_{j\to\infty} \frac{K_{D\cap U,\mu}(p_j, p_j)}{K_{D\cap U}(p_j, p_j)} = \frac{1}{\mu(p)}.$$

Hence, by Proposition 2.4 we have

$$\begin{aligned} \lim_{j\to\infty} \frac{K_{D,\mu}(p_j, p_j)}{K_D(p_j, p_j)} &= \lim_{j\to\infty} \left(\frac{K_{D,\mu}(p_j, p_j)}{K_{D\cap U,\mu}(p_j, p_j)} \cdot \frac{K_{D\cap U,\mu}(p_j, p_j)}{K_{D\cap U}(p_j, p_j)} \cdot \frac{K_{D\cap U}(p_j, p_j)}{K_D(p_j, p_j)} \right) \\ &= 1 \cdot \frac{1}{\mu(p)} \cdot 1 = \frac{1}{\mu(p)}. \end{aligned}$$

Similarly,

$$\begin{aligned} K_{D,\mu}(p_j, p_j) &= \frac{K_{D,\mu}(p_j, p_j)}{K_{D\cap U,\mu}(p_j, p_j)} K_{D\cap U,\mu}(p_j, p_j) \\ &\approx 1 \cdot \frac{1}{\mu(p)} (\delta(p_j))^{-2}. \end{aligned}$$

Hence,

$$K_{D,\mu}(z) \sim \frac{1}{\mu(p)} K_D(z) \approx \frac{1}{\mu(p)} (\delta(z))^{-2}, \tag{2.7}$$

as $z \to p$. □

This completes the proof of Theorem 1.1. □

Remark 2.5. From the proof of Theorem 1.1, observe that the boundedness assumption on D is used only in the localization. A different technique for localization from [9] can be adapted here to localize the weighted Bergman kernels for unbounded domains. The details are omitted.

Proof of Corollary 1.2. Since $\mu_1 \in C^0(D) \cap L^\infty(D)$, we have $1/\mu_1 \in L^\infty_{loc}(D)$. So,

$$\int_K \frac{1}{\sum_{i=1}^n \alpha_i \mu_i(z)} \, dz \le \int_K \frac{1}{\alpha_1 \mu_1(z)} \, dz < \infty$$

for any compact $K \subset D$. Thus, $\mu_S = \sum_{i=1}^n \alpha_i \mu_i$ is an admissible weight on D. Also, $\mu_S \in \mathcal{A}(D,p)$ as $\mu_i \in \mathcal{A}(D,p)$ and $\alpha_i > 0$ for all $i \leq n$. Therefore,

$$K_{D,\mu_S}(z) \sim \frac{1}{\mu_S(p)} K_D(z) \quad \text{as } z \to p.$$

Since $K_{D,\mu_i}(z) \sim \frac{1}{\mu_i(p)} K_D(z)$ as $z \to p$ for every $i \leq n$,

$$\lim_{z\to p} \frac{K_{\alpha_1\mu_1+\ldots+\alpha_n\mu_n}(z)}{(\alpha_1 K_{\mu_1} + \ldots + \alpha_n K_{\mu_n})(z)} = \frac{1}{\sum_{i=1}^n \alpha_i \mu_i(p)} \frac{1}{\sum_{j=1}^n \frac{\alpha_j}{\mu_j(p)}} = \frac{1}{\sum_{i,j=1}^n \alpha_i \alpha_j \frac{\mu_i(p)}{\mu_j(p)}}$$

which proves (ii). Now let $\mu_M = \mu_1 \ldots \mu_n$ and note that $\mu_M \in \mathcal{A}(D,p)$. Hence

$$K_{D,\mu_M}(z) \sim \frac{1}{\mu_M(p)} K_D(z) \quad \text{as } z \to p.$$

Since $K_{D,\mu_i}(z) \sim \frac{1}{\mu_i(p)} K_D(z)$ as $z \to p$ for every $i \leq n$,

$$\lim_{z\to p} \frac{K_{\mu_1\ldots\mu_n}(z)}{\sqrt[n]{(K_{\mu_1}\ldots K_{\mu_n})(z)}} = \frac{\sqrt[n]{\mu_1(p)}\ldots\sqrt[n]{\mu_n(p)}}{(\mu_1\ldots\mu_n)(p)} = \frac{1}{(\mu_1(p)\ldots\mu_n(p))^{\frac{n-1}{n}}}$$

which proves (i). ☐

3. Concluding Remarks and Questions

Theorem 1.1 shows that the weighted kernel $K_{D,\mu}$ and the classical one K_D have the same rate of blow up near a smooth boundary point. However, D is assumed to be simply connected in order to get estimates on the boundary behaviour of the derivatives of $K_{D,\mu}$. We believe that the simply connected assumption can be dispensed with but we have not been able to show this. Further, it is also not clear whether these estimates hold for domains in $\mathbb{C}^n$ even with additional pseudoconvexity assumptions. In the same vein, the assumption that $\mu(p) > 0$ is crucially used. This raises the question of what happens if $\mu(p) = 0$. The weights of the form $\mu_d = K_{\mathbb{D}}^{-d}$, $d \geq 0$, all vanish on the boundary. For this specific class of weights, the boundary behaviour is known as indicated. But can something be said for the weighted kernels arising from general weights that vanish on the boundary?

The connection between the weighted Bergman kernel and Green's function has been recently explored in [8]. It would be interesting to see how the weighted Green's function fits within the paradigm studied here.

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