Infinite series identities on r-Stirling numbers

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Abstract

In this paper, we study infinite series identities on the r-Stirling numbers of the first kind. In particular, we use special integrals to establish some general series identities, which reduce to various series on the Stirling numbers of the first kind, harmonic numbers and hyperharmonic numbers by specifying the parameters. We also present the relation between the r-Stirling numbers of the first kind and the multiple hyperharmonic numbers, and show that some r-Stirling series and related Euler-type sums are reducible to zeta values. It can be found that many known results in the literature, such as those due to Boyadzhiev, Dil and Boyadzhiev and Hoffman, are special cases of ours.

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Contents

1. Introduction

Infinite series involving harmonic numbers $H_n^{(r)}$ and the Stirling numbers of the first kind $\left[{n \atop k}\right]$ are closely related to multiple zeta values, and essential not only to the analysis of algorithms and data structures [16,17], but also to the higher order calculations in quantum electrodynamics (QED) and quantum chromodynamics (QCD) [3,4,6,23].

Here, the *multiple zeta values* (MZVs) [20,40] are defined by

$$\zeta(s_1,s_2,\ldots,s_k)=\sum_{n_1>n_2>\cdots>n_k\geq 1}\frac{1}{n_1^{s_1}n_2^{s_2}\cdots n_k^{s_k}}\,,$$

for $(s_1,s_2,\ldots,s_k)\in\mathbb{N}^k$ with $s_1>1$. The quantity k in the above definition is named the *depth* of a MZV, and the quantity $w=s_1+\cdots+s_k$ is known as the *weight*. Usually, when writing

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MZVs, we denote n repetitions of a substring by $\{\cdots\}_n$. The truncated MZVs (or partial sums of MZVs)

$$\zeta_n(s_1, s_2, \ldots, s_k) = \sum_{n\geq n_1>n_2>\cdots>n_k\geq 1} \frac{1}{n_1^{s_1} n_2^{s_2} \cdots n_k^{s_k}},$$

for $(s_1, s_2, \ldots, s_k) \in \mathbb{N}^k$, are known as *multiple harmonic sums*. Similarly, define the *multiple harmonic star sums* by

$$\zeta_n^*(s_1, s_2, \ldots, s_k) = \sum_{n\geq n_1\geq n_2\geq\cdots\geq n_k\geq 1} \frac{1}{n_1^{s_1} n_2^{s_2} \cdots n_k^{s_k}}.$$

By convention, $\zeta_n(s_1, s_2, \ldots, s_k) = \zeta_n^*(s_1, s_2, \ldots, s_k) = 0$ for $n < k$, and $\zeta_n(\emptyset) = \zeta_n^*(\emptyset) = 1$.

Between the classical Stirling numbers of the first kind and the multiple harmonic sums $\zeta_n(\{1\}_k)$, there holds

$$\begin{bmatrix} n \\ k \end{bmatrix} = (n-1)!\zeta_{n-1}(\{1\}_{k-1}), \tag{1.1}$$

which can be found explicitly in, e.g., [25, Section 1] and [35, Theorem 2.5], and leads us to

$$\sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} \frac{1}{n!n^m} = \sum_{n=1}^{\infty} \frac{\zeta_{n-1}(\{1\}_{k-1})}{n^{m+1}} = \zeta(m+1, \{1\}_{k-1}). \tag{1.2}$$

As remarked by Adamchik [1, Eq. (30)], the above series identity was known to Euler. Recently, by generating functions, special integrals and (alternating) multiple zeta values, numerous similar series identities on the Stirling numbers of the first kind have been established. For details, the readers may consult the works due to, Kuba and Prodinger [25], Wang and Chen [32], Wang and Lyu [33], Xu [35], and Xu et al. [37]. In particular, some of these identities are further used to find the evaluations of the classical Euler sums as well as some Euler-type sums.

On the other hand, in 2017, using symmetric functions and MZVs, Hoffman [19] proved some identities conjectured by Choi [9], and established some general series identities. For example, he obtained

$$\sum_{n=1}^{\infty} \frac{\mathcal{P}_k(H_n, H_n^{(2)}, \ldots, H_n^{(k)})}{n^2} = \frac{k+3}{2}\zeta(k+2) - \frac{1}{2}\sum_{j=2}^{k} \zeta(j)\zeta(k+2-j), \tag{1.3}$$

$$\sum_{n=1}^{\infty} \frac{\mathcal{Q}_k(H_n, H_n^{(2)}, \ldots, H_n^{(k)})}{n^2} = (k+1)\zeta(k+2), \tag{1.4}$$

$$\sum_{n=0}^{\infty} \frac{\mathcal{P}_k(H_n, H_n^{(2)}, \ldots, H_n^{(k)})\mathcal{Q}_l(H_{n+1}, H_{n+1}^{(2)}, \ldots, H_{n+1}^{(l)})}{(n+1)^2} = \binom{k+l+1}{k+1}\zeta(k+l+2), \tag{1.5}$$

$$\sum_{n=0}^{\infty} \frac{\mathcal{P}_k(H_n, H_n^{(2)}, \ldots, H_n^{(k)})}{(n+1)(n+2)\cdots(n+q)} = \frac{1}{(q-1)!(q-1)^{k+1}}, \tag{1.6}$$

for integers $k, l \geq 0$ and $q \geq 2$, and some other similar ones. According to the definitions of the multivariate polynomials $\mathcal{P}_k$, $\mathcal{Q}_k$ and the complete Bell polynomials Y_k (see [11, Sections 3.3 and 5.7], [19, Proposition 1] and [24, Lemmas 3.1–3.3]), it can be verified that

$$\mathcal{P}_k(x_1, x_2, \ldots, x_k) = \frac{(-1)^k}{k!}Y_k(-0!x_1, -1!x_2, \ldots, -(k-1)!x_k),$$

$$\mathcal{Q}_k(x_1, x_2, \ldots, x_k) = \frac{1}{k!}Y_k(0!x_1, 1!x_2, \ldots, (k-1)!x_k),$$

and

$$\mathcal{P}_k(H_n, H_n^{(2)}, \dots, H_n^{(k)}) = \zeta_n(\{1\}_k) = \frac{1}{n!}\begin{bmatrix} n+1 \\ k+1 \end{bmatrix}, \tag{1.7}$$

$$\mathcal{Q}_k(H_n, H_n^{(2)}, \dots, H_n^{(k)}) = \zeta_n^\star(\{1\}_k), \tag{1.8}$$

which coincide with the results due to Flajolet and Sedgewick [16, Example 1 and Corollary 3]. Therefore, Hoffman's results in [19] can be viewed as Stirling series identities and their variants. In 2019, Kuba and Panholzer [24] presented several similar series identities on $\zeta_n(\{1\}_k)$ (i.e., the Stirling numbers of the first kind) and $\zeta_n^\star(\{1\}_k)$, and further generalized the results in [19, 25].

The r-Stirling numbers were introduced firstly by Broder [7] in 1984, and rediscovered later by Merris [27] in 2000. In particular, the *r-Stirling number of the first kind* $\begin{bmatrix} n \\ k \end{bmatrix}_r$ counts restricted permutations of the set $\{1, 2, \dots, n+r\}$ having $k+r$ cycles such that $1, 2, \dots, r$ are in distinct cycles. The corresponding generating function is

$$\sum_{n=k}^\infty \begin{bmatrix} n \\ k \end{bmatrix}_r \frac{t^n}{n!} = \frac{1}{k!}\frac{(-\ln(1-t))^k}{(1-t)^r}. \tag{1.9}$$

By the definition, when $r = 0, 1$, the r-Stirling numbers reduce to the classical Stirling numbers: $\begin{bmatrix} n \\ k \end{bmatrix}_0 = \begin{bmatrix} n \\ k \end{bmatrix}$ and $\begin{bmatrix} n \\ k \end{bmatrix}_1 = \begin{bmatrix} n+1 \\ k+1 \end{bmatrix}$. Additionally, $\begin{bmatrix} n \\ 0 \end{bmatrix}_r = \langle r \rangle_n$, and $\begin{bmatrix} n \\ 1 \end{bmatrix}_r = n! h_n^{(r)}$, where $\langle x \rangle_n$ are the rising factorials defined by $\langle x \rangle_0 = 1$ and $\langle x \rangle_n = x(x+1)\cdots(x+n-1)$ for $n \geq 1$, and $h_n^{(r)}$ are the *hyperharmonic numbers* defined by

$$h_n^{(0)} = \frac{1}{n} \quad \text{and} \quad h_n^{(r)} = \sum_{k=1}^n h_k^{(r-1)}, \quad \text{for } n, r \geq 1.$$

The hyperharmonic numbers have the generating function

$$\sum_{n=1}^\infty h_n^{(r)} t^n = -\frac{\ln(1-t)}{(1-t)^r}, \tag{1.10}$$

and satisfy

$$h_n^{(r)} = \sum_{k=1}^n \binom{n-k+r-1}{r-1}\frac{1}{k} = \binom{n+r-1}{r-1}(H_{n+r-1} - H_{r-1}) \tag{1.11}$$

(see Benjamin et al. [2] and Conway and Guy [12, p. 258]). When $r = 1$, they reduce to the classical harmonic numbers: $h_n^{(1)} = H_n$.

Mező and Dil [28], and Dil and Boyadzhiev [13] studied the *linear Euler sums of hyperharmonic numbers*, which are of the form

$$\sigma_{r,q} = \sum_{n=1}^\infty \frac{h_n^{(r)}}{n^q}. \tag{1.12}$$

In particular, Mező and Dil presented the evaluations of some series with $r = 2, 3, 4$ [28, Section 7], and Dil and Boyadzhiev [13] established an expression of linear Euler sums of hyperharmonic numbers by zeta values, classical linear Euler sums and the Stirling numbers of the first kind. Some further results on Euler-type sums of hyperharmonic numbers and their variants can be found in Dil et al. [14], Kargın et al. [22], Xu [34], and Xu et al. [38].

Inspired by the above results, in this paper, we study infinite series identities on the r-Stirling numbers of the first kind. This paper is organized as follows.

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In Section 2, we establish r-Stirling series identities via special integrals, including two general ones related to the integral $\int_0^1 t^n(1-t)^r \ln^m(1-t)\,\mathrm{d}t$, and two ones related to the beta function. In particular, we establish a symmetric identity

$$\sum_{n=l}^{\infty} \begin{bmatrix} n \\ l \end{bmatrix}_m \frac{1}{n!(n-r+1)^{k+1}} = \sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix}_r \frac{1}{n!(n-m+1)^{l+1}},$$

for integers $l \geq r \geq 0$ and $k \geq m \geq 0$. Various special series on the Stirling numbers of the first kind, harmonic numbers and hyperharmonic numbers are listed as examples, including many known results due to Boyadzhiev [5], Dil and Boyadzhiev [13] and Hoffman [19]. In Section 3, we show that the r-Stirling numbers of the first kind are closely related to the so called multiple hyperharmonic numbers by $\begin{bmatrix} n \\ k \end{bmatrix}_r = n!\zeta_n^{(r)}(\{1\}_k)$. This fact helps us combine the r-Stirling series

$$\sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix}_r \frac{1}{n!n^{l+1}} \quad \text{and} \quad \sum_{n=k}^{\infty} \begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_r \frac{1}{n!n^{l+1}} \tag{1.13}$$

as well as related harmonic number series with the Euler-type sums $\zeta^{(r)}(q;\{1\}_k)$ of multiple hyperharmonic numbers. By establishing the recurrences, we show that the Euler-type sums $\zeta^{(r)}(q;\{1\}_k)$ and the r-Stirling series in (1.13) are finally reducible to zeta values.

2. r-Stirling series via special integrals

2.1. Two r-Stirling series related to $U_m(n,r)$

Define the integral

$$U_m(n,r) = \int_0^1 t^n(1-t)^r \ln^m(1-t)\,\mathrm{d}t\,, \quad \text{for } n,m,r \in \mathbb{N}_0\,. \tag{2.1}$$

Combining Newton's binomial theorem and the relation

$$\int_0^1 t^{n-1} \ln^m(1-t)\,\mathrm{d}t = (-1)^m m! \frac{\zeta_n^*(\{1\}_m)}{n}$$

(see [35, Eq. (2.5)]), we rewrite $U_m(n,r)$ as

$$U_m(n,r) = \sum_{i=0}^{r} (-1)^i \binom{r}{i} \int_0^1 t^{n+i} \ln^m(1-t)\,\mathrm{d}t = (-1)^m m! \sum_{i=0}^{r} (-1)^i \binom{r}{i} \frac{\zeta_{n+i+1}^*(\{1\}_m)}{n+i+1}\,.$$

By (1.8), the multiple harmonic star sums $\zeta_n^*(\{1\}_k)$ can be expressed by the complete Bell polynomials Y_k and harmonic numbers $H_n^{(r)}$, and satisfy the following recurrence:

$$\zeta_n^*(\{1\}_0) = 1\,, \quad \zeta_n^*(\{1\}_k) = \frac{1}{k} \sum_{j=0}^{k-1} H_n^{(k-j)} \zeta_n^*(\{1\}_j)\,, \quad \text{for } k \in \mathbb{N}\,.$$

Thus, the integral $U_m(n,r)$ is expressible in terms of harmonic numbers. In particular, when $m = 0, 1$, we obtain the next lemma.

Lemma 2.1. *For integers $n, r \geq 0$, the following two identities hold:*

$$U_0(n,r) = \sum_{i=0}^{r} (-1)^i \binom{r}{i} \frac{1}{n+i+1} = \frac{1}{(n+r+1)\binom{n+r}{r}}\,, \tag{2.2}$$

$$U_1(n,r) = -\sum_{i=0}^{r} (-1)^i \binom{r}{i} \frac{H_{n+i+1}}{n+i+1} = -\frac{H_{n+r+1} - H_r}{(n+1)\binom{n+r+1}{r}}\,. \tag{2.3}$$

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Proof. When $m = 0$, this integral turns into the classical beta function:

$$U_0(n,r) = \int_0^1 t^n (1-t)^r \mathrm{d}t = B(n+1, r+1) = \frac{n!r!}{(n+r+1)!} = \frac{1}{(n+r+1)\binom{n+r}{r}},$$

which gives (2.2); see also Gould's [18, Eq. (1.40)]. Now, let us verify (2.3) by induction. It is obvious that (2.3) holds when $r = 0$. Suppose that it holds for the r case, and consider the $r+1$ case. Using the recurrence of the binomial coefficients, we have

$$\begin{aligned}
&\sum_{i=0}^{r+1} (-1)^i \binom{r+1}{i} \frac{H_{n+i+1}}{n+i+1} \\
&\quad = \frac{H_{n+1}}{n+1} + \sum_{i=1}^{r} (-1)^i \left[\binom{r}{i-1} + \binom{r}{i}\right] \frac{H_{n+i+1}}{n+i+1} + (-1)^{r+1} \frac{H_{n+r+2}}{n+r+2} \\
&\quad = \sum_{i=0}^{r} (-1)^{i+1} \binom{r}{i} \frac{H_{n+i+2}}{n+i+2} + \sum_{i=0}^{r} (-1)^i \binom{r}{i} \frac{H_{n+i+1}}{n+i+1}.
\end{aligned}$$

By the induction hypothesis as well as the relation $H_r = H_{r+1} - \frac{1}{r+1}$, the above formula can be simplified as

$$\begin{aligned}
&-\frac{H_{n+r+2} - H_r}{(n+2)\binom{n+r+2}{r}} + \frac{H_{n+r+1} - H_r}{(n+1)\binom{n+r+1}{r}} \\
&\quad = \frac{n!r!}{(n+r+2)!}[(n+r+2)H_{n+r+1} - (n+1)H_{n+r+2} - (r+1)H_r] = \frac{H_{n+r+2} - H_{r+1}}{(n+1)\binom{n+r+2}{r+1}},
\end{aligned}$$

which implies that (2.3) holds in the $r+1$ case, and completes the induction. □

In the sequel of this subsection, we use generating functions and special integrals to establish two series identities involving the r-Stirling numbers of the first kind and the integral $U_m(n,r)$, and discuss their special cases in detail.

Theorem 2.2. *For integers $k, m, r \geq 0$, the following r-Stirling series identity holds:*

$$\sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix}_r \frac{U_m(n,r)}{n!} = (-1)^m \frac{(m+k)!}{k!}. \tag{2.4}$$

Proof. Multiplying the generating function (1.9) of the r-Stirling numbers of the first kind by $(1-t)^r \ln^m(1-t)$, and integrating over the interval (0,1), we have

$$\sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix}_r \frac{1}{n!} \int_0^1 t^n (1-t)^r \ln^m(1-t)\,\mathrm{d}t = \frac{(-1)^k}{k!} \int_0^1 \ln^{m+k}(1-t)\,\mathrm{d}t,$$

which, together with the integral

$$\int_0^1 \ln^k(1-t)\,\mathrm{d}t = (-1)^k k!,$$

gives the desired result. □

By specifying the parameters in the general series identity in Theorem 2.2, we obtain numerous infinite series involving harmonic numbers, hyperharmonic numbers, multiple harmonic sums $\zeta_n^*(\{1\}_m)$ and the Stirling numbers of the first kind.

Example 2.1 (The $r = 0$ case of Eq. (2.4))**.** When $r = 0$, Eq. (2.4) reduces to

$$\sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} \frac{\zeta_{n+1}^{*}(\{1\}_m)}{(n+1)!} = \binom{m+k}{m}. \tag{2.5}$$

The $m = 0, 1, 2$ cases of (2.5) are

$$\sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} \frac{1}{(n+1)!} = 1\,, \tag{2.6}$$

$$\sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} \frac{H_{n+1}}{(n+1)!} = k+1\,, \tag{2.7}$$

$$\sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} \frac{H_{n+1}^2 + H_{n+1}^{(2)}}{(n+1)!} = (k+1)(k+2)\,, \tag{2.8}$$

respectively, where (2.6) is equivalent to Hoffman's [19, Eq. (8)]. Moreover, setting directly $k = 1$ in (2.5) gives

$$\sum_{n=1}^{\infty} \frac{\zeta_{n+1}^{*}(\{1\}_m)}{n(n+1)} = m+1\,. \tag{2.9}$$

According to the relations (1.7) and (1.8), these series, as well as those in Example 2.3, provide supplements to the results of Hoffman [19] and Kuba and Panholzer [24]. □

Example 2.2 (The $m = 0, 1$ cases of Eq. (2.4))**.** By Lemma 2.1, when $m = 0, 1$, Eq. (2.4) reduces to

$$\sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix}_r \frac{1}{(n+r+1)!} = \frac{1}{r!}\,, \tag{2.10}$$

$$\sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix}_r \frac{H_{n+r+1} - H_r}{(n+r+1)!} = \frac{k+1}{r!}\,, \tag{2.11}$$

respectively. Setting $k = 1$ in (2.10) and (2.11), and using the fact $\left[\begin{smallmatrix} n \\ 1 \end{smallmatrix}\right]_r = n! h_n^{(r)}$, give two series identities on hyperharmonic numbers:

$$\sum_{n=1}^{\infty} \frac{h_n^{(r)}}{(n+1)(n+2)\cdots(n+r+1)} = \frac{1}{r!}\,, \tag{2.12}$$

$$\sum_{n=1}^{\infty} \frac{h_n^{(r)}(H_{n+r+1} - H_r)}{(n+1)(n+2)\cdots(n+r+1)} = \frac{2}{r!}\,. \tag{2.13}$$

Note that (2.12) can be found in Dil and Boyadzhiev [13, Proposition 5]. □

Replacing the integral $U_m(n, r)$ in Theorem 2.2 by $U_m(n-1, r)$, we establish another series identity, which reveals an interesting relation between the zeta values $\zeta(n)$ and the r-Stirling numbers of the first kind.

Theorem 2.3. *For integers $k \geq 1$ and $m, r \geq 0$, the following r-Stirling series identity holds:*

$$\sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix}_r \frac{U_m(n-1, r)}{n!} = (-1)^m \frac{(m+k)!}{k!} \zeta(m+k+1)\,. \tag{2.14}$$

Proof. Multiplying the generating function (1.9) of the r-Stirling numbers of the first kind by $\frac{(1-t)^r \ln^m(1-t)}{t}$, and integrating over the interval $(0,1)$, we have

$$\sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix}_r \frac{1}{n!} \int_0^1 t^{n-1}(1-t)^r \ln^m(1-t)\,\mathrm{d}t = \frac{(-1)^k}{k!} \int_0^1 \frac{\ln^{m+k}(1-t)}{t}\,\mathrm{d}t\,.$$

Since

$$\int_0^1 \frac{\ln^k(1-t)}{t}\,\mathrm{d}t = (-1)^k k! \zeta(k+1)\,,$$

the desired result can be established. □

Example 2.3 (The $r=0,1$ cases of Eq. (2.14))**.** When $r=0,1$, (2.14) reduces to

$$\sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} \frac{\zeta_n^*(\{1\}_m)}{n!n} = \binom{m+k}{m} \zeta(m+k+1)\,, \tag{2.15}$$

$$\sum_{n=k}^{\infty} \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} \frac{\zeta_n^*(\{1\}_m)}{(n+1)!n} = \binom{m+k+1}{m} \zeta(m+k+1)\,. \tag{2.16}$$

Here, to obtain (2.16), we should substitute the relation

$$\begin{aligned} \frac{\zeta_n^*(\{1\}_m)}{n} - \frac{\zeta_{n+1}^*(\{1\}_m)}{n+1} &= \frac{\zeta_n^*(\{1\}_m)}{n} - \frac{\zeta_n^*(\{1\}_m) + \frac{1}{n+1}\zeta_{n+1}^*(\{1\}_{m-1})}{n+1} \\ &= \frac{\zeta_n^*(\{1\}_m)}{n(n+1)} - \frac{\zeta_{n+1}^*(\{1\}_{m-1})}{(n+1)^2} \end{aligned}$$

into the intermediate result of $r=1$ case and combine the latter with (2.15). The further substitutions $m=0,1$ in (2.16) give

$$\sum_{n=k}^{\infty} \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} \frac{1}{(n+1)!n} = \zeta(k+1)\,, \tag{2.17}$$

$$\sum_{n=k}^{\infty} \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} \frac{H_n}{(n+1)!n} = (k+2)\zeta(k+2)\,. \tag{2.18}$$

Two similar results can be established. By setting $r=1$, $m=1$ in (2.14) directly and using Eqs. (2.17) and (2.18), we obtain

$$\sum_{n=k}^{\infty} \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} \frac{H_{n+1}}{(n+1)!n} = (k+1)\zeta(k+2) + \zeta(k+1)\,, \tag{2.19}$$

$$\sum_{n=k}^{\infty} \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} \frac{1}{(n+1)!(n+1)n} = \zeta(k+1) - \zeta(k+2)\,. \tag{2.20}$$

Additionally, the further substitution $k=1$ in (2.15) and (2.16) yields

$$\sum_{n=1}^{\infty} \frac{\zeta_n^*(\{1\}_m)}{n^2} = (m+1)\zeta(m+2)\,, \tag{2.21}$$

$$\sum_{n=1}^{\infty} \frac{H_n \zeta_n^*(\{1\}_m)}{n(n+1)} = \frac{(m+2)(m+1)}{2} \zeta(m+2)\,. \tag{2.22}$$

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Note that (2.15) is equivalent to Hoffman's [19, Theorem 4] and Wang and Lyu's [33, Theorem 3.7], (2.17) is in fact Jordan's [21, Section 68, Eq. (12)], and (2.16) and (2.21) are in fact Hoffman's [19, Theorem 2 and Theorem 3, Eq. (12)]. See also Eqs. (1.4) and (1.5) of the present paper. Some additional cases of (2.15) by specifying the parameter m can be found in [33, Corollary 3.8], including the following one in Jordan's book [21, Section 56, Eq. (6) and Section 68, Eq. (11)]:

$$\sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} \frac{1}{n!n} = \zeta(k+1)\,; \tag{2.23}$$

see also Adamchik [1, Eq. (19)], Choi and Srivastava [10, Eq. (3.4)], Hoffman [19, Corollary 4] and Shen [30, Eq. (32)]. □

Example 2.4 (The $m = 0, 1$ cases of Eq. (2.14))**.** In (2.14), setting $m = 0, 1$ gives

$$\sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix}_r \frac{1}{(n+r)!n} = \frac{1}{r!}\zeta(k+1)\,, \tag{2.24}$$

$$\sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix}_r \frac{H_{n+r} - H_r}{(n+r)!n} = \frac{k+1}{r!}\zeta(k+2)\,. \tag{2.25}$$

Setting further $k = 1$ in (2.24) and (2.25) yields two series involving hyperharmonic numbers:

$$\sum_{n=1}^{\infty} \frac{h_n^{(r)}}{n(n+1)\cdots(n+r)} = \frac{\pi^2}{6r!}\,, \tag{2.26}$$

$$\sum_{n=1}^{\infty} \frac{h_n^{(r)}(H_{n+r} - H_r)}{n(n+1)\cdots(n+r)} = \frac{2}{r!}\zeta(3)\,, \tag{2.27}$$

where (2.26) is Dil and Boyadzhiev's [13, Proposition 6]. □

Remark 2.4. By the relations between the Stirling numbers of the first kind, multiple harmonic star sums $\zeta_n^\star(\{1\}_k)$ and harmonic numbers, more special series involving only classical harmonic numbers can be evaluated from the above series. We list here some representative cases. Setting $k = 3$ in (2.6), $k = 2$ in (2.7) and $m = 2$ in (2.9), we obtain

$$\sum_{n=1}^{\infty} \frac{H_n^2 - H_n^{(2)}}{(n+1)(n+2)} = 2\,, \quad \sum_{n=1}^{\infty} \frac{H_{n-1}H_{n+1}}{n(n+1)} = 3\,, \quad \sum_{n=1}^{\infty} \frac{H_{n+1}^2 + H_{n+1}^{(2)}}{n(n+1)} = 6\,.$$

Additionally, setting $k = 2$ in (2.17), $k = 1$ in (2.19) and $k = 3$ in (2.23) yields

$$\sum_{n=1}^{\infty} \frac{H_n^2 - H_n^{(2)}}{n(n+1)} = 2\zeta(3)\,, \quad \sum_{n=1}^{\infty} \frac{H_nH_{n+1}}{n(n+1)} = 2\zeta(3) + \frac{\pi^2}{6}\,, \quad \sum_{n=1}^{\infty} \frac{H_n^2 - H_n^{(2)}}{(n+1)^2} = \frac{\pi^4}{45}\,,$$

and setting $m = 2$ in (2.21) gives

$$\sum_{n=1}^{\infty} \frac{H_n^2 + H_n^{(2)}}{n^2} = \frac{\pi^4}{15}\,.$$

It is evident that such series can also be verified from the evaluations of the classical Euler sums (see, for example, [15, 33, 35–37]). □

2.2. A symmetric r-Stirling series identity

In this subsection, we use special integrals to establish yet another two r-Stirling series identities, including a symmetric one.

Theorem 2.5. *For integers $k, r \geq 0$ and $i \geq 1$, the following r-Stirling series identity holds:*

$$\sum_{n=k}^{\infty} \frac{{n \brack k}_r}{(i+r)(i+r+1)\cdots(i+r+n)} = \frac{1}{i^{k+1}} \,. \tag{2.28}$$

Proof. By the definition of the beta function, the generating function (1.9) of the r-Stirling numbers of the first kind, and the integral

$$\int_0^1 t^{n-1} \ln^l(t)\, \mathrm{d}t = \frac{(-1)^l l!}{n^{l+1}} \,, \tag{2.29}$$

the left side of (2.28) equals

$$\mathrm{LHS} = \sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix}_r \frac{1}{n!} \int_0^1 t^{i+r-1}(1-t)^n \,\mathrm{d}t = \frac{(-1)^k}{k!} \int_0^1 t^{i-1} \ln^k(t)\, \mathrm{d}t = \frac{1}{i^{k+1}} \,,$$

which is the desired result. □

Example 2.5. When $r = 0, 1$, Eq. (2.28) reduces to two series identities due to Stirling [31, pp. 29 and 171] and Hoffman [19, Theorem 6], respectively. See also Eq. (1.6) of the present paper. When $k = 1$, Eq. (2.28) gives

$$\sum_{n=1}^{\infty} \frac{h_n^{(r)}}{(n+1)(n+2)\cdots(n+r+i)} = \frac{1}{(i+r-1)!\, i^2} \,, \tag{2.30}$$

which yields (2.12) (i.e., Dil and Boyadzhiev's [13, Proposition 5]) if $i = 1$.

Using the above result, a symmetric series identity can be established.

Theorem 2.6. *For integers $l \geq r \geq 0$ and $k \geq m \geq 0$, the following symmetric r-Stirling series identity holds:*

$$\sum_{n=l}^{\infty} \begin{bmatrix} n \\ l \end{bmatrix}_m \frac{1}{n!(n-r+1)^{k+1}} = \sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix}_r \frac{1}{n!(n-m+1)^{l+1}} \,. \tag{2.31}$$

Proof. Let $f(t) = \sum_{n=0}^{\infty} a_n t^n$ be the generating function of the sequence $(a_n)_{n \in \mathbb{N}_0}$. Multiplying both sides of Eq. (2.28) by a_{i-1}, summing over i, exchanging the order of summation, and using the beta function once again, we have

$$\sum_{i=1}^{\infty} \frac{a_{i-1}}{i^{k+1}} = \sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix}_r \sum_{i=1}^{\infty} \frac{a_{i-1}}{n!} \int_0^1 t^{i+r-1}(1-t)^n \,\mathrm{d}t = \sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix}_r \frac{1}{n!} \int_0^1 f(t) t^r (1-t)^n \,\mathrm{d}t \,. \tag{2.32}$$

Now, applying (2.32) to the special generating function

$$f(t) = \frac{(-\ln(1-t))^l}{l!(1-t)^m t^r} = \frac{1}{t^r} \sum_{n=l}^{\infty} \begin{bmatrix} n \\ l \end{bmatrix}_m \frac{t^n}{n!}$$

and its coefficient $a_i = [t^i] f(t) = {r+i \brack l}_m \frac{1}{(r+i)!}$, and using the integral (2.29), we obtain the desired symmetric identity. □

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Remark 2.7. The general transformation formula (2.32) is in fact a generalization of Boyadzhiev [5, Theorem 2.1], and it shows the relation between the Dirichlet series generating function and the ordinary power series generating function of a sequence.

In the sequel of this subsection, we present some special cases of the symmetric series identity in Theorem 2.6.

Corollary 2.8. *For integers $k \geq m \geq 1$, the following Stirling series holds:*

$$\sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} \frac{1}{n!(n-m+1)} = \sum_{j=0}^{m-1} \begin{bmatrix} m-1 \\ j \end{bmatrix} \frac{\zeta(k-j+1)}{(m-1)!} . \tag{2.33}$$

Proof. When $l = r = 0$, we rewrite Eq. (2.31) as

$$\begin{aligned} \sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} \frac{1}{n!(n-m+1)} &= \sum_{n=0}^{\infty} \frac{\langle m \rangle_n}{n!(n+1)^{k+1}} = \sum_{n=0}^{\infty} \frac{\langle n+1 \rangle_{m-1}}{(m-1)!(n+1)^{k+1}} \\ &\overset{(*)}{=} \sum_{n=1}^{\infty} \frac{1}{(m-1)!n^{k+1}} \sum_{j=0}^{m-1} \begin{bmatrix} m-1 \\ j \end{bmatrix} n^j = \sum_{j=0}^{m-1} \begin{bmatrix} m-1 \\ j \end{bmatrix} \frac{\zeta(k-j+1)}{(m-1)!} , \end{aligned}$$

where we use $\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k = \langle x \rangle_n$ to obtain $(*)$ (see [11, p. 213, Eq. (5f)]). □

Example 2.6. Setting $m = 1, 2$ in Eq. (2.33) yields (2.23) and (2.17) once again. Setting $m = 3, 4$ in (2.33) gives

$$\sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} \frac{1}{n!(n-2)} = \frac{1}{2}\zeta(k) + \frac{1}{2}\zeta(k-1) , \tag{2.34}$$

$$\sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} \frac{1}{n!(n-3)} = \frac{1}{3}\zeta(k) + \frac{1}{2}\zeta(k-1) + \frac{1}{6}\zeta(k-2) . \tag{2.35}$$

Example 2.7. More special cases of Eq. (2.31) can be obtained. For example, setting $m = r = 0$ in Eq. (2.31) yields Boyadzhiev's [5, Corollary 3.2], and setting $m = r = 1$ gives the symmetric series identity

$$\sum_{n=l}^{\infty} \begin{bmatrix} n+1 \\ l+1 \end{bmatrix} \frac{1}{n!n^{k+1}} = \sum_{n=k}^{\infty} \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} \frac{1}{n!n^{l+1}} . \tag{2.36}$$

The evaluation of the involved series will be given in Eq. (3.13). Additionally, when $l = 1$ and $m = r = 0$, we have

$$\sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} \frac{1}{(n+1)!(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n(n+1)^{k+1}} = k + 1 - \sum_{i=2}^{k+1} \zeta(i) . \tag{2.37}$$

When $l = r = 1$, we obtain from (2.31) the following relation between the Stirling series and the linear Euler sums of hyperharmonic numbers:

$$\sum_{n=k}^{\infty} \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} \frac{1}{n!(n-m+1)^2} = \sum_{n=1}^{\infty} \frac{h_n^{(m)}}{n^{k+1}} , \tag{2.38}$$

for integers $k \geq m \geq 0$. An general expression of the series in Eq. (2.38) will be presented in Remark 3.6. □

3. r-Stirling series and Euler-type sums $\zeta^{(r)}(q;\{1\}_k)$

3.1. Two r-Stirling series related to $\zeta^{(r)}(q;\{1\}_k)$

Most recently, Xu et al. [38, Section 1] introduced the *multiple hyperharmonic numbers*:

$$\zeta_n^{(r)}(s_1,\ldots,s_k) = \sum_{n\geq n_1\geq\cdots\geq n_r>n_{r+1}>\cdots>n_{r+k-1}\geq 1} \frac{1}{n_r^{s_1} n_{r+1}^{s_2}\cdots n_{r+k-1}^{s_k}},$$

where $k,r,s_i\in\mathbb{N}$ for $i=1,2,\ldots,k$, with the convention $\zeta_n^{(r)}(s_1,\ldots,s_k)=0$ for $n<k$. In particular, $\zeta_n^{(r)}(s_1)=H_n^{(s_1,r)}$, which are the *generalized hyperharmonic numbers* defined by Dil et al. [14, Section 1]:

$$H_n^{(s_1,r)} = \sum_{k=1}^{n} H_k^{(s_1,r-1)}, \quad \text{with } H_n^{(s_1,1)} = H_n^{(s_1)}.$$

Moreover, $\zeta_n^{(1)}(s_1,\ldots,s_k)=\zeta_n(s_1,\ldots,s_k)$, $\zeta_n^{(r)}(1)=h_n^{(r)}$, and $\zeta_n^{(1)}(s_1)=H_n^{(s_1)}$. Similarly to Eq. (1.12), define the *Euler-type sums of multiple hyperharmonic numbers* by

$$\zeta^{(r)}(q;s_1,\ldots,s_k) = \sum_{n=1}^{\infty} \frac{\zeta_n^{(r)}(s_1,\ldots,s_k)}{n^q}, \tag{3.1}$$

for $q,r\in\mathbb{N}$ with $q\geq r+2-s_1$ to ensure the convergence of the series (see Xu et al. [38, Section 1]). In this section, we show that two kinds of r-Stirling series are expressible in terms of $\zeta^{(r)}(q;\{1\}_k)$ (see Theorem 3.1), and finally reducible to zeta values (see Theorem 3.7).

Theorem 3.1. *For integers $k,l,r\geq 1$ with $l\geq r$, we have $\left[{n \atop k}\right]_r = n!\zeta_n^{(r)}(\{1\}_k)$. Moreover, the following two r-Stirling series identities hold:*

$$\sum_{n=k}^{\infty} \left[{n \atop k}\right]_r \frac{1}{n!n^{l+1}} = \zeta^{(r)}(l+1;\{1\}_k), \tag{3.2}$$

$$\sum_{n=k}^{\infty} \left[{n+1 \atop k+1}\right]_{r-1} \frac{1}{n!n^{l+2}} = (r-1)\zeta^{(r)}(l+2;\{1\}_{k+1}) + \zeta^{(r)}(l+2;\{1\}_k). \tag{3.3}$$

Proof. The r-Stirling numbers of the first kind satisfy the following two recurrences:

$$\left[{n \atop k}\right]_r = (n+r-1)\left[{n-1 \atop k}\right]_r + \left[{n-1 \atop k-1}\right]_r, \tag{3.4}$$

$$\left[{n \atop k}\right]_{r+1} = \left[{n \atop k}\right]_r + n\left[{n-1 \atop k}\right]_{r+1}. \tag{3.5}$$

See Broder [7, Theorem 1] and Benjamin et al. [2, Identity 1], respectively. From (3.5), we have

$$\left[{n \atop k}\right]_{r+1} = \sum_{i=k}^{n} \frac{n!}{i!}\left[{i \atop k}\right]_r,$$

which, together with the relation (1.1), gives

$$\begin{aligned}\frac{1}{n!}\left[{n \atop k}\right]_r &= \sum_{i_1=k}^{n} \frac{1}{i_1!}\left[{i_1 \atop k}\right]_{r-1} = \sum_{i_1=k}^{n}\sum_{i_2=k}^{i_1} \frac{1}{i_2!}\left[{i_2 \atop k}\right]_{r-2} = \cdots = \sum_{n\geq i_1\geq\cdots\geq i_r\geq k} \frac{1}{i_r!}\left[{i_r \atop k}\right] \\ &= \sum_{n\geq i_1\geq\cdots\geq i_r\geq k} \frac{\zeta_{i_r-1}(\{1\}_{k-1})}{i_r} = \sum_{n\geq i_1\geq\cdots\geq i_r>i_{r+1}>\cdots>i_{r+k-1}\geq 1} \frac{1}{i_r i_{r+1}\cdots i_{r+k-1}}\end{aligned}$$

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$$= \zeta_n^{(r)}(\{1\}_k)\,.$$

By considering the definition (3.1) of Euler-type sums of multiple hyperharmonic numbers, we obtain (3.2). Next, let us prove (3.3). On the one hand, using the recurrence (3.4) and Eq. (3.2), we have

$$\begin{aligned}\sum_{n=k}^{\infty} {n+1 \brack k+1}_r \frac{1}{n!n^{l+2}} &= \sum_{n=k}^{\infty} {n \brack k}_r \frac{1}{n!n^{l+2}} + \sum_{n=k}^{\infty} {n \brack k+1}_r \frac{1}{n!n^{l+1}} + r\sum_{n=k}^{\infty} {n \brack k+1}_r \frac{1}{n!n^{l+2}} \\ &= \zeta^{(r)}(l+2;\{1\}_k) + \zeta^{(r)}(l+1;\{1\}_{k+1}) + r\zeta^{(r)}(l+2;\{1\}_{k+1})\,. \end{aligned} \tag{3.6}$$

On the other hand, by the recurrence (3.5) and Eq. (3.2), we obtain

$$\begin{aligned}\sum_{n=k}^{\infty} {n+1 \brack k+1}_r \frac{1}{n!n^{l+2}} &= \sum_{n=k}^{\infty} {n+1 \brack k+1}_{r-1} \frac{1}{n!n^{l+2}} + \sum_{n=k}^{\infty} {n \brack k+1}_r \frac{1}{n!n^{l+1}} + \sum_{n=k}^{\infty} {n \brack k+1}_r \frac{1}{n!n^{l+2}} \\ &= \sum_{n=k}^{\infty} {n+1 \brack k+1}_{r-1} \frac{1}{n!n^{l+2}} + \zeta^{(r)}(l+1;\{1\}_{k+1}) + \zeta^{(r)}(l+2;\{1\}_{k+1})\,. \end{aligned}$$

Combining the above two equations gives us the desired result. □

Remark 3.2. In [35, Eq. (2.27)], Xu established the relation between the following integrals and MZVs:

$$W(k,l) = \int_0^1 \frac{\ln^l(t)\ln^k(1-t)}{t}\,\mathrm{d}t = (-1)^{k+l}k!l!\zeta(l+2,\{1\}_{k-1})\,, \tag{3.7}$$

which further gives $W(1,l) = (-1)^{l+1}l!\zeta(l+2)$ and $W(k,0) = (-1)^k k!\zeta(k+1)$. Similarly to $W(k,l)$, define

$$W_r(k,l) = \int_0^1 \frac{\ln^l(t)\ln^k(1-t)}{t(1-t)^r}\,\mathrm{d}t\,, \quad k,l\in\mathbb{N},\ r\in\mathbb{N}_0\,. \tag{3.8}$$

Then $W_0(k,l) = W(k,l)$, and for $r\geq 1$, there holds

$$W_r(k,l) = (-1)^k k! \sum_{n=k}^{\infty} {n \brack k}_r \frac{1}{n!} \int_0^1 t^{n-1}\ln^l(t)\,\mathrm{d}t = (-1)^{k+l}k!l! \sum_{n=k}^{\infty} {n \brack k}_r \frac{1}{n!n^{l+1}}\,,$$

by the generating function (1.9) and the integral (2.29). Thus, we obtain the integral representation of the r-Stirling series on the left side of (3.2). □

Let us discuss some corollaries and special cases of Theorem 3.1. In Corollary 3.3, we establish two pairs of formulas, which indicate that the multiple hyperharmonic numbers $\zeta_n^{(r)}(\{1\}_k)$ can be expressed in terms of multiple harmonic sums $\zeta_m(\{1\}_j)$.

Corollary 3.3. *For integers $k,r\geq 1$, the multiple hyperharmonic numbers $\zeta_n^{(r)}(\{1\}_k)$ satisfy*

$$\zeta_n^{(r)}(\{1\}_k) = \sum_{j=k}^{n} \binom{r+n-j-1}{n-j} \frac{\zeta_{j-1}(\{1\}_{k-1})}{j} = \sum_{j=k}^{n} \binom{j}{k} \frac{\zeta_{n-1}(\{1\}_{j-1})}{n} r^{j-k}\,, \tag{3.9}$$

$$\zeta_n^{(r)}(\{1\}_k) = \sum_{j=k}^{n} \binom{r+n-j-2}{n-j} \zeta_j(\{1\}_k) = \sum_{j=k}^{n} \binom{j}{k} \zeta_n(\{1\}_j)(r-1)^{j-k}\,, \tag{3.10}$$

where, by convention, $\zeta_0(\emptyset) := 1$ and $\zeta_0(\{1\}_i) := 0$, for $i\geq 1$.

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Proof. It is known that the following r-Stirling number identity holds:

$$\begin{bmatrix} n \\ i \end{bmatrix}_{r+s} = \sum_{k=0}^{n} \binom{n}{k} \begin{bmatrix} k \\ i \end{bmatrix}_r \langle s \rangle_{n-k} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_r \binom{k}{i} s^{k-i} \tag{3.11}$$

(see Broder [7, Theorem 12, Eq. (28)], Can and Dağlı [8, Eq. (35)], Ma and Wang [26, Eqs. (3.5) and (4.21)] and Nyul and Rácz [29, Table 1, Line 12]). To obtain the desired results, set $r = 0, 1$ in (3.11), combine the corresponding results with the relation (1.1) and the relation $\left[{n \atop k}\right]_r = n! \zeta_n^{(r)}(\{1\}_k)$ in Theorem 3.1, and do some further transformations. □

The following corollary of Theorem 3.1 shows that the Euler-type sums $\zeta^{(1)}(q; \{1\}_k)$ are reducible to zeta values.

Corollary 3.4. *For integers $k, l \geq 1$, the Euler-type sums $\zeta^{(1)}(l+1; \{1\}_k)$ satisfy*

$$\zeta^{(1)}(l+1; \{1\}_k) := \sum_{n=1}^{\infty} \frac{\zeta_n(\{1\}_k)}{n^{l+1}} = \zeta(l+2, \{1\}_{k-1}) + \zeta(l+1, \{1\}_k), \tag{3.12}$$

and are reducible to zeta values.

Proof. When $r = 1$, using the recurrence of the classical Stirling numbers of the first kind, we obtain from (3.2) and (1.2) that

$$\begin{aligned} \zeta^{(1)}(l+1; \{1\}_k) &= \sum_{n=k}^{\infty} \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} \frac{1}{n! n^{l+1}} = \sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} \frac{1}{n! n^{l+1}} + \sum_{n=k}^{\infty} \begin{bmatrix} n \\ k+1 \end{bmatrix} \frac{1}{n! n^{l}} \\ &= \zeta(l+2, \{1\}_{k-1}) + \zeta(l+1, \{1\}_k), \end{aligned} \tag{3.13}$$

as required. Combining Eq. (3.7) with the recurrence of the integrals $W(k, l)$ (see Yao and Wu [39, Theorem 2]), we obtain the recurrence of the MZVs $\zeta(l+1, \{1\}_k)$:

$$\begin{aligned} \zeta(l+1, \{1\}_k) &= \binom{l+k}{l} \frac{\zeta(l+k+1)}{k+1} \\ &\quad - \frac{1}{l} \sum_{i=1}^{l-1} \sum_{j=0}^{k-1} \binom{l+k-i-j-1}{k-j} \zeta(l+k-i-j) \zeta(i+1, \{1\}_j), \end{aligned}$$

where $l \in \mathbb{N}$, $k \in \mathbb{N}_0$. It is known that $\zeta(i, \{1\}_0) = \zeta(i)$ and $\zeta(2, \{1\}_j) = \zeta(j+2)$ for integers $i \geq 2$ and $j \geq 0$. Therefore, the MZVs $\zeta(l+1, \{1\}_k)$ are expressible in terms of zeta values, as are the Euler-type sums $\zeta^{(1)}(l+1; \{1\}_k)$. □

Example 3.1. Eq. (3.12) can be used to evaluate $\zeta^{(1)}(q; \{1\}_k)$. For example, we have

$$\begin{aligned} &\zeta^{(1)}(3; 1) = \tfrac{1}{72} \pi^4, \\ &\zeta^{(1)}(3; 1, 1) = 4\zeta(5) - \tfrac{1}{3} \zeta(3) \pi^2, \\ &\zeta^{(1)}(3; 1, 1, 1) = -\tfrac{3}{2} \zeta^2(3) + \tfrac{1}{432} \pi^6, \\ &\zeta^{(1)}(3; 1, 1, 1, 1) = 8\zeta(7) - \tfrac{1}{2} \zeta(5) \pi^2 - \tfrac{1}{40} \zeta(3) \pi^4. \end{aligned}$$

Remark 3.5. The series identity in (3.13) is in fact equivalent to a recent result due to Wang and Lyu [33, Theorem 3.2], which reduces to the Euler theorem, the Browein-Browein-Girgensohn theorem and the Flajolet-Salvy theorem of the classical Euler sums (see also Flajolet and Salvy [15, Theorems 2.2, 4.1 and 5.1(1)]. When $l = 1$, by using the relation

$$\zeta(2, \{1\}_k) = \zeta(k+2),$$

$$\zeta(3,\{1\}_k)=\zeta(k+2,1)=\frac{k+2}{2}\zeta(k+3)-\frac{1}{2}\sum_{i=1}^{k}\zeta(i+1)\zeta(k+2-i)\,,$$

we obtain from (3.13) that

$$\sum_{n=k}^{\infty}\begin{bmatrix}n+1\\k+1\end{bmatrix}\frac{1}{n!n^2}=\frac{k+3}{2}\zeta(k+2)-\frac{1}{2}\sum_{j=2}^{k}\zeta(j)\zeta(k+2-j)\,. \tag{3.14}$$

Similarly, when $l=2$, we have

$$\sum_{n=k}^{\infty}\begin{bmatrix}n+1\\k+1\end{bmatrix}\frac{1}{n!n^3}=\zeta(3,\{1\}_k)+\zeta(4,\{1\}_{k-1})=\zeta(k+2,1)+\zeta(k+1,1,1)\,. \tag{3.15}$$

It can be found that the series identities (3.14) and (3.15) are equivalent to Hoffman's [19, Theorem 3, Eq. (13)] (i.e., Eq. (1.3) of the present paper) and [19, Corollary 3, Eq. (2)]. □

Remark 3.6. Setting $k=1$ in (3.2), using the relation $\begin{bmatrix}n\\1\end{bmatrix}_r=n!h_n^{(r)}$ and the result [38, Theorem 4] due to Xu et al, we obtain an explicit expression of the linear Euler sums of hyperharmonic numbers:

$$\begin{aligned}\sigma_{r,l+1}&=\sum_{n=1}^{\infty}\frac{h_n^{(r)}}{n^{l+1}}=\zeta^{(r)}(l+1;1)\\&=\frac{1}{(r-1)!}\sum_{j=1}^{r}\begin{bmatrix}r\\j\end{bmatrix}\{\zeta(l+2-j,1)+\zeta(l+3-j)\}\\&\quad+\frac{1}{(r-1)!}\sum_{j=1}^{r}\sum_{i=j+1}^{r}(-1)^{i-j}\begin{bmatrix}r\\i\end{bmatrix}\binom{i-1}{j-1}\sum_{k=0}^{i-j-1}(-1)^k\frac{\binom{i-j}{k}}{i-j}B_k\zeta(l+k+2-i)\,,\end{aligned}$$

where B_k are the classical Bernoulli numbers defined by $t/(\mathrm{e}^t-1)=\sum_{k=0}^{\infty}B_kt^k/k!$. See also Eq. (2.38) of the present paper. The MZVs $\zeta(n,1)=\zeta(3,\{1\}_{n-2})$ are expressible in terms of zeta values, as are the linear sums $\sigma_{r,l+1}$. By computation, it can be found that

$$\begin{aligned}\sigma_{4,6}&=4\zeta(7)-\tfrac{1}{6}\zeta(5)\pi^2-\tfrac{1}{90}\zeta(3)\pi^4-\tfrac{11}{12}\zeta^2(3)+\tfrac{11}{3240}\pi^6+\tfrac{59}{36}\zeta(5)-\tfrac{1}{6}\zeta(3)\pi^2\\&\quad-\tfrac{1}{80}\pi^4-\tfrac{11}{36}\zeta(3)\,,\\ \sigma_{4,7}&=-\zeta(5)\zeta(3)+\tfrac{1}{4200}\pi^8+\tfrac{22}{3}\zeta(7)-\tfrac{11}{36}\zeta(5)\pi^2-\tfrac{11}{540}\zeta(3)\pi^4-\tfrac{1}{2}\zeta^2(3)+\tfrac{1}{2430}\pi^6\\&\quad-\tfrac{5}{6}\zeta(5)-\tfrac{1}{36}\zeta(3)\pi^2-\tfrac{11}{3240}\pi^4\,.\end{aligned}$$

Note that there are some typing errors in the evaluations of these two linear Euler-type sums in [28, Section 7]. □

3.2. Recurrence of $\zeta^{(r)}(q;\{1\}_k)$

In this subsection, by combining the recurrences of the r-Stirling numbers and the series identities in Theorem 3.1, we establish the recurrence of the Euler-type sums $\zeta^{(r)}(q;\{1\}_k)$, which in turn facilitates the computation of more special series in Theorem 3.1.

Theorem 3.7. *For integers $k\ge1$ and $l\ge r\ge2$, the Euler-type sums $\zeta^{(r)}(l+1;\{1\}_{k+1})$ satisfy the recurrence*

$$\begin{aligned}\zeta^{(r)}(l+1;\{1\}_{k+1})&=\frac{1}{r-1}\sum_{j=1}^{r-1}\{\zeta^{(j)}(l;\{1\}_{k+1})+\zeta^{(j)}(l+1;\{1\}_{k+1})\}\\&\quad+\frac{1}{r-1}\{\zeta^{(1)}(l+1;\{1\}_k)-\zeta^{(r)}(l+1;\{1\}_k)\}\,.\end{aligned}$$

Therefore they are reducible to zeta values.

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Proof. By iteratively using the recurrence (3.5), we have

$$\begin{aligned}\sum_{n=k}^{\infty}\begin{bmatrix}n+1\\k+1\end{bmatrix}_r\frac{1}{n!n^{l+2}} &= \sum_{n=k}^{\infty}\begin{bmatrix}n+1\\k+1\end{bmatrix}_{r-1}\frac{1}{n!n^{l+2}}+\sum_{n=k}^{\infty}(n+1)\begin{bmatrix}n\\k+1\end{bmatrix}_r\frac{1}{n!n^{l+2}}\\ &=\cdots=\sum_{n=k}^{\infty}\begin{bmatrix}n+1\\k+1\end{bmatrix}\frac{1}{n!n^{l+2}}+\sum_{j=1}^{r}\sum_{n=k}^{\infty}(n+1)\begin{bmatrix}n\\k+1\end{bmatrix}_j\frac{1}{n!n^{l+2}}\,,\end{aligned}$$

which, together with Eqs. (3.2) and (3.13), gives

$$\sum_{n=k}^{\infty}\begin{bmatrix}n+1\\k+1\end{bmatrix}_r\frac{1}{n!n^{l+2}}=\zeta^{(1)}(l+2;\{1\}_k)+\sum_{j=1}^{r}\{\zeta^{(j)}(l+1;\{1\}_{k+1})+\zeta^{(j)}(l+2;\{1\}_{k+1})\}\,.$$

Using Eq. (3.6) and the substitution $l\to l-1$, we obtain the desired recurrence. Now, let the Euler-type sums $\zeta^{(1)}(q;\{1\}_k)$ in Corollary 3.4 and $\zeta^{(r)}(q;1)$ in Remark 3.6 as initial values. Then it can be found that $\zeta^{(r)}(l+1;\{1\}_{k+1})$ are indeed expressible in terms of zeta values. □

Remark 3.8. Note that most recently, Xu [34, Theorems 1.1 and 1.2] and Xu et al. [38, Theorems 3 and 4] have established two different expressions of $\zeta^{(r)}(q;\{1\}_k)$ by multiple summations and showed that such Euler-type sums are reducible to zeta values. However, the recurrence established in Theorem 3.7 is more convenient to calculate these sums.

By Theorems 3.1 and 3.7, the evaluations of a large number of series involving r-Stirling numbers and harmonic numbers can be determined. We list some as examples.

Example 3.2 (The $k=2$ case of Eq. (3.2))**.** The substitutions $k=2$ and $r\to r+1$ in (3.2) yield

$$\sum_{n=2}^{\infty}\frac{\left[{n \atop 2}\right]_{r+1}}{n!n^{l+1}}=\sum_{n=1}^{\infty}\frac{\binom{n+r}{r}[(H_{n+r}-H_r)^2-(H_{n+r}^{(2)}-H_r^{(2)})]}{2n^{l+1}}=\zeta^{(r+1)}(l+1;1,1)\,,$$

where $l>r\geq 0$. Setting further $r=1$, $l=3$ and $r=2$, $l=4$, and applying Theorem 3.7, we have

$$\begin{aligned}\sum_{n=2}^{\infty}\frac{\left[{n \atop 2}\right]_2}{n!n^4}&=\sum_{n=1}^{\infty}\frac{(n+1)[(H_{n+1}-1)^2-(H_{n+1}^{(2)}-1)]}{2n^4}=\zeta^{(2)}(4;1,1)\\&=-\tfrac{3}{2}\zeta^2(3)+\tfrac{1}{432}\pi^6+4\zeta(5)-\tfrac{1}{3}\zeta(3)\pi^2-\tfrac{1}{72}\pi^4+\zeta(3)\,,\\ \sum_{n=2}^{\infty}\frac{\left[{n \atop 2}\right]_3}{n!n^5}&=\sum_{n=1}^{\infty}\frac{\binom{n+2}{2}[(H_{n+2}-\frac{3}{2})^2-(H_{n+2}^{(2)}-\frac{5}{4})]}{2n^5}=\zeta^{(3)}(5;1,1)\\&=8\zeta(7)-\tfrac{1}{2}\zeta(5)\pi^2-\tfrac{1}{40}\zeta(3)\pi^4-\tfrac{9}{4}\zeta^2(3)+\tfrac{1}{288}\pi^6-\tfrac{7}{4}\zeta(5)+\tfrac{1}{24}\zeta(3)\pi^2\\&\quad+\tfrac{1}{480}\pi^4+\tfrac{7}{8}\zeta(3)\,.\end{aligned}$$

Example 3.3 (Other special cases of Eq. (3.2))**.** Combining Theorem 3.7 with the series identity (3.2), we obtain

$$\begin{aligned}\sum_{n=5}^{\infty}\frac{\left[{n \atop 5}\right]_2}{n!n^3}&=\zeta^{(2)}(3;1,1,1,1,1)\\&=-4\zeta(5)\zeta(3)+\tfrac{1}{12}\zeta^2(3)\pi^2+\tfrac{13}{32400}\pi^8+4\zeta(7)-\tfrac{1}{6}\zeta(5)\pi^2-\tfrac{1}{90}\zeta(3)\pi^4\\&\quad+\tfrac{1}{2}\zeta^2(3)-\tfrac{1}{540}\pi^6+3\zeta(5)-\tfrac{1}{6}\zeta(3)\pi^2-\tfrac{1}{72}\pi^4+2\zeta(3)-\tfrac{1}{6}\pi^2\,,\\ \sum_{n=4}^{\infty}\frac{\left[{n \atop 4}\right]_3}{n!n^4}&=\zeta^{(3)}(4;1,1,1,1)\end{aligned}$$

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$$\begin{aligned}
&= -7\zeta(5)\zeta(3) + \tfrac{1}{4}\zeta^2(3)\pi^2 + \tfrac{47}{86400}\pi^8 + 12\zeta(7) - \tfrac{3}{4}\zeta(5)\pi^2 - \tfrac{3}{80}\zeta(3)\pi^4 \\
&\quad + \tfrac{13}{8}\zeta^2(3) - \tfrac{17}{8640}\pi^6 + \tfrac{9}{4}\zeta(5) - \tfrac{1}{4}\zeta(3)\pi^2 - \tfrac{1}{384}\pi^4 - \tfrac{27}{32}\zeta(3) + \tfrac{31}{192}\pi^2 ,
\end{aligned}$$

$$\begin{aligned}
\sum_{n=3}^{\infty} \frac{\left[{n \atop 3}\right]_4}{n!n^5} &= \zeta^{(4)}(5;1,1,1) \\
&= -7\zeta(5)\zeta(3) + \tfrac{1}{4}\zeta^2(3)\pi^2 + \tfrac{47}{86400}\pi^8 + \tfrac{55}{3}\zeta(7) - \tfrac{11}{9}\zeta(5)\pi^2 - \tfrac{11}{216}\zeta(3)\pi^4 \\
&\quad + \tfrac{13}{24}\zeta^2(3) - \tfrac{13}{15552}\pi^6 - \tfrac{97}{72}\zeta(5) + \tfrac{289}{1296}\zeta(3)\pi^2 + \tfrac{113}{29160}\pi^4 - \tfrac{25}{36}\zeta(3) - \tfrac{575}{7776}\pi^2 .
\end{aligned}$$

Example 3.4 (The $k=2$ case of Eq. (3.3))**.** Setting $k=2$ in (3.3) and specifying the parameters l and r, we have

$$\begin{aligned}
\sum_{n=2}^{\infty} \frac{\left[{n+1 \atop 3}\right]_0}{n!n^2} &= \sum_{n=2}^{\infty} \frac{H_n^2 - H_n^{(2)}}{2n^2} = \tfrac{1}{72}\pi^4 , \\
\sum_{n=2}^{\infty} \frac{\left[{n+1 \atop 3}\right]_1}{n!n^3} &= \sum_{n=2}^{\infty} \frac{n+1}{6n^3}\left(H_{n+1}^3 - 3H_{n+1}H_{n+1}^{(2)} + 2H_{n+1}^{(3)}\right) \\
&= -\tfrac{3}{2}\zeta^2(3) + \tfrac{1}{432}\pi^6 + 7\zeta(5) - \tfrac{1}{2}\zeta(3)\pi^2 , \\
\sum_{n=2}^{\infty} \frac{\left[{n+1 \atop 3}\right]_2}{n!n^4} &= 20\zeta(7) - \tfrac{4}{3}\zeta(5)\pi^2 - \tfrac{1}{18}\zeta(3)\pi^4 - 6\zeta^2(3) + \tfrac{1}{108}\pi^6 \\
&\quad - \zeta(5) + \tfrac{1}{6}\zeta(3)\pi^2 + \zeta(3) - \tfrac{1}{6}\pi^2 , \\
\sum_{n=2}^{\infty} \frac{\left[{n+1 \atop 3}\right]_3}{n!n^5} &= -21\zeta(5)\zeta(3) + \tfrac{3}{4}\zeta^2(3)\pi^2 + \tfrac{47}{28800}\pi^8 + 63\zeta(7) - \tfrac{25}{6}\zeta(5)\pi^2 - \tfrac{8}{45}\zeta(3)\pi^4 \\
&\quad - \tfrac{9}{8}\zeta^2(3) + \tfrac{1}{576}\pi^6 - \tfrac{33}{8}\zeta(5) + \tfrac{9}{16}\zeta(3)\pi^2 + \tfrac{1}{120}\pi^4 - \tfrac{5}{4}\zeta(3) - \tfrac{5}{32}\pi^2 .
\end{aligned}$$

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