Global Hopf bifurcation of a delayed diffusive predator-prey system with hunting cooperation and Holling type-III functional response

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Abstract

In this paper, we investigate the global Hopf bifurcation of a delayed diffusive predator-prey system with hunting cooperation and Holling type-III functional response. Firstly, we obtain the stability of positive steady state and the existence of local Hopf bifurcation by analyzing the characteristic equation. Secondly, we prove that the predator-prey system has the permanence properties and the positive steady state is global attractive for the system without delay by establishing the Lyapunov function. Then, according to the global Hopf bifurcation result of Wu [32], we establish the existence of global Hopf bifurcation. Finally, the results are illustrated by numerical simulations.

Keywords: Predator-prey; Diffusion; Time delay; Hunting cooperation; Global Hopf bifurcation

1. Introduction

The dynamic relationship between predators and preys is one of the crucial topics in mathematical ecology due to its ubiquity and importance. In [1, 2], Leslie proposed a predator-prey model in which the capacity of the predator in the environment is proportional to the number of prey

$$\begin{cases} \dfrac{du}{dt} = ru\left(1 - \dfrac{u}{k}\right) - b_1 uv, \\ \dfrac{dv}{dt} = \delta v\left(1 - \dfrac{hv}{u}\right), \end{cases} \tag{1}$$

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where $u = u(t)$ and $v = v(t)$ represent the densities of prey and predator at time t, respectively; and r, k, δ, h and b_1 are positive constants. The numbers r and δ are the growth rates of the prey and the predator, respectively. The parameter k stands for the carrying capacity of the prey population and b_1 is the predation rate. The value $\frac{u}{h}$ denotes the carrying capacity of the predator population, where h is the scaling coefficient.

Holling proposed the different functional response functions to describe the relation of predator-prey (see [3–5]). Population dynamics models with different dynamic functional response functions were established for different species. The Holling III functional response is suitable for the analysis of vertebrate dynamic behavior, and it has been studied by many scholars (see [6–11]). Similarly, Holling type-III functional response $\frac{bu^2}{a+u^2}$ is added to the Eq.(1), and the parameter m is introduced considering that the predator has other food sources. The model is usually written in the following form (see [12])

$$\begin{cases} \dfrac{du}{dt} = ru\left(1 - \dfrac{u}{k}\right) - \dfrac{bu^2 v}{a+u^2}, \\ \dfrac{dv}{dt} = \delta v\left(1 - \dfrac{hv}{m+u}\right). \end{cases} \tag{2}$$

In order to better understand predator-prey dynamics, many scholars introduced time delay into biological models (see [13–17]). In [18], Ma introduced maturation delay of the predator and the time delay in digesting prey. Similarly, for Eq.(2), we introduce the parameter $\tau > 0$ and $u_\tau = u(t-\tau)$, where τ denotes the time taken for digestion of the prey. Then we have

$$\begin{cases} \dfrac{du}{dt} = ru\left(1 - \dfrac{u}{k}\right) - \dfrac{bu^2 v}{a+u^2}, \\ \dfrac{dv}{dt} = \delta v\left(1 - \dfrac{hv}{m+u_\tau}\right). \end{cases} \tag{3}$$

Considering the uneven distribution of predators and their prey in different spacial locations within a fixed area, as well as the tendency of various species to spread into smaller population areas, many scholars have extensively studied the predator-prey model with diffusion (see [19–23]). Similarly, adding a diffusion factor to Eq.(3), then we obtain

$$\begin{cases} \dfrac{\partial u}{\partial t} = d_1 \Delta u + ru\left(1 - \dfrac{u}{k}\right) - \dfrac{bu^2 v}{a+u^2}, \\ \dfrac{\partial v}{\partial t} = d_2 \Delta v + \delta v\left(1 - \dfrac{hv}{m+u_\tau}\right), \end{cases} \tag{4}$$

where $u = u(x,t)$ and $v = v(x,t)$ represent the population densities of mature prey and predator at location x and time t, respectively; $u_\tau = u(x, t-\tau)$; $d_1 > 0$ and $d_2 > 0$ are the diffusion coefficient.

The cooperative behavior of some predators in finding and attacking the prey in order to improve their foraging efficiency, is called hunting cooperation and widespread in biological systems. For example, wolves, wild dogs and lions often work together to capture their preys [24]. In 2010, Berec [34] simulated a predator-prey model with foraging facilitation using ordinary differential equations, and explored the dynamic effects of different interference of different intensities on the predator-prey model. In 2017, Alves and Hilker [35] believed that the attack rate of predators is not constant, but increases with the increase of predator density, so it is essential that add a cooperation term to the attack rate of the predator population. Then, they proposed the hunting cooperation models with Holling types I, II, III and IV functional response, respectively. According to the ideas in [35], we also replace the predation rate b in Eq.(4) with $\rho(v) = \frac{bv}{a_1+v}$, where $a_1 > 0$ is a parameter describing predator cooperation in hunting. Then, in this paper, we consider this following model under Neumann boundary condition for non-negative initial values

$$\begin{cases} \dfrac{\partial u}{\partial t} = d_1\Delta u + ru\left(1-\dfrac{u}{k}\right) - \dfrac{bu^2v^2}{(a+u^2)(a_1+v)}, & \\ & x\in\Omega, t>0, \\ \dfrac{\partial v}{\partial t} = d_2\Delta v + \delta v\left(1-\dfrac{hv}{m+u_\tau}\right), & \\ \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial x} = 0, \quad x\in\partial\Omega, \quad t>0, & \\ u(x,t)\geq 0, \quad v(x,t)\geq 0, \quad x\in\bar{\Omega}, t\in[-\tau,0], & \end{cases} \tag{5}$$

where $\Omega = (0, l\pi)$ and l is a positive number. Other models with such hunting cooperation effect can be seen in [25–27].

We would like to mention that it is difficult to study the existence of global Hopf bifurcation in partial functional differential equation (PFDE) because of the existence of space factors. The global Hopf bifurcation of PFDE was investigated in [28–31] by using the global Hopf bifurcation result of Wu [32], and the authors proved the global attractiveness of the system without delay by using the method of upper and lower solutions. However, this method is not suitable for predator-prey model where the reaction function is monotone with respect to u_τ. In this paper, we use the comparison principle to obtain permanence of Eq.(5) and have the range of the positive periodic solution. In order to avoid the contradiction, the range is compressed by using the iterative method. Then, we establish Lyapunov function in this range to prove the attractiveness of positive steady state when $\tau = 0$. Finally, we obtain the conclusion about the existence of global Hopf bifurcation.

The rest of the paper is organized as follows. In section 2, we study the existence of positive steady state and local Hopf bifurcation for model (5). In section 3, we investigate permanence of model (5), and prove the attractiveness of positive steady state when $\tau = 0$ by establishing Lyapunov function. In section 4, we verified that the conclusions in section 2 and section 3 are reasonable by using numerical simulation.

2. Stability and local Hopf bifurcation

Firstly, we analyse the existence of constant steady state of Eq.(5). Obviously, the constant steady state of Eq.(5) satisfies

$$ru\left(1-\frac{u}{k}\right)-\frac{bu^2v^2}{(a+u^2)\,(a_1+v)}=0,\quad \delta v\left(1-\frac{hv}{m+u}\right)=0. \tag{6}$$

We can obtain the trivial and semi-trivial steady states of Eq.(5), which are $S_1=(0,0)$, $S_2=(0,\frac{m}{h})$ and $S_3=(k,0)$. In addition, the positive constant steady state $S_0=(u_0,v_0)$ satisfies

$$A_1u_0^4+B_1u_0^3+C_1u_0^2+D_1u_0+E_1=0,\quad v_0=\frac{m+u_0}{h},$$

where

$$\begin{aligned}&A_1=-\frac{rh}{k},\ B_1=rh-\frac{rh\,(a_1h+m)}{k}-b,\ C_1=rh\,(a_1h+m)-\frac{rah}{k}-2bm,\\&D_1=rah-\frac{rah\,(a_1h+m)}{k}-bm^2,\ E_1=rah\,(a_1h+m)\,.\end{aligned}$$

It's pretty obvious that $A_1<0$ and $E_1>0$, but the signs of B_1, C_1 and D_1 are uncertain. For convenience, we define

$$\phi(u)=A_1u^4+B_1u^3+C_1u^2+D_1u+E_1,\quad u\in\mathbb{R}_+=(0,+\infty).$$

To better analyze the conditions for the existence of positive constant steady state, we take the derivative of $\phi(u)$ with respect to u for $u\in\mathbb{R}_+$. Then we have

$$\begin{aligned}&\phi^{'}(u)=4A_1u^3+3B_1u^2+2C_1u+D_1,\\&\phi^{''}(u)=12A_1u^2+6B_1u+2C_1.\end{aligned}$$

For $\phi^{''}(u)=0$, a quadratic function of one variable, the discriminant is $\triangle=36B_1^2-96A_1C_1$, and the sign of $\triangle$ is indeterminate. By further analyzing the derivative function of $\phi(u)$, we can obtain the following lemma about the existence of positive constant steady state.

Lemma 2.1. *If (I_1) holds, Eq.(5) has a unique positive constant steady state, where*

(I_1): one of $(a)-(l)$ holds,

- *(a)* $\triangle\le 0$;
- *(b)* $\triangle>0$, $C_1>0$ *and* $D_1\ge 0$;
- *(c)* $\triangle>0$, $C_1>0$, $D_1<0$ *and* $\max\phi^{'}(u)\le 0$;
- *(d)* $\triangle>0$, $C_1>0$, $D_1<0$, $\max\phi^{'}(u)>0$, *and* $\phi(u_1)>0$ *or* $\phi(u_{11})<0$, *where* $\phi^{'}(u_1)=\phi^{'}(u_{11})=0$ *and* $0<u_1<u_{11}$;

(e) $\triangle > 0$, $B_1 > 0$, $C_1 < 0$, $D_1 > 0$, *and* $\phi^{'}(u_2) \geq 0$ *or* $\phi^{'}(u_{22}) \leq 0$, *where* $u_2 = \frac{-3B_1 - \sqrt{9B_1^2 - 24A_1C_1}}{12A_1}$, $u_{22} = \frac{-3B_1 + \sqrt{9B_1^2 - 24A_1C_1}}{12A_1}$;

(f) $\triangle > 0$, $B_1 > 0$, $C_1 < 0$, $D_1 > 0$, $\phi^{'}(u_2) < 0$, $\phi^{'}(u_{22}) > 0$, *and* $\phi(u_{33}) > 0$ *or* $\phi(u_{333}) < 0$, *where* $\phi^{'}(u_3) = \phi^{'}(u_{33}) = \phi^{'}(u_{333}) = 0$ *and* $0 < u_3 < u_{33} < u_{333}$;

(g) $\triangle > 0$, $B_1 > 0$, $C_1 < 0$, $D_1 \leq 0$ *and* $\phi^{'}(u_2) \leq 0$;

(h) $\triangle > 0$, $B_1 > 0$, $C_1 < 0$, $D_1 \leq 0$, $\phi^{'}(u_2) > 0$, *and* $\phi(u_4) > 0$ *or* $\phi(u_{44}) < 0$, *where* $\phi^{'}(u_4) = \phi^{'}(u_{44}) = 0$ *and* $0 < u_4 < u_{44}$;

(i) $\triangle > 0$, $B_1 > 0$, $C_1 = 0$ *and* $D_1 \geq 0$;

(j) $\triangle > 0$, $B_1 > 0$, $C_1 = 0$, $D_1 < 0$ *and* $\max \phi^{'}(u) \leq 0$;

(k) $\triangle > 0$, $B_1 > 0$, $C_1 = 0$, $D_1 < 0$, $\max \phi^{'}(u) > 0$, *and* $\phi(u_5) > 0$ *or* $\phi(u_{55}) < 0$, *where* $\phi^{'}(u_5) = \phi^{'}(u_{55}) = 0$ *and* $0 < u_5 < u_{55}$;

(l) $\triangle > 0$, $B_1 < 0$ *and* $C_1 \leq 0$.

The proof of this lemma is given in the appendix.

Next, taking τ as the bifurcation parameter, we study the stability of positive steady state and existence of local Hopf bifurcation. Define the real-valued Sobolev space and the abstract space, respectively,

$$X = \{(u, v) \in H^2(0, l\pi) \times H^2(0, l\pi) | u_x = v_x = 0, x = 0, l\pi\},$$
$$\mathcal{C} = C([-\tau, 0], X).$$

Let $F = (\Theta, \Upsilon)^{\mathrm{T}}$, where

$$\Theta = ru\left(1 - \frac{u}{k}\right) - \frac{bu^2v^2}{(a + u^2)(a_1 + v)}, \quad \Upsilon = \delta v\left(1 - \frac{hv}{m + u_\tau}\right).$$

The linearized system of Eq.(5) at positive constant steady state S_0 in the phase space $\mathcal{C}$ can be expressed as

$$\dot{U}(t) = D\Delta U(x, t) + AU(x, t) + BU(x, t - \tau), \tag{7}$$

where $U(x, t) = (u(x, t), v(x, t))^{\mathrm{T}}$, $D = diag(d_1, d_2)$, and

$$A = \frac{\partial F}{\partial U(x, t)}\Big|_{(u_0, v_0)} = \begin{pmatrix} -\alpha(u_0) & -\beta(u_0) \\ 0 & -\delta \end{pmatrix},$$
$$B = \frac{\partial F}{\partial U(x, t - \tau)}\Big|_{(u_0, v_0)} = \begin{pmatrix} 0 & 0 \\ \frac{\delta}{h} & 0 \end{pmatrix},$$

with

$$\alpha(u_0) = -r + \frac{2ru_0}{k} + \frac{2abu_0(m + u_0)^2}{h(a + u_0^2)^2(a_1h + m + u_0)},$$
$$\beta(u_0) = \frac{bu_0^2(m + u_0)(m + u_0 + 2a_1h)}{(a + u_0^2)(a_1h + m + u_0)^2} > 0.$$

For convenience, denote $L(U_t) = AU(x,t) + BU(x,t-\tau)$. Then the linearized system Eq.(7) is equivalent to

$$\dot{U}(t) = D\Delta U(x,t) + L(U_t). \tag{8}$$

According to Wu [32], we have that the characteristic equation for the linearized system Eq.(8) is as follow

$$\lambda g - D\Delta g - L(e^{\lambda \cdot} g) = 0, \qquad g \in dom(D\Delta), \quad g \neq 0, \tag{9}$$

where

$$dom(D\Delta) = \left\{(u,v)^T : u, v \in C^2([0,l\pi],\mathbb{R}), u_x = 0, v_x = 0 \ at \ x = 0, l\pi\right\}.$$

Considering the following eigenvalue problem under Neummann boundary condition

$$-\Delta\varphi = \mu\varphi, \ x \in (0,l\pi), \ \varphi'(0) = \varphi'(l\pi) = 0. \tag{10}$$

It is well know that Eq.(10) has eigenvalues $\mu_n = \frac{n^2}{l^2}$, $(n = 0, 1, 2\cdots)$ and corresponding eigenfunctions $\varphi_n(x) = \cos\frac{n}{l}x$. Substituting

$$g = \sum_{n=0}^{\infty} \cos\frac{n}{l}x \begin{pmatrix} g_{1n} \\ g_{2n} \end{pmatrix},$$

into Eq.(9), we obtain

$$\begin{pmatrix} d_1\frac{n^2}{l^2} + \alpha(u_0) & \beta(u_0) \\ -\frac{\delta}{h}e^{-\lambda\tau} & d_2\frac{n^2}{l^2} + \delta \end{pmatrix} \begin{pmatrix} g_{1n} \\ g_{2n} \end{pmatrix} = \lambda \begin{pmatrix} g_{1n} \\ g_{2n} \end{pmatrix}.$$

Consequently, the characteristic equation of Eq.(7) can be expressed as

$$\Delta_n(\lambda) := \lambda^2 + T_n\lambda + D_n + He^{-\lambda\tau} = 0, \quad n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}, \tag{11}$$

where

$$\begin{aligned} T_n &= \frac{(d_1+d_2)n^2}{l^2} + T_0, \\ D_n &= \frac{d_1 d_2 n^4}{l^4} + (\delta d_1 + d_2\alpha(u_0))\frac{n^2}{l^2} + D_0, \\ H &= \frac{\delta}{h}\beta(u_0) > 0, \end{aligned} \tag{12}$$

with

$$T_0 = \alpha(u_0) + \delta, \quad D_0 = \alpha(u_0)\delta.$$

Define

$$(I_2) : \alpha(u_0) > \max\left\{-\delta, -\frac{d_1\delta}{d_2}, -\frac{\beta(u_0)}{h}\right\}.$$

Then we can obtain the following results.

Lemma 2.2. *If (I_1) and (I_2) hold, then all roots of Eq.(11) have negative real parts when $\tau = 0$.*

Proof. Assume that (I_1) holds. When $\tau = 0$, Eq.(11) is equivalent to

$$\lambda^2 + T_n\lambda + D_n + H = 0. \tag{13}$$

If (I_2) holds, then we have $T_0 > 0$, $d_1\delta + d_2\alpha(u_0) > 0$ and $D_0 + H > 0$. It follows that

$$T_n = \frac{(d_1 + d_2)n^2}{l^2} + T_0 > 0,$$
$$D_n + H = \frac{d_1 d_2 n^4}{l^4} + (\delta d_1 + d_2\alpha(u_0))\frac{n^2}{l^2} + D_0 + H > 0.$$

Hence, all roots of Eq.(13) have negative real parts. The proof is complete. □

Lemma 2.3. *If (I_1) and (I_2) hold, then $\lambda = 0$ is not a root of Eq.(11) for any $\tau \geq 0$.*

Proof. Suppose (I_1) and (I_2) hold. Combined with the proof of lemma 2.2, we known that $D_n + H > 0$ for any $n \in \mathbb{N}_0$. If $\lambda = 0$ is the root of Eq.(11), there exists $n \in \mathbb{N}_0$ such that $D_n + H = 0$. There is a contradiction between the two results. Thus, $\lambda = 0$ is not a root of Eq.(11) for any $\tau \geq 0$. The proof is complete. □

By the above lemmas, we have that as time delay τ increases, the stability at (u_0, v_0) changes when the roots of Eq.(11) pass through the imaginary axis into the right. Next, we will analyze when does Eq.(11) have pure imaginary roots. Note that D_n increases monotonically with respect to n. Therefore, if $D_0 - H \geq 0$, then we have $D_n - H \geq 0$ for any $n \in \mathbb{N}_0$ and if $D_0 - H < 0$, then there exists $n_* \in \mathbb{N}_0$ such that

$$\begin{cases} D_n - H \geq 0, & n > n_*, n \in \mathbb{N}_0, \\ D_n - H < 0, & n \leq n_*, n \in \mathbb{N}_0. \end{cases}$$

For convenience, we define $\mathbb{N}_1 = \{n \mid n > n_*, n \in \mathbb{N}_0\}$ and $\mathbb{N}_2 = \{n \mid n \leq n_*, n \in \mathbb{N}_0\}$.

Lemma 2.4. *Assume that I_2 holds.*

(i) If $h\alpha(u_0) - \beta(u_0) \geq 0$, then $\Delta_n(\lambda) = 0$ has no pure imaginary root for $n \in \mathbb{N}_0$;
(ii) If $h\alpha(u_0) - \beta(u_0) < 0$, then $\Delta_n(\lambda) = 0$ has no pure imaginary root for $n \in \mathbb{N}_1$ and a unique pair of pure imaginary roots $\pm i\omega_n$ when $\tau = \tau_{n,j}$ for $n \in \mathbb{N}_2$ and $j \in \mathbb{N}_0$, where

$$\omega_n = \sqrt{\frac{-(T_n^2 - 2D_n) + \sqrt{(T_n^2 - 2D_n)^2 - 4(D_n^2 - H^2)}}{2}},$$
$$\tau_{n,j} = \frac{1}{\omega_n}\left(\arccos\frac{-\omega_n^2 + D_n}{-H} + 2j\pi\right).$$

Proof. Let $\lambda = i\omega_n$ ($\omega_n > 0$) be a root of $\Delta_n(\lambda) = 0$. Substituting it into $\Delta_n(\lambda) = 0$, we have

$$-\omega_n^2 + D_n + T_n\omega_n i + H(\cos\omega_n\tau - i\sin\omega_n\tau) = 0. \tag{14}$$

Separating the real and imaginary parts of Eq.(14), we can obtain that

$$\begin{aligned} &\omega_n^2 - D_n = H\cos\omega_n\tau, \\ &T_n\omega_n = H\sin\omega_n\tau. \end{aligned} \tag{15}$$

Squaring and adding both equations of Eq.(15), one has

$$\omega_n^4 + (T_n^2 - 2D_n)\omega_n^2 + D_n^2 - H^2 = 0. \tag{16}$$

Let $z = \omega_n^2$, then Eq.(16) is equivalent to

$$z^2 + (T_n^2 - 2D_n)z + D_n^2 - H^2 = 0. \tag{17}$$

Under the assumption (I_2), we know that $D_n + H > 0$ for any $n \in \mathbb{N}_0$. Furthermore, it is clear that

$$T_n^2 - 2D_n = (d_1^2 + d_2^2)\frac{n^4}{l^4} + \alpha^2(u_0) + \delta^2 + 2(d_1\alpha(u_0) + d_2\delta)\frac{n^2}{l^2} > 0.$$

If $h\alpha(u_0) - \beta(u_0) \geq 0$, it is easy to see that $D_0 - H \geq 0$. Then for any $n \in \mathbb{N}_0$, we have $D_n^2 - H^2 \geq 0$. It follows that Eq.(17) has no positive real root and $\Delta_n(\lambda) = 0$ has no pure imaginary root. The proof of (i) is complete.

If $h\alpha(u_0) - \beta(u_0) < 0$, then we have $D_0 - H < 0$. Note that $D_n - H \geq 0$ for $n \in \mathbb{N}_1$, which implies that $\Delta_n(\lambda) = 0$ has no pure imaginary root. For $n \in \mathbb{N}_2$, we have $D_n - H < 0$, which implies that $D_n^2 - H^2 < 0$. It follows that Eq.(17) has a positive real root and $\Delta_n(\lambda) = 0$ has a pair of pure imaginary roots $\pm i\omega_n$, where

$$\omega_n = \sqrt{z^+} = \left(\frac{-(T_n^2 - 2D_n) + \sqrt{(T_n^2 - 2D_n)^2 - 4(D_n^2 - H^2)}}{2}\right)^{\frac{1}{2}}.$$

In addition, according to Eq.(15), we have

$$\sin\omega_n\tau = \frac{T_n\omega_n}{H} > 0, \quad \cos\omega_n\tau = \frac{-\omega_n^2 + D_n}{-H}.$$

Consequently,

$$\tau_{n,j} = \frac{1}{\omega_n}\left(\arccos\frac{-\omega_n^2 + D_n}{-H} + 2j\pi\right), \quad j \in \mathbb{N}_0. \tag{18}$$

The proof of (ii) is complete. □

Next, we state the following lemma about the transversal condition.

Lemma 2.5. *Assume that* (I_2) *holds and* $h\alpha(u_0) - \beta(u_0) < 0$. *Let* $\lambda(\tau) = \alpha(\tau) \pm i\beta(\tau)$ *be the root of* $\Delta_n(\lambda) = 0$ *satisfying* $\alpha(\tau_{n,j}) = 0$ *and* $\beta(\tau_{n,j}) = \omega_n$ *for* $n \in \mathbb{N}_2$ *and* $j \in \mathbb{N}_0$. *Then the transversal condition is* $Re\frac{d\lambda}{d\tau}\big|_{\tau=\tau_{n,j}} > 0$.

Proof. Treating τ as a function of λ in $\Delta_n(\lambda) = 0$ and differentiating both sides of $\Delta_n(\lambda) = 0$ gives that

$$\frac{d\tau}{d\lambda}\bigg|_{\tau=\tau_{n,j}} = \frac{T_n + 2\omega_n i}{T_n\omega_n^2 + (\omega_n^3 - D_n\omega_n)i} - \frac{\tau}{\omega_n i}.$$

Therefore,

$$\begin{aligned}\text{Sign}\left(Re\frac{d\lambda}{d\tau}\bigg|_{\tau=\tau_{n,j}}\right) &= \text{Sign}\left\{\left(Re\frac{d\tau}{d\lambda}\bigg|_{\tau=\tau_{n,j}}\right)^{-1}\right\} \\ &= \text{Sign}\left\{\left(\frac{\sqrt{(T_n^2 - 2D_n)^2 - 4(D_n^2 - H^2)}}{T_n^2\omega_n^2 + (\omega_n^2 - D_n)^2}\right)^{-1}\right\} = 1.\end{aligned}$$

The proof is complete. □

Note that $\tau_{n,j}$ is monotonically increasing with respect to j, which implies that $\min\limits_{n\in\mathbb{N}_2, j\in\mathbb{N}_0}\{\tau_{n,j}\} = \min\limits_{n\in\mathbb{N}_2}\{\tau_{n,0}\} := \tau_0$. Thus, all roots of Eq.(11) have negative real parts when $\tau < \tau_0$, and Eq.(11) has at least two roots with positive real parts when $\tau > \tau_0$. Next we can state the following theorem on the stability of positive steady state (u_0, v_0) and the existence of local Hopf bifurcation.

Theorem 2.6. *Assume that* (I_1) *and* (I_2) *hold.*

(i) If $h\alpha(u_0) - \beta(u_0) \geq 0$, *then the positive steady state* (u_0, v_0) *of Eq.(5) is locally asymptotically stable for any* $\tau \geq 0$;

(ii) If $h\alpha(u_0) - \beta(u_0) < 0$, *then the positive steady state* (u_0, v_0) *of Eq.(5) is locally asymptotically stable for* $\tau \in [0, \tau_0)$ *and unstable for* $\tau > \tau_0$; *furthermore, Hopf bifurcation occurs at* (u_0, v_0) *when* $\tau = \tau_{n,j}$, *where* $\tau_0 = \min\{\tau_{n,0}\}$ *and* $n \in \mathbb{N}_2$, $j \in \mathbb{N}_0$.

For convenience, we define

$$(I_3) : h\alpha(u_0) - \beta(u_0) < 0.$$

Obviously, under the assumption of (I_1)-(I_3), the local Hopf bifurcation will occur near the positive steady state (u_0, v_0).

3. Global Hopf bifurcation

In this section, we investigate the global continuation of periodic solutions bifurcating from the positive steady state (u_0, v_0) by using global Hopf bifurcation theorem given by Wu [32]. Firstly, we prove permanence of Eq.(5) to obtain the range where periodic solutions exist. Secondly, we compress the range by using the iterative method to avoid contradiction. Thirdly, by establishing a Lyapunov function in the compressed range, we investigate the global attractiveness of the positive steady state (u_0, v_0) when $\tau = 0$. Finally, according to [32], we have the results about the global Hopf bifurcation.

Lemma 3.1. *If $u(x,0) \not\equiv 0$ and $v(x,0) \not\equiv 0$ for $x \in \bar{\Omega}$, then Eq.(5) has the permanence properties. To be specific,*

$$\limsup_{t\to\infty} \max_{x\in\bar{\Omega}} u(x,t) \le \tilde{u}_0, \qquad \limsup_{t\to\infty} \max_{x\in\bar{\Omega}} v(x,t) \le \tilde{v}_0,$$
$$\liminf_{t\to\infty} \min_{x\in\bar{\Omega}} u(x,t) \ge \hat{u}_0, \qquad \liminf_{t\to\infty} \min_{x\in\bar{\Omega}} v(x,t) \ge \hat{v}_0,$$

where

$$\tilde{u}_0 = k, \quad \tilde{v}_0 = \frac{m+\tilde{u}_0}{h}, \quad \hat{u}_0 = \frac{kraa_1}{raa_1 + bk\tilde{v}_0^2}, \quad \hat{v}_0 = \frac{m+\hat{u}_0}{h}.$$

Proof. Obviously, we have that $u(x,t) > 0$ and $v(x,t) > 0$ for any $x \in \bar{\Omega}$ and $t > 0$ by the maximum principle. From the first equation of Eq.(5),

$$\frac{\partial u}{\partial t} \le d_1 \Delta u + ru\left(1 - \frac{u}{k}\right), \quad x \in \Omega, t > 0.$$

Suppose that $\tilde{U}_0(x,t)$ is the solution of the following equation

$$\begin{cases} \dfrac{\partial \tilde{U}_0}{\partial t} = d_1 \Delta \tilde{U}_0 + r\tilde{U}_0 \left(1 - \dfrac{\tilde{U}_0}{k}\right), & x \in \Omega, t > 0, \\ \dfrac{\partial \tilde{U}_0}{\partial x} = 0, \quad x \in \partial\Omega,\ t > 0, \\ \tilde{U}_0(x,0) = u(x,0), \quad x \in \bar{\Omega}. \end{cases}$$

By the comparison principle, we have $u(x,t) \le \tilde{U}_0(x,t)$ for $x \in \bar{\Omega}$ and $t > 0$. Note that $\lim_{t\to\infty} \tilde{U}_0(x,t) = k$ for any $x \in \bar{\Omega}$. Thus

$$\limsup_{t\to\infty} \max_{x\in\bar{\Omega}} u(x,t) \le k := \tilde{u}_0.$$

Then, for any $\varepsilon_1 > 0$, there is $T_1 > 0$ such that $u(x, t-\tau) \le \tilde{u}_0 + \varepsilon_1$ for $x \in \bar{\Omega}$ and $t > T_1$. From the second equation of Eq.(5),

$$\frac{\partial v}{\partial t} \le d_2 \Delta v + \delta v \left(1 - \frac{hv}{m + \tilde{u}_0 + \varepsilon_1}\right), \quad x \in \Omega, t > T_1.$$

Let $\tilde{V}_0(x,t)$ be the solution of the following equation

$$\begin{cases} \dfrac{\partial \tilde{V}_0}{\partial t} = d_2\Delta \tilde{V}_0 + \delta \tilde{V}_0 \left(1 - \dfrac{h\tilde{V}_0}{m+\tilde{u}_0+\varepsilon_1}\right), \quad x\in\Omega, t>T_1, \\ \dfrac{\partial \tilde{V}_0}{\partial x} = 0, \quad x\in\partial\Omega, t>T_1, \\ \tilde{V}_0(x,T_1) = v(x,T_1), \quad x\in\bar{\Omega}. \end{cases}$$

By the comparison principle, we know that $v(x,t) \le \tilde{V}_0(x,t)$ for $x\in\bar{\Omega}$ and $t > T_1$. Moreover, it is obvious that $\lim\limits_{t\to\infty} \tilde{V}_0(x,t) = \frac{m+\tilde{u}_0+\varepsilon_1}{h}$ for any $x\in\bar{\Omega}$. Since ε_1 is arbitrary, we can obtain

$$\limsup_{t\to\infty}\max_{x\in\bar{\Omega}} v(x,t) \le \frac{m+\tilde{u}_0}{h} := \tilde{v}_0.$$

Then, for any $\varepsilon_2 > 0$, there exists $T_2 > T_1$ such that $v(x,t) \le \tilde{v}_0 + \varepsilon_2$ for $x\in\bar{\Omega}$ and $t > T_2$. Combined with the first equation of Eq.(5),

$$\frac{\partial u}{\partial t} \ge d_1\Delta u + ru\left[1 - \left(\frac{1}{k} + \frac{b(\tilde{v}_0+\varepsilon_2)^2}{aa_1 r}\right)u\right], \quad x\in\Omega, t>T_2.$$

Assume that $\hat{U}_0(x,t)$ is the solution of the following equation

$$\begin{cases} \dfrac{\partial \hat{U}_0}{\partial t} = d_1\Delta \hat{U}_0 + r\hat{U}_0 \left[1 - \left(\dfrac{1}{k} + \dfrac{b(\tilde{v}_0+\varepsilon_2)^2}{aa_1 r}\right)\hat{U}_0\right], \quad x\in\Omega, t>T_2, \\ \dfrac{\partial \hat{U}_0}{\partial x} = 0, \quad x\in\partial\Omega, t>T_2, \\ \hat{U}_0(x,T_2) = u(x,T_2), \quad x\in\bar{\Omega}. \end{cases}$$

By the comparison principle, we have $u(x,t) \ge \hat{U}_0(x,t)$ for $x\in\bar{\Omega}$ and $t > T_2$. In addition, it is easy to see that

$$\lim_{t\to\infty} \hat{U}_0(x,t) = \frac{kraa_1}{raa_1 + bk(\tilde{v}_0+\varepsilon_2)^2}, \quad x\in\bar{\Omega}.$$

It follows from the arbitrariness of ε_2 that

$$\liminf_{t\to\infty}\min_{x\in\bar{\Omega}} u(x,t) \ge \frac{kraa_1}{raa_1 + bk\tilde{v}_0^2} := \hat{u}_0.$$

Let $\varepsilon_3 > 0$ and ε_3 is sufficiently small. Then, there exists $T_3 > T_2$ such that $u(x,t-\tau) \ge \hat{u}_0 - \varepsilon_3 > 0$ for $x\in\bar{\Omega}$ and $t > T_3$. Combined with the second equation of Eq.(5),

$$\frac{\partial v}{\partial t} \ge d_2\Delta v + \delta v\left(1 - \frac{hv}{m+\hat{u}_0-\varepsilon_3}\right), \quad x\in\Omega, t>T_3.$$

Suppose that $\hat{V}_0(x,t)$ is the solution of the following equation

$$\begin{cases} \dfrac{\partial \hat{V}_0}{\partial t} = d_2 \Delta \hat{V}_0 + \delta \hat{V}_0 \left(1 - \dfrac{h\hat{V}_0}{m + \hat{u}_0 - \varepsilon_3}\right), \quad x \in \Omega, t > T_3, \\ \dfrac{\partial \hat{V}_0}{\partial x} = 0, \quad x \in \partial\Omega, t > T_3, \\ \hat{V}_0(x, T_3) = v(x, T_3), \quad x \in \bar{\Omega}. \end{cases}$$

By the comparison principle, we know that $v(x,t) \geq \hat{V}_0(x,t)$ for $x \in \bar{\Omega}$ and $t > T_3$. Notice that $\lim\limits_{t\to\infty} \hat{V}_0(x,t) = \frac{m+\hat{u}_0-\varepsilon_3}{h}$ for any $x \in \bar{\Omega}$. From the arbitrariness of ε_3, we have

$$\liminf_{t\to\infty} \min_{x\in\bar{\Omega}} v(x,t) \geq \frac{m + \hat{u}_0}{h} := \hat{v}_0.$$

The proof is complete. □

From Lemma 3.1, we can obtain that the positive periodic solutions of Eq.(5) are in $G_0 = [\hat{u}_0, \tilde{u}_0] \times [\hat{v}_0, \tilde{v}_0]$. That is, all of the positive periodic solutions of Eq.(5) are uniformly bounded. Next, We compress the region G_0 where positive periodic solutions exist into a smaller range by using the iterative method.

Lemma 3.2. *If $u(x,0) \not\equiv 0$ and $v(x,0) \not\equiv 0$ for $x \in \bar{\Omega}$, then every solution of Eq.(5) is attracted to $G_* = [\hat{u}, \tilde{u}] \times [\hat{v}, \tilde{v}]$, where*

$$\tilde{u} = \lim_{n\to\infty} \tilde{u}_n, \quad \hat{u} = \lim_{n\to\infty} \hat{u}_n, \quad \tilde{v} = \lim_{n\to\infty} \tilde{v}_n, \quad \hat{v} = \lim_{n\to\infty} \hat{v}_n, \tag{19}$$

with

$$\begin{aligned} \tilde{u}_{n+1} &= \frac{kr(a+\tilde{u}_n^2)(a_1+\hat{v}_n)}{r(a+\tilde{u}_n^2)(a_1+\hat{v}_n)+kb\hat{v}_n^2}, \quad \tilde{v}_{n+1} = \frac{m+\tilde{u}_{n+1}}{h}, \\ \hat{u}_{n+1} &= \frac{kr(a+\hat{u}_n^2)(a_1+\tilde{v}_{n+1})}{r(a+\hat{u}_n^2)(a_1+\tilde{v}_{n+1})+kb\tilde{v}_{n+1}^2}, \quad \hat{v}_{n+1} = \frac{m+\hat{u}_{n+1}}{h}. \end{aligned} \tag{20}$$

and $\tilde{u}_0$, $\tilde{v}_0$, $\hat{u}_0$ and $\hat{v}_0$ are defined in Lemma 3.1.

Proof. Let $\varepsilon_4 > 0$ and ε_4 is small enough. Then, there is $T_4 > T_3$ such that $u(x,t) \leq \tilde{u}_0 + \varepsilon_4$ and $v(x,t) \geq \hat{v}_0 - \varepsilon_4 > 0$ for $x \in \bar{\Omega}$ and $t > T_4$. According to the first equation of Eq.(5), we have

$$\frac{\partial u}{\partial t} \leq d_1 \Delta u + ru\left[1 - \left(\frac{1}{k} + \frac{b(\hat{v}_0 - \varepsilon_4)^2}{r(a + (\tilde{u}_0 + \varepsilon_4)^2)(a_1 + \hat{v}_0 - \varepsilon_4)}\right)u\right], x \in \Omega, t > T_4.$$

Assume that $\tilde{U}_1(x,t)$ is the solution of the following equation

$$\begin{cases} \dfrac{\partial \tilde{U}_1}{\partial t} = d_1 \Delta \tilde{U}_1 + r\tilde{U}_1 \left[1 - \left(\dfrac{1}{k} + \dfrac{b(\hat{v}_0 - \varepsilon_4)^2}{r(a + (\tilde{u}_0 + \varepsilon_4)^2)(a_1 + \hat{v}_0 - \varepsilon_4)}\right)\tilde{U}_1\right], \ x \in \Omega, t > T_4, \\ \dfrac{\partial \tilde{U}_1}{\partial x} = 0, \quad x \in \partial\Omega, t > T_4, \\ \tilde{U}_1(x, T_4) = u(x, T_4), \quad x \in \bar{\Omega}. \end{cases}$$

By the comparison principle, we have $u(x,t) \le \tilde{U}_1(x,t)$ for $x \in \bar{\Omega}$ and $t > T_4$. From

$$\lim_{t\to\infty} \tilde{U}_1(x,t) = \frac{kr(a+(\tilde{u}_0+\varepsilon_4)^2)(a_1+\hat{v}_0-\varepsilon_4)}{r(a+(\tilde{u}_0+\varepsilon_4)^2)(a_1+\hat{v}_0-\varepsilon_4)+kb(\hat{v}_0-\varepsilon_4)^2}, \quad x \in \bar{\Omega},$$

and the arbitrariness of ε_4, we have

$$\limsup_{t\to\infty} \max_{x\in\bar{\Omega}} u(x,t) \le \frac{kr(a+\tilde{u}_0^2)(a_1+\hat{v}_0)}{r(a+\tilde{u}_0^2)(a_1+\hat{v}_0)+kb\hat{v}_0^2} := \tilde{u}_1.$$

For any $\varepsilon_5 > 0$, there exists $T_5 > T_4$ such that $u(x,t-\tau) \le \tilde{u}_1 + \varepsilon_5$ for $x \in \bar{\Omega}$ and $t > T_5$. From the second equation of Eq.(5),

$$\frac{\partial v}{\partial t} \le d_2 \Delta v + \delta v \left(1 - \frac{hv}{m+\tilde{u}_1+\varepsilon_5}\right), \quad x \in \Omega, t > T_5.$$

Let $\tilde{V}_1(x,t)$ be the solution of the following equation

$$\begin{cases} \dfrac{\partial \tilde{V}_1}{\partial t} = d_2 \Delta \tilde{V}_1 + \delta \tilde{V}_1 \left(1 - \dfrac{h\tilde{V}_1}{m+\tilde{u}_1+\varepsilon_5}\right), \quad x \in \Omega, t > T_5, \\ \dfrac{\partial \tilde{V}_1}{\partial x} = 0, \quad x \in \partial\Omega, t > T_5, \\ \tilde{V}_1(x,T_5) = v(x,T_5), \quad x \in \bar{\Omega}. \end{cases}$$

By the comparison principle, we know that $v(x,t) \le \tilde{V}_1(x,t)$ for $x \in \bar{\Omega}$ and $t > T_5$. Moreover, it is obvious that $\lim_{t\to\infty} \tilde{V}_1(x,t) = \frac{m+\tilde{u}_1+\varepsilon_5}{h}$ for any $x \in \bar{\Omega}$. From the arbitrariness of ε_5,

$$\limsup_{t\to\infty} \max_{x\in\bar{\Omega}} v(x,t) \le \frac{m+\tilde{u}_1}{h} := \tilde{v}_1.$$

Let $\varepsilon_6 > 0$ and ε_6 is small enough. Then, there exists $T_6 > T_5$ such that $u(x,t) \ge \hat{u}_0 - \varepsilon_6 > 0$ and $v(x,t) \le \tilde{v}_1 + \varepsilon_6$ for $x \in \bar{\Omega}$ and $t > T_6$. Combined with the first equation of Eq.(5),

$$\frac{\partial u}{\partial t} \ge d_1 \Delta u + ru \left[1 - \left(\frac{1}{k} + \frac{b(\tilde{v}_1+\varepsilon_6)^2}{r(a+(\hat{u}_0-\varepsilon_6)^2)(a_1+\tilde{v}_1+\varepsilon_6)}\right) u\right], x \in \Omega, t > T_6.$$

Assume that $\hat{U}_1(x,t)$ is the solution of the following equation

$$\begin{cases} \dfrac{\partial \hat{U}_1}{\partial t} = d_1 \Delta \hat{U}_1 + r\hat{U}_1 \left[1 - \left(\dfrac{1}{k} + \dfrac{b(\tilde{v}_1+\varepsilon_6)^2}{r(a+(\hat{u}_0-\varepsilon_6)^2)(a_1+\tilde{v}_1+\varepsilon_6)}\right) \hat{U}_1\right], \ x \in \Omega, t > T_6, \\ \dfrac{\partial \hat{U}_1}{\partial x} = 0, \quad x \in \partial\Omega, t > T_6, \\ \hat{U}_1(x,T_6) = u(x,T_6), \quad x \in \bar{\Omega}. \end{cases}$$

By the comparison principle, we have $u(x,t) \geq \hat{U}_1(x,t)$ for $x \in \bar{\Omega}$ and $t > T_6$. In addition, it is easy to know that

$$\lim_{t\to\infty} \hat{U}_1(x,t) = \frac{kr(a+(\hat{u}_0-\varepsilon_6)^2)(a_1+\tilde{v}_1+\varepsilon_6)}{r(a+(\hat{u}_0-\varepsilon_6)^2)(a_1+\tilde{v}_1+\varepsilon_6)+kb(\tilde{v}_1+\varepsilon_6)^2}, \quad x \in \bar{\Omega}.$$

From the arbitrariness of ε_6,

$$\liminf_{t\to\infty} \min_{x\in\bar{\Omega}} u(x,t) \geq \frac{kr(a+\hat{u}_0^2)(a_1+\tilde{v}_1)}{r(a+\hat{u}_0^2)(a_1+\tilde{v}_1)+kb\tilde{v}_1^2} := \hat{u}_1.$$

Let $\varepsilon_7 > 0$ and ε_7 is sufficiently small. Then, there is $T_7 > T_6$ such that $u(x,t-\tau) \geq \hat{u}_1 - \varepsilon_7 > 0$ for $x \in \bar{\Omega}$ and $t > T_7$. Combined with the second equation of Eq.(5),

$$\frac{\partial v}{\partial t} \geq d_2\Delta v + \delta v\left(1 - \frac{hv}{m+\hat{u}_1-\varepsilon_7}\right), \quad x \in \Omega, t > T_7.$$

Suppose that $\hat{V}_1(x,t)$ is the solution of the following equation

$$\begin{cases} \dfrac{\partial \hat{V}_1}{\partial t} = d_2\Delta\hat{V}_1 + \delta\hat{V}_1\left(1 - \dfrac{h\hat{V}_1}{m+\hat{u}_1-\varepsilon_7}\right), \quad x \in \Omega, t > T_7, \\ \dfrac{\partial \hat{V}_1}{\partial x} = 0, \quad x \in \partial\Omega, t > T_7, \\ \hat{V}_1(x,T_7) = v(x,T_7), \quad x \in \bar{\Omega}. \end{cases}$$

By the comparison principle, we know that $v(x,t) \geq \hat{V}_1(x,t)$ for $x \in \bar{\Omega}$ and $t > T_7$. Note that $\lim\limits_{t\to\infty} \hat{V}_1(x,t) = \frac{m+\hat{u}_1-\varepsilon_7}{h}$ for any $x \in \bar{\Omega}$. From the arbitrariness of ε_7,

$$\liminf_{t\to\infty} \min_{x\in\bar{\Omega}} v(x,t) \geq \frac{m+\hat{u}_1}{h} := \hat{v}_1.$$

Note that $\frac{(m+s)^2}{a_1h+m+s}$ is an increasing function for $s > 0$. Then, by calculation, we have that

$$\hat{u}_0 = \frac{k}{1+\frac{kb(m+\tilde{u}_0)^2}{rh^2aa_1}} < \tilde{u}_1 = \frac{k}{1+\frac{kb(m+\hat{u}_0)^2}{rh(a+\tilde{u}_0^2)(a_1h+m+\hat{u}_0)}} < \tilde{u}_0 = k,$$

$$\hat{u}_0 = \frac{k}{1+\frac{kb(m+\tilde{u}_0)^2}{rh^2aa_1}} < \hat{u}_1 = \frac{k}{1+\frac{kb(m+\tilde{u}_1)^2}{rh^2(a+\hat{u}_0^2)(a_1+\tilde{v}_1)}} < \tilde{u}_0 = k,$$

$$\hat{u}_1 = \frac{k}{1+\frac{kb(m+\tilde{u}_1)^2}{rh(a+\hat{u}_0^2)(a_1h+m+\tilde{u}_1)}} < \frac{k}{1+\frac{kb(m+\hat{u}_0)^2}{rh(a+\tilde{u}_0^2)(a_1h+m+\hat{u}_0)}} = \tilde{u}_1.$$

That is to say, $\hat{u}_0 < \hat{u}_1 < \tilde{u}_1 < \tilde{u}_0$. Similarly, we can obtain that $\hat{v}_0 < \hat{v}_1 < \tilde{v}_1 < \tilde{v}_0$.

Then, define four sequences $\{\tilde{u}_n\}$, $\{\tilde{v}_n\}$, $\{\hat{u}_n\}$ and $\{\hat{v}_n\}$ for $n \in \mathbb{N}$ as follows

$$\tilde{u}_n = \frac{kr(a+\tilde{u}_{n-1}^2)(a_1+\hat{v}_{n-1})}{r(a+\tilde{u}_{n-1}^2)(a_1+\hat{v}_{n-1})+kb\hat{v}_{n-1}^2}, \quad \tilde{v}_n = \frac{m+\tilde{u}_n}{h},$$
$$\hat{u}_n = \frac{kr(a+\hat{u}_{n-1}^2)(a_1+\tilde{v}_n)}{r(a+\hat{u}_{n-1}^2)(a_1+\tilde{v}_n)+kb\tilde{v}_n^2}, \quad \hat{v}_n = \frac{m+\hat{u}_n}{h},$$

where $\tilde{u}_0$, $\tilde{v}_0$, $\hat{u}_0$ and $\hat{v}_0$ are defined in Lemma 3.1. Assume that

$$\limsup_{t\to\infty}\max_{x\in\bar{\Omega}} u(x,t) \le \tilde{u}_n, \quad \limsup_{t\to\infty}\max_{x\in\bar{\Omega}} v(x,t) \le \tilde{v}_n,$$
$$\liminf_{t\to\infty}\min_{x\in\bar{\Omega}} u(x,t) \ge \hat{u}_n, \quad \liminf_{t\to\infty}\min_{x\in\bar{\Omega}} v(x,t) \ge \hat{v}_n,$$

and $\hat{u}_{n-1} < \hat{u}_n < \tilde{u}_n < \tilde{u}_{n-1}$, $\hat{v}_{n-1} < \hat{v}_n < \tilde{v}_n < \tilde{v}_{n-1}$. Next, we will prove that

$$\limsup_{t\to\infty}\max_{x\in\bar{\Omega}} u(x,t) \le \tilde{u}_{n+1}, \quad \limsup_{t\to\infty}\max_{x\in\bar{\Omega}} v(x,t) \le \tilde{v}_{n+1},$$
$$\liminf_{t\to\infty}\min_{x\in\bar{\Omega}} u(x,t) \ge \hat{u}_{n+1}, \quad \liminf_{t\to\infty}\min_{x\in\bar{\Omega}} v(x,t) \ge \hat{v}_{n+1},$$

and $\hat{u}_n < \hat{u}_{n+1} < \tilde{u}_{n+1} < \tilde{u}_n$, $\hat{v}_n < \hat{v}_{n+1} < \tilde{v}_{n+1} < \tilde{v}_n$.

For any sufficiently small positive number ϵ_1, there is $\bar{T}_1 > 0$ such that $u(x,t) \le \tilde{u}_n + \epsilon_1$ and $v(x,t) \ge \hat{v}_n - \epsilon_1 > 0$ for $x \in \bar{\Omega}$ and $t > \bar{T}_1$. According to the first equation of Eq.(5), we have

$$\frac{\partial u}{\partial t} \le d_1\Delta u + ru\left[1 - \left(\frac{1}{k} + \frac{b(\hat{v}_n-\epsilon_1)^2}{r(a+(\tilde{u}_n+\epsilon_1)^2)(a_1+\hat{v}_n-\epsilon_1)}\right)u\right], \quad x \in \Omega, t > \bar{T}_1.$$

Then, assume that $\tilde{U}_{n+1}(x,t)$ is the solution of the following equation

$$\begin{cases} \dfrac{\partial \tilde{U}_{n+1}}{\partial t} = d_1\Delta\tilde{U}_{n+1} + r\tilde{U}_{n+1}\left[1 - \left(\dfrac{1}{k} + \dfrac{b(\hat{v}_n-\epsilon_1)^2}{r(a+(\tilde{u}_n+\epsilon_1)^2)(a_1+\hat{v}_n-\epsilon_1)}\right)\tilde{U}_{n+1}\right], \\ \qquad x \in \Omega, t > \bar{T}_1, \\ \dfrac{\partial \tilde{U}_{n+1}}{\partial x} = 0, \quad x \in \partial\Omega, t > \bar{T}_1, \\ \tilde{U}_{n+1}(x,\bar{T}_1) = u(x,\bar{T}_1), \quad x \in \bar{\Omega}. \end{cases}$$

By the comparison principle, we have $u(x,t) \le \tilde{U}_{n+1}(x,t)$ for $x \in \bar{\Omega}$ and $t > \bar{T}_1$. Furthermore, it is clear to see that

$$\lim_{t\to\infty}\tilde{U}_{n+1}(x,t) = \frac{kr(a+(\tilde{u}_n+\epsilon_1)^2)(a_1+\hat{v}_n-\epsilon_1)}{r(a+(\tilde{u}_n+\epsilon_1)^2)(a_1+\hat{v}_n-\epsilon_1)+kb(\hat{v}_n-\epsilon_1)^2}, \quad x \in \bar{\Omega}.$$

Since ϵ_1 is arbitrary,

$$\limsup_{t\to\infty}\max_{x\in\bar{\Omega}} u(x,t) \le \frac{kr(a+\tilde{u}_n^2)(a_1+\hat{v}_n)}{r(a+\tilde{u}_n^2)(a_1+\hat{v}_n)+kb\hat{v}_n^2} = \tilde{u}_{n+1}.$$

For any $\epsilon_2 > 0$, there exists $\bar{T}_2 > \bar{T}_1$ such that $u(x,t) \le \tilde{u}_{n+1} + \epsilon_2$ for $x \in \bar{\Omega}$ and $t > \bar{T}_2$. From the second equation of Eq.(5),

$$\frac{\partial v}{\partial t} \le d_2 \Delta v + \delta v \left(1 - \frac{hv}{m + \tilde{u}_{n+1} + \epsilon_2}\right), \quad x \in \Omega, t > \bar{T}_2.$$

Let $\tilde{V}_{n+1}(x,t)$ be the solution of the following equation

$$\begin{cases} \dfrac{\partial \tilde{V}_{n+1}}{\partial t} = d_2 \Delta \tilde{V}_{n+1} + \delta \tilde{V}_{n+1} \left(1 - \dfrac{h\tilde{V}_{n+1}}{m + \tilde{u}_{n+1} + \epsilon_2}\right), \quad x \in \Omega, t > \bar{T}_2, \\ \dfrac{\partial \tilde{V}_{n+1}}{\partial x} = 0, \quad x \in \partial\Omega, t > \bar{T}_2, \\ \tilde{V}_{n+1}(x, \bar{T}_2) = v(x, \bar{T}_2), \quad x \in \bar{\Omega}. \end{cases}$$

By the comparison principle, we can get $v(x,t) \le \tilde{V}_{n+1}(x,t)$ for $x \in \bar{\Omega}$ and $t > \bar{T}_2$. Since $\lim\limits_{t\to\infty} \tilde{V}_{n+1}(x,t) = \frac{m+\tilde{u}_{n+1}+\epsilon_2}{h}$ for any $x \in \bar{\Omega}$ and ϵ_2 is arbitrary, we have

$$\limsup_{t\to\infty} \max_{x\in\bar{\Omega}} v(x,t) \le \frac{m + \tilde{u}_{n+1}}{h} = \tilde{v}_{n+1}.$$

For any sufficiently small positive number ϵ_3, there exists $\bar{T}_3 > \bar{T}_2$ such that $u(x,t) \ge \hat{u}_n - \epsilon_3 > 0$ and $v(x,t) \le \tilde{v}_{n+1} + \epsilon_3$ for $x \in \bar{\Omega}$ and $t > \bar{T}_3$. Combined with the first equation of Eq.(5), we obtain

$$\frac{\partial u}{\partial t} \ge d_1 \Delta u + ru \left[1 - \left(\frac{1}{k} + \frac{b(\tilde{v}_{n+1} + \epsilon_3)^2}{r(a + (\hat{u}_0 - \epsilon_3)^2)(a_1 + \tilde{v}_1 + \epsilon_3)}\right) u\right], x \in \Omega, t > \bar{T}_3.$$

Then assume that $\hat{U}_{n+1}(x,t)$ is the solution of the following equation

$$\begin{cases} \dfrac{\partial \hat{U}_{n+1}}{\partial t} = d_1 \Delta \hat{U}_{n+1} + r\hat{U}_{n+1} \left[1 - \left(\dfrac{1}{k} + \dfrac{b(\tilde{v}_{n+1} + \epsilon_3)^2}{r(a + (\hat{u}_n - \epsilon_3)^2)(a_1 + \tilde{v}_{n+1} + \epsilon_3)}\right) \hat{U}_{n+1}\right], \\ \qquad x \in \Omega, t > \bar{T}_3, \\ \dfrac{\partial \hat{U}_{n+1}}{\partial x} = 0, \quad x \in \partial\Omega, t > \bar{T}_3, \\ \hat{U}_{n+1}(x, \bar{T}_3) = u(x, \bar{T}_3), \quad x \in \bar{\Omega}. \end{cases}$$

By the comparison principle, we have $u(x,t) \ge \hat{U}_{n+1}(x,t)$ for $x \in \bar{\Omega}$ and $t > \bar{T}_3$. In addition, for $x \in \bar{\Omega}$, it is easy to know that

$$\lim_{t\to\infty} \hat{U}_{n+1}(x,t) = \frac{kr(a + (\hat{u}_n - \epsilon_3)^2)(a_1 + \tilde{v}_{n+1} + \epsilon_3)}{r(a + (\hat{u}_n - \epsilon_3)^2)(a_1 + \tilde{v}_{n+1} + \epsilon_3) + kb(\tilde{v}_{n+1} + \epsilon_3)^2}.$$

From the arbitrariness of ϵ_3,

$$\liminf_{t\to\infty} \min_{x\in\bar{\Omega}} u(x,t) \ge \frac{kr(a + \hat{u}_n^2)(a_1 + \tilde{v}_{n+1})}{r(a + \hat{u}_n^2)(a_1 + \tilde{v}_{n+1}) + kb\tilde{v}_{n+1}^2} = \hat{u}_{n+1}.$$

For any sufficiently small positive number ϵ_4, there is $\bar{T}_4 > \bar{T}_3$ such that $u(x, t-\tau) \geq \hat{u}_{n+1} - \epsilon_4 > 0$ for $x \in \bar{\Omega}$ and $t > \bar{T}_4$. Combined with the second equation of Eq.(5),

$$\frac{\partial v}{\partial t} \geq d_2 \Delta v + \delta v \left(1 - \frac{hv}{m + \hat{u}_{n+1} - \epsilon_4}\right), \quad x \in \Omega, t > \bar{T}_4.$$

Then, suppose that $\hat{V}_{n+1}(x,t)$ is the solution of the following equation

$$\begin{cases} \dfrac{\partial \hat{V}_{n+1}}{\partial t} = d_2 \Delta \hat{V}_{n+1} + \delta \hat{V}_{n+1} \left(1 - \dfrac{h\hat{V}_{n+1}}{m + \hat{u}_{n+1} - \epsilon_4}\right), \quad x \in \Omega, t > \bar{T}_4, \\ \dfrac{\partial \hat{V}_{n+1}}{\partial x} = 0, \quad x \in \partial\Omega, t > \bar{T}_4, \\ \hat{V}_{n+1}(x, \bar{T}_4) = v(x, \bar{T}_4), \quad x \in \bar{\Omega}. \end{cases}$$

By the comparison principle, we know that $v(x,t) \geq \hat{V}_{n+1}(x,t)$ for $x \in \bar{\Omega}$ and $t > \bar{T}_4$. Since $\lim\limits_{t\to\infty} \hat{V}_{n+1}(x,t) = \frac{m+\hat{u}_{n+1}-\epsilon_4}{h}$ for any $x \in \bar{\Omega}$ and ϵ_4 is arbitrary,

$$\liminf_{t\to\infty} \min_{x\in\bar{\Omega}} v(x,t) \geq \frac{m + \hat{u}_{n+1}}{h} = \hat{v}_{n+1}.$$

Recall again that $\frac{(m+s)^2}{a_1 h + m + s}$ is an increasing function for $s > 0$. Thus, if $\hat{u}_{n-1} < \hat{u}_n < \tilde{u}_n < \tilde{u}_{n-1}$, we have

$$\begin{aligned}
\tilde{u}_{n+1} &= \frac{k}{1 + \frac{kb(m+\hat{u}_n)^2}{rh(a+\tilde{u}_n^2)(a_1 h + m + \hat{u}_n)}} < \tilde{u}_n = \frac{k}{1 + \frac{kb(m+\hat{u}_{n-1})^2}{rh(a+\tilde{u}_{n-1}^2)(a_1 h + m + \hat{u}_{n-1})}}, \\
\hat{u}_n &= \frac{k}{1 + \frac{kb(m+\tilde{u}_n)^2}{rh(a+\hat{u}_{n-1}^2)(a_1 h + m + \tilde{u}_n)}} < \tilde{u}_{n+1} = \frac{k}{1 + \frac{kb(m+\hat{u}_n)^2}{rh(a+\tilde{u}_n^2)(a_1 h + m + \hat{u}_n)}}, \\
\hat{u}_n &= \frac{k}{1 + \frac{kb(m+\tilde{u}_n)^2}{rh(a+\hat{u}_{n-1}^2)(a_1 h + m + \tilde{u}_n)}} < \hat{u}_{n+1} = \frac{k}{1 + \frac{kb(m+\tilde{u}_{n+1})^2}{rh(a+\hat{u}_n^2)(a_1 h + m + \tilde{u}_{n+1})}}, \\
\hat{u}_{n+1} &= \frac{k}{1 + \frac{kb(m+\tilde{u}_{n+1})^2}{rh(a+\hat{u}_n^2)(a_1 h + m + \tilde{u}_{n+1})}} < \tilde{u}_{n+1} = \frac{k}{1 + \frac{kb(m+\hat{u}_n)^2}{rh(a+\tilde{u}_n^2)(a_1 h + m + \hat{u}_n)}}.
\end{aligned}$$

To sum up, we know that $\hat{u}_n < \hat{u}_{n+1} < \tilde{u}_{n+1} < \tilde{u}_n$. Similarly, we can obtain that $\hat{v}_n < \hat{v}_{n+1} < \tilde{v}_{n+1} < \tilde{v}_n$.

Notice that the four sequences $\{\tilde{u}_n\}$, $\{\tilde{v}_n\}$, $\{\hat{u}_n\}$ and $\{\hat{v}_n\}$ satisfy

$$\begin{aligned}
&\hat{u}_0 < \hat{u}_1 < \cdots < \hat{u}_{n+1} < \cdots < \tilde{u}_{n+1} < \cdots < \tilde{u}_1 < \tilde{u}_0, \\
&\hat{v}_0 < \hat{v}_1 < \cdots < \hat{v}_{n+1} < \cdots < \tilde{v}_{n+1} < \cdots < \tilde{v}_1 < \tilde{v}_0,
\end{aligned}$$

which means that $\{\tilde{u}_n\}$, $\{\tilde{v}_n\}$, $\{\hat{u}_n\}$ and $\{\hat{v}_n\}$ are all monotonically bounded sequences. Hence, the limits of the four sequences exist as n approaches infinity. Denote

$$\lim_{n\to\infty} \tilde{u}_n = \tilde{u}, \quad \lim_{n\to\infty} \tilde{v}_n = \tilde{v}, \quad \lim_{n\to\infty} \hat{u}_n = \hat{u}, \quad \lim_{n\to\infty} \hat{v}_n = \hat{v}.$$

Clearly, $\tilde{u}$, $\tilde{v}$, $\hat{u}$ and $\hat{v}$ satisfy

$$\limsup_{t\to\infty}\max_{x\in\bar{\Omega}} u(x,t) \le \tilde{u}, \quad \limsup_{t\to\infty}\max_{x\in\bar{\Omega}} v(x,t) \le \tilde{v},$$
$$\liminf_{t\to\infty}\min_{x\in\bar{\Omega}} u(x,t) \ge \hat{u}, \quad \liminf_{t\to\infty}\min_{x\in\bar{\Omega}} v(x,t) \ge \hat{v}.$$

The proof is complete. □

Lemma 3.3. *If (I_1) and (I_4) hold, then the unique positive steady state (u_0, v_0) of Eq.(5) without delay is globally attractive for $(u,v) \in G_*$, where*
(I_4) : *there is a positive constant α such that*

$$\begin{aligned}
\mathbb{I}_{41} &= -\frac{r}{k} - \Phi_{\min}(u,v) + \Psi_{\max}(u,v) + \frac{\mathbb{H}_{\max}(u,v,\alpha)}{2} < 0,\\
\mathbb{I}_{42} &= -\frac{\alpha\delta h}{m+\tilde{u}} + \frac{\mathbb{H}_{\max}(u,v,\alpha)}{2} < 0,
\end{aligned} \tag{21}$$

with

$$\begin{aligned}
&\Phi(u,v) = \frac{bv^2}{(a+u^2)(a_1+v)}, \quad \Phi_{\min}(u,v) = \min_{(u,v)\in G_*} \Phi(u,v) = \Phi(\tilde{u},\hat{v}),\\
&\Psi(u,v) = \frac{r\left(1-\frac{u_0}{k}\right)(a_1+v_0)(u+u_0)}{(a+u^2)(a_1+v)}, \quad u_* = u_0 + \sqrt{u_0^2+a},\\
&\Psi_{\max}(u,v) = \max_{(u,v)\in G_*} \Psi(u,v) = \max\{\Psi(\hat{u},\hat{v}), \Psi(u_*,\hat{v}), \Psi(\tilde{u},\hat{v})\},\\
&\mathbb{H}(u,v,\alpha) = \frac{\alpha\delta}{m+u} - \frac{bu_0(v+v_0)}{(a+u^2)(a_1+v)} + \frac{r\left(1-\frac{u_0}{k}\right)}{a_1+v},\\
&\mathbb{H}_{\max}(u,v,\alpha) = \max_{(u,v)\in G_*,\alpha>0} |\mathbb{H}(u,v,\alpha)|.
\end{aligned} \tag{22}$$

Proof. According to Hsu [33], we establish the following Lyapunov function in G_*,

$$W(t) = \int_\Omega V(u(x,t), v(x,t))dx,$$

where

$$V(u,v) = (u-u_0) - u_0 \ln\left(\frac{u}{u_0}\right) + \alpha\left((v-v_0) - v_0\ln\left(\frac{v}{v_0}\right)\right), \tag{23}$$

and α is a positive constant. Taking the derivative of Eq.(23), then we have

$$\begin{aligned}
\frac{dV}{dt} =&(u-u_0)\left(r-\frac{ru}{k}-\frac{buv^2}{(a+u^2)(a_1+v)}\right)+\alpha\delta(v-v_0)\left(1-\frac{hv}{m+u}\right)\\
=&\left(-\frac{r}{k}-\frac{bv^2}{(a+u^2)(a_1+v)}\right)(u-u_0)^2-\frac{\alpha\delta h}{m+u}(v-v_0)^2\\
&+\left(\frac{\alpha\delta}{m+u}-\frac{bu_0(v+v_0)}{(a+u^2)(a_1+v)}\right)(u-u_0)(v-v_0)\\
&+\left(r-\frac{ru_0}{k}-\frac{bu_0v_0^2}{(a+u^2)(a_1+v)}\right)(u-u_0)\\
=&\left(-\frac{r}{k}-\Phi(u,v)+\Psi(u,v)\right)(u-u_0)^2-\frac{\alpha\delta h}{m+u}(v-v_0)^2\\
&+\mathbb{H}(u,v,\alpha)(u-u_0)(v-v_0)\\
\leq&\left(-\frac{r}{k}-\Phi(u,v)+\Psi(u,v)\right)(u-u_0)^2-\frac{\alpha\delta h}{m+u}(v-v_0)^2\\
&+|\mathbb{H}(u,v,\alpha)|(u-u_0)(v-v_0)\\
\leq&\left(-\frac{r}{k}-\Phi(u,v)+\Psi(u,v)+\frac{|\mathbb{H}(u,v,\alpha)|}{2}\right)(u-u_0)^2\\
&+\left(-\frac{\alpha\delta h}{m+u}+\frac{|\mathbb{H}(u,v,\alpha)|}{2}\right)(v-v_0)^2,
\end{aligned} \tag{24}$$

where $\Phi(u,v)$, $\Psi(u,v)$ and $\mathbb{H}(u,v,\alpha)$ satisfy the corresponding equation in E-q.(22).

Taking the partial derivative of $\Phi(u,v)$ with respect to u and v, respectively, we have

$$\begin{aligned}
\frac{\partial\Phi(u,v)}{\partial u}&=-\frac{2buv^2}{(a+u^2)^2(a_1+v)}<0,\\
\frac{\partial\Phi(u,v)}{\partial v}&=\frac{2a_1bv+bv^2}{(a+u^2)(a_1+v)^2}>0.
\end{aligned}$$

Obviously, $\Phi(u,v)$ decreases monotonically with respect to u and increases monotonically with respect to v for $(u,v)\in G_*$. Therefore,

$$\min_{(u,v)\in G_*}\Phi(u,v)=\Phi(\tilde{u},\hat{v}).$$

Taking the partial derivative of $\Psi(u,v)$ with respect to u and v, respectively, we obtain

$$\begin{aligned}
\frac{\partial\Psi(u,v)}{\partial u}&=\frac{r\left(1-\frac{u_0}{k}\right)(a_1+v_0)(-u^2-2u_0u+a)}{(a+u^2)^2(a_1+v)},\\
\frac{\partial\Psi(u,v)}{\partial v}&=-\frac{r\left(1-\frac{u_0}{k}\right)(a_1+v_0)(u+u_0)}{(a+u^2)(a_1+v)^2}<0.
\end{aligned}$$

Clearly, $\Psi(u,v)$ decreases monotonically with respect to v for $v \in [\hat{v}, \tilde{v}]$. Hence, we have $\max\limits_{(u,v)\in G_*} \Psi(u,v) = \max\limits_{u\in[\hat{u},\tilde{u}]} \Psi(u,\hat{v})$. Let u_* be the positive root of $\frac{\partial \Psi(u,v)}{\partial u} = 0$. Then, we obtain $u_* = u_0 + \sqrt{u_0^2 + a}$. It follows that there exists three cases for $\max\limits_{u\in[\hat{u},\tilde{u}]} \Psi(u,\hat{v})$.

Case 1. If $u_* \leq \hat{u}$, then $\Psi(u,v)$ decreases monotonically with respect to u for $u \in [\hat{u}, \tilde{u}]$, which implies that $\max\limits_{u\in[\hat{u},\tilde{u}]} \Psi(u,\hat{v}) = \Psi(\hat{u},\hat{v})$.

Case 2. If $\hat{u} < u_* < \tilde{u}$, then $\Psi(u,v)$ increases monotonically with respect to u for $u \in [\hat{u}, u_*]$ and decreases monotonically with respect to u for $u \in (u_*, \tilde{u}]$, which implies that $\max\limits_{u\in[\hat{u},\tilde{u}]} \Psi(u,\hat{v}) = \Psi(u_*,\hat{v})$.

Case 3. If $u_* \geq \tilde{u}$, then $\Psi(u,v)$ increases monotonically with respect to u for $u \in [\hat{u}, \tilde{u}]$, which implies that $\max\limits_{u\in[\hat{u},\tilde{u}]} \Psi(u,\hat{v}) = \Psi(\tilde{u},\hat{v})$.

In conclusion, we have

$$\max_{(u,v)\in G_*} \Psi(u,v) = \max\{\Psi(\hat{u},\hat{v}), \Psi(u_*,\hat{v}), \Psi(\tilde{u},\hat{v})\}.$$

If (I_4) holds, we can obtain from Eq.(24) that

$$\frac{dV}{dt} \leq \mathbb{I}_{41}(u-u_0)^2 + \mathbb{I}_{42}(v-v_0)^2 \leq 0,$$

where $\mathbb{I}_{41}$ and $\mathbb{I}_{42}$ are defined in Eq.(21). According to Hsu [33], we know that

$$\frac{dW}{dt} = \int_\Omega \left(\frac{u-u_0}{u} d_1 \Delta u + \alpha \frac{v-v_0}{v} d_2 \Delta v \right) dx + \int_\Omega V(u(x,t), v(x,t)) dx \leq 0,$$

which implies that the positive steady state (u_0, v_0) is global attractive for Eq.(5) when $\tau = 0$. The proof is complete. □

Next, we use some definitions in reference [32].

(i) $E = C(S^1, X)$ is a real isometric Banach representation of the group $G = S^1 := \{z \in \mathbb{C} : |z| = 1\}$;

(ii) Let E^G:=$\{x \in E$: $gx = x$ for all $g \in G\}$. Then $E^G = X$, and E has an isotypical direct sum decomposition $E = E^G \overset{\infty}{\underset{k=1}{\oplus}} E_k$ where $E_k = \{e^{ikt}x : x \in X\}$ for $k \geq 1$.

Then, Eq.(5) can be written as a continuously differentiable, completely continuous and G-invariant integral equation.

Suppose that $(I_1) - (I_4)$ are satisfied. Lemma 2.1 tells us that $S_0 = (u_0, v_0)$ is the unique positive constant steady state of Eq.(5). Lemma 3.2 and Lemma 3.3 show that Eq.(5) has no positive nonconstant steady state. From Lemma 2.3, we know that $\lambda = 0$ is not an eigenvalue of Eq.(11) for any $\tau \geq 0$, hence the assumption $H(1)$ in [32, Sect.6.5] is satisfied. According to Lemma 2.4, Eq.(11) has a unique pair of pure imaginary eigenvalues $\pm i\omega_n$ when $\tau = \tau_{n,j}$, so the assumption $H(2)$ in [32, Sect.6.5] is satisfied. For sufficiently small $\epsilon_0, \varsigma_0 > 0$, we define the local steady state manifold

$$M = \{(S, \tau, \omega) : |\tau - \tau_{n,j}| < \epsilon_0, |\omega - \omega_n| < \varsigma_0\} \subset E^G \times \mathbb{R} \times \mathbb{R}_+.$$

Then for

$$(\tau, \omega) \in [\tau_{n,j} - \epsilon_0, \tau_{n,j} + \epsilon_0] \times [\omega_n - \varsigma_0, \omega_n + \varsigma_0],$$

$\pm i\omega_n$ are a pair of eigenvalues of Eq.(11) if and only if $\tau = \tau_{n,j}$ and $\omega = \omega_n$, which means that $(S_0, \tau_{n,j}, \omega_n)$ is an isolated singular point in M.

Let $\mu_k(S_0, \tau_{n,j}, \omega_n)(k = 1, 2, \cdots)$ be the generalized crossing number defined in [32]. Then, from Lemma 2.5, we know that $\mu_1(S_0, \tau_{n,j}, \omega_n) = 1$. Thus we obtain that the local topological Hopf bifurcation for Eq.(5) at $\tau = \tau_{n,j}$.

Consider the global nature of the Hopf bifurcation and let

$$S = \mathrm{Cl}\{(z, \tau, \omega) \in E \times \mathbb{R} \times \mathbb{R}_+ : z = (z_1(\cdot, \omega t), z_2(\cdot, \omega t)) = (u(\cdot, t), v(\cdot, t)) \text{ is a nontrivial } \frac{2\pi}{\omega} \text{ periodic solution of Eq.(5)}\}.$$

Then according to the local bifurcation theorem, one has $(S_0, \tau_{n,j}, \omega_n) \in S$. Define the complete steady state manifold as

$$M^* = \{(U_0, \tau) : \tau \in \mathbb{R}\} \subset E^G \times \mathbb{R},$$

and the connected component that contains $(S_0, \tau_{n,j}, \omega_n)$ of S as

$$\mathfrak{C}_{n,j} = \mathfrak{C}_{n,j}(S_0, \tau_{n,j}, \omega_n).$$

Then we can use the following global Hopf bifurcation theorem given by Wu.

Lemma 3.4. *[32] For each connected component $\mathfrak{C}_{n,j}$, at least one of the following holds*

(i) $\mathfrak{C}_{n,j}$ *is unbounded, i.e.,*

$$\sup\{\max_{t \in \mathbb{R}} |z(t)| + |\tau| + \omega + \omega^{-1} : (z, \tau, \omega) \in \mathfrak{C}_{n,j}\} = \infty;$$

(ii) $\mathfrak{C}_{n,j} \cap M^* \times \mathbb{R}_+$ *is finite and for all $k \geq 1$, one has the equality*

$$\sum_{(S_0, \tau_{n,j}, \omega_n) \in \mathfrak{C}_{n,j} \cap M^* \times \mathbb{R}_+} \mu_k(S_0, \tau_{n,j}, \omega_n) = 0.$$

Now, we state our global Hopf bifurcation results as follow.

Theorem 3.5. *If (I_1)-(I_4) hold, Eq.(5) has at least one positive periodic orbit when $\tau > \tau_1$, where $\tau_1 = \min_{n \in \mathbb{N}_2} \{\tau_{n,1}\}$.*

Proof. From Lemma 3.2, we obtain that the projection of $\mathfrak{C}_{n,j}$ onto the z-space is bounded. Notice that

$$2j\pi < \omega_n \tau_{n,j} < 2(j+1)\pi, \quad j \in \mathbb{N}.$$

It follows that

$$\frac{1}{j+1} < \frac{2\pi}{\omega_n \tau_{n,j}} < \frac{1}{j}, \quad j \in \mathbb{N}.$$

Assume $(z, \tau, \omega) \in \mathfrak{C}_{n,j}$ for $j \in \mathbb{N}$, we have $\frac{\tau}{j+1} < \frac{2\pi}{\omega} < \frac{\tau}{j}$, $j \in \mathbb{N}$. It shows that the projection of $\mathfrak{C}_{n,j}$ onto the T-space is bounded if τ is bounded.

According to Lemma 3.4 and $\mu_1(S_0, \tau_{n,j}, \omega_n) > 0$ for any $\tau_{n,j}$, it is obvious that each connected component $\mathfrak{C}_{n,j}$ is unbounded. In addition, from Lemma 3.2 and Lemma 3.3, Eq.(5) has no positive periodic solutions when $\tau = 0$. Therefore, the projection of $\mathfrak{C}_{n,j}$ for $j \in \mathbb{N}$ onto the τ-space include $[\tau_{n,j}, \infty)$. The proof is complete. □

4. Numerical simulation

In this section, we verify our conclusions by taking the parameters of Eq.(5) as follows

$$\begin{aligned} &r = 0.99, \quad k = 78, \quad a_1 = 0.1, \quad a = 528, \quad b = 0.45, \quad \delta = 3.8, \\ &h = 1.13, \quad m = 44, \quad d_1 = 0.3, \quad d_2 = 0.3, \quad l = 8. \end{aligned}$$

For this set of parameters, we have that $\triangle = 0.1716 > 0$, $C_1 = 2.1765 > 0$, $D_1 = -614.5826 < 0$ and $\max \phi^{'}(u) = \phi^{'}(5.6960) = -596.8849 < 0$, which means that the case (c) of (I_1) is satisfied and Eq.(5) has a unique positive steady state $(u_0, v_0) = (29.4371, 64.9886)$. By calculation, we can obtain that $\alpha(u_0) = 0.224$ and $\beta(u_0) = -0.2797$, which implies that (I_2) holds. Moreover, we know that $D_n^2 - H^2 > 0$ for any $n > n_* = 2$ and

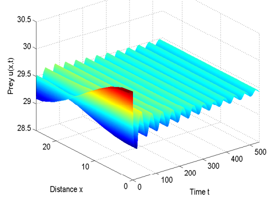
$$D_0^2 - H^2 = -0.1597, \quad D_1^2 - H^2 = -0.1272, \quad D_2^2 - H^2 = -0.0249.$$

Obviously, the assumption (I_3) holds. The corresponding Hopf bifurcation values are as follows

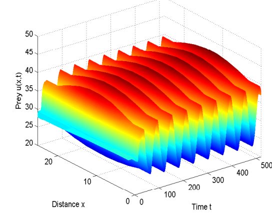
$$\begin{aligned} &\tau_{0,0} = 25.4972, \quad \tau_{0,1} = 85.3680, \quad \tau_{0,2} = 145.2387, \quad \cdots \\ &\tau_{1,0} = 29.1700, \quad \tau_{1,1} = 96.3373, \quad \tau_{1,2} = 163.5046, \quad \cdots \\ &\tau_{2,0} = 71.8044, \quad \tau_{2,1} = 224.0981, \quad \tau_{2,2} = 376.3917, \quad \cdots \end{aligned} \tag{25}$$

From the definition of G_*, we have $G_* = [21.7772, 38.5820] \times [58.2099, 73.0814]$. Choosing $\alpha = 0.177$, one has $\mathbb{I}_{41} = -0.0084 < 0$ and $\mathbb{I}_{42} = -0.0071 < 0$, which implies that (I_4) holds.

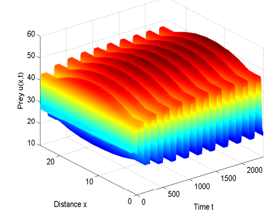
From Theorem 2.6 (ii), the positive steady state $(29.4371, 64.9886)$ of Eq.(5) is locally asymptotically stable for $\tau \in [0, \tau_0)$ (see Figure 1 (a) and (d)), unstable



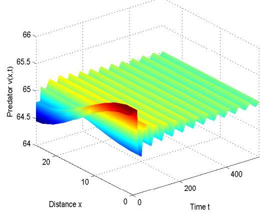
(a) $\tau = 18$



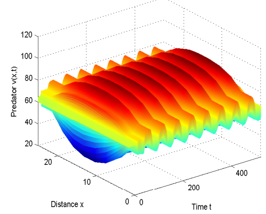
(b) $\tau = 26$



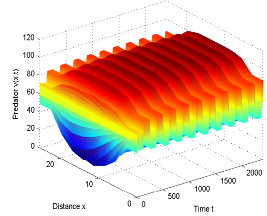
(c) $\tau = 120$



(d) $\tau = 18$



(e) $\tau = 26$



(f) $\tau = 120$

Figure 1: The unique positive steady state $(29.4371, 64.9886)$ of Eq.(5) is asymptotically stable when $\tau = 18 < \tau_0 = 25.4972$ and Eq.(5) has one positive periodic solution for $\tau = 26 > \tau_0$ and $\tau = 120 > \tau_1 = 85.3680$, respectively. (a) and (d) are the behaviours for u and v when $\tau = 18$, respectively; (b) and (e) are the behaviours for u and v when $\tau = 26$, respectively; (c) and (f) are the behaviours for u and v when $\tau = 120$, respectively.

for $\tau > \tau_0$ (see Figure 1 (b) and (e)), and Eq.(5) has at least one positive periodic orbit when $\tau > \tau_1$, where

$$\tau_0 = \min_{n=0,1,2}\{\tau_{n,0}\} = 25.4972, \quad \tau_1 = \min_{n=0,1,2}\{\tau_{n,1}\} = 85.3680.$$

In order to verify the extended existence of bifurcating periodic solutions, we choose $\tau = 120$ far away from the Hopf bifurcation values in Eq.(25). The corresponding numerical simulation results are shown in Figure 1 (c) and (f).

Conflict of interest

The authors have no conflicts to disclose.

Author's contributions

All authors participated in the writing and coordination of the manuscript and all authors read and approved the final manuscript.

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Alp publishing data sharing policy

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

Appendix The proof of Lemma 2.1

Proof. If $\triangle \leq 0$, then we have $\phi^{''}(u) \leq 0$, which means that $\phi^{'}(u)$ decreases monotonically with respect to u for $u \in \mathbb{R}_+$. It is easy to know that $\phi^{'}(0) = D_1$. And then we have the following two situations.

(i) If $D_1 > 0$, then we have $\phi(u)$ increases monotonically and then decreases monotonically with respect to u for $u \in \mathbb{R}_+$. Furthermore, we also know that $\phi(0) = E_1 > 0$. Thus, Eq.(5) has a unique positive constant steady state.

(ii) If $D_1 \leq 0$, then we have $\phi(u)$ decreases monotonically with respect to u for $u \in \mathbb{R}_+$. In addition, we also know that $\phi(0) = E_1 > 0$. Therefore, Eq.(5) has a unique positive constant steady state.

Consequently, Eq.(5) has a unique positive constant steady state for $\triangle \leq 0$. The proof of (a) in Lemma 2.1 is complete.

If $\triangle > 0$ and $\phi^{''}(0) = C_1 > 0$, then we have $\phi^{'}(u)$ increases monotonically and then decreases monotonically with respect to u for $u \in \mathbb{R}_+$.

(i) When $\phi^{'}(0) = D_1 \geq 0$, it is easy to see that $\phi(u)$ increases monotonically and then decreases monotonically with respect to u for $u \in \mathbb{R}_+$. Moreover, we also know that $\phi(0) = E_1 > 0$. Hence, Eq.(5) has a unique positive constant steady state. The proof of (b) in Lemma 2.1 is complete.

(ii) When $\phi^{'}(0) = D_1 < 0$ and $\max \phi^{'}(u) \leq 0$, it is obvious that $\phi(u)$ decreases monotonically with respect to u for $u \in \mathbb{R}_+$. Moreover, we also know that $\phi(0) = E_1 > 0$. Hence, Eq.(5) has a unique positive constant steady state. The proof of (c) in Lemma 2.1 is complete.

(iii) When $\phi^{'}(0) = D_1 < 0$ and $\max \phi^{'}(u) > 0$, we have that there exist u_1 and u_{11} such that $\phi^{'}(u_1) = \phi^{'}(u_{11}) = 0$ and $0 < u_1 < u_{11}$. It is easy to know that $\phi(u)$ decreases monotonically for $u \in (0, u_1]$ and then increases monotonically for $u \in (u_1, u_{11}]$ and then decreases monotonically for $u \in (u_{11}, +\infty)$. Furthermore, we also know that $\phi(0) = E_1 > 0$. Obviously, if $\phi(u_1) > 0$ or $\phi(u_{11}) < 0$, then Eq.(5) has a unique positive constant steady state. The proof of (d) in Lemma 2.1 is complete.

If $\triangle > 0$, $B_1 > 0$ and $\phi^{''}(0) = C_1 < 0$, then there exist $\phi^{''}(u_2) = \phi^{''}(u_{22}) = 0$, where u_2 and u_{22} are defined in Lemma 2.1. It is easy to see that $\phi^{'}(u)$ decreases monotonically for $u \in (0, u_2]$ and then increases monotonically for $u \in (u_2, u_{22}]$ and then decreases monotonically for $u \in (u_{22}, +\infty)$.

(i) When $D_1 > 0$, and $\phi^{'}(u_2) \geq 0$ or $\phi^{'}(u_{22}) \leq 0$, we have that $\phi^{'}(u)$ increases monotonically and then decreases monotonically with respect to u for $u \in \mathbb{R}_+$. In addition, we also know that $\phi(0) = E_1 > 0$. Thus, Eq.(5) has a unique positive constant steady state. The proof of (e) in Lemma 2.1 is complete.

(ii) When $D_1 > 0$, $\phi^{'}(u_2) < 0$ and $\phi^{'}(u_{22}) > 0$, there exist u_3, u_{33} and u_{333} such that $\phi^{'}(u_3) = \phi^{'}(u_{33}) = \phi^{'}(u_{333}) = 0$ and $0 < u_3 < u_{33} < u_{333}$. It is easy to know that $\phi(u)$ increases monotonically for $u \in (0, u_3]$ and then decreases monotonically for $u \in (u_3, u_{33}]$ and then increases monotonically for $u \in (u_{33}, u_{333}]$ and then decreases monotonically for $u \in (u_{333}, +\infty)$. Moreover, we also know that $\phi(0) = E_1 > 0$. Obviously, if $\phi(u_{33}) > 0$ or $\phi(u_{333}) < 0$, then Eq.(5) has a unique positive constant steady state. The proof of (f) in Lemma 2.1 is complete.

(iii) When $\phi^{'}(0) = D_1 \leq 0$ and $\phi^{'}(u_{22}) \leq 0$, then we have $\phi(u)$ decreases monotonically with respect to u for $u \in \mathbb{R}_+$. Furthermore, we also know that $\phi(0) = E_1 > 0$. Thus, Eq.(5) has a unique positive constant steady state. The proof of (g) in Lemma 2.1 is complete.

(iv) When $\phi^{'}(0) = D_1 \leq 0$ and $\phi^{'}(u_{22}) > 0$, then there exist u_4 and u_{44} such that $\phi^{'}(u_4) = \phi^{'}(u_{44}) = 0$ and $0 < u_4 < u_{44}$. It is easy to see that $\phi(u)$ decreases monotonically for $u \in (0, u_4]$ and then increases monotonically for $u \in (u_4, u_{44}]$ and then decreases monotonically for $u \in (u_{44}, +\infty)$. In addition, we also know that $\phi(0) = E_1 > 0$. Apparently, if $\phi(u_4) > 0$ or $\phi(u_{44}) < 0$, then Eq.(5) has a unique positive constant steady state. The proof of (h) in Lemma 2.1 is complete.

If $\triangle > 0$, $B_1 > 0$ and $\phi^{''}(0) = C_1 = 0$, then we have $\phi^{'}(u)$ increases monotonically and then decreases monotonically with respect to u for $u \in \mathbb{R}_+$.

(i) When $\phi^{'}(0) = D_1 \geq 0$, it is easy to know that $\phi(u)$ increases monotonically and then decreases monotonically with respect to u for $u \in \mathbb{R}_+$. Moreover, we also know that $\phi(0) = E_1 > 0$. Hence, Eq.(5) has a unique positive constant steady state. The proof of (i) in Lemma 2.1 is complete.

(ii) When $\phi^{'}(0) = D_1 < 0$ and $\max \phi^{'}(u) \leq 0$, it is obvious that $\phi(u)$ decreases monotonically with respect to u for $u \in \mathbb{R}_+$. Furthermore, we also know that $\phi(0) = E_1 > 0$. Hence, Eq.(5) has a unique positive constant steady state. The proof of (j) in Lemma 2.1 is complete.

(iii) When $\phi^{'}(0) = D_1 < 0$ and $\max \phi^{'}(u) > 0$, we have that there exist u_5 and u_{55} such that $\phi^{'}(u_5) = \phi^{'}(u_{55}) = 0$ and $0 < u_5 < u_{55}$. It is easy to know that $\phi(u)$ decreases monotonically for $u \in (0, u_5]$ and then increases monotonically for $u \in (u_5, u_{55}]$ and then decreases monotonically for $u \in (u_{55}, +\infty)$. In addition, we also know that $\phi(0) = E_1 > 0$. Obviously, if $\phi(u_5) > 0$ or $\phi(u_{55}) < 0$, then Eq.(5) has a unique positive constant steady state. The proof of (k) in Lemma 2.1 is complete.

If $\triangle > 0$, $B_1 < 0$ and $\phi^{''}(0) = C_1 \leq 0$, then we have that $\phi^{'}(u)$ decreases monotonically with respect to u for $u \in \mathbb{R}_+$. It is easy to see that $\phi^{'}(0) = D_1$. And then we have the following two situations.

(i) If $D_1 > 0$, then we have $\phi(u)$ increases monotonically and then decreases monotonically with respect to u for $u \in \mathbb{R}_+$. Furthermore, we also know that $\phi(0) = E_1 > 0$. Thus, Eq.(5) has a unique positive constant steady state.

(ii) If $D_1 \leq 0$, then we have $\phi(u)$ decreases monotonically with respect to u for $u \in \mathbb{R}_+$. In addition, we also know that $\phi(0) = E_1 > 0$. Therefore, Eq.(5) has a unique positive constant steady state.

Consequently, Eq.(5) has a unique positive constant steady state for $\triangle > 0$, $B_1 < 0$ and $C_1 \leq 0$. The proof of (l) in Lemma 2.1 is complete.

Obviously, there is no case that $\triangle > 0$ and $B_1 = 0$. Therefore, the conditions for Eq.(5) has a unique positive constant steady state are all taken into account. To sum up, If one of $(a) - (l)$ holds, Eq.(5) has a unique positive constant steady state. The proof of Lemma 2.1 is complete. □

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