AN APPROACH TO THE INITIAL VALUE PROBLEM BY THE FUZZY LAPLACE TRANSFORM WITH HEAVISIDE FUNCTION

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ABSTRACT. In this paper we investigate the solutions of second-order fuzzy initial value problem using the fuzzy Laplace transform with Heaviside function under the generalized differentiabilty. The related theorems and properties are given in detail and the method is illustrated by solving an example about electric circuit.

1. Introduction. Fuzzy differential equations (FDEs) are utilized for the purpose of the modelling problems in science and engineering. Most of the problems have uncertain structural parameters. Instead, many researches have modelled these uncertain structural parameters as fuzzy numbers in this area ([3], [9], [13]).

The concept of the fuzzy derivative was first introduced by [26]; it was followed up by [8] who used the extension principle in their approach. The properties of differentiable fuzzy set-valued functions by means of the concept of H-differentiability due to Puri and Ralescu [14] were discussed by Kaleva [13]. Seikkala [16], defined the fuzzy derivative which is the generalization of Hukuhara derivative, and showed that fuzzy initial value problem has a unique solution. Strongly generalized differentiability was introduced in Bede and Gal [7] and studied in Bede et al. [1]. The strongly generalized derivative is defined for a larger class of fuzzy-valued function than the H-derivative. So in this paper we use this differentiability method.

Fuzzy initial value problems (FIVPs) are one of the simplest FDEs which may appear in many applications. The FDEs and FIVPs (Cauchy problem) were rigorously improved by [13], [16], [2], [24]. The numerical methods for solving fuzzy differential equations are introduced in [20], [19].

The Laplace transform technique becomes truly useful when solving IVPs with discontinuous or impulsive in homogeneous terms, these terms commonly modeled using Heaviside function. This special function that often arise when the method of Laplace transforms is applied to physical problems such as electric circuits with on/off switches. Without Laplace transforms it would be much more difficult to solve differential equations that involve this function. The Heaviside step function, or the unit step function is a step function the value of which is zero for negative arguments and one for positive arguments. The function was originally developed in operational calculus for the solution of differential equations, where it represents a signal that switches on at a specified time and stays switched on indefinitely.

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Oliver Heaviside, who developed the operational calculus as a tool in the analysis of telegraphic communications [5].

Fuzzy Laplace transform is useful to solve FIVPs with Heaviside function. Allahviranloo and Barkhordari Ahmadi first introduced fuzzy Laplace transform [17]. Later, Salahshour and Allahviranloo point out that under what conditions the fuzzy-valued functions can possess the fuzzy Laplace transform and they consider the important properties and related theorems for solving FIVPs [23]. Çitil examined a solution of FIVP with fuzzy coefficient by fuzzy Laplace transform [10]. The use of the fuzzy Laplace transform with the notion of strongly generalized differentiability is very important for the solutions of FIVPs. Salgado et al. solve a fuzzy harmonic oscillator equation using this transform method with linearly correlated fuzzy numbers [22].

The aim of this article is to investigate the fuzzy solutions of the FIVPs with Heaviside function which is applied to physical problems such as electric circuits. For this purpose, we use of the fuzzy Laplace transform with the notion of generalized differentiability. Then we give comparison results of the found solutions.

2. Notation and Preliminaries. We now recall the basic definitions and the theorems utilized in this study.

Definition 1. ([9]) Let E be a universal set. A fuzzy subset $\widehat{A}$ of E is given by its membership function $\mu_{\widehat{A}} : E \to [0,1]$, where $\mu_{\widehat{A}}(t)$ represents the degree to which $t \in E$ belongs to $\widehat{A}$. We denote the class of the fuzzy subsets of E by the sembol $F(E)$.

Definition 2. ([29]) The $\alpha-$ cut of a fuzzy set $\widehat{A} \subseteq E$, denoted by $\left[\widehat{A}\right]^{\alpha}$, is defined as $\left[\widehat{A}\right]^{\alpha} = \left\{x \in E : \widehat{A}(t) \geq \alpha\right\}$, $\forall \alpha \in (0,1]$. If E is also topological space, then the $0-$ cut is defined as the closure of the support of $\widehat{A}$,that is, $\left[\widehat{A}\right]^{0} = \left\{\overline{x \in E : \widehat{A}(t) > 0}\right\}$. The $1-$ cut of a fuzzy subset $\widehat{A}$ is also called as core of $\widehat{A}$ and denoted by $\left[\widehat{A}\right]^{1} = core\left(\widehat{A}\right)$

Definition 3. ([13]) A fuzzy subset $\widehat{u}$ on $\mathbb{R}$ is called a fuzzy real number (fuzzy interval), whose $\alpha-$ cut set is denoted by $[\widehat{u}]^{\alpha}$, i.e., $[\widehat{u}]^{\alpha} = \{x : \widehat{u}(t) \geq 0\}$, if it satisfies two axioms:

i) There exists $r \in \mathbb{R}$ such that $\widehat{u}(r) = 1$,
ii) For all $0 < \alpha \leq 1$, there exist real numbers $-\infty < u_{\alpha}^{-} \leq u_{\alpha}^{+} < +\infty$ such that $[\widehat{u}]^{\alpha}$ is equal to the closed interval $[u_{\alpha}^{-}, u_{\alpha}^{+}]$.

The set of all fuzzy real numbers (fuzzy intervals) is denoted by $\mathbb{R}_F$. $F_K(\mathbb{R})$, the family of fuzzy sets of $\mathbb{R}$ whose $\alpha-$ cuts are nonempty compact convex subsets of $\mathbb{R}$.

Definition 4. ([6]) A fuzzy number $\widehat{A}$ is said to be triangular if the parametric representation of its $\alpha-$ cut is of the form for $a_1 < a_2 < a_3$ which $a_1, a_2, a_3 \in \mathbb{R}$, $\left[\widehat{A}\right]^{\alpha} = [(a_2 - a_1)\alpha + a_1, a_3 - (a_3 - a_2)\alpha]$, for all $\alpha \in [0,1]$.

Theorem 1. *([16]) Let $[u_{\alpha}^{-}, u_{\alpha}^{+}]$, $0 < \alpha \leq 1$ be a given family of non-emty intervals. If*

i) $[u_{\alpha}^{-}, u_{\alpha}^{+}] \supset \left[u_{\beta}^{-}, u_{\beta}^{+}\right]$ *for* $0 < \alpha \leq \beta$ *and*

ii) $\left[\lim\limits_{k\to\infty} u_{\alpha_k}^{-}, \lim\limits_{k\to\infty} u_{\alpha_k}^{+}\right] = [u_{\alpha}^{-}, u_{\alpha}^{+}]$,

whenever (α_k) is a non-decreasing sequence converging to $\alpha \in]0,1]$, then the family $[u_{\alpha}^{-}, u_{\alpha}^{+}]$, $0 < \alpha \leq 1$, represents the $\alpha-$cut sets of a fuzzy number $\widehat{u} \in \mathbb{R}_F$. Conversely, if $[u_{\alpha}^{-}, u_{\alpha}^{+}]$, $0 < \alpha \leq 1$, are the $\alpha-$ cut sets of a fuzzy number $\widehat{u} \in \mathbb{R}_F$ then the conditions (i) and (ii) hold true.

The Zadeh's extension principle is a mathematical method to extend classical functions to deal with fuzzy sets as input arguments [27]. For multiple fuzzy variables as arguments, the Zadeh's extension principle is defined as follows.

Definition 5. ([28], [27]) Let $f : X_1 \times X_2 \to Z$ a classical function and let $\widehat{A}_i \in F(X_i)$, for $i = 1, 2$. The Zadeh's extension $\widehat{f}$ of f, applied to $(\widehat{A}_1, \widehat{A}_2)$, is the fuzzy set $\widehat{f}\,(\widehat{A}_1, \widehat{A}_2)$ of Z, whose membership function is defined by

$$\widehat{f}(\widehat{A}_1, \widehat{A}_2)(z) = \begin{cases} \underset{(x_1,x_2)\in f^{-1}(z)}{sup} min\{\widehat{A}_1(x_1), \widehat{A}_2(x_2)\}, & \text{if } f^{-1}(z) \neq \varnothing, \\ 0 & , \text{if } f^{-1}(z) = \varnothing \end{cases}$$

where $f^{-1}(z) = \{(x_1, x_2) \in X_1 \times X_2 : f(x_1, x_2) = z\}$.

We can apply the Zadeh's extension principle to define the standard arithmetic operations for fuzzy numbers [27]. Let $[\widehat{u}]^{\alpha} = [u_{\alpha}^{-}, u_{\alpha}^{+}]$ and $[\widehat{v}]^{\alpha} = [v_{\alpha}^{-}, v_{\alpha}^{+}]$. For all $\alpha \in [0,1]$ and $\lambda \in \mathbb{R}$, we have

$$\begin{aligned} [\widehat{u} \oplus \widehat{v}]^{\alpha} &= [\widehat{u}]^{\alpha} + [\widehat{v}]^{\alpha} = [u_{\alpha}^{-} + v_{\alpha}^{-}, u_{\alpha}^{+} + v_{\alpha}^{+}] \\ [\widehat{u} - \widehat{v}]^{\alpha} &= [\widehat{u}]^{\alpha} - [\widehat{v}]^{\alpha} = [u_{\alpha}^{-} - v_{\alpha}^{+}, u_{\alpha}^{+} + v_{\alpha}^{-}] \\ [\lambda \odot \widehat{u}]^{\alpha} &= \lambda \odot [\widehat{u}]^{\alpha} = \begin{cases} [\lambda u_{\alpha}^{-}, \lambda u_{\alpha}^{+}], \text{ if } \lambda \geq 0, \\ [\lambda u_{\alpha}^{+}, \lambda u_{\alpha}^{-}], \text{ if } \lambda < 0. \end{cases} \end{aligned}$$

The diameter of $\widehat{u} \in \mathbb{R}_F$ is defined $d\left([\widehat{u}]^{\alpha}\right) = u_{\alpha}^{+} - u_{\alpha}^{-}$, where $\alpha \in [0,1]$.

The notion of interactivity among fuzzy numbers arises from a given joint possibility distribution.

Definition 6. ([25]) Let $\widehat{A}_1, ..., \widehat{A}_n \in \mathbb{R}_F$ and let J a fuzzy subset of R^n. The fuzzy subset J is called a joint possibility distribution of $\widehat{A}_1, ..., \widehat{A}_n$ if

$$\widehat{A}_i(x_i) = \sup_{x_j \in \mathbb{R}, j \neq i} J(x_1, ..., x_n),$$

for all $x_i \in \mathbb{R}$ and for all $i = 1, ..., n$.

Definition 7. ([25]) Let $\widehat{A}_1, ..., \widehat{A}_n \in \mathbb{R}_F$ and $f : \mathbb{R}_n \to \mathbb{R}$. Given a joint possibility distribution J of $\widehat{A}_1, ..., \widehat{A}_n$, the $sup - J$ extension of f at $(\widehat{A}_1, ..., \widehat{A}_n)$ is the fuzzy set $\widehat{f}(J) := f_J(A_1, ..., A_n)$ of $\mathbb{R}$ whose membership function is given by

$$\widehat{f}(J)(z) = \sup_{f(x_1,...,x_n)=z} J(x_1, ..., x_n), \forall z \in \mathbb{R}$$

for all $z \in \mathbb{R}$, where $f^{-1}(z) = \{(x_1, ..., x_n) : f((x_1, ..., x_n)) = z\}$.

We can use the sup-J extension principle to generate an arithmetic on interactive fuzzy numbers. Here we focus on the special type of interactivity called linearly correlation.

Definition 8. ([18]) Two fuzzy numbers A and B are linearly correlated if there exists $q, r \in R$, $q \neq 0$, such that $[B]^\alpha = q[A]^\alpha + r$ for each $\alpha \in [0, 1]$ In this case, we may simply write $B = qA + r$.

Remark 1. ([18]) If A and B are linearly correlated fuzzy numbers where $[B]^\alpha = q[A]^\alpha + r$, with $q, r \in R$, $q \neq 0$, $[A]^\alpha = [a_\alpha^-, a_\alpha^-]$ and $[B]^\alpha = [b_\alpha^-, b_\alpha^+]$, then the addition $(+_L)$ subtraction $(-_L)$ are given as follows

$$[B +_L A]^\alpha = (q+1)[A]^\alpha + r = \begin{cases} [b_\alpha^- + a_\alpha^-, b_\alpha^+ + a_\alpha^+] & \text{if} \quad q > 0, \\ [b_\alpha^+ + a_\alpha^-, b_\alpha^- + a_\alpha^+] & \text{if} \quad -1 \leq q < 0, \\ [b_\alpha^- + a_\alpha^+, b_\alpha^+ + a_\alpha^-] & \text{if} \quad q < -1, \end{cases}$$

$$[B -_L A]^\alpha = (q-1)[A]^\alpha + r = \begin{cases} [b_\alpha^- - a_\alpha^-, b_\alpha^+ - a_\alpha^+] & \text{if} \quad q \geq 1, \\ [b_\alpha^+ - a_\alpha^+, b_\alpha^- - a_\alpha^-] & \text{if} \quad 0 \leq q < 1, \\ [b_\alpha^- - a_\alpha^+, b_\alpha^+ - a_\alpha^-] & \text{if} \quad q < 0, \end{cases}$$

for all $\alpha \in [0; 1]$.

Remark 2. ([22]) Let $A \in \mathbb{R}_F \backslash \mathbb{R}$. Note that $B +_L A = 0$ if and only if $q = -1$ and $r = 0$.

Define $D : \mathbb{R}_F \times \mathbb{R}_F \to \mathbb{R}^+ \cup \{0\}$ by the equation

$$D(\widehat{u}, \widehat{v}) = \sup_{0<\alpha\leq 1} \{max[|u_\alpha^- - v_\alpha^-|, |u_\alpha^+ - v_\alpha^+|]\}$$

where $[\widehat{u}]^\alpha = [u_\alpha^-, u_\alpha^+]$, $[\widehat{v}]^\alpha = [v_\alpha^-, v_\alpha^+]$. Then it is easy to show that D is a metric in $\mathbb{R}_F$. [14]

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Definition 9. ([14]) Let $\widehat{u}, \widehat{v} \in \mathbb{R}_F$. If there exist $\widehat{w} \in \mathbb{R}_F$ such that $\widehat{u} = \widehat{v} \oplus \widehat{w}$, then w is called the Hukuhara difference of $\widehat{u}$ and $\widehat{v}$ and it is denoted by $\widehat{u} \ominus_h \widehat{v}$. If $\widehat{u} \ominus_h \widehat{v}$ exists, its $\alpha - cuts$ are given by

$$[\widehat{u} \ominus_h \widehat{v}]^\alpha = \left[u_\alpha^- - v_\alpha^-, u_\alpha^+ - v_\alpha^+\right]$$

for all $\alpha \in [0,1]$.

In the present work, the sign "$\ominus_h$" always stands for Hukuhara difference (H-difference) and a function $\widehat{f} : [a,b] \subseteq R \to \mathbb{R}_F$ is called fuzzy-valued function. The $\alpha - cut$ representation of fuzzy-valued function $\widehat{f}$ given by $[\widehat{f}(t)]^\alpha = [f_\alpha^-(t), f_\alpha^+(t)]$, $\forall t \in [a,b], \forall \alpha \in [0,1]$.

Definition 10. ([12]) Let $\widehat{f} : (a,b) \to \mathbb{R}_F$ and $t_0 \in [a,b]$. If there exists$\widehat{f}'(t_0) \in \mathbb{R}_F$ such that for all $h > 0$ sufficiently small, $\exists \widehat{f}(t_0 + h) \ominus_h \widehat{f}(t_0)$, $\widehat{f}(t_0) \ominus_h \widehat{f}(t_0 - h)$ and the limits hold

$$\lim_{h \to 0} \frac{\widehat{f}(t_0 + h) \ominus_h \widehat{f}(t_0)}{h} = \lim_{h \to 0} \frac{\widehat{f}(t_0) \ominus_h \widehat{f}(t_0 - h)}{h} = \widehat{f}'(t_0)$$

$\widehat{f}$ is Hukuhara differentiable at t_0.

Definition 11. ([12]) Let $\widehat{f} : (a,b) \to \mathbb{R}_F$ and $t_0 \in [a,b]$. If there exists $\widehat{f}'(t_0) \in \mathbb{R}_F$ such that for all $h > 0$ sufficiently small, $\exists \widehat{f}(t_0 + h) \ominus_h \widehat{f}(t_0)$, $\widehat{f}(t_0) \ominus_h \widehat{f}(t_0 - h)$ and the limits hold when f is $(i)-$ differentiable at t_0

$$\lim_{h \to 0} \frac{\widehat{f}(t_0 + h) \ominus_h \widehat{f}(t_0)}{h} = \lim_{h \to 0} \frac{\widehat{f}(t_0) \ominus_h \widehat{f}(t_0 - h)}{h} = \widehat{f}'(t_0),$$

If there exists $f'(t_0) \in \mathbb{R}_F$ such that for all $h > 0$ sufficiently small, $\exists \widehat{f}(t_0) \ominus_h \widehat{f}(t_0 + h)$, $\widehat{f}(t_0 - h) \ominus_h \widehat{f}(t_0)$ and the limits hold when f is $(ii)-$ differentiable at x_0

$$\lim_{h \to 0} \frac{\widehat{f}(t_0) \ominus_h \widehat{f}(t_0 + h)}{-h} = \lim_{h \to 0} \frac{\widehat{f}(t_0 - h) \ominus_h \widehat{f}(t_0)}{-h} = \widehat{f}'(t_0).$$

Theorem 2. *([11]) Let $\widehat{f} : [a,b] \to \mathbb{R}_F$ be fuzzy function and for all $\alpha \in [0,1]$ We say that*

- if $\widehat{f}$ is $(i)-$differentiable ,
  (i) $\left[\widehat{f}'(t)\right]^\alpha = \left[\left\{(f_\alpha^-)'(t), (f_\alpha^+)'(t)\right\}\right]$,
- If $\widehat{f}$ is $(ii)-$differentiable
  (ii) $\left[\widehat{f}'(t)\right]^\alpha = \left[\left\{(f_\alpha^+)'(t), (f_\alpha^-)'(t)\right\}\right]$.

Theorem 3. *([11]) Let $\widehat{f}' : [a,b] \to \mathbb{R}_F$ be fuzzy function and for all $\alpha \in [0,1]$. $\widehat{f}$ and $\widehat{f}'$ are (i) or (ii) differentiable. We say that*

- if $\widehat{f}$ and $\widehat{f}'$ are $(i)-$differentiable
  (i) $\left[\widehat{f}''(t)\right]^{\alpha} = \left[\left\{(f_{\alpha}^{-})''(t), (f_{\alpha}^{+})''(t)\right\}\right]$,
- If $\widehat{f}$ is $(i)-$differentiable and f' is $(ii)-$differentiable
  (ii) $\left[\widehat{f}''(t)\right]^{\alpha} = \left[\left\{(f_{\alpha}^{+})''(t), (f_{\alpha}^{-})''(t)\right\}\right]$,
- If $\widehat{f}$ is $(ii)-$differentiable and $\widehat{f}'$ is $(i)-$differentiable
  (iii) $\left[\widehat{f}''(t)\right]^{\alpha} = \left[\left\{(f_{\alpha}^{+})''(t), (f_{\alpha}^{-})''(t)\right\}\right]$,
- if $\widehat{f}$ and $\widehat{f}'$ are $(ii)-$differentiable
  (iv) $\left[\widehat{f}''(t)\right]^{\alpha} = \left[\left\{(f_{\alpha}^{-})''(t), (f_{\alpha}^{+})''(t)\right\}\right]$.

Definition 12. ([23]) Let $\widehat{f}(t)$ be a fuzzy-valued function on $[a,\infty)$ represented by $\left[\widehat{f}(t)\right]^{\alpha} = [(f_{\alpha}^{-}(t)), (f_{\alpha}^{+}(t))]$ and $\alpha \in [0,1]$. If $f_{\alpha}^{-}(t)$ and $f_{\alpha}^{+}(t)$ are Riemann-integrable on $[a,\infty[$ for each $\alpha \in [0,1]$, then the improper fuzzy Riemann integral is the fuzzy number given by

$$\int_a^{\infty} \widehat{f}(t)dt = \left(\int_a^{\infty} f_{\alpha}^{-}(t)dt, \int_a^{\infty} f_{\alpha}^{+}(t)dt\right)$$

Theorem 4. *([24]) Let $\widehat{f}(t)$ be a fuzzy-valued function on $[a,\infty[$ represented by $((f_{\alpha}^{-}(t)), (f_{\alpha}^{+}(t)))$. For any fixed $\alpha \in [0,1]$,assume $f_{\alpha}^{-}(t)$ and $f_{\alpha}^{+}(t)$ are Riemann-integrable on $[a,b]$ for every $b \geq a$, and assume there are two positive functions M_{α}^{-} and M_{α}^{+} such that $\int_a^b |f_{\alpha}^{-}(t)|\,dt \leq M_{\alpha}^{-}$ and $\int_a^b |f_{\alpha}^{+}(t)|\,dt \leq M_{\alpha}^{+}$ for every $b \geq a$. Then $\widehat{f}(t)$ is improper fuzzy Riemann-integrable on $[a,\infty[$*

Definition 13. ([17]) Let $\widehat{f} : [a,\infty[\to \mathbb{R}_F$ be a continuous function with $\left[\widehat{f}(t)\right]^{\alpha} = [(f_{\alpha}^{-}(t)), (f_{\alpha}^{+}(t))]$ for all $t \in [0,\infty[$ and $\alpha \in [0,1]$. If for every $s > 0$ the function $\widehat{f}(t) \odot e^{-st}$ is improper fuzzy Riemann integrable on $[0,\infty[$, then

$$L\left(\widehat{f}(t)\right) = \int_0^{\infty} \widehat{f}(t) \odot e^{-st}dt$$

is called fuzzy Laplace transform of the fuzzy function f.

Under the conditions of the Definition (Üst), we can show that ref yap

$$\left[L\left(\widehat{f}(t)\right)\right]^{\alpha} = \left[L\left(f_{\alpha}^{-}(t)\right), L\left(f_{\alpha}^{+}(t)\right)\right]$$

where L denotes the usual Laplace transform, that is, if $g : [0,\infty[\to R$, then

$$L(g(t)) = \int_0^{\infty} g(t)\,e^{-st}dt.$$

Theorem 5. *([17]) Let $\widehat{f},\widehat{g} : [a,\infty[\rightarrow \mathbb{R}_F$ be continuous functions such that their fuzzy Laplace transform exist and let $c_1, c_2 \in \mathbb{R}$. We have that* $L\left(c_1 \odot \widehat{f}(t) + c_2 \odot \widehat{g}(t)\right) = c_1 \odot L\left(\widehat{f}(t)\right) + c_2 \odot L\left(\widehat{g}(t)\right).$

Definition 14. ([17]) If $\widehat{f},\widehat{f}' : [a,\infty[\rightarrow \mathbb{R}_F$ are continuous functions nad their fuzzy Laplace transform exist, then

$$L\left(\widehat{f}'(t)\right) = s \odot L\left(\widehat{f}(t)\right) \ominus_h \widehat{f}(0),$$

if $\widehat{f}$ is $(i)-$differentiable,

$$L\left(\widehat{f}'(t)\right) = \left(-\widehat{f}(0)\right) \ominus_h \left(-s \odot L\left(\widehat{f}(t)\right)\right),$$

if $\widehat{f}$ is $(ii)-$differentiable.

Theorem 6. *([23]) If $\widehat{f},\widehat{f}'' : [a,\infty[\rightarrow \mathbb{R}_F$ are continuous functions and$\odot$ their fuzzy Laplace transform exist, then*

$$L\left(\widehat{f}''(t)\right) = s^2 \odot L\left(\widehat{f}(t)\right) \ominus_h s \odot \widehat{f}(0) \ominus_h \widehat{f}'(0), \tag{1}$$

if $\widehat{f}$ and $\widehat{f}'$ are $(i)-$differentiable,

$$L\left(\widehat{f}''(t)\right) = (-1) \odot \widehat{f}'(0) \ominus_h \left(-s^2\right) \odot L\left(\widehat{f}(t)\right) \oplus (-s) \odot \widehat{f}(0), \tag{2}$$

if $\widehat{f}$ is $(i)-$differentiable and f' is $(ii)-$differentiable,

$$L\left(\widehat{f}''(t)\right) = (-s) \odot \widehat{f}(0) \ominus_h \left(-s^2\right) \odot L\left(\widehat{f}(t)\right) \ominus \widehat{f}'(0), \tag{3}$$

if $\widehat{f}$ is $(ii)-$differentiable and $\widehat{f}'$ is $(i)-$differentiable,

$$L\left(\widehat{f}''(t)\right) = s^2 \odot L\left(\widehat{f}(t)\right) \ominus_h s \odot \widehat{f}(0) \oplus (-1) \odot \widehat{f}'(0), \tag{4}$$

if $\widehat{f}$ and $\widehat{f}'$ are $(ii)-$differentiable.

Now, we present two useful results for determining the fuzzy Laplace transforms. An important function occurring in electrical system is the (delayed) unit step function.

Definition 15. ([4])The Heaviside or unit step function, denoted here by $u_c(t)$, is zero for $t < c$ and is one for $t \geq c$; that is,

$$u_c(t) = \begin{cases} 0, & t < c \\ 1, & t \geq c \end{cases}$$

The Heaviside function can be used to represent a translation of a function $f(t)$ a distance c in the positive t direction. We have

$$u_c(t)f(t-c)=\begin{cases}0, & t<c\\ f(t-c), & t\geq c\end{cases}$$

Theorem 7. *([23]) (First translation theorem) If* $\widehat{F}(s)=L\left(\widehat{f}(t)\right)$ *for* b *real* $s>b$, *then*

$$\widehat{F}(s-b)=L\left(e^{bt}\widehat{f}(t)\right).$$

Theorem 8. *([23]) (Second translation theorem) If* $\widehat{F}(s)=L\left(\widehat{f}(t)\right)$ *then*

$$L\left(u_c(t)\widehat{f}(t-b)\right)=e^{-bt}\widehat{F}(s),\ b\geq 0$$

2.1. Solution of FIVPs. In this section, we are going to researche the solution of FIVPs with Heaviside function under generalized H-differentiability which has been proposed in Salahshour and Allahviranloo ([23]).

We consider a fuzzy inital value problem of the form

$$\widehat{u}''(t)+\lambda\widehat{u}(t)=g(t),\quad \widehat{u}(0)=\widehat{u}_0,\quad \widehat{u}'(0)=\widehat{z}_0, \tag{5}$$

where $\widehat{u},\widehat{u}'':[0,\infty[\,\to\mathbb{R}_F$, $[\widehat{u}_0]^\alpha=[u_\alpha^-,u_\alpha^+]$ and $[\widehat{z}_0]^\alpha=[z_\alpha^-,z_\alpha^+]$ for all $\alpha\in[0,1]$ and $\lambda=k^2$, $k\geq 0$ is a real number and $g(t)$ is a real Heaviside function (discontinuous function) such that its Laplace transform exists. Here for every $t\in[0,\infty[$, $\widehat{u}''(t)$ and $\lambda\widehat{u}(t)$ are fuzzy numbers whose the sum is a real number $g(t)$. This assumption is not restrictive, since the sum of two fuzzy numbers is a real number if, and only if, they are linearly-correlated as we pointed out in Remark 2. So from Remark 2 ,this sum make sense if $\widehat{u}''(t)$ and $\lambda\widehat{u}(t)$ are linearly correlated with $q=-1$ and $r=g(t)$. In other words, we have $\widehat{u}''(t)+_L\lambda\widehat{u}(t)=g(t)\Leftrightarrow\widehat{u}''(t)=-\lambda\widehat{u}(t)+g(t)$ for all $t\in[0,\infty[$. From above comments, the FIVP (5) can be rewritten as follows:

$$\widehat{u}''(t)=-\lambda\widehat{u}(t)+g(t),\quad \widehat{u}(0)=\widehat{u}_0,\quad \widehat{u}'(0)=\widehat{z}_0 \tag{6}$$

Thus, using the Theorem 6, we have the following alternatives:

case[1]

If $\widehat{u}$ and $\widehat{u}'$ are $(i)-$differentiability, we have

$$\left(s^2\odot L(\widehat{u}(t))\right)\ominus_h(s\odot\widehat{u}(0))\ominus_h\widehat{u}'(0)=-k^2\odot L(\widehat{u}(t))\oplus L(g(t)), \tag{7}$$

and we get the $\alpha-$cut representation of Eq. (7) as following

$$\begin{aligned} L\left\{u_\alpha^-(t)\right\} &= \frac{s^3}{s^4-k^4}u_\alpha^- + \frac{s^2}{s^4-k^4}z_\alpha^- \\ &-\frac{k^2 s}{s^4-k^4}u_\alpha^+ - \frac{k^2}{s^4-k^4}z_\alpha^+ + \frac{1}{(s^2+k^2)}L\left(g\left(t\right)\right), \end{aligned}$$

and

$$\begin{aligned} L\left\{u_\alpha^+(t)\right\} &= \frac{s^3}{s^4-k^4}u_\alpha^+ + \frac{s^2}{s^4-k^4}z_\alpha^+ \\ &-\frac{k^2 s}{s^4-k^4}u_\alpha^- - \frac{k^2}{s^4-k^4}z_\alpha^- + \frac{1}{(s^2+k^2)}L\left(g\left(t\right)\right). \end{aligned}$$

case[2]

If $\widehat{u}$ is $(i)-$differentiability and $\widehat{u}'$ are $(ii)-$differentiability, we have

$$(-1\odot\widehat{u}'(0))\ominus_h\left(-s^2\odot L(\widehat{u}(t))\right)\oplus(-s\odot\widehat{u}(0)) = -k^2\odot L(\widehat{u}(t))\oplus L\left(g(t)\right) \tag{8}$$

and we get the $\alpha-cut$ representation of Eq. (8) as following

$$L\left\{u_\alpha^-(t)\right\} = \frac{1}{s^2+k^2}z_\alpha^- + \frac{s}{s^2+k^2}u_\alpha^+ + \frac{L\left(g\left(t\right)\right)}{s^2+k^2}$$

and

$$L\left\{u_\alpha^+(t)\right\} = \frac{1}{s^2+k^2}z_\alpha^+ + \frac{s}{s^2+k^2}u_\alpha^- + \frac{L\left(g\left(t\right)\right)}{s^2+k^2}$$

case[3]

If $\widehat{u}$ is $(ii)-$differentiability and $\widehat{u}'$ are $(i)-$differentiability, we have

$$(-s\odot\widehat{u}(0))\ominus_h\left(-s^2\odot L(\widehat{u}(t))\right)\ominus_h\widehat{u}'(0) = -k^2\odot L(\widehat{u}(t))\oplus L\left(g(t)\right) \tag{9}$$

and we get the $\alpha-cut$ representation of Eq. (9) as following

$$L\left\{u_\alpha^-(t)\right\} = \frac{1}{s^2+k^2}z_\alpha^- + \frac{s}{s^2+k^2}u_\alpha^- + \frac{L\left(g\left(t\right)\right)}{s^2+k^2}$$

and

$$L\left\{u_\alpha^+(t)\right\} = \frac{1}{s^2+k^2}z_\alpha^+ + \frac{s}{s^2+k^2}u_\alpha^+ + \frac{L\left(g\left(t\right)\right)}{s^2+k^2}$$

case[4]

If $\widehat{u}$ and $\widehat{u}'$ are $(ii)-$differentiability, we have

$$\left(s^2\odot L(\widehat{u}(t))\right)\ominus_h(s\odot\widehat{u}(0))\oplus(-1\odot\widehat{u}'(0)) = -k^2\odot L(\widehat{u}(t))\oplus L\left(g(t)\right) \tag{10}$$

and we get the $\alpha - cut$ representation of Eq. (10) as following

$$\begin{aligned} L\left\{u_{\alpha}^{-}(t)\right\} &= \frac{s^3}{s^4-k^4}u_{\alpha}^{-} + \frac{s^2}{s^4-k^4}z_{\alpha}^{+} \\ &\quad -\frac{k^2 s}{s^4-k^4}u_{\alpha}^{+} - \frac{k^2}{s^4-k^4}z_{\alpha}^{-} + \frac{1}{(s^2+k^2)}L\left(g\left(t\right)\right), \end{aligned}$$

and

$$\begin{aligned} L\left\{u_{\alpha}^{+}(t)\right\} &= \frac{s^3}{s^4-k^4}u_{\alpha}^{+} + \frac{s^2}{s^4-k^4}z_{\alpha}^{-} \\ &\quad -\frac{k^2 s}{s^4-k^4}u_{\alpha}^{-} - \frac{k^2}{s^4-k^4}z_{\alpha}^{+} + \frac{1}{(s^2+k^2)}L\left(g\left(t\right)\right). \end{aligned}$$

Then, the authors of [12] state that if we ensure that the solution $[\widehat{u}\left(t\right)]^{\alpha} = [u_{\alpha}^{-}\left(t\right), u_{\alpha}^{+}\left(t\right)]$ of the problem (5) are valid cut sets of a fuzzy number valued function and if $[\widehat{u}\left(t\right)]^{\alpha} = [u_{\alpha}^{-}\left(t\right), u_{\alpha}^{+}\left(t\right)]$ are valid cut sets of a fuzzy valued function, then by the Theorem 1, it is possible to construct the solution of FIVP (5).

Now we give an example to solve second-order FIVP under generalized H-differentiability.

Example 1. Consider the following FIVP: (Electric circuit)

$$\widehat{u}''(t) = -4\widehat{u}(t) \oplus g(t),\ \widehat{u}(0) = \widehat{0},\ \ \widehat{u}'(0) = \widehat{0} \tag{11}$$

where $\widehat{0} = (-0.5, 0, 0.5)$ is initial condition given by linearly correlated fuzzy numbers whose $\alpha-$cuts are given by $\left[\widehat{0}\right]^{\alpha} = [0.5\alpha - 0.5, 0.5 - 0.5\alpha]$, $g\left(t\right) = 1 - 2U\left(t-1\right) + U\left(t-2\right)$ is a real Heaviside funcion. This problem is a fuzzy version of the electric circuit problem which is given by Nagle et al. [21]. Now we consider this example by using fuzzy laplace method in four cases (gh-differentiability) as following:

case[1]

Let us consider $\widehat{u}(t)$ and $\widehat{u}'(t)$ are $(i)-$differentiable; then by applying (1), we have

$$\left(s^2 \odot L(\widehat{u}(t))\right) \ominus_h \left(s \odot \widehat{u}(0)\right) \ominus_h \widehat{u}'(0) = -2^2 \odot L(\widehat{u}(t)) \oplus L\left(g\left(t\right)\right),$$

then we get the $\alpha-$cut representation of solution as following

$$\begin{aligned} u_{\alpha}^{-}(t) &= \left(0.5\alpha - 0.5\right)/4\left(2\left(cosh(2t) + cos(2t)\right) + sinh(2t) + sin(2t)\right) \\ &\quad - \left(0.5 - 0.5\alpha\right)/4\left(2\left(cosh(2t) - cos(2t)\right) + sinh(2t) - sin(2t)\right) \\ &\quad + \left(1/4 - 1/4cos(2t)\right) - \left(1/2 - 1/2cos(2t-2)\right)U(t-1) \\ &\quad + \left(1/4 - 1/4cos(2t-4)\right)U(t-2) \end{aligned} \tag{12}$$

and

$$\begin{aligned} u_\alpha^+(t) &= (0.5-0.5\alpha)/4\left(2\left(cosh(2t)+cos(2t)\right)+sinh(2t)+sin(2t)\right) \\ &\quad -(0.5\alpha-0.5)/4\left(2\left(cosh(2t)-cos(2t)\right)+sinh(2t)-sin(2t)\right) \\ &\quad +\left(1/4-1/4cos(2t)\right)-\left(1/2-1/2cos(2t-2)\right)U(t-1) \\ &\quad +\left(1/4-1/4cos(2t-4)\right)U(t-2) \end{aligned} \tag{13}$$

The diameter of 0−cut is

$$dt = cosh(2t) + sinh(2t)/2$$

which is positive for all $t \in [0,\infty[$. So the solution (12)-(13) yields a fuzzy function $\widehat{u} : [0,\infty[\to \mathbb{R}_F$. Note that the conditions (i) and (ii) of Theorem 1 are indeed satisfied:

i) by analysing the signs of the terms of Eq. (12)-(13), one can verify that $u_\alpha^-(t) < u_\beta^-(t)$ and $u_\alpha^+(t) > u_\beta^+(t)$ for all $t > 0$ and $\beta > \alpha$.
ii) since $\widehat{0}$ is a fuzzy number, we have that $u_{\alpha_k}^-(t) \to u_\alpha^-(t)$ and $u_{\alpha_k}^+(t) \to u_\alpha^+(t)$ whenever (α_k) is a non-decreasing sequence converging to $\alpha \in]0,1]$.

Hence, from Theorem 1, $\widehat{u} : [0,\infty[\to \mathbb{R}_F$ is a fuzzy function, that is, $\widehat{u}(t)$ is a fuzzy number for every $t \geq 0$. Similarly, one can show that $\widehat{u}' : [0,\infty[\to \mathbb{R}_F$ is well defined and $\widehat{u}$, $\widehat{u}'$ are differentiable in the sense (i) (that is, $\widehat{u}$ and $\widehat{u}'$ are Hukuhara differentiable).

So the function $\widehat{u}$, given by (12)-(13), is a solution of the problem (5) and Figure 1 illustrates the geometric behavior of this solution.$\widehat{u}(t)$ is a valid fuzzy function for $t \in (0,\infty)$ in Figure 1.

case[2]

Let us consider $\widehat{u}(t)$ is $(i)-$differentiable and $\widehat{u}'(t)$ are $(ii)-$differentiable; then by applying (2), we have

$$-\widehat{u}'(0;\alpha) \ominus_h \left(-s^2\right) L(\widehat{u}(t;\alpha)) - s\widehat{u}(0;\alpha) \oplus 4\left(L\widehat{u}(t;\alpha)\right) = L\left(tU(t-3)\right),$$

then we get the $\alpha-$cut representation of solution as following

$$\begin{aligned} u_\alpha^-(t) &= (0.5\alpha-0.5)\left(cos(2t)+sin(2t)\right) \\ &\quad +\left(1/4-1/4cos(2t)\right)-\left(1/2-1/2cos(2t-2)\right)U(t-1) \\ &\quad +\left(1/4-1/4cos(2t-4)\right)U(t-2) \end{aligned} \tag{14}$$

and

$$\begin{aligned} u_0^+(t) &= (0.5-0.5\alpha)\left(cos(2t)+sin(2t)\right) \\ &\quad +\left(1/4-1/4cos(2t)\right)-\left(1/2-1/2cos(2t-2)\right)U(t-1) \\ &\quad +\left(1/4-1/4cos(2t-4)\right)U(t-2) \end{aligned} \tag{15}$$

The diameter of $0-$cut is

$$dt = cos(2t) + sin(2t)$$

which is not positive for all $t \in [0, \infty[$. Therefore, $\alpha-$cut of $\widehat{u}$ (14)-(15) is not a solution of problem (11). This indicates that the condition of case (2) do not hold true in the domain $[0, \infty[$. But at some intervals conditions of fuzzy solution are satisified such as $t \in [0, 1] \cup [3, 4]$.

case[3]

Let us consider $\widehat{u}(t)$ is $(ii)-$differentiable and $\widehat{u}'(t)$ are $(i)-$differentiable; then by applying (3), we have

$$-s\widehat{u}(0;\alpha) \ominus_h \left(-s^2\right) L(\widehat{u}(t;\alpha)) \ominus_h \widehat{u}'(0;\alpha) \oplus 4\left(L\widehat{u}(t;\alpha)\right) = L\left(tU(t-3)\right),$$

then we get the $\alpha-$cut representation of solution as following

$$\begin{aligned} u_\alpha^-(t) \quad = \quad & (0.5\alpha - 0.5)\,cos(2t) + (0.5 - 0.5\alpha)\,sin(2t) \\ & + (1/4 - 1/4cos(2t)) - (1/2 - 1/2cos(2t-2))\,U(t-1) \\ & + (1/4 - 1/4cos(2t-4))\,U(t-2) \end{aligned} \tag{16}$$

and

$$\begin{aligned} u_0^+(t) \quad = \quad & (0.5 - 0.5\alpha)\,cos(2t) + (0.5\alpha - 0.5)\,sin(2t) \\ & + (1/4 - 1/4cos(2t)) - (1/2 - 1/2cos(2t-2))\,U(t-1) \\ & + (1/4 - 1/4cos(2t-4))\,U(t-2) \end{aligned} \tag{17}$$

The diameter of $0-$cut is

$$dt = cos(2t) - sin(2t)$$

which is not positive for all $t \in [0, \infty[$. Therefore, $\alpha-$cut of $\widehat{u}$ (16)-(17) is not a solution of problem (11). This indicates that the condition of case (3) do not hold true in the domain $[0, \infty[$. But at some intervals conditions of fuzzy solution are satisified such as $t \in [0, 0.3] \cup (2, 3.5)$.

case[4]

Let us consider $\widehat{u}(t)$ and $\widehat{u}'(t)$ are $(ii)-$differentiable; then by applying (4), we have

$$s^2 L(\widehat{u}(t;\alpha)) \ominus_h s\widehat{u}(0;\alpha) - \widehat{u}'(0;\alpha) \oplus 4\left(L\widehat{u}(t;\alpha)\right) = L\left(tU(t-3)\right),$$

then we get the $\alpha-$cut representation of solution as following

$$\begin{aligned} u_\alpha^-(t) \quad = \quad & (0.5\alpha - 0.5)\,/4\,(2\,(cosh(2t) + cos(2t)) - (sinh(2t) - sin(2t))) \\ & - (0.5 - 0.5\alpha)\,/4\,(2\,(cosh(2t) - cos(2t)) - (sinh(2t) + sin(2t))) \\ & + (1/4 - 1/4cos(2t)) - (1/2 - 1/2cos(2t-2))\,U(t-1) \\ & + (1/4 - 1/4cos(2t-4))\,U(t-2) \end{aligned} \tag{18}$$

and

$$
\begin{aligned}
u_\alpha^+(t) &= (0.5-0.5\alpha)/4\left(2\left(cosh(2t)+cos(2t)\right)-\left(sinh(2t)-sin(2t)\right)\right) \\
&\quad -(0.5\alpha-0.5)/4\left(2\left(cosh(2t)-cos(2t)\right)-\left(sinh(2t)+sin(2t)\right)\right) \\
&\quad +\left(1/4-1/4cos(2t)\right)-\left(1/2-1/2cos(2t-2)\right)U(t-1) \\
&\quad +\left(1/4-1/4cos(2t-4)\right)U(t-2)
\end{aligned} \tag{19}
$$

The diameter of $0-$cut is

$$dt = cosh(2t) - sinh(2t)/2$$

which is not positive for all $t \in [0,\infty[$. Therefore, $\alpha-$cut of $\widehat{u}$ (18)-(19) is not a solution of problem (11). This indicates that the condition of case (4) do not hold true in the domain $[0,\infty[$.

Note that the crisp solution of the problem (11) is

$$
\begin{aligned}
u(t) &= (\frac{1}{4}-\frac{1}{4}\cos(2t))-(\frac{1}{2}-\frac{1}{2}\cos(2(t-1))U(t-1) \\
&\quad +(\frac{1}{4}-\frac{1}{4}\cos(2(t-2))U(t-2).
\end{aligned} \tag{20}
$$

So it is easy to see that if the initial conditions are real numbers, that is, $u_\alpha^-(0) = u_\alpha^+(0) = \left(u_\alpha^-\right)'(0) = \left(u_\alpha^+\right)'(0) = 0$, for all $\alpha \in (0,1]$, then Eq. (20) becomes the classical solution of Electric circuit problem for all the four cases. But fuzzy solution of the problem (11) is satisfied only case 1 for $t \in (0,\infty)$. In case 2, case 3 and case 4, the fuzzy solution is provided only at certain intervals. So there are no fuzzy solutions in these cases for $t \in (0,\infty)$.

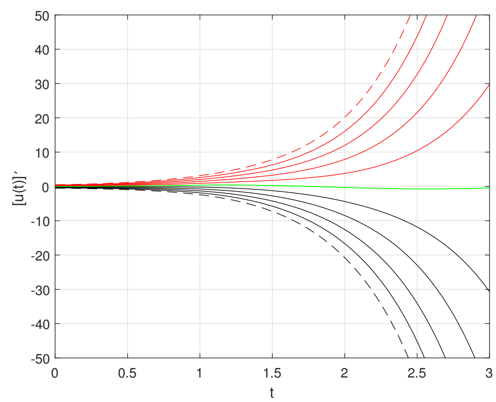

FIGURE 1. Fuzzy solution of FIVP (11) for Cases 1. For $\alpha = 0, 0.2, 0.4, 0.6, 0.8, 1$, $\alpha-$cut of $\widehat{u}(t)$. the red and black dashed-line illustrate the 0-cut of $\widehat{u}(t)$ and green line illustrate the $1-$cut for all t.

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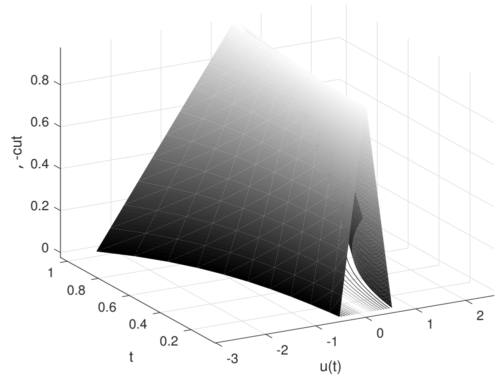

FIGURE 2. Fuzzy solution of FIVP (11) in the $ut\alpha-$space. the gray-scale lines varying from black to white represent the $\alpha-$cuts of $\widehat{u}(t)$, with α varying from 0 to 1

3. Conclusion. In this study, the fuzzy Laplace transform method was used to obtain the fuzzy solution for second order FIVP under strongly generalized differentiability with the forcing function which is Heaviside function. Moreover, itt was discussed how the sum of two fuzzy numbers is real. and gived some remark. To illustrate the transform method, an example was solved by generalized differentiability and four different solutions for the FIVP were obtained. But it is shown that just for one case the valid fuzzy solution was found for each α-cut from Theorem 1.

As seen in this work, the uniqueness of the solution of a fuzzy initial value problem is lost when the strongly generalized derivative concept was used. This situation is looked on as a disadvantage, but actually it is not [1], because researchers can choose the best solution which better reflects the behavior of the problem under consideration, from multiple solutions. So a definite analysis of the physical system is required to find the best solution which under study.

REFERENCES

1. B. Bede, I. J. Rudas, A. L. Bencsik, *First order linear fuzzy differential equations under generalized differentiability*, Fuzzy Sets and Syst. **177** (2007), no. 7, 1648–1662.

2. J. J. Buckley, T. Feuring , *Fuzzy initial value problem for Nth-order linear differentia equations* , Fuzzy Sets and Syst. **121** (2001), 247–255.

3. B. Bede, L. Stefanini, *Generalized differentiability of fuzzy-valued functions*, Fuzzy Sets and Syst. **230** (2013), 119–141.

4. R. Bracewell, *Heaviside's Unit Step Function, H(x)*, The Fourier Transform and Its Applications, 3rd ed. , New York: McGraw-Hill, 2000.

5. B. J. Hunt, *Oliver Heaviside: A first-rate oddity*, Physics Today **65** (2012), no. 11, 48–54.

6. M. Friedman, M. Ma, A. Kandel, *Numerical solutions of fuzz differential and integral equations*, Fuzzy Sets and Syst. **106** (1999), no. 1, 35–48.

7. B. Bede, G. S. Gal, *Generalizations of the differentiability of fuzzy-number-valued functions with applications to fuzzy differential equations*, Fuzzy Sets and Syst. **151** (2005), no. 3, 581–599.

8. D. Dubios, H. Prade, *Towards fuzzy differential calculus* , Fuzzy Sets Syst **8** (1982), 225–233.

9. L. T. Gomes, L. C. Barros, B. Bede, *Fuzzy Differential Equations in Various Approaches*, Springer Cham, London, 2015.

10. H. Gültekin Çitil, *Solutions of Fuzzy Differential Equation with Fuzzy Number Coefficient by Fuzzy Laplace Transform*, Comptes rendus de l'Acade'mie bulgare des Sciences **73** (1999), no. 9, 1191-1200.

11. A. Khastan, F. Bahrami, K. Ivaz, *New Results on Multiple Solutions for Nth-Order Fuzzy Differential Equations under Generalized Differentiability*, Boundary Value Problems **70** (2009), 1–13.

12. A. Khastan, J. J. Nieto, *A boundary value problem for second order fuzzy differential equations*, Nonlinear Analysis **72** (2010), no. 9-10, 3583–3593.

13. O. Kaleva, *Fuzzy Differential equations*, Fuzzy Sets and Systems **24** (1987), 301-317.

14. M. L. Puri, D. A. Ralescu, *Differentials of fuzzy functions*, Journal of Mathematical Analysis and Applications **91** (1983), 552–558.

15. M. L. Puri, D. A. Ralescu, *Fuzzy random variables*, J. Math. Anal. Appl. **114** (1986), 409–422.

16. S. Seikkala, *On the fuzzy initial value problem*, Fuzzy Set and Syst. **24** (1987), no. 3, 319–330.

17. T. Allahviranloo, M. Barkhordari Ahmadi, *Fuzzy Laplace transforms*, Soft Computing **14** (2010), no. 3, 235–243.

18. L.C. Barros, F. Santo Pedro , *Fuzzy differential equations with interactive derivative*, Fuzzy Sets Syst **309** (2017), 64-80.

19. T. Allahviranloo, N. Ahmady, E. Ahmady , *Numerical solution of fuzzy differential equations by predictor-corrector method* , Inform. Sci. **177** (2007), no. 7, 1633–1647.

20. S. Abbasbandy, T, Allahviranloo, O. Lopez-Pouso, J. J. Nieto , *Numerical methods for fuzzy differential inclusions* , Comput. and Math. Appl. **48** (2004), no. 10-11, 1633–1641.

21. R. K. Nagle, E. B. Saff, A. D. Snider, *Fundamentals of differential equations and boundary value problems*, Pearson Education, 2011.

22. S.A.B. Salgado, L.C. Barros, E. Esmi, D.E. Sanchez , *Solution of a fuzzy differential equation with interactivity via Laplace transform* , J Intell Fuzzy Syst **37** (2019), no. 2, 2495–2501.

23. S. Salahshour, T. Allahviranloo, *Applications of fuzzy Laplace transforms*, Soft Computing **17** (2013), no. 1, 145- 158.

24. H. C. Wu, *The improper fuzzy Riemann integral and its numerical integration*, Theory of Probability **111** (1999), no. 4, 109–137.

25. C. Carlsson, R. Full´er, P. Majlender , *Additions of completely correlated fuzzy numbers* , IEEE International Conference **1** (2004), 535–539.

26. L. Zadeh, *A Fuzzy set theoretic interpretation of linguistic hedges* , Journal of Cybernetics **3** (1972), 4-34.

27. L.C. Barros, R.C. Bassanezi, P.A. Tonelli , *On the continuity of the Zadeh's extension* , Seventh IFSA World Congress **2** (1997), 3-8.

28. L.C. Barros, R.C. Bassanezi, W.A. Lodwick , *, First Course in Fuzzy Logic, Fuzzy Dynamical Systems, and Biomathematics* , Springer 2016.

29. G. J. Klir, B. Yuan , *Fuzzy Sets and Fuzzy Logic* , New Jersey: Prentice Hall , 1995.

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