A ROBUST STUDY ON THE EXISTENCE AND MULTIPLICITY OF SOLUTIONS FOR A CLASS OF VARIABLE-EXPONENT EQUATIONS OF KIRCHHOFF TYPE

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Abstract. The multiplicity and existence of solutions for a class of variable-exponent equations include Kirchhoff term in variable-exponent Sobolev spaces has been proven under certain conditions. This is achieved by utilizing the sub-super solution method in conjunction with the mountain pass theory.

1. Introduction and main results

The solutions to mathematical models enhance our capacity to predict, understand, and influence the universe. They are crucial for tackling complex issues, decision making, and advancing in different fields and application [1, 2, 3, 4]. In this research, our main concentration will be on existence of solutions for Kirchhoff $p(q)-$Laplacian problem

$$\begin{cases} -\mathcal{Q}\left(\sum_{\beth=1}^{\mathcal{R}} \int\limits_{\mathcal{Z}} \frac{1}{p(q)} \left|\frac{\partial \upsilon}{\partial q_\beth}\right|^{p(q)} dq\right) \sum_{\beth=1}^{\mathcal{R}} \frac{\partial}{\partial q_\beth}\left(\left|\frac{\partial \upsilon}{\partial q_\beth}\right|^{p(q)-2} \frac{\partial \upsilon}{\partial q_\beth}\right) = \varpi(q)\upsilon^{\aleph(q)-1} + j(q,\ \upsilon) & , \text{in } \mathcal{Z} \\ \upsilon \geq 0 & , \text{in } \mathcal{Z} \\ \upsilon = 0 & , \text{on } \partial\mathcal{Z}, \end{cases} \tag{1.1}$$

where $\mathcal{Z} \subset \mathrm{R}^{\mathcal{R}}$ is a bounded domain having smooth boundary, $\sum_{\beth=1}^{\mathcal{R}} \frac{\partial}{\partial q_\beth}\left(\left|\frac{\partial \upsilon}{\partial q_\beth}\right|^{p(q)-2} \frac{\partial \upsilon}{\partial q_\beth}\right)$ is $p(q)-$Laplacian operator, where $p \in \mathcal{M}^1(\overline{\mathcal{Z}})$, with $2 \leq p^- \leq p^+ < \mathcal{R}$, where $p^- := \text{ess inf}_{\mathcal{Z}}\, p$, $p^+ := \text{ess sup}_{\mathcal{Z}}\, p, \aleph \in \mathcal{M}(\overline{\mathcal{Z}},\ (1,\ +\infty))$ and $\varpi \in L^\infty(\mathcal{Z})$ in which $\varpi(q) > 0$.a.e., $q \in \mathcal{Z}$. Define the function $p^\star(q) := \frac{Np(q)}{\mathcal{R}-p(q)}$ if $p(q) < \mathcal{R}$ and $p^\star(q) := +\infty$ if $\mathcal{R} \geq p(q)$.

Over the past decade, extensive research have been done on the equations with variable exponent growth conditions, with significant advancements documented in recent works such as [8, 10, 12, 18]. The extensive literature on problems having variable exponent growth conditions is motivated by the understanding that these equations can effectively model various

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phenomena in field of image processing [20], theory of electrorheological fluids [6, 17], and theory of elasticity [21]. The elliptic equations having variable exponent growth conditions often employ the so-called $p(q)$-Laplace operator, i.e., $\sum_{\daleth=1}^{\mathcal{R}} \frac{\partial}{\partial q_\daleth}\left(\left|\frac{\partial \upsilon}{\partial q_\daleth}\right|^{p(q)-2}\frac{\partial \upsilon}{\partial q_\daleth}\right) := \triangle_{p(q)}$,, where $p(q)$ represents a function and $\forall\ q,\ 1 < p\,(q)$.

Problem (1.1) represents the $p(q)-$ version associated with

$$\alpha\frac{\partial^2 \upsilon}{\partial t^2} - \left(\frac{\alpha_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial \upsilon}{\partial q}\right|^2 dq\right)\frac{\partial^2 \upsilon}{\partial q^2} = 0.$$

Kirchhoff [13] introduced this concept first as a generalization of classical D'Alembert wave equation, considering changes in the length of strings due to transverse vibrations. Additionally, evolution equation of Kirchhoff-type was presented by Woinowsky-Krieger [22] which is stated as

$$\upsilon_{tt} + \Delta^2 \upsilon - \mathcal{Q}(||\nabla \upsilon||_2^2)\Delta \upsilon = g(q,\ \upsilon). \tag{1.2}$$

It act as a model for deflection of an extensible beam; for details on physics background and related models, see [5, 7]. Mathematically, multiplicity and existence of solutions for Kirchhoff-type problems having $p(q)-$ Laplacian have been extensively studied in [12]. The authors in [12], demonstrated existence of solutions for a wide range of problems with variable exponents. To obtain the multiplicity of solutions they utilized additional conditions. The paper also provides illustrative examples to demonstrate applicability of the results. Methodology is based on the use of sub-super solutions and appropriate L^∞ estimates within the context of variable spaces.

Main objective of this research is to examine multiplicity and existence of solutions for problem (1.1). The outcomes of this research is an extension of previous findings in [12], which concentrated on $p(q)-$Laplacian problem with $\mathcal{Q} \equiv 1$. Our paper explores Kirchhoff-type problems having variable exponent, with focus on conditions where $\mathcal{Q}$ is not fixed. Utilizing sub-super solution method and specialized weak comparison principle, we demonstrate existence of solution for problem (1.1). Additionally, utilizing mountain pass theorem, we establish multiplicity of solutions for problem (1.1). These outcomes represent significant new contributions to Kirchhoff-type variable-exponent boundary value problems.

In this context, we consider nonlinearity j and Kirchhoff function $\mathcal{Q}$ under certain assumptions.

(K_0) Let $\mathfrak{Q} : [0,\ +\infty) \to [k_0,\ +\infty)$ be a continuous and nondecreasing function for some positive constant k_0;

(K_1) One can find $\chi \in (0,1)$ with

$$\widehat{\mathfrak{Q}}(t) := \int_0^t \mathfrak{Q}(r)dr \geq (1-\chi)\mathfrak{Q}(t)t \ \forall\ t \geq 0;$$

(j_1) $j \in \mathcal{M}(\mathcal{Z} \times [0,\ +\infty), \mathrm{R})$ and $\exists\ \vartheta > 0$ with

$$j(q,\ t) \geq \varpi(q)(1 - t^{\aleph(q)-1})\ \forall\ (q,\ t) \in \mathcal{Z} \times [0,\ \vartheta];$$

(j_2) One can find $s \in \mathcal{M}(\overline{\mathcal{Z}},\ (1,\ +\infty))$ such that

$$|j(q,\ t)| \leq \varpi(q)(1 + t^{s(q)-1})\ \forall\ (q,\ t) \in \mathcal{Z} \times [0,\ +\infty);$$

(j_3) One can find $\mu > \frac{p^+}{1-\chi}$ such that

$$0 < \mu G(q,\ t) := \mu \int_0^t j(q,r)dr \leq j(q,\ t)t \text{ i.e. } q \in \mathcal{Z}\ \forall\ 0 < T < t.$$

Theorem 1.1. *Assume that (K_0) and $(j_1) - (j_2)$ holds. Then, we can get a $\digamma_\star > 0$ in such manner that problem (1.1) has at one solution with $||\varpi||_\infty < \digamma_\star$.*

Theorem 1.2. *Assume that $(K_0) - (K_1)$ and $(j_1) - (j_3)$ holds. If $\aleph^+, r^+ < (p^\star)^-$ and $\left(\aleph^- > \frac{p^+}{1-\chi} \text{ or } \aleph^+ < p^-\right)$, then we can get $\digamma^\star > 0$ such that (1.1) has at one solution with $||\varpi||_\infty < \digamma^\star$.*

Our paper is structured as follows: Section 2 introduces some outcomes related to variable exponentiated distances, Section 3 provides auxiliary L^∞ estimate, and Sections 4 and 5 presents the proofs of Theorems 1.1 and 1.2, respectively.

2. Fundamental theory

This section, showcase some fundamental ideas and concepts concerning variable exponent Lebesgue spaces, which will be used to prove the main results (see [11]). Let us indicate the set of all continuous function by $\mathcal{M}_+(\overline{\mathcal{Z}})$ and $\aleph : \overline{\mathcal{Z}} \to (1,\ +\infty)$. For $\aleph \in \mathcal{M}_+(\overline{\mathcal{Z}})$, we have

$$\aleph^+ := \max_{\overline{\mathcal{Z}}} \aleph(x) \text{ and } \aleph^- := \min_{\overline{\mathcal{Z}}} \aleph(x).$$

Variable exponent Lebesgue space is stated as

$$L^{\aleph(q)}(\mathcal{Z}) = \left\{ \upsilon : \mathcal{Z} \to \mathrm{R} \text{ measurable} : \int_{\mathcal{Z}} |\upsilon|^{\aleph(q)} dq < \infty \right\}.$$

having norm

$$||\upsilon||_{\aleph(q)} = \inf\left\{\nu > 0 \ : \ \int_{\mathcal{Z}} \left|\frac{\upsilon}{\nu}\right|^{\aleph(q)} dq \leq 1\right\}.$$

Let $L^{\aleph'(q)}(\mathcal{Z})$ be the conjugate space of $L^{\aleph(q)}(\mathcal{Z})$ such that $\frac{1}{\aleph(q)} + \frac{1}{\aleph'(q)} = 1$. Then, the below stated inequality of Holder-type satisfies.

([11]): Let $\upsilon \in L^{\aleph(q)}(\mathcal{Z})$ and $\varkappa \in L^{\aleph'(q)}(\mathcal{Z})$. Then

$$\int_{\Omega} |\upsilon\varkappa| dq \leq \left(\frac{1}{\aleph-} + \frac{1}{(\aleph'-)}\right) ||\upsilon||_{\aleph(q)} ||\varkappa||_{\aleph'(q)}.$$

Modular function in space $L^{\aleph(x)}$ is consider in the following way

$$\alpha_{\aleph(q)}(\upsilon) = \int_{\mathcal{Z}} |\upsilon|^{\aleph(q)} dq.$$

([11]): For any $\upsilon \in L^{\aleph(q)}(\mathcal{Z})$, we get

$$\min\left(||\upsilon||_{\aleph(q)}^{\aleph-},\ ||\upsilon||_{\aleph(q)}^{\aleph+}\right) \leq \alpha_{\aleph(q)}(\upsilon) \leq \max\left(||\upsilon||_{\aleph(q)}^{\aleph-},\ ||\upsilon||_{\aleph(q)}^{\aleph+}\right) .$$

([11]): Let $\upsilon \in L^{\aleph(q)}(\mathcal{Z})$ *and* $\{\upsilon_n\} \subset L^{\aleph(q)}(\mathcal{Z})$. Then, the below stated properties are equivalent:

(1) $\lim_{n\to+\infty} ||\upsilon_n - \upsilon||_{\aleph(q)} = 0$;

(2) $\lim_{n\to+\infty} \alpha_{\aleph(q)}(\upsilon_n - \upsilon) = 0$.

Sobolev space in generalized form as $\mathcal{W}^{1,p(q)}$ is

$$\mathcal{W}^{1,p(q)}(\mathcal{Z}) = \left\{\upsilon \in L^{p(q)}(\mathcal{Z}) : \frac{\partial \upsilon}{\partial q_{\daleth}} \in L^{p(q)}(\mathcal{Z}), \daleth = 1, ..., \mathcal{R}\ \right\},$$

with norm

$$||\upsilon||_{1,p(q)} = \|\upsilon\|_{p(q)} + \sum_{\daleth=1}^{\mathcal{R}} \left\|\frac{\partial \upsilon}{\partial q_{\daleth}}\right\|_{p_{(q)}}$$

$(\mathcal{W}^{1,p(q)}(\mathcal{Z}),\ ||\cdot||_{1,p(x)})$ represents a Banach reflexive space [11]. Suppose $\mathbf{X}_0 := \mathcal{W}_0^{1,p(q)}(\mathcal{Z})$ be the closure of $\mathcal{M}_0^{\infty}(\mathcal{Z})$ in $\mathcal{W}^{1,p(q)}$ $(\mathcal{Z})$. After all $p(q) < p^{\star}(q)\ \forall\ q \in \overline{\mathcal{Z}}$,

$$||\upsilon||_{p(q)} \leq \mathcal{M} \sum_{\daleth=1}^{\mathcal{R}} \left\|\frac{\partial \upsilon}{\partial q_{\daleth}}\right\|_{p_{(q)}} \quad \text{for all } \upsilon \in \mathbf{X}_0 \text{ (Poincaré-typeinequality)},$$

in which $\mathcal{M}$ represents positive constant independent of $\Im$ and $||\cdot||_{\mathbf{X}_0}$ represents norm of space $\mathbf{X}_0$, stated as

$$||\upsilon||_{\mathbf{X}_0} := \sum_{\daleth=1}^{\mathcal{R}} \left\|\frac{\partial \upsilon}{\partial q_{\daleth}}\right\|_{p_{(q)}} \quad \text{for all } \upsilon \in \mathbf{X}_0,$$

Lemma 2.1. *Assume* $\aleph \in \mathcal{M}_+(\overline{\mathcal{Z}})$ *such that* $p^{\star}(q) > \aleph(q)\ \forall\ q \in \overline{\mathcal{Z}}$*. Then, one can achieve compact and continuous embedding* $\mathbf{X}_0 \hookrightarrow L^{\aleph(x)}(\mathcal{Z})$.

3. Comparison principle

This section will present an estimate for weak comparison principle for (1.1) and L^∞ estimate, which will be utilized in the formation of appropriate sub and super solutions.

Definition 3.1. Take $\upsilon, \varkappa \in \mathbf{X}_0$. We say that

$$-\mathcal{Q}(\mathcal{S}(\upsilon))\sum_{\gimel=1}^{\mathcal{R}}\left|\frac{\partial\upsilon}{\partial q_\gimel}\right|^{p(q)-2}\frac{\partial\upsilon}{\partial q_\gimel} \leq -\mathcal{Q}(\mathcal{S}(\varkappa))\sum_{\gimel=1}^{\mathcal{R}}\left|\frac{\partial\varkappa}{\partial q_\gimel}\right|^{p(q)-2}\frac{\partial\varkappa}{\partial q_\gimel}$$

if $\forall$ nonnegative function $\varsigma \in \mathbf{X}_0$.

$$\mathcal{Q}(\mathcal{S}(\upsilon))\sum_{\gimel=1}^{\mathcal{R}}\int_{\mathcal{Z}}\left|\frac{\partial\upsilon}{\partial q_\gimel}\right|^{p(q)-2}\frac{\partial q}{\partial q_\gimel}.\frac{\partial\varsigma}{\partial q_\gimel}dq \leq \mathcal{Q}(\mathcal{S}(\varkappa))\sum_{\gimel=1}^{\mathcal{R}}\int_{\mathcal{Z}}\left|\frac{\partial\varkappa}{\partial q_\gimel}\right|^{p(q)-2}\frac{\partial\varkappa}{\partial q_\gimel}.\frac{\partial\varsigma}{\partial q_\gimel}dq,$$

in which $\mathcal{S}(\upsilon) = \sum_{\gimel=1}^{\mathcal{R}}\int\limits_{\mathcal{Z}}\frac{1}{p(q)}\left|\frac{\partial\upsilon}{\partial q_\gimel}\right|^{p(q)}dq$.

Lemma 3.2. *Let (K_0) satisfies. Then, $\tau : \mathbf{X}_0 \to \mathbf{X}_0^*$ is given as*

$$\langle\tau(\upsilon), \varsigma\rangle = \mathcal{Q}(\mathcal{S}(\upsilon))\sum_{\gimel=1}^{\mathcal{R}}\int_{\mathcal{Z}}\left|\frac{\partial\upsilon}{\partial q_\gimel}\right|^{p(q)-2}\frac{\partial\upsilon}{\partial q_\gimel}.\frac{\partial\varsigma}{\partial q_\gimel}dq, \tag{3.1}$$

is strictly monotone and continuous.

Proof. Clearly the operator τ is continuous. Let $\upsilon \neq \varkappa \in \mathbf{X}_0$, and $\mathcal{S}(\upsilon) \geq \mathcal{S}(\varkappa)$. Moreover, nondecreasing property of $\mathcal{Q}$ yields

$$\mathcal{Q}(\mathcal{S}(\upsilon)) \geq \mathcal{Q}(\mathcal{S}(\varkappa)). \tag{3.2}$$

Moreover, we get

$$\frac{\partial\upsilon}{\partial q_\gimel}.\frac{\partial\varkappa}{\partial q_\gimel} \leq \frac{1}{2}\left(\left|\frac{\partial q}{\partial q_\gimel}\right|^2 + \left|\frac{\partial\varkappa}{\partial q_\gimel}\right|^2\right). \tag{3.3}$$

Thus

$$\int_{\mathcal{Z}}\left(\left|\frac{\partial\upsilon}{\partial q_\gimel}\right|^{p(q)} - \left|\frac{\partial\upsilon}{\partial q_\gimel}\right|^{p(q)-2}\frac{\partial\upsilon}{\partial q_\gimel}.\frac{\partial\varkappa}{\partial q_\gimel}\right)dq \geq \int_{\mathcal{Z}}\frac{1}{2}\left|\frac{\partial\upsilon}{\partial q_\gimel}\right|^{p(q)-2}\left(\left|\frac{\partial\upsilon}{\partial q_\gimel}\right|^2 - \left|\frac{\partial\varkappa}{\partial q_\gimel}\right|^2\right)dq \tag{3.4}$$

and

$$\int_{\mathcal{Z}}\left(\left|\frac{\partial\varkappa}{\partial q_\gimel}\right|^{p(q)} - \left|\frac{\partial\varkappa}{\partial q_\gimel}\right|^{p(q)-2}\frac{\partial\upsilon}{\partial q_\gimel}.\frac{\partial\varkappa}{\partial q_\gimel}\right)dq \geq \int_{\mathcal{Z}}\frac{1}{2}\left|\frac{\partial\varkappa}{\partial q_\gimel}\right|^{p(q)-2}\left(\left|\frac{\partial\varkappa}{\partial q_\gimel}\right|^2 - \left|\frac{\partial\upsilon}{\partial q_\gimel}\right|^2\right)dq \tag{3.5}$$

Assume $\gimel = 1, \ldots, \mathcal{R}$ be fixed and set

$$\mathcal{Z}_{a_\gimel} = \left\{q \in \mathcal{Z} : \left|\frac{\partial\upsilon}{\partial q_\gimel}\right| \geq \left|\frac{\partial\varkappa}{\partial q_\gimel}\right|\right\}$$

and

$$\mathcal{Z}_{b_\gimel} = \left\{q \in \mathcal{Z} : \left|\frac{\partial\upsilon}{\partial q_\gimel}\right| < \left|\frac{\partial\varkappa}{\partial q_\gimel}\right|\right\}.$$

From (3.2), (3.4)-(3.5) and (K_0), we have

$$
\begin{aligned}
A_\daleth \quad : \quad & = \mathcal{Q}(\mathcal{S}(\upsilon)) \int_{\mathcal{Z}_{a\daleth}} \left(\left| \frac{\partial \upsilon}{\partial q_\daleth} \right|^{p(q)} - \left| \frac{\partial \upsilon}{\partial q_\daleth} \right|^{p(q)-2} \frac{\partial \upsilon}{\partial q_\daleth} . \frac{\partial \varkappa}{\partial q_\daleth} \right) dq \\
& + \mathcal{Q}(\mathcal{S}(\varkappa)) \int_{\mathcal{Z}_{a_\daleth}} \left(\left| \frac{\partial \varkappa}{\partial q_\daleth} \right|^{p(q)} - \left| \frac{\partial \varkappa}{\partial q_\daleth} \right|^{p(q)-2} \frac{\partial \upsilon}{\partial q_\daleth} . \frac{\partial \varkappa}{\partial q_\daleth} \right) dq \\
\geq \quad & \frac{1}{2} \mathcal{Q}(\mathcal{S}(\upsilon)) \int_{\mathcal{Z}_{a_\daleth}} \left| \frac{\partial \upsilon}{\partial q_\daleth} \right|^{p(q)-2} \left(\left| \frac{\partial \upsilon}{\partial q_\daleth} \right|^2 - \left| \frac{\partial \varkappa}{\partial q_\daleth} \right|^2 \right) dq \\
& - \frac{1}{2} \mathcal{Q}(\mathcal{S}(\varkappa)) \int_{\mathcal{Z}_{a_\daleth}} \left| \frac{\partial \varkappa}{\partial q_\daleth} \right|^{p(q)-2} \left(\left| \frac{\partial \upsilon}{\partial q_\daleth} \right|^2 - \left| \frac{\partial \varkappa}{\partial q_\daleth} \right|^2 \right) dq \\
\geq \quad & \frac{1}{2} \mathcal{Q}(\mathcal{S}(\varkappa)) \int_{\mathcal{Z}_{a_\daleth}} \left(\left| \frac{\partial \upsilon}{\partial q_\daleth} \right|^{p(q)-2} - \left| \frac{\partial \varkappa}{\partial q_\daleth} \right|^{p(q)-2} \right) \left(\left| \frac{\partial \upsilon}{\partial q_\daleth} \right|^2 - \left| \frac{\partial \varkappa}{\partial q_\daleth} \right|^2 \right) dq \\
\geq \quad & \frac{k_0}{2} \int_{\mathcal{Z}_{a\daleth}} \left(\left| \frac{\partial \upsilon}{\partial q_\daleth} \right|^{p(q)-2} - \left| \frac{\partial \varkappa}{\partial q_\daleth} \right|^{p(q)-2} \right) \left(\left| \frac{\partial \upsilon}{\partial q_\daleth} \right|^2 - \left| \frac{\partial \varkappa}{\partial q_\daleth} \right|^2 \right) dq \\
\geq \quad & 0.
\end{aligned}
\tag{3.6}
$$

Similarly, we obtain

$$
\begin{aligned}
B_\daleth \quad : \quad & = \mathcal{Q}(\mathcal{S}(\upsilon)) \int_{\mathcal{Z}_{b\daleth}} \left(\left| \frac{\partial \upsilon}{\partial q_\daleth} \right|^{p(q)} - \left| \frac{\partial \upsilon}{\partial q_\daleth} \right|^{p(q)-2} \frac{\partial \upsilon}{\partial q_\daleth} . \frac{\partial \varkappa}{\partial q_\daleth} \right) dq \\
& + \mathcal{Q}(\mathcal{S}(\varkappa)) \int_{\mathcal{Z}_{b_\daleth}} \left(\left| \frac{\partial \varkappa}{\partial q_\daleth} \right|^{p(q)} - \left| \frac{\partial \varkappa}{\partial q_\daleth} \right|^{p(q)-2} \frac{\partial \upsilon}{\partial q_\daleth} . \frac{\partial \varkappa}{\partial q_\daleth} \right) dq \\
\geq \quad & \frac{1}{2} \mathcal{Q}(\mathcal{S}(\upsilon)) \int_{\mathcal{Z}_{b_\daleth}} \left| \frac{\partial \upsilon}{\partial q_\daleth} \right|^{p(q)-2} \left(\left| \frac{\partial \upsilon}{\partial q_\daleth} \right|^2 - \left| \frac{\partial \varkappa}{\partial q_\daleth} \right|^2 \right) dq \\
& - \frac{1}{2} \mathcal{Q}(\mathcal{S}(\varkappa)) \int_{\mathcal{Z}_{b_\daleth}} \left| \frac{\partial \varkappa}{\partial q_\daleth} \right|^{p(q)-2} \left(\left| \frac{\partial \upsilon}{\partial q_\daleth} \right|^2 - \left| \frac{\partial \varkappa}{\partial q_\daleth} \right|^2 \right) dq \\
\geq \quad & \frac{1}{2} \mathcal{Q}(\mathcal{S}(\varkappa)) \int_{\mathcal{Z}_{b_\daleth}} \left(\left| \frac{\partial \upsilon}{\partial q_\daleth} \right|^{p(q)-2} - \left| \frac{\partial \varkappa}{\partial q_\daleth} \right|^{p(q)-2} \right) \left(\left| \frac{\partial \upsilon}{\partial q_\daleth} \right|^2 - \left| \frac{\partial \varkappa}{\partial q_\daleth} \right|^2 \right) dq \\
\geq \quad & \frac{k_0}{2} \int_{\mathcal{Z}_{b\daleth}} \left(\left| \frac{\partial \upsilon}{\partial q_\daleth} \right|^{p(q)-2} - \left| \frac{\partial \varkappa}{\partial q_\daleth} \right|^{p(q)-2} \right) \left(\left| \frac{\partial \upsilon}{\partial q_\daleth} \right|^2 - \left| \frac{\partial \varkappa}{\partial q_\daleth} \right|^2 \right) dq \\
\geq \quad & 0.
\end{aligned}
\tag{3.7}
$$

It yields

$$\begin{aligned}\langle \tau(\upsilon)-\tau(\varkappa),\ \upsilon-\varkappa\rangle &= \langle \tau(\upsilon),\ \upsilon-\varkappa\rangle - \langle \tau(\varkappa),\ \upsilon-\varkappa\rangle\\ &= \langle \Xi(\upsilon),\ \upsilon\rangle - \langle \Xi(\upsilon),\ \varkappa\rangle + \langle \Xi(\varkappa),\ \varkappa\rangle - \langle \Xi(\varkappa),\ \upsilon\rangle\\ &= \mathcal{Q}(\mathcal{S}(\upsilon))\sum_{\beth=1}^{\mathcal{R}}\int_{\mathcal{Z}}\left(\left|\frac{\partial \upsilon}{\partial q_\beth}\right|^{p(q)} - \left|\frac{\partial \upsilon}{\partial q_\beth}\right|^{p(q)-2}\frac{\partial \upsilon}{\partial q_\beth}.\frac{\partial \varkappa}{\partial q_\beth}\right)dq\\ &\quad +\mathcal{Q}(\mathcal{S}(\varkappa))\sum_{\beth=1}^{\mathcal{R}}\int_{\mathcal{Z}}\left(\left|\frac{\partial \varkappa}{\partial q_\beth}\right|^{p(q)} - \left|\frac{\partial \varkappa}{\partial q_\beth}\right|^{p(q)-2}\frac{\partial \upsilon}{\partial q_\beth}.\frac{\partial \varkappa}{\partial q_\beth}\right)dq\\ &= \sum_{\beth=1}^{\mathcal{R}}(A_\beth + B_\beth)\geq 0.\end{aligned}$$

Which shows that $\langle \tau(\upsilon)-\tau(\varkappa),\ \upsilon-\varkappa\rangle > 0$. Also from (3.6)-(3.7), we get

$$0=\langle \tau(\upsilon)-\tau(\varkappa),\ \upsilon-\varkappa\rangle = \sum_{\beth=1}^{\mathcal{R}}(A_\beth + B_\beth) \tag{3.8}$$

$$\begin{aligned}&\geq \frac{k_0}{2}\sum_{\beth=1}^{\mathcal{R}}\int_{\mathcal{Z}}\left(\left|\frac{\partial \upsilon}{\partial q_\beth}\right|^{p(q)-2} - \left|\frac{\partial \varkappa}{\partial q_\beth}\right|^{p(q)-2}\right)\left(\left|\frac{\partial \upsilon}{\partial q_\beth}\right|^{2} - \left|\frac{\partial \varkappa}{\partial q_\beth}\right|^{2}\right)dq \qquad (3.9)\\ &\geq 0,\end{aligned}$$

which gives the following

$$\sum_{\beth=1}^{\mathcal{R}}\int_{\mathcal{Z}}\left(\left|\frac{\partial \upsilon}{\partial q_\beth}\right|^{p(q)-2} - \left|\frac{\partial \varkappa}{\partial q_\beth}\right|^{p(q)-2}\right)\left(\left|\frac{\partial \upsilon}{\partial q_\beth}\right|^{2} - \left|\frac{\partial \varkappa}{\partial q_\beth}\right|^{2}\right)dq,$$

hence $\left|\frac{\partial \upsilon}{\partial q_\beth}\right| = |\frac{\partial \varkappa}{\partial q_\beth}|$ for $\beth = 1, ..., \mathcal{R}$. After that, $\mathcal{Q}(\mathcal{S}(\upsilon)) = \mathcal{Q}(\mathcal{S}(\varkappa))$ and from (3.9), we have

$$\begin{aligned}0 &= \langle \tau(\upsilon)-\tau(\varkappa),\ \upsilon-\varkappa\rangle\\ &= \mathcal{Q}(\mathcal{S}(\upsilon))\sum_{\beth=1}^{\mathcal{R}}\int_{\mathcal{Z}}\left|\frac{\partial \upsilon}{\partial q_\beth}\right|^{p(q)-2}\left(\frac{\partial \upsilon}{\partial q_\beth}-\frac{\partial \varkappa}{\partial q_\beth}\right)^2 dq,\end{aligned}$$

so, for $\beth = 1, ..., \mathcal{R}. \frac{\partial \upsilon}{\partial q_\beth} = \frac{\partial \varkappa}{\partial q_\beth}$. a.e., in $\mathcal{Z}$, as a result $\upsilon = \varkappa$ in $\mathbf{X}_0$. Which is a contradiction, and $\langle \tau(\upsilon)-\tau(\varkappa),\ \upsilon-\varkappa\rangle > 0$. As a result of this, it can be affirmed that τ is strictly monotonic. □

Lemma 3.3. *(Comparison principle): Let* (K_0) *satisfies and assume* $\Im, \varkappa \in \mathbf{X}_0$ *verify*

$$-\mathcal{Q}(\mathcal{S}(\upsilon))\sum_{\beth=1}^{\mathcal{R}}\frac{\partial}{\partial q_\beth}\left(\left|\frac{\partial \upsilon}{\partial q_\beth}\right|^{p(q)-2}\frac{\partial \upsilon}{\partial q_\beth}\right) \leq -\mathcal{Q}(\mathcal{S}(\varkappa))\sum_{\beth=1}^{\mathcal{R}}\frac{\partial}{\partial q_\beth}\left(\left|\frac{\partial \varkappa}{\partial q_\beth}\right|^{p(q)-2}\frac{\partial \varkappa}{\partial q_\beth}\right) \tag{3.10}$$

and $\upsilon \leq \varkappa$ on $\partial\mathcal{Z}$, i.e., $(\upsilon-\varkappa)^+ \in \mathbf{X}_0$. Then, $\upsilon \leq \varkappa$ i.e., in $\mathcal{Z}$.

Proof. Consider a test function $\varsigma=(\upsilon-\varkappa)^{+}$ in (3.10), then, by (3.1), we achieve

$$\begin{aligned}\langle\tau(\upsilon)-\tau(\varkappa),\ (\upsilon-\varkappa)^{+}\rangle &= \mathcal{Q}(\mathcal{S}_0(\upsilon))\sum_{\daleth=1}^{\mathcal{R}}\int_{\mathcal{Z}\cap[\upsilon>\varkappa]}\left|\frac{\partial\upsilon}{\partial q_\daleth}\right|^{p(q)-2}\frac{\partial\upsilon}{\partial q_\daleth}\frac{\partial(\upsilon-\varkappa)}{\partial q_\daleth}dq\\ &\quad-\mathcal{Q}(\mathcal{S}_0(\varkappa))\sum_{\daleth=1}^{\mathcal{R}}\int_{\mathcal{Z}\cap[\upsilon>\varkappa]}\left|\frac{\partial\varkappa}{\partial q_\daleth}\right|^{p(q)-2}\frac{\partial\varkappa}{\partial q_\daleth}\frac{\partial(\upsilon-\varkappa)}{\partial q_\daleth}dq\\ &\leq 0.\end{aligned}$$

Additionally, from (3.9) we achieve:

$$\begin{aligned}&\langle\tau(\upsilon)-\tau(\varkappa),\ (\upsilon-\varkappa)^{+}\rangle\\ \geq\ &\frac{k_0}{2}\sum_{\daleth=1}^{\mathcal{R}}\int_{\mathcal{Z}\cap[\upsilon>\varkappa]}\left(\left|\frac{\partial\upsilon}{\partial q_\daleth}\right|^{p(q)-2}-\left|\frac{\partial\varkappa}{\partial q_\daleth}\right|^{p(q)-2}\right)\left(\left|\frac{\partial\upsilon}{\partial q_\daleth}\right|^{2}-\left|\frac{\partial\varkappa}{\partial q_\daleth}\right|^{2}\right)dq\\ \geq\ &0.\end{aligned}$$

Hence, $\langle\tau(\upsilon)-\tau(\varkappa),\ (\upsilon-\varkappa)^{+}\rangle=0$. Utilizing Lemma 3.1, we get that $(\upsilon-\varkappa)^{+}=0$ which completes the proof. □

Lemma 3.4. *Let* (K_0) *satisfies and* $\varpi\in L^{\infty}(\mathcal{Z})$ *. Then, there exists a unique solution of*

$$\begin{cases}-\mathcal{Q}(\mathcal{S}(\upsilon))\sum_{\daleth=1}^{\mathcal{R}}\frac{\partial}{\partial q_\daleth}\left(\left|\frac{\partial\upsilon}{\partial q_\daleth}\right|^{p(q)-2}\frac{\partial\upsilon}{\partial q_\daleth}\right)=\varpi(q) & \text{in } \mathcal{Z}\\ \upsilon=0 & \text{on } \partial\mathcal{Z}\end{cases}\tag{3.11}$$

in the space $\mathbf{X}_0$.

Proof. To complete the proof by Lemma 3.1, we take the strictly monotone operator

$$\langle\tau(\upsilon)),\ \varsigma\rangle=\mathcal{Q}(\mathcal{S}(\upsilon))\sum_{\daleth=1}^{\mathcal{R}}\int_{\mathcal{Z}}\left|\frac{\partial\upsilon}{\partial q_\daleth}\right|^{p(q)-2}\frac{\partial\upsilon}{\partial q_\daleth}\frac{\partial\varsigma}{\partial q_\daleth}dq.$$

We will prove that τ is coercive to prove the result. Assume $\{\upsilon_n\}\subset\mathbf{X}_0$ be a sequence such that $||\upsilon_n||_{\mathbf{X}_0}\to+\infty$. We set

$$\mathcal{P}_n=\left\{\daleth=1,\ \ldots,\mathcal{R}:\left\|\frac{\partial\upsilon_n}{\partial q_\daleth}\right\|_{p(q)}\leq 1\right\}.$$

By (K_0) and Lemma 2.2, the following is obtained

$$
\begin{aligned}
\langle \tau(\upsilon_n)),\ \upsilon_n\rangle &= \mathcal{Q}(\mathcal{S}_0(\upsilon_n))\sum_{\beth=1}^{\mathcal{R}}\int_{\mathcal{Z}}\left|\frac{\partial \upsilon_n}{\partial q_\beth}\right|^{p(q)} dq \geq k_0\sum_{\beth=1}^{\mathcal{R}}\int_{\mathcal{Z}}\left|\frac{\partial \upsilon_n}{\partial q_\beth}\right|^{p(q)} dq \qquad (3.12)\\
&\geq k_0\sum_{\beth\notin\mathcal{P}_n}\left\|\frac{\partial \upsilon_n}{\partial q_\beth}\right\|_{p(q)}^{p^-}\\
&= k_0\left(\sum_{\beth=1}^{\mathcal{R}}\left\|\frac{\partial \upsilon_n}{\partial q_\beth}\right\|_{p(q)}^{p^-} - \sum_{\beth\in\mathcal{P}_n}\left\|\frac{\partial q_n}{\partial q_\beth}\right\|_{p(q)}^{p^-}\right)\\
&\geq k_0\mathcal{M}_{p^-}\left(\sum_{\beth=1}^{\mathcal{R}}\left\|\frac{\partial \upsilon_n}{\partial q_\beth}\right\|_{p(q)}\right)^{p^-} - k_0\mathcal{R}\\
&= k_0\mathcal{M}_{p^-}\,\|\upsilon_n\|_{\mathbf{X}_0}^{p^-}\, k_0\mathcal{R},
\end{aligned}
$$

where $\mathcal{M}_{p^-}$ is independent of n and positive constant. Hence,

$$\lim_{n\to+\infty}\frac{\langle \tau(\upsilon_n)),\ \upsilon_n\rangle}{||\upsilon_n||_{\mathbf{X}_0}} = +\infty,$$

which demonstrates the coerciveness of τ. By applying the Theorem of Minty-Browder Theorem [23], we get that equation $\tau(\upsilon) = \varpi$ has a unique solution in $\mathbf{X}_0$. □

Let us denote the optimal constant of continuous embedding $\mathcal{W}_0^{1,1}(\mathcal{Z}) \hookrightarrow L^{\frac{\mathcal{R}}{\mathcal{R}-1}}$ by $\mathcal{M}_0$. Then,

$$||\upsilon||_{L^{\frac{\mathcal{R}}{\mathcal{R}-1}}(\mathcal{Z})} \leq \mathcal{M}_0||\upsilon||_{\mathcal{W}_0^{1,1}(\mathcal{Z})} \text{ for all } \upsilon\in\mathcal{W}_0^{1,1} \qquad (3.13)$$

Lemma 3.5. *Let (K_0) satisfies, $\varrho > 0$ and υ_ϱ be the unique solution of*

$$\begin{cases} -\mathcal{Q}(\mathcal{S}_0(\upsilon))\sum_{\beth=1}^{\mathcal{R}}\frac{\partial}{\partial q_\beth}(|\frac{\partial \upsilon}{\partial q_\beth}|^{p(q)-2}\frac{\partial \upsilon}{\partial q_\beth}) = \varrho & \text{in } \mathcal{Z}\\ \upsilon = 0 & \text{on } \partial\mathcal{Z}. \end{cases} \qquad (3.14)$$

Put $\delta = \frac{k_0 p^-}{2\mathcal{M}_0|\mathcal{Z}|^{\frac{1}{\mathcal{R}}}}$. Then, with $\varrho \geq \delta, \upsilon_\varrho \in L^\infty(\mathcal{Z})$ we have

$$||\upsilon_\varrho||_\infty \leq \mathcal{M}_1^\star\mathcal{Q}(\mathcal{M}_2^\star\varrho^{(p^-)'})\varrho^{\frac{1}{p^--1}},$$

with $\varrho < \delta$, we get

$$||\upsilon_\varrho||_\infty \leq \mathcal{M}_\star\varrho^{\frac{1}{p^+-1}},$$

in which $\mathcal{M}_1^\star, \mathcal{M}_2^\star$ and $\mathcal{M}_\star$ represents positive constants which depends on $\mathcal{Z}, k_0, \mathcal{R}$ and p.

Proof. Let $\psi \geq 0$ be fixed and put $\mathcal{Z}_\psi = \{q \in \mathcal{Z} :\ \upsilon_\varrho(q) > \psi\}$ and $\upsilon_\varrho \geq 0$ and using comparison principle. By testing equation (3.14) with $(\upsilon_\varrho - \psi)^+$, and using (3.13) along with Young's inequality, it follows that

$$\begin{aligned}\sum_{\daleth=1}^{\mathcal{R}}\int_{\mathcal{Z}}\left|\frac{\partial\upsilon_\varrho}{\partial q_\daleth}\right|^{p(q)}dq &= \frac{\varrho}{\mathcal{Q}(\mathcal{S}(\upsilon_\varrho))}\int_{\mathcal{Z}_\psi}(\upsilon_\varrho-\psi)dq \\ &\le \frac{\varrho|\mathcal{Z}_\psi|^{\frac{1}{\mathcal{R}}}}{\mathcal{Q}(\mathcal{S}_0(\upsilon_\lambda))}\|(\upsilon_\varrho-\psi)^+\|_{L^{\frac{\mathcal{R}}{\mathcal{R}-1}}(\mathcal{Z})} \\ &\le \frac{\varrho|\mathcal{Z}_\psi|^{\frac{1}{\mathcal{R}}}\mathcal{M}_0}{k_0}\int_{\mathcal{Z}_\psi}|\nabla\upsilon_\varrho|\,dq \\ &\le \frac{\varrho|\mathcal{Z}_\psi|^{\frac{1}{\mathcal{R}}}\mathcal{M}_0}{k_0}\sum_{\daleth=1}^{\mathcal{R}}\int_{\mathcal{Z}_\psi}\left|\frac{\partial\upsilon_\varrho}{\partial q_\daleth}\right|^{p(q)}dq\end{aligned} \tag{3.15}$$

$$\le \frac{\varrho|\mathcal{Z}_\psi|^{\frac{1}{\mathcal{R}}}\mathcal{M}_0}{k_0}\left(\sum_{\daleth=1}^{\mathcal{R}}\int_{\mathcal{Z}_\psi}\frac{\varepsilon^{p(q)}\left|\frac{\partial\upsilon_\varrho}{\partial q_\daleth}\right|^{p(q)}}{p(q)}dq+\sum_{\daleth=1}^{\mathcal{R}}\int_{\mathcal{Z}_\psi}\frac{\varepsilon^{-p'(q)}}{p'(q)}dq\right) \tag{3.16}$$

For $\varrho\ge\delta$, we have

$$\epsilon=\left(\frac{k_0p^-}{2\varrho|\mathcal{Z}|^{\frac{1}{\mathcal{R}}}\mathcal{M}_0}\right)^{\frac{1}{p^-}}=\left(\frac{\delta}{\varrho}\right)^{\frac{1}{p^-}}, \tag{3.17}$$

we have $\epsilon\le 1$, thus

$$\begin{aligned}\frac{\varrho|\mathcal{Z}_\psi|^{\frac{1}{\mathcal{R}}}\mathcal{M}_0}{k_0}\sum_{\daleth=1}^{\mathcal{R}}\int_{\mathcal{Z}_\psi}\frac{\varepsilon^{p(q)}\left|\frac{\partial\upsilon_\varrho}{\partial q_\daleth}\right|^{p(q)}}{p(q)}dq &\le \frac{\lambda|\Omega|^{\frac{1}{\mathcal{R}}}\mathcal{M}_0\epsilon^{p^-}}{k_0p^-}\sum_{\daleth=1}^{\mathcal{R}}\int_{\mathcal{Z}_\psi}\left|\frac{\partial\upsilon_\varrho}{\partial q_\daleth}\right|^{p(q)}dq, \\ &= \frac{1}{2}\sum_{\daleth=1}^{\mathcal{R}}\int_{\mathcal{Z}_\psi}\left|\frac{\partial\upsilon_\varrho}{\partial q_\daleth}\right|^{p(q)}dq.\end{aligned} \tag{3.18}$$

Combining (3.16) and (3.18), we arrive at

$$\sum_{\daleth=1}^{\mathcal{R}}\int_{\mathcal{Z}_\psi}\left|\frac{\partial\upsilon_\varrho}{\partial q_\daleth}\right|^{p(q)}dq\le\frac{2\varrho|\mathcal{Z}_\psi|^{\frac{1}{\mathcal{R}}}\mathcal{M}_0}{k_0(p^+)'}\sum_{\daleth=1}^{\mathcal{R}}\int_{\mathcal{Z}_\psi}\epsilon^{-(p^-)'}dq=\frac{2\mathcal{R}\varrho\mathcal{M}_0\epsilon^{-(p^-)'}}{k_0\,(p^+)'}|\mathcal{Z}_\psi|^{1+\frac{1}{\mathcal{R}}}. \tag{3.19}$$

In the same way, test function in (3.14) with $\Im_\varrho$, yields that

$$\sum_{\daleth=1}^{\mathcal{R}}\int_{\mathcal{Z}}\left|\frac{\partial\upsilon_\varrho}{\partial q_\daleth}\right|^{p(q)}dq\le\frac{2\mathcal{R}\varrho\mathcal{M}_0\epsilon^{-(p^-)'}}{k_0(p^+)'}|\mathcal{Z}|^{1+\frac{1}{\mathcal{R}}}.$$

From (3.15), (3.18) and monotonicity of $\mathcal{Q}$, we have

$$\begin{aligned}\int_{\mathcal{Z}_\psi}(\upsilon_\varrho-\xi)dq &= \frac{\mathcal{Q}(\mathcal{S}(\upsilon))}{\varrho}\sum_{\daleth=1}^{\mathcal{R}}\int_{\mathcal{Z}_\psi}\left|\frac{\partial\upsilon_\varrho}{\partial q_\daleth}\right|^{p(q)}dq \\ &\le \mathcal{Q}\left(\frac{2\mathcal{R}\varrho\mathcal{M}_0\epsilon^{-(p^-)'}}{k_0p^-(p^+)'}|\mathcal{Z}_\psi|^{1+\frac{1}{\mathcal{R}}}\right)\frac{2NC_0\epsilon^{-(p^-)'}}{k_0(p^+)}|\mathcal{Z}_\psi|^{1+\frac{1}{\mathcal{R}}}.\end{aligned}$$

Through Lemma 5.1 in [14], the below is achieved

$$||\upsilon_\varrho||_\infty \leq \mathcal{Q}\left(\frac{2\mathcal{R}\varrho\mathcal{M}_0\epsilon^{-(p^-)'}}{k_0 p^-(p^+)'}|\mathcal{Z}_\psi|^{1+\frac{1}{\mathcal{R}}}\right))\frac{2\mathcal{R}(\mathcal{R}+1)\mathcal{M}_0\epsilon^{-(p^-)'}}{k_0(p^+)'}|\mathcal{Z}_\psi|^{\frac{1}{\mathcal{R}}}. \tag{3.20}$$

It follows from (3.17) and (3.20) that

$$||\upsilon_\varrho||_\infty \leq \mathcal{M}_1^\star \mathcal{Q}\left(\mathcal{M}_2^\star \varrho^{(p^-)'}\right)\varrho^{\frac{1}{p^--1}},$$

where

$$\mathcal{M}_1^\star := \frac{\mathcal{R}(\mathcal{R}+1)(2\mathcal{M}_0)^{(p^-)'}}{(p^+)k_0^{(p^-)'}(p^-)^{\frac{1}{p^--1}}}|\mathcal{Z}|^{\frac{(p^-)'}{\mathcal{R}}},$$

and

$$\mathcal{M}_2^\star := \frac{\mathcal{R}(2\mathcal{M}_0)^{(p^-)'}}{(p^+)'k_0^{(p^-)'}(p^-)^{p^-}}|\mathcal{Z}|^{1+\frac{(p^-)'}{\mathcal{R}}}.$$

For $\lambda < \delta$, we have

$$\varepsilon = \left(\frac{k_0 p^-}{2\varrho|\mathcal{Z}|^{\frac{1}{\mathcal{R}}}\mathcal{M}_0}\right)^{\frac{1}{p^+}} = \left(\frac{\delta}{\varrho}\right)^{\frac{1}{p^+}}$$

we have $\epsilon < 1$. Utilizing the same argument we can prove the following:

$$||\upsilon_\varrho||_\infty \leq \mathcal{M}_\star \varrho^{\frac{1}{p^+-1}},$$

where

$$\mathcal{M}_* = \frac{\mathcal{R}(\mathcal{R}+1)(2\mathcal{M}_0)^{(p^+)'}}{(p^+)'k_0^{(p^+)'}(p^-)^{\frac{1}{p^+-1}}}|\mathcal{Z}|^{\frac{(p^+)'}{\mathcal{R}}}\mathcal{Q}\left(\frac{\mathcal{R}(2\delta\mathcal{M}_0)^{(p^+)'}}{(p^+)'k_0^{(p^+)'}(p^-)^{(p^+)'}}|\mathcal{Z}|^{1+\frac{(p^+)'}{\mathcal{R}}}\right).$$

□

4. Proof of Theorem 1.1

Take pair of sub-super solution $(\underline{\upsilon}, \overline{\upsilon})$ of problem (1.1), if $\underline{\upsilon}, \overline{\upsilon} \in L^\infty(\mathcal{Z})$, $\underline{\upsilon} \leq \overline{\upsilon}$ a.e., in $\mathcal{Z}$ and $\forall$ nonnegative function $\varsigma \in \mathbf{X}_0$, the following satisfies

$$\begin{cases} \mathcal{Q}(\mathcal{S}(\underline{\upsilon}))\sum_{\daleth=1}^{\mathcal{R}}\int_{\mathcal{Z}}\left|\frac{\partial\underline{\upsilon}}{\partial q_\daleth}\right|^{p(q)-2}\frac{\partial\underline{\upsilon}}{\partial q_\daleth}\frac{\partial\varsigma}{\partial q_\daleth}dq \leq \int_{\mathcal{Z}}\varpi(q)\underline{\upsilon}^{\aleph(q)-1}\varsigma dq + \int_{\mathcal{Z}} j(q,\underline{\upsilon})\varsigma dq \\ \\ \mathcal{Q}(\mathcal{S}(\overline{\upsilon}))\sum_{\daleth=1}^{\mathcal{R}}\int_{\mathcal{Z}}\left|\frac{\partial\overline{\upsilon}}{\partial q_\daleth}\right|^{p(q)-2}\frac{\partial\overline{\upsilon}}{\partial q_\daleth}\frac{\partial\varsigma}{\partial q_\daleth}dq \geq \int_{\mathcal{Z}}\varpi(q)\overline{\upsilon}^{\aleph(q)-1}\varsigma dq + \int_{\mathcal{Z}} j(q,\overline{\upsilon})\varsigma dq, \end{cases} \tag{4.1}$$

Lemma 4.1. *Assume that* (K_0) *and* $(j_1)-(j_2)$ *satisfy. Then, there is* $\digamma_\star > 0$ *in a way that (1.1) has a pair of sub-super solution* $(\underline{\Im}, \overline{\Im}) \in (\mathbf{X}_0 \cap L^\infty(\mathcal{Z}) \times (\mathbf{X}_0 \cap L^\infty(\mathcal{Z}))$ *with* $\|\Im\|_\infty \leq \vartheta$, *provided that* $||\varpi||_\infty < \digamma_\star$, *where* ϑ *is stated in* (j_1).

Proof. Utilizing the Lemmas 3.2, 3.3, and 3.4, we get that $\underline{\Im}, \overline{\Im} \in \mathbf{X}_0 \cap L^\infty(\mathcal{Z})$ a unique non-negative solution of the following

$$\begin{cases} -\mathcal{Q}(\mathcal{S}(\underline{\upsilon}))\sum_{\gimel=1}^{\mathcal{R}} \frac{\partial}{\partial q_\gimel}(|\frac{\partial \underline{\upsilon}}{\partial q_\gimel}|^{p(q)-2}\frac{\partial \underline{\upsilon}}{\partial q_\gimel}) = \varpi(q) & \text{in } \mathcal{Z} \\ \underline{\upsilon} = 0 & \text{on } \partial\mathcal{Z} \end{cases}$$

and

$$\begin{cases} -\mathcal{Q}(\mathcal{S}(\overline{\upsilon}))\sum_{\gimel=1}^{\mathcal{R}} \frac{\partial}{\partial q_\gimel}(|\frac{\partial \overline{\upsilon}}{\partial q_\gimel}|^{p(q)-2}\frac{\partial \overline{\upsilon}}{\partial q_\gimel}) = \varpi(q)+1 & \text{in } \mathcal{Z} \\ \overline{\upsilon} = 0 & \text{on } \partial\mathcal{Z} \end{cases} \tag{4.2}$$

such that

$$||\underline{\upsilon}||_\infty \leq \max(\mathcal{M}_1^\star\mathcal{Q}(\mathcal{M}_2^\star||\varpi||_\infty^{(p^-)'})||\varpi||^{\frac{1}{p^--1}},\ \mathcal{M}_\star||\varpi||^{\frac{1}{p^+-1}})$$

where $\mathcal{M}_1^\star, \mathcal{M}_2^\star$ and $\mathcal{M}_\star$ are stated in Lemma 3.4. Next, consider that $\mathcal{Q}$ is nondecreasing, then $\exists\, F > 0$ relying only on $\mathcal{M}_1^\star, \mathcal{M}_2^\star$ and $\mathcal{M}_\star$ with $||\underline{\upsilon}||_\infty \leq \vartheta$, given that $||\varpi||_\infty < F$. Additionally, $\underline{\upsilon} \leq \overline{\upsilon}$ by Lemma 3.2.

For any arbitrary nonnegative function ς in $\mathbf{X}_0$. The above (4.2) and (j_1) implies that

$$\begin{aligned} &\mathcal{Q}(\mathcal{S}(\underline{\upsilon}))\sum_{\gimel=1}^{\mathcal{R}}\int_{\mathcal{Z}}\left|\frac{\partial \underline{\upsilon}}{\partial q_\gimel}\right|^{p(q)-2}\frac{\partial \underline{q}}{\partial q_\gimel}\frac{\partial \varsigma}{\partial q_\gimel}dq - \int_{\mathcal{Z}}\varpi(q)\underline{\upsilon}^{\aleph(q)-1}\varsigma dq - \int_{\mathcal{Z}} j(q,\underline{\upsilon})\varsigma dq \\ \leq\ & \int_{\mathcal{Z}}\varpi(q)\varsigma dq - \int_{\mathcal{Z}}\varpi(q)\underline{\upsilon}^{\aleph(q)-1}\varphi dq - \int_{\mathcal{Z}}\varpi(q)(1-\underline{\upsilon}^{\aleph(q)-1})\varsigma dq \\ =\ & 0. \end{aligned}$$

From (4.2) and (j_2) , we obtain

$$\begin{aligned} &\mathcal{Q}(\mathcal{S}(\overline{\upsilon}))\sum_{\gimel=1}^{\mathcal{R}}\int_{\mathcal{Z}}\left|\frac{\partial \overline{\upsilon}}{\partial q_\gimel}\right|^{p(q)-2}\frac{\partial \overline{\upsilon}}{\partial q_\gimel}\frac{\partial \varsigma}{\partial q_\gimel}dq - \int_{\mathcal{Z}}\varpi(q)\overline{\upsilon}^{\aleph(q)-1}\varsigma dq - \int_{\mathcal{Z}} j(q,\overline{\upsilon})\varsigma dq \\ \geq\ & \int_{\mathcal{Z}}(1 - B_\infty||\varpi||_\infty)\varsigma dq, \end{aligned}$$

where

$$B_\infty := \max(||\upsilon||_\infty^{\aleph^+-1},\ ||\upsilon||_\infty^{\aleph^--1}) + \max(||\upsilon||_\infty^{s^+-1},\ ||\upsilon||_\infty^{s^--1})$$

Selecting $F_\star = \min\left(F, \frac{1}{B_\infty}\right)$, yields

$$\int_{\mathcal{Z}}(1 - B_\infty||\varpi||_\infty)\varsigma dx \geq 0 \text{ for } ||\varpi||_\infty < F_\star.$$

Which completes the proof. □

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Proof of Theorem 1.1: Let $\underline{\upsilon}, \overline{\upsilon} \in \mathbf{X}_0 \cap L^\infty(\mathcal{Z})$ as stated in the above lemma and introduce

$$h(q,\ t) = \begin{cases} \varpi(q)\overline{\upsilon}(q)^{\aleph(q)-1} + j(q, \overline{\upsilon}(q)) & \text{if } t > \overline{\upsilon}(q) \\ \varpi(q)t^{\aleph(q)-1} + j(q,\ t) & \text{if } \underline{\upsilon}(q) \le t \le \overline{\upsilon}(q) \\ \varpi(q)\underline{\upsilon}(q)^{\aleph(q)-1} + j(q, \underline{\upsilon}(q)) & \text{if } t < \underline{\upsilon}(q). \end{cases}$$

Consider the problem

$$\begin{cases} -\mathcal{Q}(\mathcal{S}(\upsilon)) \sum_{\beth=1}^{\mathcal{R}} \frac{\partial}{\partial q_\beth} \left(\left| \frac{\partial \upsilon}{\partial q_\beth} \right|^{p(q)-2} \frac{\partial \upsilon}{\partial q_\beth} \right) = h(q, \upsilon) & \text{in } \mathcal{Z} \\ \upsilon = 0 & \text{on } \partial\mathcal{Z} \end{cases} \tag{4.3}$$

where functional $\mathcal{I} : \mathbf{X}_0 \to \mathrm{R}$ defined as

$$\mathcal{I}(\upsilon) = \widehat{\mathcal{Q}}(\mathcal{S}(\upsilon)) - \int_{\mathcal{Z}} \mathcal{H}(q,\ \upsilon)dq,$$

where $G(q,t) = \int_0^t h(q,r)dr$. Then $\mathcal{I}$ is a member of class $\mathcal{M}^1$, and its critical points correspond to solutions of problem (4.3). From (K_0), it is clear that $\mathcal{I}$ is coercive and sequentially weakly lower semicontinuous. Thus, $\mathcal{I}$ attains its minimum within the weakly closed subset $[\underline{\upsilon}, \overline{\upsilon}] \cap \mathbf{X}_0$ at some υ_0, proving it as a critical point of $\mathcal{I}$.

5. Proof of Theorem 1.2

To demonstrate the theorem, we will introduce the following

$$g(q,\ t) = \begin{cases} \varpi(q)t^{\aleph(q)-1} + j(q,\ t) & \text{if } \underline{\upsilon}(q) \le t, \\ \varpi(q)\underline{\upsilon}(q)^{\aleph(q)-1} + j(q, \underline{\upsilon}(q)) & \text{if } \underline{\upsilon}(q) \ge t, \end{cases}$$

also, consider

$$\begin{cases} -\mathcal{Q}(\mathcal{S}(\upsilon)) \sum_{\beth=1}^{\mathcal{R}} \frac{\partial}{\partial q_\beth} \left(\left| \frac{\partial \upsilon}{\partial q_\beth} \right|^{p(q)-2} \frac{\partial \upsilon}{\partial q_\beth} \right) = g(q,\ \upsilon) & \text{in } \mathcal{Z} \\ \upsilon = 0 & \text{on } \partial\mathcal{Z}. \end{cases} \tag{5.1}$$

Our method to finding solutions of (5.1) involves determining critical points of functional $\mathcal{G} : \mathbf{X}_0 \to \mathrm{R}$, stated as:

$$\mathcal{G}(\upsilon) = \widehat{\mathcal{Q}}(\mathcal{S}(\upsilon)) - \int_{\mathcal{Z}} \mathcal{B}(q,\ \upsilon)dq,$$

where $\mathcal{B}(q,\ t) = \int_0^t g(q,r)dr$. Clearly, $\mathcal{G}$ is of class $\mathcal{M}^1$.

Lemma 5.1. *Functional $\mathcal{G}$ holds the Palais-Smale condition for the stated assumptions of Theorem 1.2.*

Proof. Let $\{\upsilon_n\} \subset \mathbf{X}_0$ be a sequence, with

$$\mathcal{G}(\upsilon_n) \to c \in \mathrm{R} \text{ and } \mathcal{G}'(\upsilon_n) \to 0 \text{ in } \mathbf{X}_0^*.$$

Here, the boundedness of $\{\upsilon_n\}$ in $\mathbf{X}_0$ claimed.

Case 1: $\aleph^- > \frac{p^+}{1-\chi}$. Let $\mu_0 \in \left(\frac{p^+}{1-\chi},\ \min(\mu,\ \aleph)\right)$. By $(K_0)-(K_1), (j_3)$, (3.12) and embedding theorem, for n large enough, yields that

$$\begin{aligned}
1 + c + ||\upsilon_n||_{\mathbf{X}_0} &\geq \mathcal{G}(\upsilon_n) - \frac{1}{\mu_0}\langle \mathcal{G}'(\upsilon_n),\ \upsilon_n\rangle \\
&\geq (1-\chi)\mathcal{Q}(\mathcal{S}(\upsilon_n))\mathcal{S}(\upsilon_n) - \frac{1}{\mu_0}\mathcal{Q}(\mathcal{S}(\upsilon_n))\sum_{\gimel=1}^{\mathcal{R}}\int_{\mathcal{Z}}\left|\frac{\partial \upsilon_n}{\partial q_\gimel}\right|^{p(q)} dq \\
&\quad + \int_{\mathcal{Z}}\left(\frac{1}{\mu_0} g(q,\ \upsilon_n)\upsilon_n - \mathcal{B}(q,\ \upsilon_n)\right) dq \\
&\geq k_0\left(\frac{1-\chi}{p^+} - \frac{1}{\mu_0}\right)\left(\mathcal{M}_{p^-}||\upsilon_n||_{\mathbf{X}_0}^{p^-} - \mathcal{R}\right) + \int_{[\upsilon_n > \underline{\upsilon}]}\left(\frac{1}{\mu_0} j(q,\ q_n)q_n - \mathcal{H}(q,\ \upsilon_n)\right) dq \\
&\quad + \int_{[\upsilon_n > \underline{\upsilon}]}\left(\frac{1}{\mu_0} - \frac{1}{\aleph(q)}\right)\varpi(q)\upsilon_n^{\aleph(q)} dq - \mathcal{M}_3||\upsilon_n||_{\mathbf{X}_0} - \mathcal{M}_2 \\
&\geq k_0\left(\frac{1-\chi}{p_M^+} - \frac{1}{\mu_0}\right)\left(\mathcal{M}_{p^-}||\upsilon_n||_{\mathbf{X}_0}^{p^-} - \mathcal{R}\right) - \mathcal{M}_4\left(\|\upsilon_n\|_{\mathbf{X}_0}^{\aleph^+} + \|\upsilon_n\|_{\mathbf{X}_0}^{\aleph^-}\right) \\
&\quad -\mathcal{M}_3||\upsilon_n||_{\mathbf{X}_0} - \mathcal{M}_2,
\end{aligned}$$

where $\mathcal{M}_1$ and $\mathcal{M}_2$ independent of n and are positive constants. Therefore, sequence $\{\upsilon_n\}$ is bounded in $\mathbf{X}_0$ as $p^- > 1$.

Case 2: $\aleph^+ < p^-$. Utilizing $(K_0)-(K_1), (j_3)$, (3.12) and embedding theorem, yields that

$$\begin{aligned}
1 + c + ||\upsilon_n||_{\mathbf{X}_0} &\geq \mathcal{G}(\upsilon_n) - \frac{1}{\mu}\langle \mathcal{G}'(\upsilon_n),\ \upsilon_n\rangle \\
&\geq k_0\left(\frac{1-\chi}{p^+} - \frac{1}{\mu}\right)(\mathcal{M}_{p^-}||\upsilon_n||_{\mathbf{X}_0}^{p^-} - \mathcal{R}) + \int_{[\upsilon_n > \underline{\upsilon}]}\left(\frac{1}{\mu} j(q,\ \upsilon_n)\upsilon_n - \mathcal{H}(q,\ \upsilon_n)\right) dq \\
&\quad + \int_{[\upsilon_n > \underline{\upsilon}]}\left(\frac{1}{\mu} - \frac{1}{\aleph(q)}\right)\varpi(q)\upsilon_n^{\aleph(q)} dq - \mathcal{M}_3||\upsilon_n||_{\mathbf{X}_0} - \mathcal{M}_2 \\
&\geq k_0\left(\frac{1-\chi}{p^+} - \frac{1}{\mu}\right)(\mathcal{M}_{p^-}||\upsilon_n||_{\mathbf{X}_0}^{p^-} - \mathcal{R}) - \mathcal{M}_4(||\upsilon_n||_{\mathbf{X}_0}^{\aleph^+} + ||\upsilon_n||_{\mathbf{X}_0}^{\aleph^-}) \\
&\quad -\mathcal{M}_3||\upsilon_n||_{\mathbf{X}_0} - \mathcal{M}_2.
\end{aligned}$$

where $\mathcal{M}_3$ and $\mathcal{M}_4$ are independent of n and are positive constants. Therefore, the boundedness of $\{\upsilon_n\}$ in $\mathbf{X}_0$ is proved. Further, we have

$$\begin{cases} \upsilon_n \rightharpoonup \upsilon \ \text{ in } \mathbf{X}_0 \\ \upsilon_n \to \upsilon \ \text{ a.e. in } \ \mathcal{Z} \\ \upsilon_n \to \upsilon \ \text{ in } L^{\varkappa(q)}(\mathcal{Z}) \text{ with } 1 < \varkappa^- \leq \varkappa^+ < (p^\star)^- \end{cases} \tag{5.2}$$

Thus,

$$\begin{aligned} o_n(1) &= \langle \mathcal{G}'(\upsilon_n),\ \upsilon_n - \upsilon\rangle \\ &= \mathcal{Q}(\mathcal{S}(\upsilon_n)) \sum_{\daleth=1}^{\mathcal{R}} \int_{\mathcal{Z}} \left(\left|\frac{\partial \upsilon_n}{\partial q_\daleth}\right|^{p(q)-2} \frac{\partial \upsilon_n}{\partial q_\daleth} \frac{\partial(\upsilon_n - \upsilon)}{\partial q_\daleth}\right) dq - \int_{\mathcal{Z}} g(q,\ \upsilon_n)(\upsilon_n - \upsilon) dq \end{aligned}$$

Now, from (j_2), (5.2), Lemmas 2.1 and 2.4, we can prove the following

$$\int_{\mathcal{Z}} g(q,\ \upsilon_n)(\upsilon_n - \upsilon) \to 0.$$

So that

$$\mathcal{Q}(\mathcal{S}(\upsilon_n)) \sum_{\daleth=1}^{\mathcal{R}} \int_{\mathcal{Z}} \left(\left|\frac{\partial \upsilon_n}{\partial q_\daleth}\right|^{p(q)-2} \frac{\partial \upsilon_n}{\partial q_\daleth} \frac{\partial(\upsilon_n - \upsilon)}{\partial q_\daleth}\right) \to 0.$$

Through the assumption of (K_0), we have

$$\sum_{\daleth=1}^{\mathcal{R}} \int_{\mathcal{Z}} \left(\left|\frac{\partial \upsilon_n}{\partial q_\daleth}\right|^{p(q)-2} \frac{\partial \upsilon_n}{\partial q_\daleth} \frac{\partial(\upsilon_n - \upsilon)}{\partial q_\daleth}\right) \to 0.$$

Similarly

$$\sum_{\daleth=1}^{\mathcal{R}} \int_{\mathcal{Z}} \left(\left|\frac{\partial \upsilon}{\partial q_\daleth}\right|^{p(q)-2} \frac{\partial \upsilon}{\partial q_\daleth} \frac{\partial(\upsilon_n - \upsilon)}{\partial q_\daleth}\right) \to 0.$$

It holds that

$$\begin{aligned} &\frac{1}{2^{p^+-2}} \sum_{\daleth=1}^{\mathcal{R}} \int_{\mathcal{Z}} \left|\frac{\partial(\upsilon_n - \upsilon)}{\partial q_\daleth}\right|^{p(q)} dq \\ \leq \ & \sum_{\daleth=1}^{\mathcal{R}} \int_{\mathcal{Z}} \left(\left|\frac{\partial \upsilon_n}{\partial q_\daleth}\right|^{p(q)-2} \frac{\partial \upsilon_n}{\partial q_\daleth} - \left|\frac{\partial \upsilon}{\partial q_\daleth}\right|^{p(q)-2} \frac{\partial \upsilon}{\partial q_\daleth}\right) \left(\frac{\partial \upsilon_n}{\partial q_\daleth} - \frac{\partial \upsilon}{\partial q_\daleth}\right) dq \to 0, \end{aligned}$$

combine with Lemma 2.3, we get $\upsilon_n \to \upsilon$ in $\mathbf{X}_0$. □

Lemma 5.2. *For $||\varpi||_\infty$ sufficiently small and under assumptions of Theorem 1.2, the following holds*

(i) We can get $\gamma > 0$ and $\alpha > ||\underline{\upsilon}||_{\mathbf{X}_0}$ such that

$$\mathcal{G}(\underline{\upsilon}) < 0 < \gamma \leq \inf_{\upsilon \in \partial B_\alpha(0)} \mathcal{G}(\upsilon)\ ;$$

(ii) We can get $e \in \mathbf{X}_0$ such that $||e||_{\mathbf{X}_0} > 2\alpha$ and $\mathcal{G}(e) < \gamma$.

Proof. (i) Take $\varsigma = \underline{\upsilon}$ in first inequality of (4.1) and apply that $\mathcal{Q}$ is nondecreasing, we get

$$\begin{aligned} \mathcal{G}(\underline{\upsilon}) &= \mathcal{Q}(\mathcal{S}(\underline{\upsilon})) - \int_{\mathcal{Z}} \mathcal{B}(q,\ \upsilon)dq \\ &\leq \mathcal{Q}(\mathcal{S}(\underline{\upsilon}))\mathcal{S}(\underline{\upsilon}) - \int_{\mathcal{Z}} \varpi(q)\underline{\upsilon}^{\aleph(q)}dq - \int_{\mathcal{Z}} j(q,\underline{\upsilon})\underline{\upsilon}dq \\ &< \mathcal{Q}(\mathcal{S}(\underline{\upsilon}))\sum_{\daleth=1}^{\mathcal{R}}\int_{\mathcal{Z}}\left|\frac{\partial\underline{\upsilon}}{\partial q_{\daleth}}\right|^{p(q)}dq - \int_{\mathcal{Z}}\beta(q)\underline{\upsilon}^{\aleph(q)}dq - \int_{\mathcal{Z}} j(q,\underline{\upsilon})\underline{\upsilon}dq \\ &\leq 0, \end{aligned}$$

therefore, $\mathcal{G}(\underline{\upsilon}) < 0$. Let $\upsilon \in \mathbf{X}_0$ with $||\upsilon||_{\mathbf{X}_0} \geq 1$. From $(K_0), (j_2)$, (3.12) and embedding theorem, one have

$$\mathcal{G}(\upsilon) \geq \frac{k_0}{p^+}(\mathcal{M}_{p^-}||\upsilon||^{p^-}_{\mathbf{X}_0} - \mathcal{R}) - \mathcal{M}_5||\varpi||_\infty(||\upsilon||_{\mathbf{X}_0} + ||\upsilon||^{\aleph^+}_{\mathbf{X}_0} + ||\upsilon||^{s^+}_{\mathbf{X}_0}) - \mathcal{M}_6,$$

where $\mathcal{M}_5, \mathcal{M}_6 > 0$. We can pick $\gamma > 0$ and $\alpha > ||\underline{\upsilon}||_{\mathbf{X}_0}$ with

$$\frac{k_0}{p^+}(\mathcal{M}_{p^-}||\upsilon||^{p^-}_{\mathbf{X}_0} - \mathcal{R}) - \mathcal{M}_6 \geq 2\gamma.$$

Then, letting $||\varpi||_\infty \leq \frac{\gamma}{\mathcal{M}_5(\alpha+\alpha^{\aleph^+}+\alpha^{s^+})}$, this implies that $\mathcal{G}(\upsilon) \geq \gamma$ for $||\Im||_{\mathbf{X}_0} = \alpha$.

(ii) By (K_1) , there is $\mathcal{M}_7 > 0$ with

$$\mathcal{Q}(t) \leq \mathcal{M}_7 t^{\frac{1}{1-\chi}} \text{ for all } t > 1. \tag{5.3}$$

From (5.3) and (j_3), $\forall\ t > 1$, we get

$$\mathcal{G}(t\underline{\upsilon}) = \mathcal{Q}(\mathcal{S}(t\underline{\upsilon})) - \int_{\mathcal{Z}} \mathcal{B}(q,\ t\underline{\upsilon})dq$$

$$\leq \mathcal{M}_7 t^{\frac{P^+}{1-\chi}}(\mathcal{S}(\underline{\upsilon}))^{\frac{1}{1-\chi}} - t^{\aleph^-}\int_{\mathcal{Z}} \varpi(q)\underline{\upsilon}^{\aleph(q)}dq - \mathcal{M}_8 t^\mu \int_{\mathcal{Z}} \underline{\upsilon}^\mu dq + \mathcal{M}_9.$$

Then, for some $t_0 > 1$ large enough, $\mathcal{G}(t_0\underline{\upsilon}) < 0$ and $||t_0\underline{\upsilon}||_{\mathbf{X}_0} > 2\alpha$, due to $\frac{p^+}{1-\chi} < \mu$. Thus, we take $e = t_0\underline{\upsilon}$, which completes the proof. □

Proof*:* Let υ_0 be the solution of problem (1.1) stated in Theorem 1.1 which holds

$$\mathcal{I}(\upsilon_0) = \inf_{\upsilon\in\Lambda} \mathcal{I}(\upsilon)\ ,$$

with $\upsilon_0 \in \Lambda := [\underline{\upsilon}, \overline{\upsilon}] \cap \mathbf{X}_0$. In a standard way, by mountain pass theorem [19] and Lemmas 5.1, 5.2, mountain pass level is stated as

$$c^* := \inf_{\beta\in\Gamma}\max_{t\in[0,1]} \mathcal{G}(\beta(t)),$$

with

$$\Gamma := \{\beta \in \mathcal{M}([0,1], \mathbf{X}_0); \beta(0) = \underline{\upsilon},\ \beta(1) = e\},$$

where $\mathcal{G}$ represent critical value. Which shows the existence of $\Im_1 \in \mathbf{X}_0$ such that $\mathcal{G}'(\Im_1) = 0$ and $\mathcal{G}(\Im_1) = c^*$. Considering that $\mathcal{I}(\Im) = \mathcal{G}(\Im)\ \forall\ \Im \in [0, \overline{\Im}] \cap \mathbf{X}_0$, it follows that $\mathcal{G}(\Im_0) \leq \mathcal{G}(\underline{\Im})$. Let $\Im_1 \geq \underline{\Im}$ i.e. in $\mathcal{Z}$. Utilizing $(\underline{\Im} - \Im_1)^+$ as a test function in $\mathcal{G}'(\Im_1) = 0$ and in the first inequality of (4.1), we get:

$$\begin{aligned}
\mathcal{Q}(\mathcal{S}(\upsilon_1)) \sum_{\gimel=1}^{\mathcal{R}} \int_{\mathcal{Z}} \left| \frac{\partial \upsilon_1}{\partial q_\gimel} \right|^{p(q)-2} \frac{\partial \upsilon_1}{\partial q_\gimel} \frac{(\underline{\upsilon} - \upsilon_1)^+}{\partial q_\gimel} dq &= \int_{\mathcal{Z}} g(q,\ \upsilon_1)(\underline{\upsilon} - \upsilon_1)^+ dq \\
&= \int_{\mathcal{Z}} \varpi(q)\underline{\upsilon}(q)^{\aleph(q)-1} + j(q, \underline{\upsilon}))(\underline{\upsilon} - \upsilon_1)^+ dq \\
&\geq \mathcal{Q}(\mathcal{S}(\underline{\upsilon})) \sum_{\gimel=1}^{\mathcal{R}} \int_{\mathcal{Z}} \left| \frac{\partial \underline{\upsilon}}{\partial q_\gimel} \right|^{p(q)-2} \frac{\partial \underline{\upsilon}}{\partial q_\gimel} \frac{\partial(\underline{\upsilon} - \upsilon_1)^+}{\partial q_\gimel} dq
\end{aligned}$$

that is

$$\langle \tau(\underline{\upsilon}) - \tau(\upsilon_1),\ (\underline{\upsilon} - \upsilon_1)^+ \rangle \leq 0.$$

If τ is strictly monotone (see Lemma 3.1), then $(\underline{\upsilon} - \upsilon_1)^+$ is zero almost every where in $\mathcal{Z}$. This implies $\upsilon_1 \geq \underline{\upsilon}$ almost every where in $\mathcal{Z}$. Consequently, υ_0 and υ_1 are two nonnegative solutions to problem (1.1) such that

$$\mathcal{G}(\upsilon_0) \leq \mathcal{G}(\underline{\upsilon}) < 0 < \gamma \leq c^* = \mathcal{G}(\upsilon_1)$$

Which completes the proof.

Declarations

Ethical Approval: There is no applicable.

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