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WHEN IS $R \ltimes M$ AN APPROXIMATELY COHEN-MACAULAY LOCAL RING?

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ABSTRACT. Let $(R, \mathfrak{m})$ be a Noetherian local ring and M a finitely generated R-module. In this paper, we give a complete answer to the question of when the idealization $R \ltimes M$ of M over R is an approximately Cohen-Macaulay local ring.

1. Introduction

Throughout this paper, let $(R, \mathfrak{m})$ denote a Noetherian local ring of dimension r with the maximal ideal $\mathfrak{m}$ and M a finitely generated R-module of dimension d. It is well-known that R is Gorenstein if and only if there is an element $a \in \mathfrak{m}$ such that R/a^nR is a Gorenstein ring of dimension $r-1$ for all $n \geq 1$ (see [19]). However, this is not true in the Cohen-Macaulay case. Since such rings are close to Cohen-Macaulay rings, S. Goto introduced the notion of approximately Cohen-Macaulay local rings (see [14]).

Definition 1.1. The local ring $(R, \mathfrak{m})$ is called *approximately Cohen-Macaulay* if either $r = 0$ or if there is an element $a \in \mathfrak{m}$ such that R/a^nR is a Cohen-Macaulay ring of dimension $r-1$ for all $n \geq 1$.

We consider a multiplication on the additive group $R \oplus M$ as follows:

$$(a,x)(b,y) = (ab, ax + by)$$

for all $(a,x),(b,y) \in R \oplus M$. This multiplication results in $R \oplus M$ forming a Noetherian local ring with the unique maximal ideal $\mathfrak{m} \times M$. This special local ring is called the *idealization* of M over R and is denoted by $R \ltimes M$. Notably, it is important to observe that $\dim(R \ltimes M) = \dim(R)$. The structure of the idealization and its applications have piqued the interest of numerous mathematicians, as evidenced in works such as [2, 13, 15, 16, 21, 26, 30].

It is well-established that $R \ltimes M$ is a Gorenstein ring if and only if there exists an isomorphism between M and the canonical module K_R of R as R-modules (see [26]). S. Goto et al. in [15] delve into the investigation of the idealization $R \ltimes M$ to ascertain the circumstances under which it qualifies as an almost Gorenstein local ring. Specifically, they focus on scenarios where R is a Cohen-Macaulay local ring and M denotes a maximal Cohen-Macaulay R-module. In [15, Section 6], the authors gave a complete answer to the question in the case, where M is a faithful R-module, that is, the case $\mathrm{Ann}_R(M) = 0$. However, in the case where M is not a faithful module it has been left open. Recently,

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S. Goto and S. Kumashiro answered in the special case where R is a Gorenstein local ring and $M = I$ is an ideal of R such that R/I is a Cohen-Macaulay ring with $\dim(R/I) = \dim R$. For the case, where $\dim(R/I) = \operatorname{depth}(R/I) + 1$ the question remains open (see [16, Remark 2.6]).

Inspired by the notion of approximately Cohen-Macaulay rings, we introduce the concept of approximately Cohen-Macaulay modules, which is a generalization of the one presented by N.T. Cuong et al. (see [8, Definition 4.4]).

Definition 1.2. An R-module M is called an *approximately Cohen-Macaulay* module if either $\dim(M) = 0$ or there exists an element $a \in \mathfrak{m}$ such that M/a^nM is Cohen-Macaulay of dimension $\dim(M) - 1$, for every integer $n \geq 1$.

The aim of this paper is to explore the question of when the idealization $R \ltimes M$ is an approximately Cohen-Macaulay local ring. In more detail, the following theorem is the main result of this paper.

Theorem 1.3. *Let R be a local ring of dimension r and M a finitely generated R-module of dimension d. The following assertions are equivalent.*

(i) *$R \ltimes M$ is approximately Cohen-Macaulay which is not Cohen-Macaulay.*

(ii) *One of the following conditions is satisfied.*

(a) R is Cohen-Macaulay and M is approximately Cohen-Macaulay of dimension r which is not Cohen-Macaulay.

(b) R is approximately Cohen-Macaulay which is not Cohen-Macaulay and M is maximal Cohen-Macaulay.

(c) R and M are both approximately Cohen-Macaulay of the same dimension which are not Cohen-Macaulay.

(d) R is Cohen-Macaulay and M is Cohen-Macaulay of dimension $d = r - 1$.

(e) R is approximately Cohen-Macaulay which is not Cohen-Macaulay and M is Cohen-Macaulay of dimension $d = r - 1$.

The proof of Theorem 1.3 relies on the Theorem 3.4, which is a parametric characterization of the idealization as an approximately Cohen-Macaulay local ring. We also describe the approximate Cohen-Macaulayness of the idealization $R \ltimes I$ in the case where R is a Cohen-Macaulay local ring and I is an ideal of R (Corollary 3.5 and Corollary 3.6).

In the next section, we provide some preliminary results on the good system of parameters and the almost p-standard system of parameters of the idealization. In Section 3, we present the proof of Theorem 1.3.

2. Preliminaries

From now on, we always assume that $(R, \mathfrak{m})$ is a Noetherian local ring of dimension r and M is a finitely generated R-module with $d = \dim_R(M)$. The notion of almost p-standard systems of parameters is introduced by D.T. Cuong and the first author in [3]. We recall that a system of parameter (s.o.p for short) $x_1, \ldots, x_d$ of M is called *almost p-standard* if there exist non-negative integers $\lambda_0, \ldots, \lambda_d$ such that

$$\ell(M/(x_1^{n_1}, \ldots, x_d^{n_d})M) = \sum_{i=0}^{d} \lambda_i n_1 \ldots n_i$$

for all $n_1, \ldots, n_d \geq 1$.

Following [11, Theorem 1.2], the ring R possesses an almost p-standard s.o.p if and only if it is a quotient of a Cohen-Macaulay local ring, if and only if every finitely generated R-module admits an almost p-standard s.o.p. This concept extends the notion of a standard s.o.p for generalized Cohen-Macaulay modules which are not generalized Cohen-Macaulay. In general, every p-standard s.o.p in the sense of [7] is an almost p-standard s.o.p. However, the converse statement does not hold true even for Buchsbaum local rings (also see [22, Example 1]). Almost p-standard systems of parameters are useful in the studies of sequentially Cohen-Macaulay and sequentially generalized Cohen-Macaulay modules. The fact that an almost p-standard s.o.p is a (strong) d-sequence which is crucial in applications (see [3, 4, 5, 8, 9, 10, 11, 17, 18]). Recently, in [6] D.T. Cuong et al. constructed almost p-standard systems of parameters of idealizations and gave several applications (also see [22, 23, 24, 25]).

Note that every almost p-standard system of parameters is a good s.o.p (see [8, Corollary 2.7], [3, Proposition 2.5]). The latter concept was introduced by N.T. Cuong et al. (see [8, Definition 2.2]) which is a useful tool for studying the sequentially Cohen-Macaulay modules. Taking ideas from [6, Theorem 2.5], in the next part of this section, we will construct good systems of parameters for the idealization $R \ltimes M$.

From now on, let $\mathfrak{F}_M : M_0 \subseteq M_1 \subseteq \ldots \subseteq M_d = M$ be the dimension filtration of M, i.e. M_i is the largest submodule of M such that $\dim(M_i) \leq i$ for all $i = 0, 1, \ldots, d$ (see [27, Definition 2.1]). Such M_i's exist uniquely since M is Noetherian. Moreover, $M_0 = H^0_{\mathfrak{m}}(M)$ is the 0-th local cohomology module of M with respect to the maximal ideal $\mathfrak{m}$.

Definition 2.1. A s.o.p $x_1, \ldots, x_d$ of M is called a *good s.o.p* of M if $M_i \cap (x_{i+1}, \ldots, x_d)M = 0$ for all $i = 0, \ldots, d-1$.

From now on, we denote by $A = R \ltimes M$ the idealization of M over R. From the definition of dimension filtration, we can describe the dimension filtration of idealizations.

Lemma 2.2. *Let $\mathfrak{F}_M : M_0 \subseteq M_1 \subseteq \ldots \subseteq M_d = M$ and $\mathfrak{F}_R : R_0 \subseteq R_1 \subseteq \ldots \subseteq R_r = R$ be the dimension filtrations of M and R, respectively.*

(i) *If $d = r$, we put $A_i = R_i \times M_i$ for $i = 0, \ldots, r$. Then, we have*
$$\mathfrak{F}_A : A_0 \subseteq A_1 \subseteq \ldots \subseteq A_r = A$$
is the dimension filtration of A.

(ii) *If $d < r$, we put $A_i = R_i \times M_i$ for $i = 0, \ldots, d$ and $A_j = R_j \times M$ for $j = d+1, \ldots, r$. Then, we have $\mathfrak{F}_A : A_0 \subseteq A_1 \subseteq \ldots \subseteq A_r = A$ is the dimension filtration of A.*

In the following proposition, we construct a good system of parameters of the idealization $R \ltimes M$ (also see [25, Proposition 2.7]).

Proposition 2.3. *Let $\underline{x} = x_1, \ldots, x_r$ be elements in $\mathfrak{m}$. Set $u_i = (x_i, 0)$ for $i = 1, \ldots, r$ and $\underline{u} = u_1, \ldots, u_r$. The following statements are equivalent.*

(i) *$\underline{u}$ is a good s.o.p of A.*

(ii) *$\underline{x}$ is a good s.o.p of R and $x_1, \ldots, x_d$ is a good s.o.p of M. If $d < r$, then $x_{d+1}, \ldots, x_r \in \mathrm{Ann}_R(M)$.*

Proof. $(i) \Rightarrow (ii)$. Since $\underline{u}$ is a s.o.p of A, it follows that $\underline{x}$ is a s.o.p of R and $\underline{x}$ is a multiplicity system of M (i.e. $\ell(M/(x_1,\dots,x_r)M) < \infty$).

• If $d = r$, then $x_1,\dots,x_d$ is a s.o.p of M. Since $u_1,\dots,u_d$ is a good s.o.p of A, we have

$$\begin{aligned} 0\times 0 &= A_i \cap (u_{i+1},\dots,u_d)A \\ &= (R_i \cap (x_{i+1},\dots,x_d)R) \times (M_i \cap (x_{i+1},\dots,x_d)M) \end{aligned}$$

for all $i = 0,\dots,d-1$. By Definition 2.1, $\underline{x}$ is a good s.o.p of both M and R.

• If $d < r$, then we have $\dim_A(0\times M) = d < r$ and $0\times M \subseteq A_d = R_d \times M$. Since $\underline{u}$ is a good s.o.p of A, we get by Definition 2.1 that

$$\begin{aligned} 0\times (x_{d+1},\dots,x_r)M &= (0\times M)\cap (u_{d+1},\dots,u_r)(R\ltimes M) \\ &\subseteq A_d \cap (u_{d+1},\dots,u_r)A = 0\times 0. \end{aligned}$$

Hence $x_{d+1},\dots,x_r \in \mathrm{Ann}_R(M)$. Since $u_1,\dots,u_r$ is a good s.o.p of A, we have

$$\begin{aligned} 0\times 0 &= A_i \cap (u_{i+1},\dots,u_r)A \\ &= (R_i \cap (x_{i+1},\dots,x_r)R) \times (M_i \cap (x_{i+1},\dots,x_r)M) \\ &= (R_i \cap (x_{i+1},\dots,x_r)R) \times (M_i \cap (x_{i+1},\dots,x_d)M) \end{aligned}$$

for all $i = 0,\dots,r-1$. By Definition 2.1, $x_1,\dots,x_r$ is a good s.o.p of R and $x_1,\dots,x_d$ is a good s.o.p of M.

$(ii) \Rightarrow (i)$. Since $x_1,\dots,x_d$ is a good s.o.p of M and $x_1,\dots,x_r$ is a good s.o.p of R and $x_{d+1},\dots,x_r \in \mathrm{Ann}_R(M)$, we get by Definition 2.1 that

$$\begin{aligned} A_i \cap (u_{d_i+1},\dots,u_r)A &= (R_i \cap (x_{d_i+1},\dots,x_r)R) \times (M_i \cap (x_{d_i+1},\dots,x_r)M) \\ &= (R_i \cap (x_{d_i+1},\dots,x_r)R) \times (M_i \cap (x_{d_i+1},\dots,x_d)M) \\ &= 0\times 0 \end{aligned}$$

for all $i = 0,\dots,r-1$. By Definition 2.1, $\underline{u}$ is a good s.o.p of A. □

Following Proposition 2.3 we have the following interesting corollary.

Corollary 2.4. *There always exists a good s.o.p of A of the form $(x_1,0),\dots,(x_r,0)$, where $x_1,\dots,x_r$ is a good s.o.p of R and $x_1,\dots,x_d$ is a good s.o.p of M. Moreover, if $d < r$ then $x_{d+1},\dots,x_r \in \mathrm{Ann}_R(M)$.*

Proof. Let $\mathfrak{F}_M : M_0 \subseteq M_1 \subseteq \ldots \subseteq M_d = M$ and $\mathfrak{F}_R : R_0 \subseteq R_1 \subseteq \ldots \subseteq R_r = R$ be the dimension filtrations of M and R, respectively. We set $d_i = \dim(R_i)$ and $d'_j = \dim(M_j)$ for $i = 0,1,\dots,r$ and $j = 0,1,\dots,d$. By [8, Remark 2.3(i)], we have

$$M_j = \bigcap_{\dim(R/\mathfrak{p})\geq d'_{j+1}} L(\mathfrak{p}), R_i = \bigcap_{\dim(R/\mathfrak{p})\geq d_{i+1}} N(\mathfrak{p})$$

where $\bigcap_{\mathfrak{p}\in \mathrm{Ass}(M)} L(\mathfrak{p}) = 0$ and $\bigcap_{\mathfrak{p}\in \mathrm{Ass}(M)} N(\mathfrak{p}) = 0$ are the reduced primary decompositions of submodules 0 of M and R, respectively. We put

$$L_j = \bigcap_{\dim(R/\mathfrak{p})\leq d'_j} L(\mathfrak{p}), N_i = \bigcap_{\dim(R/\mathfrak{p})\leq d_i} N(\mathfrak{p}).$$

Then $\dim(L_j) = d'_j$ and $\dim(N_i) = d_i$. We divide it into two cases.

• Let $d = r$. By the Prime Avoidance Theorem, there exists a s.o.p $x_1,\ldots,x_r$ of R such that $x_1,\ldots,x_d$ is a s.o.p of M and $x_{d_i+1},\ldots,x_r \in \mathrm{Ann}_R(R/N_i), x_{d'_j+1},\ldots,x_d \in \mathrm{Ann}_R(M/L_j)$. Therefore, we have

$$(x_{d'_j+1},\ldots,x_d)M \cap M_j \subseteq L_j \cap M_j = 0$$

and $(x_{d_i+1},\ldots,x_r)R \cap R_i \subseteq N_i \cap R_i = 0$. Therefore, $x_1,\ldots,x_r$ is a good s.o.p of R and $x_1,\ldots,x_d$ is a good s.o.p of M. By Proposition 2.3, $(x_1,0),\ldots,(x_r,0)$ is a good s.o.p of A.

• Let $d < r$. By the Prime Avoidance Theorem, there exists a s.o.p $x_1,\ldots,x_r$ of R such that $x_{d+1},\ldots,x_r \in \mathrm{Ann}_R(M)$, $x_1,\ldots,x_d$ is a s.o.p of M and $x_{d_i+1},\ldots,x_r \in \mathrm{Ann}_R(R/N_i)$, $x_{d'_j+1},\ldots,x_d \in \mathrm{Ann}_R(M/L_j)$. Therefore, we have

$$(x_{d'_j+1},\ldots,x_d)M \cap M_j \subseteq L_j \cap M_j = 0$$

and $(x_{d_i+1},\ldots,x_r)R \cap R_i \subseteq N_i \cap R_i = 0$. Therefore, $x_1,\ldots,x_r$ is a good s.o.p of R and $x_1,\ldots,x_d$ is a good s.o.p of M and $x_{d+1},\ldots,x_r \in \mathrm{Ann}_R(M)$. By Proposition 2.3, $(x_1,0),\ldots,(x_r,0)$ is a good s.o.p of A. □

Almost p-standard systems of parameters of the form $(x_1,0),\ldots,(x_r,0)$ were used to construct Cohen-Macaulay Rees algebras for idealizations and Cohen-Macaulay Rees modules for unmixed modules; to compute Hilbert coefficients of the idealization and partial Euler-Poincaré characteristics (see [6, 24]); to bound for the reducibility index (see [22]); to compute the length function of saturation of powers ideals (see [23]).

In the next section, we continuously use almost p-standard and good s.o.p of the form $(x_1,0),\ldots,(x_r,0)$ to characterize the approximate Cohen-Macaulayness of idealization.

3. Approximate Cohen-Macaulayness for idealization

Let $\mathfrak{F}_M : M_0 \subseteq M_1 \subseteq \ldots \subseteq M_d = M$ be the dimension filtration of M and $\underline{y} = x_1,\ldots,x_d$ be a good s.o.p of M. It is clear that $x_1,\ldots,x_i$ is a multiplicity system of M_i for $i = 0,\ldots,d$. Therefore, the following difference is well-defined

$$I_{\mathfrak{F}_M}(\underline{y}) = \ell(M/\underline{y}M) - \sum_{i=0}^{d} e(x_1,\ldots,x_i;M_i)$$

where $e(x_1,\ldots,x_i;M_i)$ is the multiplicity of M_i with respect to $x_1,\ldots,x_i$, for $i = 0,1,\ldots,d$. The function $I_{\mathfrak{F}_M}(\underline{y})$ was studied by N.T. Cuong and D.T. Cuong in [8]. Note that we have $e(x_1,\ldots,x_i;M_i) = 0$ if and only if $\dim(M_i) < i$. Therefore, the above concept of $I_{\mathfrak{F}_M}(\underline{y})$ is identical to the concept of $I_{\mathfrak{F}_M}(\underline{y})$ by N.T. Cuong et al. in [8]. However, for the convenience of calculations, we will use the above definition of $I_{\mathfrak{F}_M}(\underline{y})$. For any integers $\underline{m} = n_1,\ldots,n_d$, we denote

$$I_{\mathfrak{F}_M}(\underline{y}(\underline{m})) = \ell(M/\underline{y}(\underline{m})M) - \sum_{i=0}^{d} n_1\ldots n_i e(x_1,\ldots,x_i;M_i)$$

where $\underline{y}(\underline{m}) = x_1^{n_1},\ldots,x_d^{n_d}$. By [8, Lemma 2.7, Proposition 2.9], we have the following lemma.

Lemma 3.1. *Let $\underline{y} = x_1,\ldots,x_d$ be a good s.o.p of M. Then the function $I_{\mathfrak{F}_M}(\underline{y}(\underline{m}))$ is non-decreasing and non-negative for all integers $n_1,\ldots,n_d \geq 1$.*

From now on, let $\mathfrak{F}_M : M_0 \subseteq M_1 \subseteq \ldots \subseteq M_d = M$, $\mathfrak{F}_R : R_0 \subseteq R_1 \subseteq \ldots \subseteq R_r = R$, and $\mathfrak{F}_A$ be the dimension filtrations of M, R and A, respectively. Then, we have the following lemma (also see [25, Lemma 3.2]).

Lemma 3.2. *Let $\underline{x} = x_1, \ldots, x_r$ be a good s.o.p of R. Set $\underline{u} = u_1, \ldots, u_r$, where $u_i = (x_i, 0)$ for $i = 1, \ldots, r$, and $\underline{u}(\underline{n}) = u_1^{n_1}, \ldots, u_r^{n_r}$ for $n_1, \ldots, n_r \geq 1$.*

(i) *Let $d = r$. Suppose that $\underline{x}$ is a good s.o.p of M. Then, we have*

$$I_{\mathfrak{F}_A}(\underline{u}(\underline{n})) = I_{\mathfrak{F}_R}(\underline{x}(\underline{n})) + I_{\mathfrak{F}_M}(\underline{x}(\underline{n}))$$

for all integers $n_1, \ldots, n_r \geq 1$.

(ii) *Let $d < r$. Suppose that $\underline{y} = x_1, \ldots, x_d$ is a good s.o.p of M and $x_{d+1}, \ldots, x_r \in \mathrm{Ann}_R(M)$. Then, we have*

$$I_{\mathfrak{F}_A}(\underline{u}(\underline{n})) = I_{\mathfrak{F}_R}(\underline{x}(\underline{n})) + I_{\mathfrak{F}_M}(\underline{y}(\underline{m}))$$

for all integers $n_1, \ldots, n_r \geq 1$.

Recall that a local ring R is called *generalized Cohen-Macaulay* if the i-th local cohomology module $H^i_{\mathfrak{m}}(R)$ has finite length for all $i = 0, 1, \ldots, \dim(R) - 1$ (see [29]). Note that if $\dim(A) = 1$, then either A is Cohen-Macaulay or A is generalized Cohen-Macaulay with the dimension filtration $0 \neq H^0_{\mathfrak{m} \times M}(A) \subsetneq A$. Following [8, Proposition 4.5], A is an approximately Cohen-Macaulay ring. From now on, we always assume that A is not Cohen-Macaulay and $r \geq 2$. Then, we have the following lemma.

Lemma 3.3. *Suppose that $r \geq 2$. The following statements are equivalent.*

(i) *A is approximately Cohen-Macaulay which is not Cohen-Macaulay.*

(ii) *$\ell(A/\underline{u}A) = e(\underline{u}; A) + e(u_1, \ldots, u_{r-1}; A_{r-1})$, where $e(u_1, \ldots, u_{r-1}; A_{r-1}) \neq 0$ for all good s.o.p $\underline{u} = u_1, \ldots, u_r$ of A.*

(iii) *There exists a s.o.p $\underline{u} = u_1, \ldots, u_r$ of A such that $\underline{u}$ is an almost p-standard s.o.p of A and*

$$\ell(A/\underline{u}A) = e(\underline{u}; A) + e(u_1, \ldots, u_{r-1}; A_{r-1})$$

where $e(u_1, \ldots, u_{r-1}; A_{r-1}) \neq 0$.

(iv) *There exists a good s.o.p $\underline{u} = u_1, \ldots, u_r$ of A such that*

$$\ell(A/\underline{u}(\underline{n})A) = n_1 \ldots n_r e(\underline{u}; A) + n_1 \ldots n_{r-1} e(u_1, \ldots, u_{r-1}, A_{r-1})$$

for all integers $n_1, \ldots, n_r \geq 1$, where $e(u_1, \ldots, u_{r-1}; A_{r-1}) \neq 0$.

Proof. $(i) \Rightarrow (ii)$. Since A is approximately Cohen-Macaulay but not Cohen-Macaulay, we get by [8, Proposition 4.3] that A is a sequentially Cohen-Macaulay module with the dimension filtration

$$\mathfrak{F}_A : 0 = A_0 \subseteq A_1 \subseteq \ldots \subseteq A_{r-1} \subsetneq A$$

where $\dim(A_{r-1}) = r - 1$ and $A_i = 0$ for all $i = 0, \ldots, r-2$. Hence $u_1, \ldots, u_{r-1}$ is a s.o.p of A_{r-1} and

$$I_{\mathfrak{F}_A}(\underline{u}) = \ell(A/\underline{u}A) - e(\underline{u}; A) - e(u_1, \ldots, u_{r-1}; A_{r-1})$$

for all good s.o.p $\underline{u} = u_1, \ldots, u_r$ of A. Therefore, $e(u_1, \ldots, u_{r-1}; A_{r-1}) \neq 0$. Following [8, Theorem 4.2], we have

$$\ell(A/\underline{u}A) - e(\underline{u}; A) - e(u_1, \ldots, u_{r-1}; A_{r-1}) = 0$$

for every good s.o.p $\underline{u} = u_1, \ldots, u_r$ of A.

$(ii) \Rightarrow (iii)$. Let $\underline{u} = u_1, \ldots, u_r$ be a good s.o.p of A. By [8, Remark 3.11], we have $\underline{u}(\underline{x}) = u_1^{n_1}, \ldots, u_r^{n_r}$ is also a good s.o.p of A for all positive integers $\underline{n} = n_1, \ldots, n_r$. From the assumption (ii), we have

$$\ell(A/\underline{u}(\underline{n})A) - n_1 \ldots n_r e(\underline{u}; A) - n_1 \ldots n_{r-1} e(u_1, \ldots, u_{r-1}; A_{r-1}) = 0$$

for all $n_1, \ldots, n_r \geq 1$, where $e(u_1, \ldots, u_{r-1}; A_{r-1}) \neq 0$. Therefore, $\underline{u}$ is an almost p-standard s.o.p of A.

$(iii) \Rightarrow (iv)$. Assume that $\underline{u} = u_1, \ldots, u_r$ is an almost p-standard s.o.p of A and

$$\ell(A/\underline{u}A) - e(\underline{u}; A) - e(u_1, \ldots, u_{r-1}; A_{r-1}) = 0$$

where $e(u_1, \ldots, u_{r-1}; A_{r-1}) \neq 0$. By [8, Corollary 3.7], we have $x_1, \ldots, x_d$ is a good s.o.p of M. Since $\underline{u}$ is almost p-standard, we get by [3, Theorem 3.7] that

$$\ell(A/\underline{u}A) = n_1 \ldots n_r e(\underline{u}; A) + \sum_{i=0}^{r-1} n_1 \ldots n_i e(u_1, \ldots, u_i, U_A^{ir})$$

for all integers $n_1, \ldots, n_r \geq 1$. Therefore, we have

$$e(u_1, \ldots, u_{r-1}; A_{r-1}) = \sum_{i=0}^{r-1} e\left(u_1, \ldots, u_i, U_A^{ir}\right).$$

Following [3, Remark 3.6], $U_A^{r-1,r} = A_{r-1}$ is the biggest submodule of M of dimension less than or equal to $r-1$. Hence

$$e(u_1, \ldots, u_{r-1}; U_A^{r-1,r}) = e(x_1, \ldots, x_{r-1}; A_{r-1})$$

and $e(u_1, \ldots, u_i; U_A^{i,r}) = 0$ for all $i = 0, \ldots, r-2$, the result follows.

$(iv) \Rightarrow (i)$. Let $\underline{u} = u_1, \ldots, u_r$ be a good s.o.p of A such that

$$\ell(A/\underline{u}(\underline{n})A) = n_1 \ldots n_r e(\underline{u}; A) + n_1 \ldots n_{r-1} e(u_1, \ldots, u_{r-1}, A_{r-1})$$

for all integers $n_1, \ldots, n_r \geq 1$, where $e(u_1, \ldots, u_{r-1}; A_{r-1}) \neq 0$. Therefore, $\underline{u}$ is an almost p-standard s.o.p of M. Clearly,

$$\ell(A/\underline{u}(\underline{n})A) - n_1 \ldots n_r e(\underline{u}; A) - n_1 \ldots n_{r-1} e(u_1, \ldots, u_{r-1}, A_{r-1}) \geq I_{\mathfrak{F}_A}(\underline{u}(\underline{n})) \geq 0.$$

Therefore, by the hypothesis (iv) we have implied that $I_{\mathfrak{F}_A}(\underline{u}(\underline{n})) = 0$. By [8, Theorem 4.2], A is sequentially Cohen-Macaulay. Now, we will prove that $A_i = 0$ for all $i = 0, 1, \ldots, r-2$. Suppose that there exists an integer $i \in \{0, 1, \ldots, r-2\}$ such that $A_i \neq 0$. Put $j = \dim_A(A_i)$. Then $0 \leq j \leq i$. Hence $\dim_A(A_j) = j \geq 0$. Therefore, $e(u_1, \ldots, u_j; A_j) > 0$. Since A is sequentially Cohen-Macaulay, we get by [3, Proposition 2.9(2)] that $A_j \cong U_A^{j,r}$. Hence

$$e(u_1, \ldots, u_j; U_A^{j,r}) = e(u_1, \ldots, u_j; A_j) > 0.$$

Since $\underline{u}$ is an almost p-standard s.o.p of A, we get by [3, Theorem 3.7] that

$$\ell(A/\underline{u}(\underline{n})A) = \sum_{i=0}^{r} n_1 \ldots n_i e(u_1, \ldots, u_i; U_A^{i,r})$$

for all integers $n_1, \ldots, n_r \geq 1$, where $e_j(u_1, \ldots, u_j; U_A^{j,r}) > 0$ with $0 \leq j \leq r-2$ a contradiction. So $A_i = 0$ for all $i = 0, 1, \ldots, r-2$. Thus A is an approximately Cohen-Macaulay local ring but not a Cohen-Macaulay local ring. □

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The first main result of this section is the following theorem.

Theorem 3.4. *Suppose that $r \geq 2$. The following statements are equivalent.*

(i) *A is approximately Cohen-Macaulay which is not Cohen-Macaulay.*

(ii) *There exists an almost p-standard s.o.p $\underline{u} = (x_1, 0), \ldots, (x_r, 0)$ of A such that*

$$\ell(A/\underline{u}(\underline{n})A) = n_1 \ldots n_r e(\underline{u}; A) + n_1 \ldots n_{r-1} e(u_1, \ldots, u_{r-1}; A_{r-1})$$

for all integers $n_1, \ldots, n_r \geq 1$, where $e(u_1, \ldots, u_{r-1}; A_{r-1}) > 0$.

(iii) *There exists a good s.o.p $\underline{u} = (x_1, 0), \ldots, (x_r, 0)$ of A such that*

$$\ell(A/\underline{u}(\underline{n})A) = n_1 \ldots n_r e(\underline{u}; A) + n_1 \ldots n_{r-1} e(u_1, \ldots, u_{r-1}; A_{r-1})$$

for all integers $n_1, \ldots, n_r \geq 1$, where $e(u_1, \ldots, u_{r-1}; A_{r-1}) > 0$.

(iv) *One of the following conditions is satisfied.*

(a) There exists a good s.o.p $\underline{x} = x_1, \ldots, x_r$ of both R and M such that

$$\ell(R/\underline{x}(\underline{n})R) = n_1 \ldots n_r e(\underline{x}; R),$$

$$\ell(M/\underline{x}(\underline{n})M) = n_1 \ldots n_r e(\underline{x}; M) + n_1 \ldots n_{r-1} e(x_1, \ldots, x_{r-1}; M_{r-1})$$

for all integers $n_1, \ldots, n_r \geq 1$, where $e(x_1, \ldots, x_{r-1}; M_{r-1}) > 0$.

(b) There exists a good s.o.p $\underline{x} = x_1, \ldots, x_r$ of both R and M such that

$$\ell(R/\underline{x}(\underline{n})R) = n_1 \ldots n_r e(\underline{x}; R) + n_1 \ldots n_{r-1} e(x_1, \ldots, x_{r-1}; R_{r-1})$$

$$\ell(M/\underline{x}(\underline{n})M) = n_1 \ldots n_r e(\underline{x}; M)$$

for all integers $n_1, \ldots, n_r \geq 1$, where $e(x_1, \ldots, x_{r-1}; R_{r-1}) > 0$.

(c) There exists a good s.o.p $\underline{x} = x_1, \ldots, x_r$ of both R and M such that

$$\ell(R/\underline{x}(\underline{n})R) = n_1 \ldots n_r e(\underline{x}; R) + n_1 \ldots n_{r-1} e(x_1, \ldots, x_{r-1}; R_{r-1}),$$

$$\ell(M/\underline{x}(\underline{n})M) = n_1 \ldots n_r e(\underline{x}; M) + n_1 \ldots n_{r-1} e(x_1, \ldots, x_{r-1}; M_{r-1})$$

for all integers $n_1, \ldots, n_r \geq 1$, where $e(x_1, \ldots, x_{r-1}; R_{r-1}) > 0$, $e(x_1, \ldots, x_{r-1}; M_{r-1}) > 0$.

(d) There exists a good s.o.p $\underline{x} = x_1, \ldots, x_r$ of R so that $\underline{y} = x_1, \ldots, x_{r-1}$ is a good s.o.p of M and $x_r \in \mathrm{Ann}_R(M)$ such that

$$\ell(R/\underline{x}(\underline{n})R) = n_1 \ldots n_r e(\underline{x}; R)$$

and

$$\ell(M/\underline{y}(\underline{m})M) = n_1 \ldots n_{r-1} e(\underline{y}; M)$$

for all integers $n_1, \ldots, n_r \geq 1$.

(e) There exists a good s.o.p $\underline{x} = x_1, \ldots, x_r$ of R so that $\underline{y} = x_1, \ldots, x_{r-1}$ is a good s.o.p of M and $x_r \in \mathrm{Ann}_R(M)$ such that

$$\ell(R/\underline{x}(\underline{n})R) = n_1 \ldots n_r e(\underline{x}; R) + n_1 \ldots n_{r-1} e(x_1, \ldots, x_{r-1}; R_{r-1}),$$

$$\ell(M/\underline{y}(\underline{m})M) = n_1 \ldots n_{r-1} e(\underline{y}; M)$$

for all integers $n_1, \ldots, n_r \geq 1$, where $e(x_1, \ldots, x_{r-1}; R_{r-1}) > 0$.

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Proof. $(iii) \Rightarrow (i)$ is obvious by Lemma 3.3(iv).
$(i) \Rightarrow (ii)$. By Corollary 2.4, there is always a good s.o.p $\underline{u} = (x_1,0),\dots,(x_r,0)$ of A. Following [8, Remark 2.3], $\underline{u}(\underline{n}) = (x_1,0)^{n_1},\dots,(x_r,0)^{n_r}$ also is good s.o.p for all integers $n_1,\dots,n_r$. Since A is approximately Cohen-Macaulay but not Cohen-Macaulay, we get by Lemma 3.3(ii) that

$$\ell(A/\underline{u}(\underline{n})A) = n_1\dots n_r e(\underline{u};A) + n_1\dots n_{r-1}e(u_1,\dots,u_{r-1};A_{r-1})$$

for all integers $n_1,\dots,n_r \geq 1$, where $e(u_1,\dots,u_{r-1};A_{r-1}) > 0$. Therefore, $\underline{u}$ is an almost p-standard s.o.p of A.
$(iii) \Rightarrow (i)$. By [8, Corollary 2.7] and [3, Proposition 2.5], the statement follows.
$(iii) \Rightarrow (iv)$. We divide it into two cases.
• Let $d = r$. Let $\underline{u} = (u_1,0),\dots,(u_r,0)$ be a good s.o.p of M such that

$$\ell(A/\underline{u}(\underline{n})A) = n_1\dots n_r e(\underline{u};A) + n_1\dots n_{r-1}e(u_1,\dots,u_{r-1};A_{r-1})$$

for all positive integers $n_1,\dots,n_r$, where $e(u_1,\dots,u_{r-1};A_{r-1}) > 0$. Hence $(u_1,0),\dots,(u_r,0)$ is an almost p-standard s.o.p of M. Let

$$\mathfrak{F}_M : M_0 \subseteq M_1 \subseteq \dots \subseteq M_r = M$$

and

$$\mathfrak{F}_R : R_0 \subseteq R_1 \subseteq \dots \subseteq R_r = R$$

be the dimension filtrations of M and R, respectively. For $0 \leq i \leq r$, we put $A_i = R_i \ltimes M_i$. By Lemma 2.2, $A_0 \subseteq A_1 \subseteq \dots \subseteq A_r$ is the dimension filtration of A. Following the proof of Lemma 3.3 $(iv) \Rightarrow (i)$, we have $A_i = R_i \times M_i = 0$ for all $i = 0,1,\dots,r-2$ and $\dim(A_{r-1}) = r-1$ where $A_{r-1} = R_{r-1} \times M_{r-1}$. Hence

$$I_{\mathfrak{F}_A}(\underline{u}(\underline{n}) = \ell(A/\underline{u}(\underline{n})A) - n_1\dots n_r e(\underline{u};A) - n_1\dots n_{r-1}e(u_1,\dots,u_{r-1};A_{r-1}),$$

and $R_i = 0, M_i = 0$ for all $i = 0,1,\dots,r-2$. Therefore, one of the following assertions is true.
(a) $\dim(R_{r-1}) < r-1$ and $\dim(M_{r-1}) = r-1$. Hence $R_{r-1} = 0$ because if otherwise $R_{r-1} \neq 0$ then we have

$$\begin{aligned}\ell(A/\underline{u}(\underline{n})A) &= n_1\dots n_r e(u_1,\dots,u_r;A) + n_1\dots n_{r-1}e(u_1,\dots,u_{r-1};A_{r-1})\\ &\quad + n_1\dots n_{d_i}e(u_1,\dots,u_{d_i};R_{r-1}\times 0)\end{aligned}$$

where $\dim(R_{r-1}) = d_i < r-1$, a contradiction. Therefore, the dimension filtrations of R and M is $\mathfrak{F}_M : 0 \subsetneq M_{r-1} \subsetneq M$ and $\mathfrak{F}_R : 0 \subsetneq R$, respectively. Hence

$$I_{\mathfrak{F}_R}(\underline{x}(\underline{n})) = \ell(R/\underline{x}(\underline{n})R) - n_1\dots n_r e(\underline{x};R)$$

and

$$I_{\mathfrak{F}_M}(\underline{x}(\underline{n})) = \ell(M/\underline{x}(\underline{n})M) - n_1\dots n_r e(\underline{x};M) - n_1\dots n_{r-1}e(x_1,\dots,x_{r-1};M_{r-1}).$$

By the assumption and Lemma 3.2, we have

$$I_{\mathfrak{F}_A}(\underline{u}(\underline{n})) = I_{\mathfrak{F}_R}(\underline{x}(\underline{n})) + I_{\mathfrak{F}_M}(\underline{x}(\underline{n})) = 0$$

for all $n_1,\dots,n_r \geq 1$. By Lemma 3.1, $I_{\mathfrak{F}_R}(\underline{x}(\underline{n}))$ and $I_{\mathfrak{F}_M}(\underline{x}(\underline{n}))$ are non-negative functions. Therefore, we have $I_{\mathfrak{F}_R}(\underline{x}(\underline{n})) = I_{\mathfrak{F}_M}(\underline{x}(\underline{n})) = 0$, which means

$$\ell(R/\underline{x}(\underline{n})R) = n_1\dots n_r e(\underline{x};R)$$

and

$$\ell(M/\underline{x}(\underline{n})M) = n_1 \dots n_r e(\underline{x};M) + n_1 \dots n_{r-1} e(x_1,\dots,x_{r-1};M_{r-1})$$

for all positive integers $n_1,\dots,n_r$, where $e(x_1,\dots,x_{r-1};M_{r-1}) > 0$.
(b) $\dim(R_{r-1}) = r-1$ and $\dim(M_{r-1}) < r-1$. Similar to the proof in case (a), we have

$$\ell(R/\underline{x}(\underline{n})R) = n_1 \dots n_r e(\underline{x};R) + n_1 \dots n_{r-1} e(x_1,\dots,x_{r-1};R_{r-1})$$

where $e(x_1,\dots,x_{r-1};R_{r-1}) > 0$, and

$$\ell(M/\underline{x}(\underline{n})M) = n_1 \dots n_r e(\underline{x};M)$$

for all positive integers $n_1,\dots,n_r$.
(c) $\dim(R_{r-1}) = \dim(M_{r-1}) = r-1$. Therefore, the dimension filtrations of R and M are $\mathfrak{F}_M : 0 \subsetneq M_{r-1} \subsetneq M$ and $\mathfrak{F}_R : 0 \subsetneq R_{r-1} \subsetneq R$, respectively. Similar to the proof in case (a), the result follows.
• Let $d < r$. Let $\underline{u} = (x_1,0),\dots,(x_r,0)$ be a good s.o.p of A such that

$$\ell(A/\underline{u}(\underline{n})A) = n_1 \dots n_r e(\underline{u};A) + n_1 \dots n_{r-1} e(u_1,\dots,u_{r-1};A_{r-1})$$

for all positive integers $\underline{n} = n_1,\dots,n_r$, where $e(u_1,\dots,u_{r-1};A_{r-1}) > 0$. Hence $\underline{u}$ is an almost p-standard s.o.p of M. By [8, Remark 2.3], $(x_1,0)^{n_1},\dots,(x_r,0)^{n_r}$ also is good s.o.p for all integers $n_1,\dots,n_r$. We set $\underline{y} = x_1,\dots,x_d$. By Proposition 2.3, $x_1^{n_1},\dots,x_r^{n_r}$ is a good s.o.p of R, $x_1^{n_1},\dots,x_d^{n_d}$ is a good s.o.p of M and $x_{d+1}^{n_{d+1}},\dots,x_r^{n_r} \in \mathrm{Ann}_R(M)$. Let

$$\mathfrak{F}_M : M_0 \subseteq M_1 \subseteq \dots \subseteq M_d = M$$

and

$$\mathfrak{F}_R : R_0 \subseteq R_1 \subseteq \dots \subseteq R_r = R$$

be the dimension filtrations of M and R, respectively. For $0 \le i \le r$, we put $A_i = R_i \times M_i$ and $A_j = R_j \times M$ for $j = d+1,\dots,r$. By Lemma 2.2, $A_0 \subseteq A_1 \subseteq \dots \subseteq A_r$ is the dimension filtration of A. Following the proof of Lemma 3.3 $(iv) \Rightarrow (i)$, we have $A_i = R_i \times M_i = 0$ for all $i = 0,1,\dots,r-2$ and $\dim(A_{r-1}) = r-1$ where $A_{r-1} = R_{r-1} \times M$. Hence $R_i = 0, M_i = 0$ for all $i = 0,1,\dots,r-2$, and $d = r-1$. Indeed, suppose that $d < r-1$ then $\dim_A(0 \times M) = d$. Therefore, we have

$$\begin{aligned}
0 = \ell(A/\underline{u}(\underline{n})A) &- n_1 \dots n_r e(\underline{u};A) - n_1 \dots n_{r-1} e(u_1,\dots,u_{r-1};A_{r-1})\\
&\ge \ell(A/\underline{u}(\underline{n})A) - n_1 \dots n_r e(\underline{u};A)\\
&- n_1 \dots n_{r-1} e(u_1,\dots,u_{r-1};A_{r-1}) - n_1 \dots n_d e(u_1,\dots,u_d;A_d)\\
&\ge I_{\mathfrak{F}_A}(\underline{u}(\underline{n})) \ge 0
\end{aligned}$$

for all positive integers $n_1,\dots,n_r$. Hence

$$\begin{aligned}
\ell(A/\underline{u}(\underline{n})A) &= n_1 \dots n_r e(\underline{u};A) + n_1 \dots n_{r-1} e(u_1,\dots,u_{r-1};A_{r-1})\\
&+ n_1 \dots n_d e(u_1,\dots,u_d;A_d)
\end{aligned}$$

where $e(u_1,\dots,u_d;A_d) > 0$, a contradiction. Thus one of the following assertions is true.
(d) $\dim(R_{r-1}) < r-1$. Similar to the proof in case (a), we have $R_{r-1} = 0$. Therefore, the dimension filtrations of R and M are $\mathfrak{F}_M : 0 \subsetneq M$ and $\mathfrak{F}_R : 0 \subsetneq R$, respectively. By the assumption and Lemma 3.2, we have

$$I_{\mathfrak{F}_A}(\underline{u}(\underline{n})) = I_{\mathfrak{F}_R}(\underline{x}(\underline{n})) + I_{\mathfrak{F}_M}(\underline{y}(\underline{m})) = 0$$

for all $n_1,\dots,n_r \geq 1$, where $\underline{y}(\underline{m}) = x_1^{n_1},\dots,x_{r-1}^{n_{r-1}}$. By Lemma 3.1, $I_{\mathfrak{F}_R}(\underline{x}(\underline{n}))$ and $I_{\mathfrak{F}_M}(\underline{y}(\underline{m}))$ are non-negative functions. Hence $I_{\mathfrak{F}_R}(\underline{x}(\underline{n})) = I_{\mathfrak{F}_M}(\underline{y}(\underline{m})) = 0$. Therefore,

$$\ell(R/\underline{x}(\underline{n})R) = n_1\dots n_r e(\underline{x};R)$$

and

$$\ell(M/\underline{y}(\underline{m})M) = n_1\dots n_{r-1} e(\underline{y};M)$$

for all positive integers $n_1,\dots,n_r$, where $\underline{y} = x_1,\dots,x_{r-1}$.

(e) $\dim(R_{r-1}) = r-1$. Therefore, the dimension filtrations of M and R are $\mathfrak{F}_M : 0 \subsetneq M$ and $\mathfrak{F}_R : 0 \subsetneq R_{r-1} \subsetneq R$, respectively. Similar to the proof in case (d), the result follows.

$(iv) \Rightarrow (iii)$. We consider the following two cases.

• Suppose that one of the conditions (a) or (b) or (c) is satisfied. Since $x_1,\dots,x_r$ is a good s.o.p of both R and M, we get by Proposition 2.3(i) that $\underline{u} = (x_1,0),\dots,(x_r,0)$ is a good s.o.p of A. It is clear

$$\ell(A/\underline{u}(\underline{n})A) = \ell(R/\underline{x}(\underline{n})R) + \ell(M/\underline{x}(\underline{n})M)$$

for all $n_1,\dots,n_r \geq 1$ where $\underline{u}(\underline{n}) = u_1^{n_1},\dots,u_r^{n_r}$, $\underline{x}(\underline{n}) = x_1^{n_1},\dots,x_r^{n_r}$ and

$$e(u_1,\dots,u_r;A) = e(x_1,\dots,x_r;R) + e(x_1,\dots,x_r;M),$$

and

$$e(u_1,\dots,u_{r-1};A_{r-1}) = \begin{cases} e(x_1,\dots,x_{r-1};M_{r-1}), & \text{if } (a) \text{ is satisfied,} \\ e(x_1,\dots,x_{r-1};R_{r-1}), & \text{if } (b) \text{ is satisfied,} \\ e(x_1,\dots,x_{r-1};R_{r-1}\times M_{r-1}), & \text{if } (c) \text{ is satisfied.} \end{cases}$$

According to the assumption (iv), we have

$$\ell(A/\underline{u}(\underline{n})A) = n_1\dots n_r e(\underline{u};A) + n_1\dots n_{r-1} e(u_1,\dots,u_{r-1};A_{r-1})$$

for all $n_1,\dots,n_r \geq 1$, where

$$A_{r-1} = \begin{cases} 0\times M_{r-1}, & \text{if the condition } (a) \text{ is satisfied,} \\ R_{r-1}\times 0, & \text{if the condition } (b) \text{ is satisfied,} \\ R_{r-1}\times M_{r-1}, & \text{if the condition } (c) \text{ is satisfied.} \end{cases}$$

• Suppose that one of the conditions (d) or (e) is satisfied. Since $x_1,\dots,x_r$ is a good s.o.p of R, $x_1,\dots,x_{r-1}$ is a good s.o.p of M and $x_r \in \mathrm{Ann}_R(M)$, we get by Proposition 2.3(ii) that $\underline{u} = (x_1,0),\dots,(x_r,0)$ is a good s.o.p of A. Since $x_r \in \mathrm{Ann}_R(M)$, we get

$$\ell(A/\underline{u}(\underline{n})A) = \ell(R/\underline{x}(\underline{n})R) + \ell(M/\underline{y}(\underline{m})M)$$

for all $n_1,\dots,n_r \geq 1$, where $\underline{u}(\underline{n}) = u_1^{n_1},\dots,u_r^{n_r}$, $\underline{x}(\underline{n}) = x_1^{n_1},\dots,x_r^{n_r}$, $\underline{y}(\underline{m}) = x_1^{n_1},\dots,x_{r-1}^{n_{r-1}}$. Moreover, we have $e(u_1,\dots,u_r;A) = e(x_1,\dots,x_r;R)$ and

$$e(u_1,\dots,x_{r-1};A_{r-1}) = \begin{cases} e(x_1,\dots,x_{r-1};M), & \text{if } (d) \text{ is satisfied,} \\ e(x_1,\dots,x_{r-1};R_{r-1}\times M), & \text{if } (e) \text{ is satisfied.} \end{cases}$$

Combining with the assumption (iv), we have

$$\ell(A/\underline{u}(\underline{n})A) = n_1\dots n_r e(\underline{u};A) + n_1\dots n_{r-1} e(u_1,\dots,u_{r-1};A_{r-1})$$

where

$$A_{r-1} = \begin{cases} 0 \times M, & \text{if the condition } (d) \text{ is satisfied,} \\ R_{r-1} \times M, & \text{if the condition } (e) \text{ is satisfied.} \end{cases}$$

□

Now we are ready to present the proof of Theorem 1.3.

Proof of Theorem 1.3. We divide into two cases.
• The case $d = r$. Following Theorem 3.4(iv), A is approximately Cohen-Macaulay which is not Cohen-Macaulay if and only if one of the three conditions $(a),(b),(c)$ of Theorem 3.4 is satisfied. By [8, Proposition 4.5], the statements follow.
• The case $d < r$. Following Theorem 3.4(iv), A is approximately Cohen-Macaulay which is not Cohen-Macaulay if and only if one of the two conditions $(d),(e)$ of Theorem 3.4 is satisfied. By [8, Proposition 4.5], the statements follow. □

Theorem 1.3 immediately gives us the following interesting corollary.

Corollary 3.5. *Suppose that R is a Cohen-Macaulay local ring of dimension $r \geq 2$. Let I be an ideal of R. We put $d = \dim_R(I)$. Then the following assertions are equivalent.*

(i) *$R \ltimes I$ is a Cohen-Macaulay local ring.*
(ii) *$R \ltimes I$ is an approximately Cohen-Macaulay local ring.*
(iii) *Either $I = (0)$ or I is a maximal Cohen-Macaulay R-module.*

Proof. $(i) \Rightarrow (ii)$ is trivial.
$(ii) \Rightarrow (iii)$. We consider the following two cases.
• $R \ltimes I$ is Cohen-Macaulay. If $d = r$, then by [22, Corollary 1(i)], I is a maximal Cohen-Macaulay R-module. If $d < r$, then by [22, Corollary 1(ii)], we have $I = (0)$.
• $R \ltimes I$ is approximately Cohen-Macaulay but not Cohen-Macaulay. If $r = d$, then following Theorem 1.3, I is approximately Cohen-Macaulay but not Cohen-Macaulay. Since R is Cohen-Macaulay, the largest submodule of R of dimension less than r equals (0). So, the largest submodule of I of dimension less than r equals (0), a contradiction. If $d < r$, then following Theorem 1.3, I is Cohen-Macaulay of dimension $d = r - 1 > 0$. Since I is a submodule of R of dimension less than r and R is Cohen-Macaulay, we get $I = (0)$, a contradiction.
$(iii) \Rightarrow (i)$ is trivial by [22, Corollary 1(ii)]. □

The proof of Corollary 3.5 immediately gives us the following corollary.

Corollary 3.6. *Suppose that R is a Cohen-Macaulay local ring of dimension $r \geq 2$. Then there does not exist an ideal I of R such that $R \ltimes I$ is an approximately Cohen-Macaulay local ring which is not Cohen-Macaulay.*

We end this paper with the following example.

Example 3.7. Let $S = k[[X,Y,Z,T]]$ be the formal power series ring over a field k and $\mathfrak{a} = (X) \cap (Y,Z)$. Let $R = S/\mathfrak{a}$. Then $\dim(R) = 3$. Denote by x,y,z,t are the image of X,Y,Z,T in R, respectively.

Then the largest submodule of R of dimension less than 3 is $I = (x)S \cong (X,Y,Z)/(Y,Z)$. Hence I is Cohen-Macaulay of dimension $\dim_R(I) = 2$. Let $x_1 = t+z, x_2 = x+y, x_3 = t(y+z)$. Then we have

$$\ell\left(R/(x_1^{n_1}, x_2^{n_2}, x_3^{n_3})\right) = 2n_1n_2n_3 + n_1n_2,$$

for all $n_1,n_2,n_3 \geq 1$. Hence x_1,x_2,x_3 is an almost p-standard s.o.p of R. By [3, Theorem 4.7] we have

$$\ell\left(R/(x_1^{n_1}, x_2^{n_2}, x_3^{n_3})\right) = 2n_1n_2n_3 + e(x_1,x_2;I)n_1n_2,$$

for all integers $n_1,n_2,n_3 \geq 1$. Therefore, R is an approximately Cohen-Macaulay local ring. By Theorem 1.3, $R \ltimes I$ is an approximately Cohen-Macaulay local ring which is not Cohen-Macaulay.

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