Geometric Properties and Iterates of Hesse Derivatives of Cubic Curves $\overline{\mathcal{L}}$. The potator and two divides of Hospe Derivatives of Cubic Curves \mathbf{P} is a set of \mathbf{P} is the parameter of \mathbf{P} Geometric Properties and Iterates o Seometric 1 roperties and relates o id Itopates of H_{QSG} operties and Iterates of Hesse \mathcal{L}_{P} \mathcal{L}_{P} and \mathcal{L}_{P} is discussed by the model of \mathcal{L}_{P} \overline{a} and Iterates of Hesse and Iterates of Hesse \liminf \limsup \overline{a} and \overline{b} α \bigcup urves

Sayan Dutta \mathcal{C}_{2} \mathcal{C}_{2} \mathcal{D}_{3} \mathcal{D}_{4} \mathcal{D}_{5} \mathcal{D}_{6} \mathcal{D}_{7} \mathcal{D}_{8} \mathcal{D}_{9} \mathcal{D}_{7} \mathcal{D}_{8} \mathcal{D}_{9} \mathcal{D}_{9} \mathcal{D}_{9} \mathcal{D}_{9} \mathcal{D}_{9} \mathcal{D}_{9} \mathcal{D}_{9} \mathcal{D}_{9} \mathcal{D}_{9} ω ₂ ω ₂ ω ₂ ω ₂ ω ₂ ω ₂ $S_{avan} Dutta$ $\mathbf{H}_{\mathbf{B}}$ Dutta

Department of Mathematics and Statistics, IISER Kolkata, Mohanpur, West Bengal 741246, India sd19ms148@iiserkol.ac.in Department of Mathematics and Statistics, IISER Kolkata, Mohanpur, West Bengal ${\rm and\ Statistics,\,HSER\ Kolkata,\,Mohanpur,\, West\,Bengal\ 741246,\, India}$ b^{a} and b^{a} in c in b^{a} is c^{a} in d^{a} is a set of i.e. by c is a set of i.e. by d^{a} is a set of i.e. by d^{a} is a set of i.e. by $\text{d}^{\text{a$ \mathbf{a} \mathbf{b} \mathbf{a} \mathbf{b} \mathbf{c} \mathbf{c} \mathbf{d} \mathbf{c} \mathbf{d} \mathbf{c} \mathbf{d} \mathbf{c} \mathbf{c} \mathbf{c} \mathbf{d} \mathbf{c} \mathbf{d} \mathbf{d} \mathbf{c} \mathbf{d} \mathbf{c} \mathbf{d} \mathbf{d} \mathbf{d} $\mathbf{$ H^2 i) \rightarrow H^2 i) \rightarrow H^2 in P^2 is a $2B$ d $2B$ d $2B$ h.Bb b2i B is \rightarrow B and B ; if B ; if B ; if B Kolkata, Mohanpur, West Bengal 741246, India Serkol.ac.in $\mathbf{1}$ is a set of the parameter $\mathbf{2}$ is a set of the parameter $\mathbf{2}$

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Department of Mathematics, ETH Zentrum, Rämistrasse 101, 8092 Zürich, Switzerland norbert.hungerbuehler@math.ethz.ch $\sum_{i=1}^{n}$ \mathbb{R} ratio for the BM is the BM in BM in BM i. $Depa$ ement of Machematics, $B1H$ Zentrum, reamstrasse for, $602Z$ Zurich, Switzerfa \mathbf{p}_{1}^{n+1} rad \mathbf{p}_{2}^{n+1} rad \mathbf{p}_{2}^{n+1} mematics, ETH Zentrum, Ramistrasse 101, 0022 Zurich, Switzerland rbuenier@matn.etnz.cn $\texttt{P}\texttt{B}\texttt{R}$. The $\texttt{P}\texttt{B}\texttt{R}$ ratio $\texttt{P}\texttt{R}$ is a $2\text{H}\texttt{R}$ that is a set of $\texttt{P}\texttt{R}$ is a R_{2} \sim R_{2} \sim R_{2} \sim R_{3} \sim R_{4} \sim R_{4} Department of Mathematics, $E[H]$ Zentrum, Kamistrasse 101, 8092 Zurich, Switzerland $\overline{\text{NOT}}$ bett $\overline{\text{H}}$ in $\overline{\text{H}}$

key-words: Elliptic curves, Hessian curves, geometry of cubic curves, halving formulae, discrete dynamical system Solution Fig. 2. Support the 2 a 2 of the Tag and the U.S. participal in the anti- $\mathbf{H} = \mathbf{H} \times \mathbf{H}$ k_{ℓ^2} en-words: Ellintic curves Hessian curves geometry of cubic curves halying formulae discrete dynamical system \mathcal{L} , for a set of \mathcal{L} is \mathcal{L} is \mathcal{L} . Δ \mathbf{b} ${level.}$ worde: Elliptic curves. Hessian curves, geometry of cubic curves, halying formulae, discrete dy \mathcal{L} and \mathcal{L} reads for \mathcal{L}

 2020 Mathematics Subject Classification: $11G05$ 37N99 \overline{C} ² \overline{C} $\$ 2020 Mathematics Subject Classification: 11G05 37N99 $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$

Abstract Δ bstrac \emph{tract} Λ batnoat $\frac{1}{2}$ Λ betract $\overline{2}$ $\overline{$ \overline{M}

The Hesse curve or Hesse derivative $\overline{\mathfrak{S}}\Gamma_f$ of a cubic curve Γ_f given by a homogeneous matrix of f evaluated at P. Then $\overline{\xi}I_f$ is again a cubic curve. We show that for any point of ℓ_1^P and ℓ_2^P is the degenerate polar conic of Γ_f at P. Furthermore, the operator $\overline{\zeta}$ defines points of this system, investigate orbits that end in the fixed points, and discuss the closed orbits of the dynamical system. polynomial f is the set of points P such that $\det(H_f(P)) = 0$, where $H_f(P)$ is the Hesse $P \in \overline{\mathfrak{D}}_f$, all contact points of tangents from P to both curves Γ_f and $\overline{\mathfrak{D}}_f$ are intersection
points of two straight lines ℓ^P and ℓ^P (mooting on $\overline{\mathfrak{D}}_f$) with Γ_f and $\overline{\mathfrak{D}}_f$, w an iterative discrete dynamical system on the set of cubic curves. We identify the two fixed points of two straight lines ℓ_1^P and ℓ_2^P (meeting on $\overline{\mathfrak{D}}_f$) with Γ_f and $\overline{\mathfrak{D}}_f$, where the product $\overline{\mathfrak{D}}$ _{*F_f*}, all contact points of tangents from *P* to both curves Γ_f and $\overline{\mathfrak{D}}$ _{*F_f*} are intersection is of two straight lines ℓ_1^P and ℓ_2^P (meeting on $\overline{\mathbf{S}}\Gamma_f$) with Γ_f and $\overline{\mathbf{S}}\Gamma_f$, where the product The Hesse curve or Hesse derivative ${}^2\Gamma_f$ of a cubic curve Γ_f given by a homogeneous $P \in \mathcal{R}$, all contact points of tangents from P to both curves Γ_f and \mathcal{R} are intersection points of two straight lines ℓ_1^P and ℓ_2^P (meeting on $\overline{\xi} \Gamma_f$) with Γ_f and $\overline{\xi} \Gamma_f$, where the product points of this system, investigate orbits that end in the fixed points, and discuss the closed points or this system, investigate or shis that end in the fire

1 Introduction \bf{I} introduction

.2`BpiBp2 Q7 b i?2 +m#B+ +m`p2^R

We will work with cubic curves in the real projective plane RP². Points $X = (x_1, x_2, x_3)^T \in \mathbb{R}^3 \setminus \{0\}$ will be denoted by capital letters, the components with small letters, and the equivalence class by (W_1, \ldots, W_n) $[X] := {\lambda X \mid \lambda \in \mathbb{R} \setminus \{0\}}$. However, since we mostly work with representatives, we often omit the square brackets in the notation. We will work with cubic curves in the real projective plane \mathbb{RP}^2 . Points $X = (x_1, x_2, x_3)^T \in \mathbb{R}^3 \setminus \{0\}$

Let f be a homogeneous polynomial in the variables x_1, x_2, x_3 of degree 3. Then f defines the projective cubic curve $(5\pi)^{-2}$ Let f be a homogeneous polynomial in the variables x_1, x_2, x_3 of degree 3. Then f defines the $\text{Dec}[f]$ be a nonogeneous polynomial in the variables x_1, x_2, x_3 of

$$
\Gamma_f := \left\{ [X] \in \mathbb{RP}^2 \mid f(X) = 0 \right\}.
$$

The Hesse matrix of f is the symmetric 3×3 matrix $H_f = \left(\frac{\partial^2 f}{\partial x_i \partial y_j} + \frac{\partial^2 f}{\partial y_j \partial z_j}\right)$ $\partial x_i\partial x_j$ The Hesse matrix of f is the symmetric 3×3 matrix $H_f = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)$.

Observe that $\det(H_f)$ is again a homogeneous cubic polynomial. Therefore, we can define the Hesse derivative of Γ_f , denoted Σ_f^{-1} Σ_f^{-1} Σ_f^{-1} , as the cubic curve Observe that $\det(H_f)$ is again a homogeneous cubic polynomial. Therefore, we can define the

$$
\overline{\mathfrak{D}}\Gamma_f := \Gamma_{\det(H_f)} = \left\{ [X] \in \mathbb{RP}^2 \mid \det(H_f(X)) = 0 \right\}.
$$

symbols beyond the scope of English, Greek and Latin, we have been left with no choice but to embark on this $\frac{1}{\text{In an extraordinary twist of mathematical fate, we find ourselves compelled to introduce the Bengal alphabet}}$ $\bar{\mathfrak{R}}$ (pronounced "Haw") as a notation in this paper to denote the Hesse derivative. In a desperate search for suitable linguistic expedition. Fun Fact: "Hesse" in Bengali means "to laugh"!

The polar conic of Γ_f with respect to the pole P is given by the equation with respect to the pole P is given by the equation

$$
\mathcal{C}_f(P): \ \langle X, H_f(P)X \rangle = 0. \tag{1}
$$

Here $H_f(P)X$ denotes the product of the Hesse matrix of f evaluated at P with the vector X, First $H_f(T)X$ denotes the Product of the Hesse matrix of f evaluated at T with the vector X , and $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product. An equivalent formulation is given by $\frac{1}{\sqrt{2}}$ $\frac{1}{\sqrt{2}}$ $\frac{1}{\sqrt{2}}$ $\frac{1}{\sqrt{2}}$ $\frac{1}{\sqrt{2}}$ $\frac{1}{\sqrt{2}}$ $\frac{1}{\sqrt{2}}$ \mathcal{L} and product of international matrix of f and \mathcal{L} the matrix of \mathcal{L} means \mathcal{L} and \mathcal{L} and \mathcal{L} is a control to the vector \mathcal{L} , \mathbf{F} behaved respectively represented by \mathbf{F} and \mathbf{F} $\lim_{\Delta} \frac{\partial f}{\partial x}$ contained at I with the vector Δt , $\frac{1}{2}$ by

$$
\mathcal{C}_f(P): \langle \nabla f(X), P \rangle = 0,\tag{2}
$$

where ∇ denotes the gradient operator. Indeed, a simple but somewhat tedious calculation shows that $\langle X, H_f(P)X \rangle = 2\langle \nabla f(X), P \rangle$ (see [\[4](#page-17-0)] for details and a general formula of this type for homogeneous polynomials of any degree). It is clear from [\(2\)](#page-1-0) that the contact points of the tangents geneous polynomials of any degree). It is clear from (2) that the contact points of from P to Γ_f are precisely the intersection points of $C_f(P)$ with Γ_f (see Figure [1\)](#page-1-1). \mathcal{L} and \mathcal{L} and \mathcal{L} by \mathcal{L} by \mathcal{L} by \mathcal{L} \mathcal{L} S`QT2`iB2b Q7 >2bb2 /2`BpiBp2b Q7 +m#B+ +m`p2b

If there is no danger of confusion, we will omit the index and briefly write Γ instead of Γ_f . Moreover, we will use the notation H_{Γ} instead of H_f , and $C_{\Gamma}(P)$ instead of $C_f(P)$ if the polynomial f is determined by the context or if a general but unique polynomial is meant. We would like to mention that the Hesse derivative $\overline{\mathfrak{L}}\Gamma$ is also known as *Hessian curve*, denoted Hess(Γ) (see, e.g.,[5, whose iterations we want to study. Whenever convenient, we will use x, y, z instead of x_1, x_2, x_3 $\S 4.12$, p. 111]). However, we prefer the notation $\S \Gamma$ because we want to interpret \S as an operator for the coordinates. The figures below of the various projective curves show images of the curves in the affine plane $x_3 = 1$ embedded in \mathbb{RP}^2 . \mathcal{L}_{1} is a discussed for fair \mathcal{L}_{2} . The filter \mathcal{L}_{3} is decomposited as \mathcal{L}_{3} in the \mathcal{L}_{4} If there is no danger of confusion, we will omit the index and briefly write Γ instead of Γ_f . Moreover, \lim_{ω} \lim_{ω} \lim_{ω} \lim_{ω} \lim_{ω} \lim_{ω} \lim_{ω} $,$

Figure 1: A cubic curve Γ_f , its polar conic $\mathcal{C}_f(P)$ with respect to the pole P, and the tangents from P to the curve Γ_f . Figure 1: A cubic curve Γ_f , its polar conic $C_f(P)$ with respect to the pole P, $\frac{1}{\sqrt{2}}$ ratio $\frac{1}{\sqrt{2}}$ ratio $\frac{1}{\sqrt{2}}$ is the set of th $\frac{1}{2}$ radial r

It is well known that the polar conic is the product of two projective lines ℓ_1^P and ℓ_2^P , if and only
if the determinant of the Hegge matrix evaluated at B is equal to 0 i.e. if the determinant of the Hesse matrix evaluated at P is equal to 0, i.e., It is well kn $\mathbf d$ only $\begin{bmatrix} \text{to } 0, \text{ i.e.,} \end{bmatrix}$

$$
\mathcal{C}_{\Gamma}(P) = \langle X, \ell_1^P \rangle \langle X, \ell_2^P \rangle \iff \det \left(H_{\Gamma}(P) \right) = 0.
$$

In particular, we obtain the following well-known result (see Figure 2):

Fact 1. For any point $P \in \overline{R} \Gamma$, the polar conic is the product of $\frac{2}{3}$ To $\frac{2}{3}$ is the $\frac{2}{3}$ respectively. The processing as the position $\frac{2}{3}$ in the $\frac{2}{3}$. Γ , the polar conic is the product of the two lines ℓ_1^P and ℓ_2^P , and the $tangents from P$ to Γ touch Γ precisely at the points $\Gamma \cap \ell_1^P$ and $\Gamma \cap \ell_2^P$. *and* **2** + + σ ₂. \mathbf{P} below \mathbf{P} by \mathbf{P} by \mathbf{P} by \mathbf{P} and \mathbf{P} P_{max} P_{max} by the two times ℓ_1 and ℓ_2 , and the

 $\mathcal{C}_e(P)$ and that the intersection point of the two lines of the conic $\mathcal{C}_e(P)$ lies on the curve $\mathfrak{F} \Gamma$ (see In the next section we compute the contact points of the tangents from a point $P \in \overline{P} \Gamma$ to the curve Σ . In Section [3](#page-3-0) we show that these contact points lie also on the degenerate polar conic $\mathcal{C}_f(P)$ and that the intersection point of the two lines of the conic $\mathcal{C}_f(P)$ lies on the curve $\overline{\mathfrak{D}}$ (see gents from a point $P \in \mathcal{I}$ to the

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on the Hesse derivative $\overline{\mathfrak{S}}\Gamma_f$, and the tangents from P to the curve Γ_f . Figure 2: A cubic curve Γ_f , its degenerate polar conic $\mathcal{C}_f(P)$ with respect to the pole P

Figure 3). In Section 4 and 5 we [i](#page-7-0)nvestigate iterated Hesse derivatives of cubic curves in Hesse in Hesse form can be transformed into curves in other normal forms. form, and then compute in Section [6](#page-10-0) the number of chains and loops of given length of iterated Hesse derivatives. Finally, in Section 7 it is shown how the results from Section 6 for cubic curves

on the Hesse derivative $\overline{\mathfrak{L}}\Gamma_f$, and the tangents from P to the curves Γ_f and $\overline{\mathfrak{L}}\Gamma_f$. Figure 3: A cubic curve Γ_f , its degenerate polar conic $\mathcal{C}_f(P)$ with respect to the pole P

2 Halving formulae for points on Σ হ Γ \mathbf{p} হ Γ P#b2`p2 i?i /2i Bb ;BM ?QKQ;2M2Qmb +m#B+ TQHvMQKBHX h?2`27Q`2 r2 +M /2}M2 i?2 *>2bb2* $k = 1$ Δ for points on $\overline{2}$ Γ $\sum_{\alpha=1}^{\infty}$ points on α

In this section, we compute the contact points of the tangents from a point $P \in \overline{\mathcal{F}}\Gamma$ to the curve $\overline{\mathfrak{D}}\Gamma$. By a suitable projective transformation, we may assume that the curve is of the form $E_{a,b}$
defined by defined by

$$
E_{a,b}: y^2 = x^3 + a x^2 + b x \tag{3}
$$

in the affine plane, where $a, b \in \mathbb{R}$.

Proposition 2. Let $E_{a,b}$ be a non-singular elliptic curve over $\mathbb C$ defined by

$$
E_{a,b}: y^2 = x^3 + a x^2 + b x
$$

where a, $b \in \mathbb{C}$, and let $P = (x_0, y_0)$ be a point on $E_{a,b}$. $\frac{1}{2}$ where $a, b \in \mathbb{C}$, and let $P = (x_0, y_0)$ be a point on $E_{a,b}$. $\begin{pmatrix} w_0, y_0 \end{pmatrix}$ be a point Ω a point on F

$$
Let
$$

$$
e_1 = \frac{-a + \sqrt{a^2 - 4b}}{2}, \qquad e_2 = \frac{-a - \sqrt{a^2 - 4b}}{2},
$$

and let

and let
$$
\gamma = \sqrt{x_0}, \qquad \alpha = \sqrt{x_0 - e_1}, \qquad \beta = \sqrt{x_0 - e_2}.
$$

Then, $E_{a,b}$ is of the form

Then, $E_{a,b}$ is of the form

$$
y^2 = x(x - e_1)(x - e_2)
$$

and the x-coordinates of the contact points of the tangents of P with $E_{a,b}$ are

$$
x_{11} = (\alpha + \gamma)(\beta + \gamma),
$$

\n
$$
x_{12} = (\alpha - \gamma)(\beta - \gamma),
$$

\n
$$
x_{21} = (\alpha + \gamma)(-\beta + \gamma),
$$

\n
$$
x_{22} = (\alpha - \gamma)(-\beta - \gamma).
$$

Proof. Since $-e_1 - e_2 = a$ and $e_1e_2 = b$, the curve $E_{a,b}$ is of the form $y^2 = x(x - e_1)(x - e_2)$. In T_{100} . Since $-e_1 - e_2 = u$ and $e_1e_2 = v$, the curve $E_{a,b}$ is of the form $y = x(x - e_1)(x - e_2)$. In
order to show that $x_{11}, x_{12}, x_{21}, x_{22}$ are the x-coordinates of the contact points of the tangents of P with $E_{a,b}$, it is enough to show that the x-coordinate of the points $2 * Q_{ij}$, where $i, j \in \{1, 2\}$, $Q_{ij} := (x_{ij}, y_{ij})$ and $y_{ij}^2 = x_{ij}^3 + a x_{ij}^2 + b x_{ij}$, is equal to x_0 . In other words, for the four points Q_{ij} we have $2 * Q_i = -P$, i.e., $\frac{P}{2} = -Q_{ij}$. Here $2 * Q_{ij} = Q_{ij} + Q_{ij}$ is the usual elliptic curve operation on $E_{a,b}$ (see, e.g., [\[3](#page-17-2)]). Now, the *x*-coordinate x_{2ij} of the point $2 * Q_{ij}$ is given by the formula $P_{\text{test}}(S_{\text{test}})$ is a small in the sum $F_{\text{test}}(S_{\text{test}})$ of $\mathcal{L}_{\text{test}}(S_{\text{test}})$. $\frac{1}{2}$ To $\frac{1}{2}$ is the contracted in $\frac{1}{2}$ to $\frac{1}{2}$ to $\frac{1}{2}$ in $\frac{1}{2}$

$$
x_{2ij} = \frac{x_{ij}^4 - 2b x_{ij}^2 + b^2}{4(x_{ij}^3 + a x_{ij}^2 + b x_{ij})} = \frac{(x_{ij}^2 - b)^2}{4x_{ij}(x_{ij}^2 + a x_{ij} + b)}
$$

Furthermore, we have $a = \alpha^2 + \beta^2 - 2\gamma^2$ and $b = (\alpha^2 - \gamma^2)(\beta^2 - \gamma^2)$, and if we write x_{ij}, a, b in terms of γ, α, β , it is not hard to verify that α being that α X >Qr2p2`- bBM+2 r2 KQbiHv rQ`F rBi? `2T`2b2MiiBp2b- r2 Q7i2M QKBi i?2 $\frac{1}{2}$ $\frac{1}{2}$

$$
(x_{ij}^2 - b)^2 = 4x_{ij}\gamma^2(x_{ij}^2 + a x_{ij} + b),
$$

which shows that $x_{2ij} = x_0$. $q.e.d.$ P ws that $x_{2ij} = x_0$. $q.e.d.$

 Φ is a set of $q.e.d.$

.

3 Intersection of $C_{\Gamma}(P)$ with $\Sigma \Gamma$ for $P \in \Sigma \Gamma$ হ Γ P#b2`p2 i?i /2i Bb ;BM ?QKQ;2M2Qmb +m#B+ TQHvMQKBHX h?2`27Q`2 r2 +M /2}M2 i?2 *>2bb2*

In this section, we combine the results from Section 2 with the property that for every point $P \in \mathcal{F} \Gamma$, the polar conic of Γ with respect to the pole P is the product of two lines, ℓ_1^P and ℓ_2^P (see

 $\overline{22 \cdot 24}$.2.BpiBp2 $\overline{2}$ bppe GiBM- r2 ?p2 #22M H27i rBi? MQ +?QB+2 #mi iQ 2K#`F QM i?Bb HBM;mBbiB+ 2tT2/BiBQMX amiel (2000-01) by "2MAMb dziel" by Theory $G =$ Submitted to Bocky Mountain J Math 6mM 6+i, dz>2bb2Ǵ BM "2M;HB K2Mb dziQ Hm;?Ǵ5 AM M 2tit Δ and Δ and Δ and Δ is all Δ is all Δ is and Δ is and Δ is and Δ is and Δ MQiiBQM BM i?Bb TT2` iQ `2T`2b2Mi >2bb2 .2`BpiBp2X AM /2bT2`i2 b2`+? 7Q` bmBi#H2 bvK#QHb #2vQM/ i?2 `2HKb Q7 1M;HBb?- :`22F M/ **7 Aug 2024 06:22:24 PDT** $240328 -$ Hungarbuhlar Varsion 3 - Subt 240328-Hungerbuhler Version 3 - Submitted to Rocky Mountain J. Math.

Fact [1\)](#page-1-2). In particular, we will show that the intersection of the two lines ℓ_1^P and ℓ_2^P lies on the curve *.2`BpiBp2* Q7 b i?2 +m#B+ +m`p2^R X ₹Γ and that the points x_{ij} , $i, j = 1, 2$ correspond to the other intersection points of ℓ_1^P and ℓ_2^P with $\overline{\mathfrak{D}}$ (see Figure 4). P#b2`p2 i?i /2i Bb ;BM ?QKQ;2M2Qmb +m#B+ TQHvMQKBHX h?2`27Q`2 r2 +M /2}M2 i?2 *>2bb2* Fact 1). In particular, we will show that the intersection of the two lines ℓ_1^P and ℓ_2^P lies on the Γ particular, we will show that the intersection of the two fines ϵ_1 and ϵ_2 hes of ind that the points x_{ij} , $i,j = 1,2$ correspond to the other intersection points of ℓ_1^* and \overline{P}_1 bim/v \overline{P}_2 in \overline{P}_2 is the \overline{P}_2 of \overline{P}_2 and \overline{P}_2 at \overline{P}_2 $\overline{$ Fact 1). In particular, we will show that the intersection of the two lines ℓ_1 and ℓ_2 lie curve $\mathcal{L}I$ and that the points x_{ij} , $i, j = 1, 2$ correspond to the other intersection points of ℓ_2^P with $\overline{\mathfrak{D}}$ (see Figure 4). iq is 2 μ above μ above μ is 20 i.2 Tqhsmid μ is 2M μ is 2M μ is 2M μ is 2M μ σ rticular that the points x_{ij} , $i, j = 1, 2$ correspond to the other intersection points of t_1 and

For this, we consider the projective cubic curve For this, we consider the projective cubic curve

$$
\Gamma_{a,b}: ax^3 + 3xy^2 + 3bx^2z - b^2z^3 = 0
$$

iq i \mathcal{L} b2² b2² b2² t \mathcal{L} and \mathcal{L} is \mathcal{L} and \mathcal{L} is \mathcal{L}

with $a, b \in \mathbb{R}, b \neq 0$. We first show that every regular cubic curve Γ can be transformed to the curve $\Gamma_{a,b}$. By a suitable projective transformation, we may assume that Γ F_{α} \cdots $\alpha^{2}=\alpha^{3}+\Lambda \alpha^{2}+P_{\alpha}$. Then the projective transformation given by the mat $\mathcal{L}_{A,B} \cdot g = x + \mathcal{L}_{A,B} + \mathcal{L}_{A,B}$. Then, the nai
Sa $E_{A,B}:y^2=x^3+Ax^2+Bx.$ Then, the projective transformation given by the matrix $M_{\rm H}$ is a time 2 time μ and μ because μ by μ and μ is a value μ is a time μ is a time μ the curve $\Gamma_{a,b}$. By a suitable projective transformation, we may assume that Γ is of the form $E_{A,B}: y^2 = x^3 + Ax^2 + Bx$. Then, the projective transformation given by the matrix arve $\Gamma_{a,b}$. By a suitable projective transformation, we may assume that $E_{A,B}: y^2 = x^3 + Ax^2 + Bx$. Then, the projective transformation given by the matrix $B = x^3 + A x^2 + B x$. **T**

$$
\begin{pmatrix}\nA & 0 & -(A^2 - 3B)^2 \\
0 & 1 & 0 \\
-3 & 0 & 0\n\end{pmatrix}
$$

transforms the curve $E_{A,B}$ to the curve $\Gamma_{a,b}$ with $a = -2A^3 + 9AB$ and $b = (A^2 - 3B)^3$. transforms the curve E_{AB} to the curve Γ_{AB} with $a = -2A^3 + 9AB$ and $b = (A^2 - 3B)^3$.

 $\Gamma_{2,k}$ we get

Now, for
$$
\Gamma_{a,b}
$$
, we get
\n
$$
H_{\Gamma_{a,b}} := \begin{pmatrix} 6ax + 6bz & 6y & 6bx \\ 6y & 6x & 0 \\ 6bx & 0 & -6b^2z \end{pmatrix}
$$

and hence \mathbf{p} and \mathbf{b} hence

$$
\mathfrak{D}_{a,b}: \; y^2z=x^3+ax^2z+bxz^2.
$$

In other words, $\overline{\mathfrak{D}}_{a,b} = E_{a,b}$, as introduced in [\(3\)](#page-3-2) in the previous section.

 $\mathcal{C}_{\mathbf{p}}$ By definition, if $P = (x_0, y_0, z_0) \in \overline{\mathfrak{D}}_{a,b}$, then det $(H_{\Gamma_{a,b}}(P)) = 0$, which implies that the conic section

$$
\mathcal{C}_{\Gamma_{a,b}}(P): (ax_0 + bz_0)x^2 + x_0y^2 + 2y_0xy + 2bx_0xz - b^2z_0z^2 = 0
$$

 \mathbb{R}^n iraq in terms of a and a defined in the previous section these two lines in terms of e_1 and e_2 defined in the previous section. $\cos \theta$ can be written as the product of two lines ℓ_1^P and ℓ_2^P . In the following lemma, we will compute these two lines in terms of e_1 and e_2 defined in the previous section

 P_1 . The 22M H27i rate P_2 **Lemma 3.** The lines ℓ_1^P and ℓ_2^P are given by **Lemma 3.** The lines ℓ_1^P and ℓ_2^P are given by

$$
\ell_1^P: ux + vy + wz = 0
$$

$$
\ell_2^P: rx + sy + tz = 0
$$

where

$$
u = -x_0 - \sqrt{(e_1 - x_0)(e_2 - x_0)}
$$

\n
$$
v = -\sqrt{x_0}
$$

\n
$$
w = e_1 e_2
$$

\n
$$
r = -x_0 + \sqrt{(e_1 - x_0)(e_2 - x_0)}
$$

\n
$$
s = \sqrt{x_0}
$$

\n
$$
t = e_1 e_2.
$$

Proof. If we replace y_0 by $\sqrt{x_0^3 + ax_0^2 + bx_0}$, and a, b by $-e_1 - e_2$ and e_1e_2 , respectively, then we have

$$
\mathcal{C}_{\Gamma_{a,b}}: (e_1e_2 - e_1x_0 - e_2x_0) x^2 + x_0 y^2 + 2\sqrt{x_0^3 - e_1x_0^2 - e_2x_0^2 + e_1e_2x_0} xy + 2e_1e_2x_0 x - e_1^2e_2^2.
$$

Now, in order to show that $\mathcal{C}_{\Gamma_{a,b}} = \ell_1^P \cdot \ell_2^P$, we just have to check that

$$
ur = -e_1e_2 + e_1x_0 + e_2x_0
$$

\n
$$
us + vr = -2\sqrt{x_0^3 - e_1x_0^2 - e_2x_0^2 + e_1e_2x_0}
$$

\n
$$
vs = -x_0^2
$$

\n
$$
ut + wr = -2e_1e_2x_0
$$

\n
$$
wt = e_1^2e_2^2
$$

\n
$$
vt + ws = 0
$$

which is easy to see. $q.e.d.$

Before we compute the intersection points of ℓ_1^P and ℓ_2^P with the curve $E_{a,b}$, we show that the two lines intersect on the curve $E_{a,b}$. *.2`BpiBp2* Q7 b i?2 +m#B+ +m`p2^R X P#b2`p2 i?i /2i Bb ;BM ?QKQ;2M2Qmb +m#B+ TQHvMQKBHX h?2`27Q`2 r2 +M /2}M2 i?2 *>2bb2* $\mathcal{L}_{a,b}$ $\mathcal{L}_{a,b}$ $\mathcal{L}_{a,b}$ $\mathcal{L}_{a,b}$ in $\mathcal{L}_{a,b}$ α ²

Lemma 4. Let $P = (x_0, y_0, z_0)$ be a point on $\overline{\mathfrak{D}}_{a,b}$. Then the point $\ell_1^P \cap \ell_2^P =: S = (x_S, y_S, z_S)$ $\overline{u}_a, \overline{v}_b$ $\overline{u}_a, \overline{v}_b$ lies on the same curve $\overline{\mathfrak{D}}_{a,b} = E_{a,b}$. **Lemma 4.** Let $P = (x_0, y_0, z_0)$ be a point on \overline{z} $\Gamma_{a,b}$. Then the point $\ell_1^P \cap \ell_2^P =: S = (x_S, y_S, y_S, z_0)$ \mathcal{E} on the same curve $\mathfrak{F} \Gamma$, \overline{E} **4.** Let $P=(x_0, y_0, z_0)$ be a point on $\overline{\mathfrak{L}}_{\alpha,b}$. Then the point $\ell_1^P \cap \ell_2^P =: S = (x_S, y_S, z_S)$ \mathcal{E} is a particle \mathcal{E} \mathbf{F} i. \mathcal{E} i.e. \mathcal{E}

Proof. First, let us assume $x_0 \neq 0$ and rewrite the lines as $\frac{1}{2}$ $\frac{1}{2}$. The $\frac{1}{2}$ for an assume $\frac{1}{2}$ / $\frac{1}{2}$ and for iteration in mass as First, lot us assume $x_2 \neq 0$ and rewrite the lines as $\frac{1}{2}$ H₂), to do dobtain w_0 / σ and 2 emitted the integration

$$
\ell_1^P: y = \frac{-u}{v}x + \frac{-w}{v} \n\ell_2^P: y = \frac{-r}{s}x + \frac{-t}{s}.
$$

 $S = (x_2, y_2, 1)$ of the two lines is given by $\mathbf{m} \in \left(\omega_{\mathcal{S}}, y_{\mathcal{S}}, \mathbf{1}\right)$ or the two filled in AM M 2tit Φ irak Φ ir B is a regular Φ indicated by Φ is a regular Φ indicated by Φ Then the intersection $S = (x_S, y_S, 1)$ of the two lines is given by

$$
x_S = \frac{tv - ws}{us - rv} = \frac{e_1 e_2}{x_0} = \frac{b}{x_0}
$$

$$
y_S = \frac{-u}{v} x_S + \frac{-w}{v} = -\frac{by_0}{x_0^2}.
$$

Now, we use the fact that references the β -table i.

$$
y_0^2 = x_0^3 + ax_0^2 + bx_0
$$

to show that

$$
\left(\frac{b}{x_0}\right)^3 + a\left(\frac{b}{x_0}\right)^2 + b\left(\frac{b}{x_0}\right) = \left(\frac{-by_0}{x_0^2}\right)^2
$$

which means that $S \in E_{a,b}$, as claimed.

On the other hand, if $x_0 = 0$, then $P = (0, 0, 1)$ or $P = (0, 1, 0)$. In the first case, we have C_{Γa,b} : $x^2 - bz^2 = (x + \sqrt{bz})(x - \sqrt{bz}) = 0$, and the two lines intersect in $S = (0, 1, 0)$ on $E_{a,b}$. In the second case, we have $\mathcal{C}_{\Gamma_{a,b}}$: $xy = 0$, and the two lines intersect in $S = (0,0,1)$ on $E_{a,b}$. On the other hand, if $x_0 = 0$, then $P = (0, 0, 1)$ or $P = (0, 1, 0)$. In the

q.e.d.

Lemma 5. The map $\overline{\mathfrak{D}}_{a,b} \to \overline{\mathfrak{D}}_{a,b}$, $P = (x_0, y_0, z_0) \mapsto S = (x_S, y_S, z_S)$, is an involution.

 \mathcal{L}^2 to indicate it is a function of indicate indicate indicate indicate indicate indicate indicate indicate in *Proof.* For $x_0 \neq 0$, we just have to check that

$$
\frac{b}{\frac{b}{x_0}} = x_0 \quad \text{and} \quad \frac{-b \cdot \frac{-by_0}{x_0^2}}{\frac{b^2}{x_0^2}} = y_0,
$$

which is easy to see. For the case $x_0 = 0$ see the end of the proof of Lemma [4.](#page-5-0) $q.e.d.$

Now, we show that the other intersection points of ℓ_1^P and ℓ_2^P with $E_{a,b}$ are exactly the points x_{ij} , $i, j = 1, 2$ from Proposition [2.](#page-3-3)

h?2 +2 bb2 KirBt Q7 Bb2 KirBt Q7 Bb2 KirBt Q7 Bb2 KirBt Q7 Bb2 KirBt XirBt XirBt XirBt XirBt XirBt XirBt XirBt

Lemma 6. Besides the point S, the intersection points of ℓ_1^P and ℓ_2^P with $E_{a,b}$ are exactly the points Q_i , $i = 1, 2, 3, 4$, with $2 \times Q_i = -P$. More precisely, the points x_{11} and x_{12} are on the line ℓ_1^P and the points x_{21} and x_{22} are on ℓ_2^P . 2, 3, 4, with $2 * Q_i = -P$. More precisely, the points x_{11} and x_{12} are on the line Ω are on ℓ_2^P .

Proof. To find the x-coordinate of the intersection points of ℓ_i^P with $E_{a,b}$ we eliminate y form the equations for $E_{a,b}$ and ℓ_1^P , and ℓ_2^P , respectively. The resulting equations are of degree 3 in x, but since we already know the root $\frac{b}{x_0}$, the problem reduces to a quadratic equation equations for $E_{a,b}$ and ℓ_1^P , and ℓ_2^P , respectively. The resulting equations are of degree 3 in x, but Example 2.2 The set of $\frac{b}{2}$, the problem reduces to a quadratic equation $\overline{\text{P}}$ is a resulting equations are of degree $\overline{3}$ in x, but From the root $\frac{b}{\epsilon}$, the problem reduces to a quadratic equation x_0 in a minimized by x_0 and x_0 arbitrary \mathbb{R} arbitrary.

$$
x^{2} - 2(x_{0} \pm \alpha \beta) x + e_{1}e_{2} = 0.
$$

The solutions are

$$
x_0 + \alpha \beta \pm \sqrt{(x_0 + \alpha \beta)^2 - e_1 e_2}
$$
 and $x_0 - \alpha \beta \pm \sqrt{(x_0 - \alpha \beta)^2 - e_1 e_2}$

and one checks easily that these expressions agree with the formulas for x_{ij} from Proposition 2. nd one checks easily that these expressions agree with the formulas for x_{ij} from Proposition 2. \mathcal{L}^2 T \mathcal{L}^2 and $2\mathcal{L}^2$ bi2^2 bi2^2 bi2^2 bi2^2 bi2^2bb2 /2bb2 is the formulas for x_{ii} from Proposition 2. \mathcal{L} and \mathcal{L} and \mathcal{L} bizon distributible models with \mathcal{L} is easily that these expressions agree with the formulas for x_{ij} from Proposition 2.

It remains to show that the y-coordinates match as well. To see that, let us denote by A and B the points at which the tangents from P to $E_{a,b}$ meet $E_{a,b}$. So, for our claim to be true, we have

$$
A + B = -S
$$

2 * A = -P
2 * B = -P

which implies

$$
2*P=2*S
$$

and hence it is enough to show that this is indeed the case. To do so, we note that a formula for doubling the point $P = (x_2, y_2)$ on F_{\perp} is given by doubling the point $P = (x_0, y_0)$ on $E_{a,b}$ is given by μ ₀, μ ₀) σ is σ ₁, σ ₂ σ is σ ₂. σ - σ ₂ σ \mathcal{B}

$$
2 * P = \left(\frac{\left(x_0^2 - e_1 e_2\right)^2}{4y_0^2}, \frac{\left(x_0^2 - e_1 e_2\right)\left(e_1 e_2 - 2e_1 x_0 + x_0^2\right)\left(e_1 e_2 - 2e_2 x_0 + x_0^2\right)}{8y_0^3}\right).
$$

On the other hand, since $S = \left(\frac{e_1 e_2}{x_0}, \frac{-e_1 e_2 y_0}{x_0^2}\right)$, we have $\mathcal{L}_{\mathcal{L}}$, and i.e. points the H2b d7 /2 points in the H2b i? On the other hand, since $S = \left(\frac{e_1e_2}{x_0}, \frac{e_1e_2g_0}{x_0^2}\right)$, we have \Box then been defined α θ θ = e_1e_2 \mathcal{L}

$$
2*S = \left(\frac{\left(\frac{e_1^2e_2^2}{x_0^2} - e_1e_2\right)^2}{4 \cdot \frac{e_1^2e_2^2y_0^2}{x_0^4}}, \frac{\left(\frac{e_1^2e_2^2}{x_0^2} - e_1e_2\right)\left(e_1e_2 - 2e_1 \cdot \frac{e_1e_2}{x_0} + \frac{e_1^2e_2^2}{x_0^2}\right)\left(e_1e_2 - 2e_2 \cdot \frac{e_1e_2}{x_0} + \frac{e_1^2e_2^2}{x_0^2}\right)}{-8 \cdot \frac{e_1^3e_2^3y_0^3}{x_0^6}}\right).
$$

These two expressions for $2 * P$ and $2 * S$ obviously coincide. $q.e.d.$ These two expressions for $2 * F$ and $2 * S$ obviously coincide. $q.e.d.$ γ ^p concrue.

Theorem 7. Let Γ be a cubic curve and let $P \in \mathcal{F}\Gamma$. Then, all the contact points of tangents from $\frac{d}{dx}$ is the intersection points by $\frac{d}{dx}$ and $\frac{d}{dx}$ is an intersection, in P to the curves Γ and $\Sigma \Gamma$ are intersection points of ℓ_1^P and ℓ_2^P with Γ and $\Sigma \Gamma$. In addition, the *.2`BpiBp2* Q7 b i?2 +m#B+ +m`p2^R intersection Q of ℓ_1^P and ℓ_2^P lies on $\overline{\mathfrak{R}}\Gamma$ (see Figure [4\)](#page-7-1). *.2`BpiBp2* Q7 b i?2 +m#B+ +m`p2^R $\mathcal{O}(\gamma)$. **n 7.** Let Γ be a cubic curve and let P

4 Hesse Form of Cubic Curves

 \mathcal{L} is the contract of the contract of the contract of the contract of the 2v \mathcal{L} He \mathcal{L} In this section, we consider a cubic curve in its Hesse form \mathbf{m} and becausity we consider a case can re \mathbf{m} to \mathbf{m} beso form In this section, we consider a cubic curve in its Hesse form $\mathbf e \text{ form}$ \mathcal{L}_{SDE} .2.Be2 .2.Be2

 $\mathcal{L}^{\mathcal{L}}$

$$
\Gamma_c: \ x^3 + y^3 + z^3 + c \, xyz = 0
$$

with $c \in \mathbb{R}$. Notice Γ_{-3} is a degenerate curve. Formally, we put

$$
\Gamma_{\infty}: xyz = 0.
$$

 Range (see [1]). The Hesse derivative $\Sigma \Gamma$ has the same symmetry. The curve $2\sqrt{3x^3} + 9(\sqrt{3}+1)(x^2+y^2)z - 6\sqrt{3xy^2 - 9z^3} = 0$ and has the property that Figure 4: Illustration for Theorem 7 for a cubic curve Γ with the symmetry group of an equilateral triangle (see [1]). The Hesse derivative $\Sigma \Gamma$ has the same symmetry. The cu $\frac{2}{\sqrt{N^2 + 2\pi N^2 + 2}}$ or $\frac{2}{\sqrt{N^2 + 2\pi N^2}}$ is $\frac{2}{\sqrt{N^2 + 2\pi N^2}}$ in $\frac{2}{\sqrt{N^2 + 2\pi N^2}}$ equilateral triangle (see [1]). The Hesse derivative $\Sigma \Gamma$ has the same symmetry. The curve Γ is given by $2\sqrt{3}x^3 + 9(\sqrt{3}+1)(x^2+y^2)z - 6\sqrt{3}xy^2 - 9z^3 = 0$ and has the property that Figure 4: Illustration for Theorem 7 for a cubic curve Γ with the symmetry group of an equilateral triangle (see [1]). The Hesse derivative 21 has the same symmetry. The curve
 Γ is given by $2\sqrt{3}x^3 + 9(\sqrt{3} + 1)(x^2 + y^2)z - 6\sqrt{3}xy^2 - 9z^3 = 0$ and has the property that $\overline{\mathbf{z}}^2 \Gamma = \Gamma$ (see Section [6\)](#page-10-0). $+$ m $+$ m $+$ m $+$ m $+$ $\frac{1}{2}$ illet are regarded (see [1]). The Hesse derivative $\overline{2}$ has the same symmetry. The curve $\frac{1}{\sqrt{2}}$ i.e. $\frac{1}{\sqrt{2}}$ $\frac{2}{\sqrt{2}}$ i.e. $\frac{1}{\sqrt{2}}$ i.e. $\frac{2}{\sqrt{2}}$ i.e. $\frac{2}{\sqrt{2}}$ b $\frac{2}{\sqrt{2}}$ b $\frac{3}{\sqrt{2}}$ b $\frac{3}{\sqrt{2}}$ b $\frac{1}{\sqrt{2}}$ i.e. $\frac{1}{\sqrt{2}}$ i.e. $\frac{1}{\sqrt{2}}$ b $\frac{1}{\sqrt{2}}$ b $\frac{1}{\sqrt{2$ S given by $2\nabla \partial x + \vartheta(\nabla \partial + 1)(x + y) = \nabla \partial \partial x$ $y = \vartheta z$ $\neq 0$ and has the property that

Lemma 8. Let $c_0 \neq 0$. Then the Hesse derivative of Γ_{c_0} is $\overline{\mathbb{R}} \Gamma_{c_0} = \Gamma_{c_1}$ where h?2 >2bb2 Ki`Bt Q7 Bb i?2 bvKK2i`B+ Ki`Bt X h?2 >2bb2 Ki`Bt Q7 Bb i?2 bvKK2i`B+ Ki`Bt X

$$
c_1 = -\frac{108 + c_0^3}{3c_0^2}.
$$

 $\overline{}$ if $\overline{}$ if $\overline{}$ is in the matrix of the ratio $\overline{}$ is in the $\overline{}$ The Hesse derivative of Γ_0 is $\Sigma_0 = \Gamma_\infty$, and the Hesse derivative of Γ_∞ is $\Sigma_0 =$ The Hesse derivative of Γ_0 is $\overline{\mathfrak{D}}\Gamma_0 = \Gamma_{\infty}$, and the Hesse derivative of Γ_{∞} is $\overline{\mathfrak{D}}\Gamma_{\infty} = \Gamma_{\infty}$. \mathcal{L} /2MQi2/ \mathcal{L} is the state in the value of \mathcal{L} is the value of \mathcal{L}

Proof. We have

$$
H_{\Gamma_{\!c_0}}(x,y,z):=\begin{pmatrix} 6x & c_0z & c_0y \\ c_0z & 6y & c_0x \\ c_0y & c_0x & 6z \end{pmatrix}.
$$

This yields det $H_{\Gamma_0}(x, y, z) = -6c^2(x^3 + y^3 + z^3) + 2(108 + c^3)xyz$, and the claim follows for $c_0 \in \mathbb{R} \setminus \{0\}$. The cases Γ_0 and Γ_{∞} are also easily checked. $q.e.d.$ M_g is the extended.

An immediate corollary is μ bezondare coronary is

Corollary 9. Let $c_0 \neq 0$. Then, the $(n + 1)$ -th Hesse derivative of Γ_{c_0} is given by

$$
\xi^{n+1}\Gamma_{c_0}: x^3 + y^3 + z^3 + c_{n+1}xyz = 0
$$

where $c_{n+1} = -\frac{108+c_n^3}{3c_n^2}$ for every $n \ge 0$, as long as $c_n \ne 0$.

5 Analysis of iterates

AM M 2tit Δ and Δ is a proposition of the distribution o Motivated by Lemma [8,](#page-7-2) we consider the function

$$
h(x) = \frac{a + x^3}{bx^2} \tag{4}
$$

for $a \in \mathbb{R}$ and $b \in \mathbb{R} \setminus \{0\}.$

Fact 10. The function h defined in [\(4\)](#page-7-3) has a pole at $x = 0$, and an oblique asymptote $y = \frac{x}{b}$. For $b \neq -1$

$$
\varphi:=\sqrt[3]{\frac{a}{b-1}}
$$

is the unique real fixed point of h. The function h has the unique critical point $\kappa := \sqrt[3]{2a}$ with critical value

$$
h(\kappa) = \frac{3a^{1/3}}{2^{2/3}b}.
$$

The proof is elementary.

Remark: In our case, $b = -3$, we have

$$
h(\varphi) = \varphi = -\sqrt[3]{\frac{a}{4}} = -\sqrt[3]{\frac{a^3}{4a^2}} = h(\kappa).
$$

This case also gives us the crucial property for the partition of $\mathbb{R} \setminus \{\varphi\}$ into the intervals $N =$ $(-\infty, \varphi)$ and $P = (\varphi, \infty)$. Namely, we have $x \in P \setminus \{0\}$ iff $h(x) \in N$, and $x \in N$ iff $h(x) \in P$. For the next propositions, we will assume $b = -3$ and $a \neq 0$, and hence $\varphi \neq 0$. For now, we refrain from imposing a specific value of a, since, as we will see below, most of the results are independent of the value of a. We will specify $a = 108$ in the next section.

Proposition 11. Let $b = -3$ and $a \neq 0$. If we define

$$
h^{(n)} := \underbrace{h \circ h \circ \dots h}_{n \ times}
$$

then, $y = \frac{x}{b^n}$ is an oblique asymptote of $h^{(n)}$. Furthermore, if κ_n is a critical point of $h^{(n)}$, we have

$$
h^{(n)}(\kappa_n) = \varphi = -\sqrt[3]{\frac{a}{4}}.
$$

Conversely, if $h^{(n)}(x) = \varphi$, then either $x = \varphi$ or $\frac{d}{dx}h^{(n)}(x) = 0$.

Proof. Since we already know the oblique asymptote of h from Fact [10,](#page-8-0) we can now inductively argue that

$$
\lim_{x \to \pm \infty} \left| h^{(n+1)}(x) - \frac{x}{b^{n+1}} \right| = \lim_{x \to \pm \infty} \left| \frac{a + (h^{(n)}(x))^3}{b (h^{(n)}(x))^2} - \frac{x}{b} \right|
$$

$$
= \lim_{x \to \pm \infty} \left| \frac{a}{b (h^{(n)}(x))^2} + \frac{h^{(n)}(x) - \frac{x}{b^n}}{b} \right| = 0
$$

using the fact that $(h^{(n)}(x))^{2} \to \infty$ for $x \to \pm \infty$. Note that in this part we did not need the assumption $b = -3$.

For the next part, observe first that we have

$$
h(x) = \varphi \iff (x - \varphi)(x - \kappa)^2 = 0.
$$

This equation has only two solutions, namely $x_1 = \varphi$ and $x_2 = \kappa$. By chain rule we obtain

$$
\frac{d}{dx}h^{(n)}(x) = \prod_{r=0}^{n-1} h'\left(h^{(r)}(x)\right) = 0
$$
\n
$$
\iff h'\left(h^{(r)}(x)\right) = 0 \text{ for some } r \in \{0, 1, \dots, n-1\}
$$
\n
$$
\iff h^{(r)}(x) = \kappa \text{ for some } r \in \{0, 1, \dots, n-1\}.
$$
\n(5)

So, it follows immediately that $\frac{d}{dx}h^{(n)}(x) = 0$ implies $h^{(n)}(x) = \varphi$.

Now we prove the converse, as stated in the lemma. For $n = 0$, the statement is trivially true. Assume that for some $n \geq 1$ we have that $h^{(n)}(x) = \varphi$ implies that either $x = \varphi$ or $\frac{d}{dx}h^{(n)}(x) = 0$. Then we have for $h^{(n+1)}(x) = \varphi$ that $h^{(n)}(x) = \kappa$ or $h^{(n)}(x) = \varphi$. On the other hand, $\frac{d}{dx}h^{(n+1)}(x) =$ $h'(h^{(n)}(x))\frac{d}{dx}h^{(n)}(x)$. If $h^{(n)}(x) = \kappa$, then the first factor in this product is zero and the derivative of $\frac{d}{dx}h^{(n+1)}(x)$ vanishes. If $h^{(n)}(x) = \varphi$, then, by induction, $x = \varphi$ or $\frac{d}{dx}h^{(n)}(x) = 0$, and again the derivative of $\frac{d}{dx}h^{(n+1)}(x)$ is zero. $q.e.d.$

Proposition 12. Let χ_n be the number of critical points of $h^{(n)}$. Then, the sequence $\{\chi_n\}$ is given by

 $\chi_{2r+1} = 2 \times 3^r - 1$ and $\chi_{2r} = 3^r - 1$

for all integers $r \geq 0$. This corresponds to OEIS A062318.

Proof. We only carry out the case $\varphi < 0$. The proof for $\varphi > 0$ is essentially the same.

Let $N = (-\infty, \varphi)$ and $P = (\varphi, \infty)$. Observe first that for given $y \in N$ the equation

 $y = h(x)$ or equivalently $x^3 + 3yx^2 - 4\varphi^3 = 0$

has three distinct real roots. Indeed, the discriminant $\Delta = 27 \times 16\varphi^3 \left(y^3 - \varphi^3 \right)$ is strictly positive for $y \in N$. Moreover, if $x, y \in N$ then the expression $x^3 + 3yx^2 - 4\varphi^3$ is strictly negative, hence the three solutions of $y = h(x)$ must lie in P. Thus, the preimage $h^{-1}(y)$ of a point $y \in N$ has cardinality 3, and lies in P. Similarly, the preimage $h^{-1}(y)$ of a point $y \in P$ has cardinality 1, and lies in N.

Now, for $n = 1$, the set of critical points of h is $C_1 = \{\kappa\} \subset P$. Let $S_1 := h^{-1}(C_1)$, and $S_k := h^{-1}(S_{k-1})$ for $k > 1$. In other words, S_k is the preimage of $\{\kappa\}$ under $h^{(k)}$. Observe that $S_n \subset N$ if n is odd, and $S_n \subset P$ if n is even. Also note that two sets S_k and S_j , $k > j$ are disjoint. Indeed, if $x \in S_j$, then $h^{(j)}(x) = \kappa$, and hence $h^{(k)}(x) = \varphi \neq \kappa$. It follows that card $S_{2n} = \text{card } S_{2n+1} = 3^n$. For $n > 1$ we can read off from equation [\(5\)](#page-8-1) that the set of critical points of $h^{(n)}$ is the set $C_n = C_{n-1} \cup S_{n-1}$. In particular, the sets C_n and S_n are disjoint for all n since the sets S_n are disjoint. Hence, we have card $C_{2n} = \text{card } C_{2n-1} + 3^{n-1}$, and card $C_{2n+1} = \text{card } C_{2n} + 3^n$. This corresponds to the sequence OEIS A062318. $q.e.d.$

Proposition 13. Let Φ_n be the number of fixed points of $h^{(n)}$. Then, the sequence $\{\Phi_n\}$ is given by

$$
\Phi_{2r+1} = 1
$$
 and $\Phi_{2r} = 2\chi_{2r} - 1 = 2 \times 3^r - 3$

for all integers $r \geq 0$.

Proof. Since h maps N to P and vice versa, the only fixed point of $h^{(2r+1)}$ is φ .

For the fixed points of $h^{(2r)}$, we begin by assuming without loss of generality that $a > 0$, $\varphi < 0$, and recalling that h has a pole at $x = 0$, it is decreasing in the intervals $(-\infty, 0)$, (κ, ∞) and increasing in $(0, \kappa)$. We also know that the only critical point of h is at κ and it has a local maximum there. So, for convenience, we define

$$
\tilde{h}(x) := h(x + \varphi) - \varphi
$$

i.e., we shift the point (φ, φ) to the origin. The number of fixed points of $h^{(n)}$ is the same as the number of fixed points of $\tilde{h}^{(n)}$. Consider the sets $\tilde{N} = (-\infty, 0), \tilde{P} = (0, \infty)$, and $A = (0, -\varphi), B =$ $(-\varphi, -\varphi + \kappa), C = (-\varphi + \kappa, \infty)$. Then, \tilde{h} maps \tilde{N} bijectively to \tilde{P} , and A, B and C each bijectively to \tilde{N} . Hence \tilde{h} maps \tilde{N} , \tilde{P} to \tilde{P} and tree copies of \tilde{N} . So after 2r iterations, the range of $\tilde{h}^{(2r)}$

Figure 5: Schematic profile of the function $\tilde{h}^{(2r)}$: We have $(3r-1)/2$ spikes on the positive x-axis and $(3^r - 1)/2$ on the negative x-axis.

consists of 3^r copies of \tilde{N} and 3^r copies of \tilde{P} . Figure [5](#page-10-1) shows schematically the behaviour of $\tilde{h}^{(2r)}$.

Observe that the oblique asymptote of $\tilde{h}^{(2r)}$ is given by $y = x/3^{2r}$, hence the line $y = x$ does not intersect the leftmost and the rightmost branch of the graph of $\tilde{h}^{(2r)}$. Hence the number of fixed points of $\tilde{h}^{(2r)}$ is $2(3^r - 1) - 1 = 2 \times 3$ $r-3.$ q.e.d.

The number of zeros of $h^{(n)}$ is now calculated in the same way as the number of fixed points.

Proposition 14. Let ρ_n be the number of zeros of $h^{(n)}$. Then, the sequence $\{\rho_n\}$ is given by

 $\rho_{2r} = \rho_{2r+1} = 3^r$

for all integers $r \geq 0$.

Proof. We still assume $\varphi < 0$, the case $\varphi > 0$ is similar. The number of zeros of $h^{(n)}$ equals the number of solutions of the equation $\tilde{h}^{(n)} = -\varphi$. Let us first assume that $n = 2r$ is even. Then, as in the proof of Proposition [13,](#page-9-0) $\tilde{h}^{(2r)}$ maps \tilde{N} , \tilde{P} to 3^r copies of \tilde{N} and 3^r copies of \tilde{P} . Since $-\varphi \in \tilde{P}$, the number of solutions of $\tilde{h}^{(n)} = -\varphi$ is 3^r.

If $n = 2r + 1$ is odd, then, $\tilde{h}^{(2r+1)}$ maps \tilde{N} , \tilde{P} to 3^{r+1} copies of \tilde{N} and 3^r copies of \tilde{P} . Hence, we have again 3^r solutions of $\tilde{h}^{(n)} = -\varphi$. q.e.d.

6 Loops and chains of Hesse derivatives

We return to considering the curves in Hesse form, i.e.,

$$
\Gamma_c: x^3 + y^3 + z^3 + c \, xyz = 0, \qquad \Gamma_\infty: xyz = 0 \tag{6}
$$

for $c \in \mathbb{R}$. Depending on the value of c, the curve Γ_c has either one or two components in the affine plane $z = 1$ (see Figure [6\)](#page-11-0). Notice that Γ_c is unchanged under the transformation $(x, y) \mapsto (y, x)$,

Figure 6: Hesse curves Γ_5 and Γ_{-9} . \mathcal{L} α radius α radius α Figure 6: Hesse curves Γ_5 and Γ_{-9} .

and hence it is symmetric with respect to the line $y = x$. For $c = -3$ the curve degenerates to the line $x + y + 1 = 0$ and the point (1,1), otherwise it is smoothly embedded. If the curve has one component, it intersects the line $y = x$ in one point, if it has two components we have three intersection points. \mathcal{D}^2 and \mathcal{D}^2 and \mathcal{D}^2 and \mathcal{D}^2 matrix \mathcal{D}^2 and hence it is symmetric with respect to the line $y = x$. For $c = -3$ the curve degenerates to the line $x + y + 1 = 0$ and the point (1,1), otherwise it is smoothly embedded. If the curve has i.2 T`QT2`ib2 \mathcal{L} iv i?i 7 \mathcal{L} to \mathcal{L} is \mathcal{L} is 7 \mathcal{L} in to \mathcal{L} in TqBMib 7. \mathcal{L} and \mathcal{L} and \mathcal{L} and \mathcal{L} is and \mathcal{L} is and \mathcal{L} is and \mathcal{L} $\frac{d}{dx}$ is $\frac{d}{dx} + \frac{d}{dx} + \frac{d}{dx} = 0$ and the point $(1, 1)$, otherwise it is simplointy embedded. If the i_M i) $\frac{1}{2}$ 0 and the point (1.1) otherwise it is smoothly embedded. If the gyvys hes $y + 1 = 0$ and the point (x, t) , otherwise it is sincoving embedded. If the early has

In this section, while referring to the function h from the previous section, we will assume $a = 108$, $b = -3, \, \varphi = -3, \, \kappa = 6$ (see Lemma 8 and Fact [10\)](#page-8-0). $B_{\rm eff}$ /2² μ /2² μ i2+. μ i2+ μ In this section, while referring to the function h from the previous section, we will assume $a = 108$, $\frac{1}{\sqrt{2}}$ i.e. $\frac{2}{\sqrt{2}}$ i.e. $\frac{2}{\sqrt{2}}$ i.e. $\frac{2}{\sqrt{2}}$ i.e. $\frac{2}{\sqrt{2}}$ b $\frac{2}{\sqrt{2}}$ i.e. $\frac{2}{\sqrt$ $\hat{\theta} = -\partial, \hat{\varphi} = -\partial, \hat{\kappa} = 0$ (see hemina $\hat{\theta}$ and rate $\hat{\theta}$). $\frac{1}{\sqrt{2}}$ is $\frac{2}{\sqrt{2}}$ in $\frac{2}{\sqrt{2}}$ in $\frac{2}{\sqrt{2}}$ in $\frac{2}{\sqrt{2}}$ in $\frac{2}{\sqrt{2}}$ in $\frac{2}{\sqrt{2}}$ is $\frac{2}{\sqrt{2}}$ in $\frac{2}{\sqrt{2}}$ in $\frac{2}{\sqrt{2}}$ in $\frac{2}{\sqrt{2}}$ is $\frac{2}{\sqrt{2}}$ in $\frac{2}{\sqrt{2}}$ in $\frac{2}{\sqrt{2}}$ $\lambda^2 = -3$, $\kappa = 0$ (see Lemma 0 and Pact 10). r^2 be r^2 by r^2 (respectively in \mathbb{F}_{2} in \mathbb{F}_{2} of \mathbb{F}_{2} i \mathbb{F}_{2} + \mathbb{F}_{2}) δ -b, $\varphi = -\partial$, $\kappa = 0$ (see Benina ∂ and Tact To). reaches $2 \text{ and } \text{F}_2 + 10$ $\lim_{\Delta t \to \infty}$ and rate $\lim_{\Delta t \to \infty}$.

We will begin with the following observation. iQ i?2 μ 2b2 μ iQ i?2 μ 2b2 μ $M_{\rm eff}$ is 20 TQH $_{\rm 2}$ and r2 $_{\rm 2}$ bi2^ μ and 2 bi2^ μ and μ $>$ 2 μ

Lemma 15. Let $c \in \mathbb{R}$. Then the curve Γ_c has two components in the affine plane $z = 1$ if $c < -3$ and only one component if $c > -3$. Moreover, for $c \neq 0$, if Γ_c has one component, then Σ_c has *.2`BpiBp2* Q7 b i?2 +m#B+ +m`p2^R $\frac{2}{3}$ $two\ components,$ and if Γ_c has two components, then $\overline{\mathfrak{D}}_c$ has one component. P#b2`p2 i?i /2i Bb ;BM ?QKQ;2M2Qmb +m#B+ TQHvMQKBHX h?2`27Q`2 r2 +M /2}M2 i?2 *>2bb2* **Lemma 15.** Let $c \in \mathbb{R}$. Then the curve Γ_c has two components /2`BpiBp2bX $\overline{\mathbf{r}}$ and $\overline{\mathbf{r}}$ ∞ components, and if Γ_c has two components, then $\overline{\mathfrak{D}}\Gamma_c$ has one component. **Thima 15.** Let $c \in \mathbb{R}$. Then the curve $\mathbf{1}_c$ has two components in the affine plane $z = 1$ if $c < -3$ i.2 T \mathcal{P} i \mathcal{P} is the interval to the i.2 T`QT2`iB2b \mathcal{I} iv i?i 7 \mathcal{I} to i?i 7 \mathcal{I} i.3 \mathcal{I} is \mathcal{I} if \mathcal{I} is \mathcal{I} is \mathcal{I} if \mathcal{I} is \mathcal{I} if \mathcal{I} is a set of imaginary \mathcal{I} is a set of imaginary \mathcal{I} is a

number of intersections with the line $y = x$, i.e., by counting the zeros of the equation *Proof.* As explained above, we can determine the number of components of Γ_c by computing the $\mathcal{L} = \mathcal{L} = \mathcal$ *oof.* As explained above, we can determine the number of components of Γ_c by computing the

$$
2x^3 + cx^2 + 1 = 0.
$$

The discriminant of this cubic equation is

 $-27 \cdot 4 - 4c^3$ $-2i \cdot 4 - 4c$

and hence the equation has three real roots if $c < -3$, and one if $c > -3$. and hence the equation has three real roots if $c < -3$, and one if $c > -3$.

Recall that $\overline{\mathfrak{D}_c} = \Gamma_{h(c)}$ if $c \neq 0$ (see Lemma 8), and that h maps $N = (-\infty, -3)$ to $P = (-3, \infty)$ and hence $h(c) \in P$, which means that $\overline{\epsilon} \Gamma_c = \Gamma_{h(c)}$ has one component—and the same applies the and P to N (see the proof of Proposition [12\)](#page-9-1). Hence, if Γ_c has two components, we have $c \in N$ P#b2`p2 i?i /2i Bb ;BM ?QKQ;2M2Qmb +m#B+ TQHvMQKBHX h?2`27Q`2 r2 +M /2}M2 i?2 *>2bb2* $\frac{2}{\pi}$ TQ is considered by indicated by iQ iQ iQ iQ iQ iQ iQ iQ iQ i $\frac{2}{\pi}$ other way around. $q.e.d.$ *.2`BpiBp2* Q7 b i?2 +m#B+ +m`p2^R \mathbb{R}^p is a set of \mathbb{R}^p in \mathbb{R}^p ratio \mathbb{R}^p ratio \mathbb{R}^p *a.e.d. .2`BpiBp2* Q7 b i?2 +m#B+ +m`p2^R \mathcal{L}^2 ratio between \mathcal{L}^2 ratio between \mathcal{L}^2 ratio \mathcal{L}^2 ratio \mathcal{L}^2 Recall that $\mathcal{L}_c = I_{h(c)}$ if $c \neq 0$ (see Lemma 8), and that h and hence $h(c) \in P$, which means that $\overline{\mathbf{C}}_c = \Gamma_{h(c)}$ has one component—and the same applies the $\mathcal{L}^2=\mathcal{L}^2$ radiation is a set of radiation of radiation in the set of radiation \mathcal{L}^2

JI ∪
== + points, namely Γ_{-3} and Γ_{∞} . We are now interested in orbits of a given length which end in one of μ ₁ for d points, and in algoed onlite of a given langth. We call the forman Here, sheine the fixed points, and in closed orbits of a given length. We call the former *Hesse chains* and the latter *Hesse loops*. So, a Hesse chain is given by The operator \times defines via 1.4% and netative discrete dynamical system on the set of the cube
curves [\(6\)](#page-10-2) in Hesse form. The dynamics is given by Lemma [8.](#page-7-2) The system has exactly two fixed
paints around E and E and E and \mathcal{L} is a substitution of the state \mathcal{L} because \mathcal{L} and \mathcal{L} and \mathcal{L} is a state \mathcal{L} and \mathcal{L} and The operator $\overline{\xi}$ defines via $\Gamma \mapsto \overline{\xi} \Gamma$ an iterative discrete dynamical system on the set of the cubic

$$
\mathbf{\bar{R}}^n \Gamma_{\!c_0} = \Gamma_{\!c_n} = \Gamma_{\!-3} \qquad \text{or} \qquad \mathbf{\bar{R}}^n \Gamma_{\!c_0} = \Gamma_{\!c_n} = \Gamma_{\!\infty}
$$

where we call the minimal n with this property the length of the chain. Similarly, a Hesse loop is given by P#b2`p2 i?i /2i Bb ;BM ?QKQ;2M2Qmb +m#B+ TQHvMQKBHX h?2`27Q`2 r2 +M /2}M2 i?2 *>2bb2* where we call the minimal n with this property the length of the chain. Similarly, a Hess \mathbb{R}^{n} = \mathbb{R}^{n} = \mathbb{R}^{n} = \mathbb{R}^{n}

$$
\mathfrak{T}^n\Gamma_{\!c_0}=\Gamma_{\!c_n}=\Gamma_{\!c_0}
$$

where the minimal $n > 0$ with this property is the length of the loop. $\frac{1}{2}$ and $\frac{1}{2}$ is $\frac{1}{2}$ in $\frac{1}{2}$ in $\frac{1}{2}$ is the property of $\frac{1}{2}$ and $\frac{1}{2}$ and $\frac{1}{2}$ is $\frac{1}{2}$ in $\frac{1}{2}$ and $\frac{1}{2}$ is $\frac{1}{2}$ in $\frac{1}{2}$ in $\frac{1}{2}$ is $\frac{1}{2}$ in $\frac{1}{$ where the minimal $n > 0$ with this property is the length of the leap where the minimal $n > 0$ with this property is the length of the loop.

The number of Hesse chains ending in Γ_{-3} of length n is easy to calculate as shown in the following $Lemma.$ $$ I ne number of resse chains ending in $1-3$ of length n is easy to calculate B ₂ B ₂ μ ² μ in μ ₂ μ ² $\overline{1}$ I de number of resse chains ending in $1-3$ of length n is easy to calculate as shown in the B ₂ B _p B ₂ μ ² μ in 2 ⁿ μ in 2 ⁿ μ μ ² μ $a.$

Lemma 16. If $\mathbf{D}_n^{(-3)}$ denotes the number² of Hesse chains ending in Γ_{-3} of length n, then S^2 ib2b \mathcal{L}^2 ib2b \mathcal{L}^2 pi \mathcal{L}^2

$$
\overline{\mathbf{b}}_{2r}^{(-3)} = \overline{\mathbf{b}}_{2r-1}^{(-3)} = 3^{r-1}
$$

for integers $r \geq 1$. *r*_b₂ b i.e. b in Eq. b i.e. $\frac{1}{2}$

Proof. Notice that $\bar{\epsilon}^n \Gamma_{c_0} = \Gamma_{c_n} = \Gamma_{-3}$ implies $h^{(n)}(c_0) = \varphi = -3$. By Proposition 11 it follows $\frac{1}{\sqrt{2}}$ $h^{(n)}$ contains the set of critical points of $h^{(n-1)}$ (see the proof of Proposition [12\)](#page-9-1). Hence, by the minimality of n , it follows from Proposition 12 that that either $c_0 = -3$ or that c_0 is a critical point of $h^{(n)}$. Recall that the set of critical points of *Proof.* Notice that $\bar{\mathbf{z}}^n \Gamma_{\alpha} = \Gamma_{\alpha} = \Gamma_{\alpha}$ implies $h^{(n)}(c_0) = \varphi = -3$. By Prop at either $c_0 = -3$ or that c_0 is a critical point of $h^{(n)}$. Recall that the set *Proof.* Notice that $\bar{\mathbf{z}}^n \Gamma_{\alpha} = \Gamma_{\alpha} = \Gamma_{\beta}$ implies $h^{(n)}(c_0) = \varnothing = -3$. By Proposition 11 it foll *Proof.* Notice that $\langle 1 \rangle_{c_0} = 1_{c_n} = 1_{-3}$ implies $h^{(n)}(c_0) = \varphi = -3$. By Proposition 11 it follows that either $c_0 = -3$ or that c_0 is a critical point of $h^{(n)}$. Recall that the set of critical point that either $c_0 = -3$ or that c_0 is a critical point of $h^{(n)}$. Recall that the set of critic \mathbb{R}^n is the contribution of \mathbb{R}^n in the following from \mathbb{R}^n Decomposition 10 that $\mathbf{1}$ $\theta(c_0) = \varphi = -3$. By Proposition 11 it follows

$$
\mathbf{5}_{2r}^{(-3)} = \chi_{2r} - \chi_{2r-1} = 3^{r-1}
$$

and

$$
\mathbf{5}_{2r-1}^{(-3)} = \chi_{2r-1} - \chi_{2r-2} = 3^{r-1}.
$$

$$
q.e.d.
$$

and

G2KK RdX *A7* x

 \mathcal{L}^2 is 2th in \mathcal{L}^2 at the distribution of \mathcal{L}^2 at the distribution of \mathcal{L}^2 at the distribution of \mathcal{L}^2 \mathcal{L}^2 is 2th in \mathcal{L}^2 at the interval distribution of the interval \mathcal{L}^2 at the interval distribution of \mathcal{L}^2 q.e.d.

Lemma 17. For any positive $B \in \mathbb{R}$, there exists a $c > B$ and a $d < -B$ and an $n \in \mathbb{R}$ *M b*emina 17. $\mathbb{R}^n I_c = I_{-3}$ and $\mathbb{R}^n I_d = I_{-3}$. **EXECUTE: FOR any positive** $D \in \mathbb{R}$, there exists a $\bar{\mathbf{z}}^n \Gamma_c = \Gamma_{-3}$ and $\bar{\mathbf{z}}^n \Gamma_d = \Gamma_{-3}$. **Lemma 17.** For any positive $B \in \mathbb{R}$, there exists a $c > B$ and a $d < -B$ and an $n \in \mathbb{N}$ such that $\mathcal{A} \mathcal{A} \mathcal{A} \geq \mathcal{A} \mathcal{A}$ and $\mathcal{A} \mathcal{A} \leq -\mathcal{B}$ and an $n \in \mathbb{N}$ such that

Proof Observe first that $\Gamma_2 - \Gamma_2$. Also note that for $c > 6$ the solution \bar{c} of the equation $h(\bar{c}) = c$ satisfies $\bar{c} > -3c - 1$. Hence by backward iteration and choosing always the so with $c > B$ or $d < -B$.
 $q.e.d.$ b(\bar{c}) = c satisfies \bar{c} > -3c - 1. Hence by backward iteration and choosing always the solution with *Proof.* Observe first that $\Gamma_6 = \Gamma_{-3}$. Also note that for $c \geq 6$ the solution \bar{c} of the equation $h(\bar{c}) = c$ the largest absolute value, we can construct an orbit ending in Γ_{-3} and starting at some Γ_c or Γ_d
with $\Gamma_c \geq R \geq \Gamma_d \leq R$ Thought the mass charge $\overline{16} - \overline{1} - \overline{3}$. Also note that for $c \le 0$ the solution c of the equation $h(c) = c$ satisfies $\overline{c} < -3c$. On the other hand, for $c \le -6$, the largest of the three solutions \overline{c} of the $\begin{array}{c}\n n(c) \\
 \text{the }c\n \end{array}$ $U(t) = c$ satisfies $t > 0$. It is trially backward iteration and choosing aways the solution with the largest absolute value, we can construct an orbit ending in Γ_{-3} and starting at some Γ_c or Γ_d

G2KK RNX *A7* x */2MQi2b i?2 MmK#2` Q7 >2bb2 +?BMb Q7 H2M;i? - i?2M* **G2KK RNX** *A7* x */2MQi2b i?2 MmK#2` Q7 >2bb2 +?BMb Q7 H2M;i? - i?2M* G_2 G_3 G_4 G_5 G_6 G_7 G_8 G_9 G_9 *_2K`FX* h?Bb `2bmHi Bb HbQ BMimBiBp2Hv +H2` bBM+2 BM S`QTQbBiBQM Rk- r2 ?p2 H`2/v T`Qp2/ i?i MQ/2 Similarly as before, we consider the Hesse chains ending in Γ_{∞} . $\mathcal{L}_{\mathcal{D}}$ because $\mathcal{L}_{\mathcal{D}}$ because $\mathcal{L}_{\mathcal{D}}$ Similarly as before, we consider the Hesse chains ending in $\Gamma_\infty.$

Lemma 18. If $\overline{\mathfrak{b}_n^{\infty}}$ denotes the number of Hesse chains of length n ending in Γ_{∞} , then be chains of ichgin h chaing

$$
\overline{\mathtt{D}}_{2r}^{\infty}=\overline{\mathtt{D}}_{2r-1}^{\infty}=3^{r-1}
$$

 i,j *for integers* $r \geq 1$. *M/ M bm+? i?i X* S^2 *7Q` X* j *b* is integer $\theta = -1$

 $P_{\text{max}} f$ $\Omega'_{\text{max}} = \overline{B}$ zeros of $h^{(n-1)}$. Therefore the claim follows from Lemma 14. *Proof.* Since $\overline{\mathfrak{A}}l_0 = \Gamma_\infty$, the number of Hesse chains ending in Γ_∞ of length *n* is the number of A/2MiB+HHv b #27Q`2- r2 +QMbB/2` i?2 +?BMb 2M/BM; rBi? X 6Q` 2t+i bK2 `2bQMb b #27Q`2- r2 ?p2 *S`QQ7X* *H2` 7`QK i?2 /Bb+mbbBQM BM T`QQ7 Q7 S`QTQbBiBQM RjX [X2X/X *Proof.* Since $\overline{\xi}I_0 = \Gamma_{\infty}$, the number of Hesse chains ending in Γ_{∞} of length *n* is the number of zeros of $h^{(n-1)}$. Therefore the claim follows from Lemma 14. q.e.d.

> Now we turn our attention towards Hesse loops.
 \quad Now we turn our attention towards Hesse loops.

Proposition 19. The only Hesse loop of odd length is the tra **Proposition 19.** The only Hesse loop of odd length is the trivial loop $\overline{C_1} = \Gamma_{-3}$. $\mathbf F$

IS SOMETHING CONSULTED IN A 1983 OF SUPPLESSIONS TO HOURD THE UP SOURCE THE UNION TO START THE START IS SUCH THE START TO START THE UP ISSUE TO THE UP IN THE UP ISSUE TO THE UP ISSUE TO THE UP ISSUE TO THE UP ISSUE TO THE pronounced as "chaw". ²We again find ourselves at a loss of expressions to notate the "ch" sound used in "chain" as such an alphabet $\frac{1}{2}$ J. is not present in English, Latin or Greek script. We will use this excuse to use another Bengali alphabet, namely \bar{b} , 2 We again find ourselves at a loss of expressions to notate the "ch" sound used in "chain" as such an alphabet $\frac{1}{2}$

Proof. This follows immediately from Lemma [15.](#page-11-1) $q.e.d.$

Now, we want to determine the number of Hesse loops of length n for even n . We start with the following observation.

Proposition 20. For every even n, there is at least one Hesse loop of length n.

Proof. Recall that Φ_n denotes the number of fixed points of $h^{(n)}$ (see Proposition [13\)](#page-9-0). Now, note that the value of Φ_n already includes the trivial fixed point -3. So let $\Phi'_n := \Phi_n - 1$ denote the number of non-trivial fixed points of $h^{(n)}$. Furthermore, for any r, a fixed point of $h^{(r)}$ is also a fixed point of $h^{(mr)}$ for any m. Also, any loop of length r consists of r elements and contributes this number to Φ'_n . It is enough to show that the quantity

$$
\Phi'_{2r}-\sum_{k=1}^{r-1}\Phi'_{2k}
$$

is strictly positive. This follows from the fact that for $r > 1$, we have

$$
\sum_{k=1}^{r-1} \Phi'_{2k} = \sum_{k=1}^{r-1} (2 \times 3^k - 4) = 3^r - 4r + 1,
$$

and this is indeed strictly smaller than $\Phi'_{2r} = 2 \times 3^r - 4$. q.e.d.

Remark. The above calculation also shows that we must at least have

$$
\left\lceil \frac{1}{2r} \left(\Phi'_{2r} - \sum_{k=1}^{r-1} \Phi'_{2k} \right) \right\rceil = \left\lceil \frac{3^r - 5}{2r} \right\rceil + 2
$$

loops of length 2r.

Proposition 21. If Λ_n denotes the number of Hesse loops of length n, then the sequence $\{\Lambda_{2r}\}$ is strictly increasing.

Proof. Since Φ'_{2r} includes the two elements of the only 2-loop, the value of Λ_{2r} can be at most

$$
\left\lfloor \frac{\Phi_{2r}' - 2}{2r} \right\rfloor = \left\lfloor \frac{3^r - 3}{r} \right\rfloor
$$

and hence

$$
\left\lceil \frac{3^r - 5}{2r} \right\rceil + 2 \le \Lambda_{2r} \le \left\lfloor \frac{3^r - 3}{r} \right\rfloor
$$

for $r > 1$. So, to prove that the sequence $\{\Lambda_{2r}\}\$ is strictly increasing, it is enough to show that

$$
\left\lceil \frac{3^r - 5}{2r} \right\rceil > \left\lfloor \frac{3^{r-1} - 3}{r - 1} \right\rfloor.
$$

This follows from

$$
\frac{3^x - 5}{2x} > \frac{3^{x-1} - 3}{x - 1}
$$

which is true for $x \geq 3$ as then we have

 $3^{x-1} \cdot x + 5 + x > 3^x$

which completes the proof for $r \geq 3$.

The cases $r = 1, 2$ can be checked by hand. q.e.d.

Theorem 22. The number of loops of length $2r$ is

$$
\Lambda_{2r} = \frac{1}{2r} \sum_{d|r} \mu\left(\frac{r}{d}\right) \Phi'_{2d}
$$

where $\Phi'_{2d} = 2 \times 3^d - 4$, and μ is the Möbius function.

Proof. Let the even divisors of 2r be $d_1 = 2, d_2, \ldots, d_k = 2r$. Since each loop of length d_m contains exactly d_m elements, the total number of fixed points $\neq -3$ of $h^{(2r)}$ is given by

$$
\Phi'_{2r} = \sum_{\substack{d \mid 2r \\ d \text{ even}}} d \cdot \Lambda_d.
$$

The even divisors of $2r$ are twice the divisors of r . Hence we may write

$$
\Phi'_{2r} = \sum_{d|r} 2d \cdot \Lambda_{2d}.
$$

Using the Möbius inversion formula, we obtain

$$
2r \cdot \Lambda_{2r} = \sum_{d|r} \mu\left(\frac{r}{d}\right) \Phi'_{2d}
$$

hence completing the proof. $q.e.d.$

The sequence (Λ_{2r}) starts as follows:

 $\Lambda_2 = 1, \ \Lambda_4 = 3, \ \Lambda_6 = 8, \ \Lambda_8 = 18, \ \Lambda_{10} = 48, \ \Lambda_{12} = 116, \ \Lambda_{14} = 312, \ \Lambda_{16} = 810, \ \ldots$

The Hesse loop of length 2 is shown in Figure [4.](#page-7-1)

Remark. Let $\lambda_r := \Lambda_{2r}$. Then we have for $r \geq 1$

$$
\lambda_r = \frac{1}{2r} \sum_{d|r} \mu\left(\frac{r}{d}\right) (2 \cdot 3^d - 4) = \underbrace{\frac{1}{r} \sum_{d|r} \mu\left(\frac{r}{d}\right) \cdot 3^d}_{=:a_r} - 2 \cdot \underbrace{\frac{1}{r} \sum_{d|r} \mu\left(\frac{r}{d}\right)}_{=:\delta_{1,r}},
$$

where $\delta_{1,r} = 1$ if $r = 1$ and $\delta_{1,r} = 0$ for all other values of r. Notice that the sequence a_r represents the number of aperiodic necklaces with r beads of 3 colors, see OEIS A027376. Another sequence that agrees with λ_r for $r \geq 2$ is OEIS A185171.

7 Hesse derivatives of other normal forms

So far, we just considered Hesse derivatives of cubic curves in Hesse form, i.e., of curves Γc. The reason was that the Hesse derivative of a curve in Hesse form is again a curve in Hesse form, which is in general not the case for curves, for example, in Weierstrass normal form (WNF).

Below, we first provide examples of curves in WNF such that their Hesse derivatives are not in WNF, and then we provide cubic curves with a D_3 -symmetry, whose Hesse derivatives have the same symmetry— like the cubics in Figure [4.](#page-7-1)

Curves in Weierstrass normal form \mathbf{u} ves in weight ass normal form

Let $\Gamma_c: x^3 + y^3 + z^3 + cxyz = 0$ be a cubic curve in Hesse form. Then, as described in [2, Sec. 3], by a projective transformation, for $\text{Ext } \mathbf{r}_c : x \mapsto y \mapsto z \text{Ext } \mathbf{r}_c \mathbf{$ \mathbb{R} as described in [2, Sec. 3]. R^2

$$
c=-\frac{2q^3+1}{q^2}\,,
$$

the curve Γ_c can be transformed to the curve \mathcal{L}

$$
E_{a,b}: y^2 = x^3 + a x^2 + b x
$$

where \mathcal{L}

where
$$
b = \frac{(q-1)^3}{q+q^2+q^3}
$$
 and $a = \frac{b^2 - 6b - 3}{4}$.

For example, for $c_0 = -3(\sqrt{3} + 1)$ we obtain For example, for $\zeta_0 = -\partial(\nabla \partial + 1)$ we obtain

$$
q_0 = \frac{\sqrt{3} - 1}{2}
$$
, $b_0 = 3 - 2\sqrt{3}$, $a_0 = 0$.

As a matter of fact we would like to mention that over the quadratic field $\mathbb{Q}(\sqrt{3})$, the torsion group of E_{a_0,b_0} is $\mathbb{Z}/6\mathbb{Z}$ with the generating point $(-2\sqrt{3}+3,5\sqrt{3}-9)$. Furthermore, we have of fact we would like to mention that over the quadratic field $\mathbb{Q}(\sqrt{3})$, the torsion group $\mathbb{Z}/\sqrt{2}$ with the generating point $(-2\sqrt{3} + 3, 0\sqrt{3} - 3)$. Furthermore, we have

$$
\mathfrak{E} E_{a_0,b_0} = 7 - 4\sqrt{3} + (-3 + 2\sqrt{3})x^2 - xy^2 = 0
$$

and $\bar{\mathcal{R}}^2 E_{a_0,b_0} = E_{a_0,b_0}$. b_0^2 $\mu_{20,00}^2$.

Curves in D_3 -symmetric form

In [\[1](#page-16-0), Sec. 2], a D_3 -symmetric form of cubic curves was introduced. With a suitable projective
transformation every regular cubic curve can be brought into the form $\frac{1}{2}$ transformation every regular cubic curve can be brought into the form transformation, every regular cubic curve can be brought into the form $\overline{\text{Cov}}$ $\frac{1}{2}$ /2 $\frac{1}{2}$ introduced. With a suitable projective **R AMi`Q/m+iBQM** \mathcal{L}^2 radius \mathcal{L}^2 respectively. Hence it is the matrix \mathcal{L}^2

$$
\Gamma_b: x^3 - 3xy^2 + b(x^2 + y^2)z + z^3 = 0,\tag{7}
$$

with $h \in \mathbb{R}$, $h \neq -\frac{3}{2}$. The corresponding curve in R with $b \in \mathbb{R}, b \neq -\frac{3}{\sqrt[3]{4}}$. The corresponding curve in the affine plane $z = 1$ has the symmetry of an equilateral triangle, like the curves in Figure [4.](#page-7-1) For $b = -\frac{3}{\sqrt[3]{4}}$ the curve degenerates to three lines: α effine plane $z=1$ has the symmetry of an $\frac{1}{3}$ +m² $\in \mathbb{R}, b \neq -\frac{3}{\sqrt[3]{4}}.$ The corresponding $\frac{3}{\sqrt[3]{4}}$

$$
x^3 - 3xy^2 - \frac{3}{\sqrt[3]{4}}(x^2 + y^2)z + z^3 = \frac{1}{6}(2x + \sqrt[3]{2})(\sqrt{3}x + 3y - \sqrt[3]{2}\sqrt{3})(\sqrt{3}x - 3y - \sqrt[3]{2}\sqrt{3}).
$$

This class of cubic curves shows a very similar behaviour under the operator $\overline{\mathcal{R}}$ as the class of Hesse curves which we investigated before. *.2`BpiBp2* Q7 b i?2 +m#B+ +m`p2^R

Proposition 23. The class of cubic curves [\(7\)](#page-15-0) is invariant under the operator $\bar{\mathbf{x}}$. We have *<u>Biblion</u>* **20.** *The cluss of cable carves*

$$
\overline{\epsilon}_{\Gamma_b} = \Gamma_{\tilde{b}}, \ \ with \ \tilde{b} = -\frac{27 + b^3}{3b^2}.
$$

Proof. For $f(x, y, z) = x^3 - 3xy^2 + b(x^2 + y^2)z + z^3$, we have $\frac{3}{2}$ range here $\frac{1}{2}$ ϵ^2 , we have

$$
H_f = \frac{1}{2} \begin{pmatrix} 3x + bz & -3y & bx \\ -3y & -3x + bz & by \\ bx & by & 3z \end{pmatrix}.
$$

 ϕ the determinant gives the desired result Taking the determinant gives the desired result. $q.e.d.$

In order to find a Hesse loop of length 2 in D_3 -symmetric form, we have to find b_0 and b_1 , such that

$$
b_1 = -\frac{27 + b_0^3}{3b_0^2}
$$
 and $b_0 = -\frac{27 + b_1^3}{3b_1^2}$,

which holds for

$$
b_0 = -3\sqrt[3]{\frac{3\sqrt{3}+5}{2}}
$$
 and $b_1 = 3\sqrt[3]{\frac{3\sqrt{3}-5}{2}}$.

Figure [7](#page-16-2) shows the two cubics Γ_{b_0} and Γ_{b_1} of the Hesse loop of length 2 in D₃-symmetric form together with degenerate polar conics at point $P = (0, 1, 0)$, which is one of the three real intersecting points of the two cubics. Notice that since the intersecting points of the curves are also the points of inflection of the curves, the two polar conics share a line which meets the two conics in four points, and the other two lines of the two conics are the tangent lines at the inflection point at Γ_{b_0} and Γ_{b_1} , respectively, and are tangent to Γ_{b_1} and Γ_{b_0} , respectively.

Figure 7: The Hesse loop of length 2 of two curves in D_3 -symmetric form, together with the two degenerate polar conics at the intersection point $P = (0, 1, 0)$.

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References

- [1] Andrin Halbeisen, Lorenz Halbeisen, and Norbert Hungerbühler. Configurations on elliptic curves. Innov. Incidence Geom., 19(3):111–135, 2022.
- [2] Lorenz Halbeisen and Norbert Hungerb¨uhler. An elementary approach to Hessian curves with torsion group $\mathbb{Z}/6\mathbb{Z}$. Int. Electron. J. Pure Appl. Math., 13(1):1-30, 2019.

- [3] Lorenz Halbeisen and Norbert Hungerbühler. Constructing cubic curves with involutions. Beitr. Algebra Geom., 63(4):921–940, 2022.
- [4] Lorenz Halbeisen, Norbert Hungerbühler, and Vera Stalder. Three conics determine a cubic. Matematica, https://doi.org/10.1007/s44007-024-00094-1, 2024.
- [5] Audun Holme. A royal road to algebraic geometry. Springer, Heidelberg, 2012.