Minimizing the number of edges in $(P_k \cup S_\ell)$-saturated graphs *

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Abstract

For a graph H, a graph G is *H-saturated* if it contains no copy of H as a (not necessarily induced) subgraph, but the addition of any edge missing from G creates a copy of H in the resultant graph. The *saturation number* $sat(n, H)$ is defined as the minimum number of edges in H-saturated graphs on n vertices. Let P_k and S_k be path and star on k vertices, respectively. In this paper we consider the $(P_k \cup S_\ell)$-saturated graphs on n vertices and focus on the determination of $sat(n, P_k \cup S_\ell)$. We prove the upper bounds of $sat(n, P_k \cup S_\ell)$ for $k \geq 6$ and $\ell \geq 4$. Moreover, we get $sat(n, P_k \cup S_\ell) = n - \lfloor \frac{n-3(\ell-4)}{a_k} \rfloor$ on certain conditions and $sat(n, P_k \cup S_4) = n - \lfloor \frac{n}{a_k} \rfloor$ for $k \geq 6$, where a_k is the order of the minimum P_k-saturated tree. We also give a conjecture about the exact value of $sat(n, P_k \cup S_\ell)$ if ℓ is not less than some positive integer.

Keywords: graph saturation, saturation number, path, star

AMS Classification (2020): 05C35, 05C38

1. Introduction

All graphs considered in this paper are finite, undirected and simple. Let $G = (V(G), E(G))$ be a graph, as usual, denote by $V(G)$, $E(G)$, $|G|$, $m(G)$ and $\overline{G}$ the vertex set, edge set, the number of vertices, the number of edges and the complement of G, respectively. For any $v \in V(G)$, let $d_G(v)$ and $N_G(v)$ denote the degree and the set of neighbors of v in G, respectively. As usual, let $\Delta(G)$ be the maximum degree of graph G. Denote by K_n, P_n, C_n and S_n the complete graph, path, cycle and the star graph on n vertices, respectively. For a vertex $u \in V(G)$, denote by S_u the star subgraph of G with central vertex u. Let T be a tree with $u \in V(T)$, we write by T_u the subtree of T consisting of root vertex u and all descendants of u in the tree T. For a path P with n vertices, denote by uPv, uP and Pv the path P starting at u and ending at v, the path P starting at u and the path P ending at v, respectively. The *distance* $d_G(x, y)$ of two vertices x, y is the length of a shortest (x, y)-path in G. The

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diameter of G, denoted by $diam(G)$, is the greatest distance between any two vertices in G. The *eccentricity* $ecc_G(v)$ of a vertex v in a graph G is $\max\{d(u,v) \mid u \in V(G)\}$. For brevity, we write $[t] = \{1, 2, \ldots, t\}$ for positive integer t.

For $A \subseteq V(G)$, let $G[A]$ be the subgraph of G induced by A. For any edge $e \in E(\overline{G})$, we write by $G + e$ the graph obtained from G by adding the new edge e. For any graph H and any positive integer $t > 1$, let tH be the graph composed of t vertex-disjoint copies of H. Given any two vertex-disjoint graphs G and H, their *union* $G \cup H$ is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$, and their *join* $G \vee H$ is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{gh | g \in V(G), h \in V(H)\}$.

Given a graph H, a graph G is *H-free* if G does not contain H as a (not necessarily induced) subgraph. A graph G is *H-saturated* if G is H-free, but the addition of any edge missing from G creates a copy of H in the resultant graph. The *saturation number* $sat(n, H)$ is defined as the minimum number of edges in H-saturated graphs on n vertices. This can be viewed as the dual of the celebrated *Turán number* $ex(n, H)$, the maximum number of edges in H-saturated graphs on n vertices. Let G be an H-free graph and e be a non-edge of G, we say e is an *H-saturating edge* of G if $G + e$ contains a copy of H. A graph G is *H-oversaturated* if for any non-edge e of G, $G + e$ contains a copy of H with $e \in E(H)$ (Note that an H-oversaturated graph is not necessarily H-free). For a graph G, let e be a non-edge of G, we say e is an *H-oversaturating edge* of G if $G + e$ contains a copy of H with $e \in E(H)$.

Saturation number was first studied by P. Erdős, A. Hajnal and J. Moon [12], who proved that $sat(n, K_p) = (p-2)(n-p-2) + \binom{p-2}{2}$ with the extremal graphs $K_{p-2} \vee \overline{K_{n-p+2}}$. L. Kászonyi and Z. Tuza in [23] considered $sat(n, H)$ for $H \in \{S_k, mK_2, P_m\}$ and determined the extremal graphs, respectively. As a generalization, R. Faudree, M. Ferrara, R. Gould and M. Jacobson [17] proved $sat(n, tK_p) = (t-1)\binom{p+1}{2} + \binom{p-2}{2} + (p-2)(n-p+2)$ and constructed the extremal graphs $K_{p-2} \vee \big((t-1)K_{p+1} \cup \overline{K_{n-pt-t+3}}\big)$. Moreover, they also determined $sat(n, K_p \cup K_q)$ and $sat(n, F_{t,p,\ell})$ with the extremal graphs, where $F_{t,p,\ell}$ is the generalized friendship graph composed of t copies of K_p intersecting in a common K_ℓ for positive integers t, p and ℓ. F. Chen and X. Yuan [6] proved $sat(n, K_p \cup (t-1)K_q) = (t-1)\binom{q+1}{2} + \binom{p-2}{2} + (p-2)(n-p+2)$ with $2 \le p < q$ and the extremal graphs were determined when $t = 3$. Moreover, they also determined $sat(n, K_p \cup K_q \cup K_r) = \binom{r+1}{2} + \binom{q+1}{2} + \binom{p-2}{2} + (p-2)(n-p+2)$ with $2 \le p \le q \le r-2$ and the corresponding extremal graphs.

For the path, L. Kászonyi and Z. Tuza in [23] found $sat(n, P_m) = n - \lfloor \frac{n}{a_m} \rfloor$ where $a_m = 3 \cdot 2^{k-1} - 2$ if $m = 2k$ and $a_m = 2^{k+1} - 2$ if $m = 2k+1$, and they also characterized the family of extremal graphs. M. Frick and J. Singleton [18] proved $sat(n, P_n) = \lceil \frac{3n-2}{2} \rceil$ for $n \ge 54$ and several small order cases. For $t \ge 2$, let $F = P_{k_1} \cup P_{k_2} \cup \cdots \cup P_{k_t}$ be a linear forest with $k_1 \ge k_2 \ge \cdots \ge k_t$. G. Chen, J. Faudree, R. Faudree, G. Gould and C. Magnant [7] investigated the saturation numbers for forests and provided the upper and lower bounds on $sat(n, H)$ with $H \in \{F, tP_k, P_k \cup P_\ell\}$. Furthermore, they obtained the exact values of $sat(n, P_m \cup tP_2)$ with $m \in \{3, 4, 5\}$. S. Cao and H. Lei et al. [5] improved the lower bound on $sat(n, tP_3)$ in [7] and determined the exact values of $sat(n, tP_3)$ for $t = 4$, $n \ge 3t + 2$

and $t = 5$, $n \geq 3t + 1$. Moreover, they gave some counterexamples for the conjecture in [7] for $k \in \{4, 5\}$. Z. He, M. Lu and Z. Lv [22] improved the lower bound on $sat(n, tP_3)$ for $t \geq 1$ and $n \geq 10t$ in [7] and presented reasons to support the conjectures in [7]. Moreover, they gave some tP_3-saturated graphs that attained the upper bound in [7]. Q. Fan and C. Wang [14] proved that $sat(n, P_5 \cup tP_2) = min\{\lceil \frac{5n-4}{6} \rceil, 3t + 12\}$ for $n \geq 3t + 8$ with the extremal graphs $K_6 \cup (t-1)K_3 \cup \overline{K_{n-3t-3}}$ for $n > \frac{18t+76}{5}$. Recently J. Yan [29] showed that $sat(n, P_6 \cup tP_2) = min\{n - \lfloor \frac{n}{10} \rfloor, 3t + 18\}$ with the extremal graphs $K_7 \cup (t-1)K_3 \cup \overline{K_{n-3t-4}}$ for $n > \frac{10}{3}t + 20$. The known results about C_k-saturated graphs are mainly for small values of k. Please refer to [8, 12, 26, 27] for the exact values of $sat(n, C_k)$ with $k \leq 5$ and $sat(n, C_n)$. For $k \geq 6$, some lower bounds and upper bounds on $sat(n, C_k)$ are established in [1, 19, 20, 24, 30]. Please see an informative survey [11] for some detailed results in graph saturation.

By now there are some results on $sat(n, H_1 \cup H_2)$ if H_1 and H_2 have a same type such as the above cases when they are both paths or complete graphs. But for all we know, there are few results on $sat(n, H_1 \cup H_2)$ when H_1 and H_2 are of distinct types. In [25] we proved the bounds on connected saturation number $sat'(n, P_k \cup K_3)$ and gave the exact values of $sat'(n, P_k \cup K_3)$ with $k \in \{2, 3, 4\}$. In this paper we consider the $(P_k \cup S_\ell)$-saturated graphs on n vertices and prove the bounds on $sat(n, P_k \cup S_\ell)$. The paper is organized as follows. In Section 2 we prove the upper bounds on $sat(n, P_k \cup S_\ell)$ for $k \geq 6$ and $\ell \geq 4$. In Section 3 we discuss the structural properties of minimum $(P_k \cup S_\ell)$-saturated graph and get $sat(n, P_k \cup S_\ell) = n - \lfloor \frac{n-3(\ell-4)}{a_k} \rfloor$ on certain conditions. Moreover, we give a conjecture about the exact value of $sat(n, P_k \cup S_\ell)$ if ℓ is not less than some positive integer. In Section 4 we determine the value of $sat(n, P_k \cup S_4)$ for $k \geq 6$.

2. Upper bounds on $sat(n, P_k \cup S_\ell)$ with $k \geq 6$, $\ell \geq 4$

Firstly, we recall the P_k-saturated trees described in [23]. If $k \geq 5$, let T_k be a rooted (or double rooted) tree with $\lfloor \frac{k}{2} \rfloor$ levels in which every vertex has degree 3, except for the lowest level, and the highest level contains $k + 1 - 2\lfloor \frac{k}{2} \rfloor$ vertices. (see Figure 1 for $k = 6, 7$). Then

$$|T_k| = a_k = (3 + i)2^{\lfloor \frac{k}{2} \rfloor - 1} - 2, \quad k \equiv i \ (mod\ 2), \quad i \in \{0, 1\}.$$

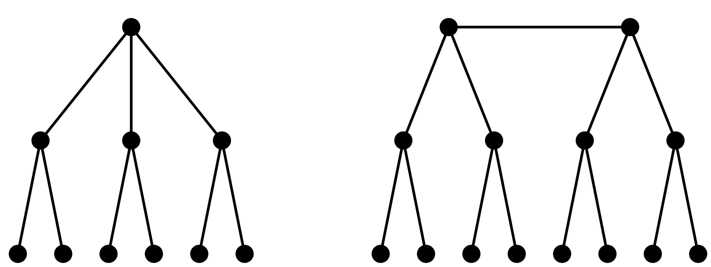

Figure 1 : T_6 (left) and T_7 (right).

In the following we denote a_k as the order of T_k.

Theorem 2.1 ([23]). *If T is a P_k-saturated tree, then $T_k \subset T$.*

Moreover, every P_k-saturated tree can be obtained from T_k by multiplying some branches or by adding more pendant vertices to the neighbors of leaves or by adding a single pendant vertex to other vertices of degree at least 3 as pointed out in [23].

Theorem 2.2 ([23]). *If $n \geq a_k$ and $k \geq 6$, then $sat(n, P_k) = n - \lfloor \frac{n}{a_k} \rfloor$ and every minimum P_k-saturated graph on n vertices consists of a forest with $\lfloor \frac{n}{a_k} \rfloor$ components each of which is P_k-saturated tree containing T_k as a subgraph.*

Let $\hat{T}_k$ be a rooted tree with root p and $\lfloor \frac{k}{2} \rfloor - 1$ levels when $k \geq 8$. If $k = 8$ or 9 and $k = 10$ or 11, let $\hat{T}_k$ be the trees shown in Figure 2. For $k \geq 12$, the structure of $\hat{T}_k$ is as follows: Let $\hat{T}_k$ be the tree obtained by attaching a pendant edge to the leaves in the lowest level of $\hat{T}_{k-2}$ and attaching a pendant edge to the vertices with degree 2 in the penultimate level of $\hat{T}_{k-2}$. Moreover, if $k = 8m$ or $8m + 1$, we also need to attach a P_4 to each leaf in the penultimate level of $\hat{T}_{k-2}$ in which the leaf is a central vertex of P_4.

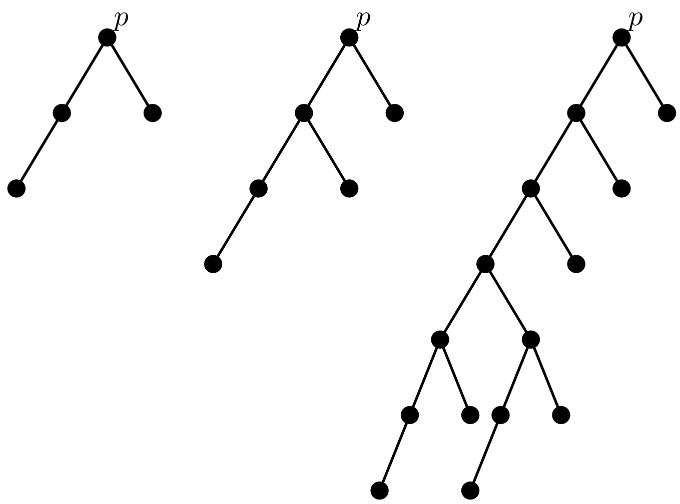

Figure 2 : $\hat{T}_k$ with $k = 8$ or 9 (left), $k = 10$ or 11 (center) and $k = 16$ or 17 (right).

Proposition 2.1. *For any non-edge $e = xy$ of $\hat{T}_k$ with $p \notin \{x, y\}$ and $1 \leq d_{\hat{T}_k}(x), d_{\hat{T}_k}(y) \leq 2$, the order of a longest path in $\hat{T}_k + e$ containing $e = xy$ and p as an endpoint is at least $\lfloor \frac{k}{2} \rfloor$.*

Proof. Let P be a longest path of $\hat{T}_k$ with an endpoint p and P' be a path in $\hat{T}_k + e$ containing $e = xy$ and p as an endpoint, which such that $(V(P) \cup V(P')) \backslash (V(P) \cap V(P')) \subset V(C)$ for a cycle C in $\hat{T}_k + e$. Then $|V(P)| = \lfloor \frac{k}{2} \rfloor - 1$ since $\hat{T}_k$ is a rooted tree with root p and $\lfloor \frac{k}{2} \rfloor - 1$ levels. Moreover, $V(P) \cap V(C)$ form an inferior arc and $V(P') \cap V(C)$ form a superior arc of cycle C. Thus $|V(P')| > |V(P)| = \lfloor \frac{k}{2} \rfloor - 1$, our result follows immediately. □

Here we construct a rooted tree $T_k^5 \subset T_k$ with root v and $\lceil \frac{k}{2} \rceil$ levels for $k \geq 6$. The structures of T_6^5 and T_7^5 are as follows:

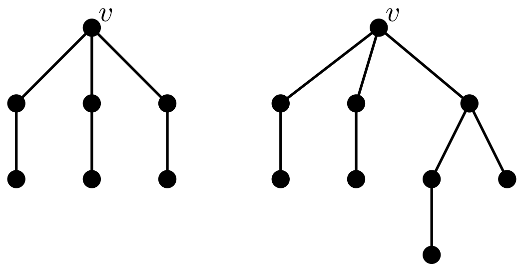

Figure 3 : T_6^5 (left) and T_7^5 (right).

For $k \geq 8$, if k is even, let T_k^5 be the tree obtained by attaching a $\hat{T}_k$ to each leaf of star S_4. If k is odd, let T_k^5 be the tree obtained by attaching a $\hat{T}_k$ to each of two leaves of star S_4 and attaching a $\hat{T}_{k+1}$ to the third leaf of star S_4. (see Figure 4 for $k = 15, 16$).

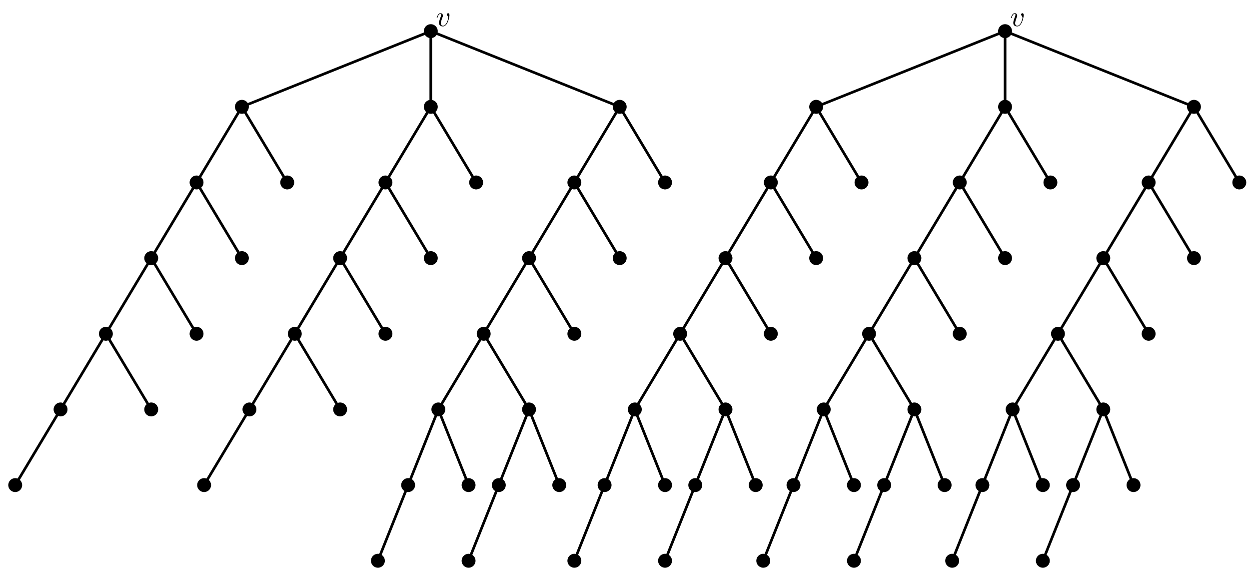

Figure 4 : T_{15}^5 (left) and T_{16}^5 (right).

By the construction of T_k^5 and calculation, for $k \geq 8$ and $m = \lfloor \frac{k}{8} \rfloor$, we can get

$$|T_k^5| = b_k = \begin{cases} [33 + 2(\lceil \frac{i}{2} \rceil + 2\lfloor \frac{i}{2} \rfloor)]2^{m-1} - 20, & \text{if } k = 8m + i,\ i \in \{0, 1, 2, 3, 4, 5, 6\}, \\ 56 \cdot 2^{m-1} - 20, & \text{if } k = 8m + 7. \end{cases}$$

Lemma 2.1. *Any non-edge e of T_k^5 is a P_k-saturating edge or S_5-saturating edge in it.*

Proof. Firstly T_k^5 is P_k-free and S_5-free since $diam(T_k^5) = k - 2$ and $\Delta(T_k^5) = 3$. This result is evident by routine verification for $k = 6$ or 7. For $k \geq 8$, let $N_{T_k^5}(v) = \{v_1, v_2, v_3\}$ with root v of T_k^5. We can see $T_{v_i} = \hat{T}_k$ for $i \in [3]$ if k is even, without loss of generality, we assume $T_{v_i} = \hat{T}_k$ for $i = 1, 2$ and $T_{v_3} = \hat{T}_{k+1}$ if k is odd. For any non-edge $e = xy$ of T_k^5, we divide into three cases based on the position of e and the symmetry of x and y.

Case 1. There is an endpoint of e, say x, with degree 3.

Clearly, e is a S_5-saturating edge of T_k^5 with $S_5 = S_x$ in $T_k^5 + e$ since $d_{T_k^5}(x) = 3$.

In the following we assume $1 \leq d_{T_k^5}(x), d_{T_k^5}(y) \leq 2$.

Case 2. x and y both belong to the same T_{v_i} for $i \in [3]$.

Without loss of generality, we say $\{x, y\} \subset T_{v_1}$. Let P^1 be a longest path in $T_{v_1} + e$ with an endpoint v_1 and $e \in P^1$, then $|P^1| \geq \lfloor \frac{k}{2} \rfloor$ by Proposition 2.1. Let P^2 be a longest path in T_{v_3} with an endpoint v_3, then P^2 has $\frac{k-1}{2}$ vertices if k is odd or has $\frac{k}{2} - 1$ vertices if k is even by the construction of T_k^5. Thus e is a P_k-saturating edge of T_k^5 with $P_k \subset P^1 v_1 v v_3 P^2$ in $T_k^5 + e$.

Similarly as above, we can also get that e is a P_k-saturating edge of T_k^5 if $\{x, y\} \subset T_{v_2}$ or $\{x, y\} \subset T_{v_3}$.

Case 3. x and y belong to different T_{v_i} for $i \in [3]$.

Without loss of generality, we say $x \in T_{v_1}$ and $y \in T_{v_2}$. Let P^3 be a longest path in T_{v_1} with x and v_1 as endpoints , P^4 be a longest path in T_{v_1} with an endpoint x and $v_1 \notin P^4$, $P^{3'}$ be a longest path in T_{v_2} with y and v_2 as endpoints, $P^{4'}$ be a longest path in T_{v_2} with an endpoint y and $v_2 \notin P^{4'}$. Then we can know $max\{|V(P^3)| + |V(P^{4'})|, |V(P^{3'})| + |V(P^4)|\} \geq \lfloor \frac{k}{2} \rfloor + 2$ since $|V(P^3)| + |V(P^4)| \geq \lfloor \frac{k}{2} \rfloor + 2$ and $|V(P^{3'})| + |V(P^{4'})| \geq \lfloor \frac{k}{2} \rfloor + 2$. Without loss of generality, we assume $|V(P^3)| + |V(P^{4'})| \geq \lfloor \frac{k}{2} \rfloor + 2$. Thus e is a P_k-saturating edge of T_k^5 with $P_k \subset P^{4'} y x P^3 v_1 v v_3 P^2$ in $T_k^5 + e$.

Similarly as above, we can also get that e is a P_k-saturating edge of T_k^5 if $x \in T_{v_1}$ and $y \in T_{v_3}$ or $x \in T_{v_2}$ and $y \in T_{v_3}$. □

Let T^* be the following tree on 5 vertices.

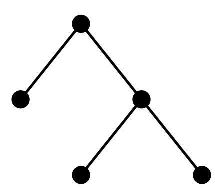

Figure 5 : T^*.

Here we construct a rooted tree $T'_k \subset T_k$ with root u and $\lceil \frac{k}{2} \rceil$ levels for $k \geq 9$. If k is odd, let T'_k be the tree obtained by attaching $\hat{T}_k$ to each leaf of T^*. If k is even, let T'_k be the tree obtained by attaching $\hat{T}_k$ to the leaf in the penultimate level of T^* and attaching $\hat{T}_{k-1}$ to the other two leaves of T^*. (see Figure 6 for $k = 16, 17$).

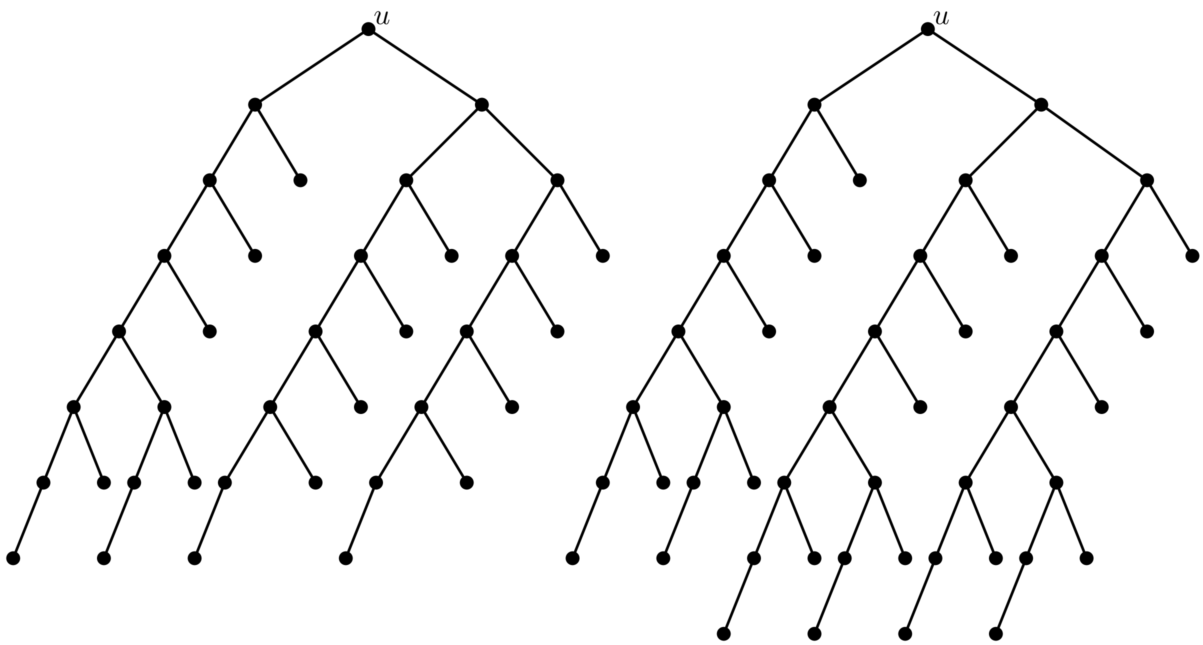

Figure 6 : T'_{16} (left) and T'_{17} (right).

Lemma 2.2. *Any non-edge e of T'_k with $u \notin e$ is P_k-saturating or S_5-saturating in it.*

Proof. Let $N_{T'_k(u)} = \{u_1, u_2\}$ with root u of T'_k and $N_{T'_k}(u_2) = \{w_1, w_2\}$. T'_k is P_k-free and S_5-free since $diam(T'_k) = k-2$ and $\Delta(T'_k) = 3$. We can see $T_{u_1} = \hat{T}_k$ and $T_{w_i} = \hat{T}_{k-1}$ for $i = 1, 2$ if k is even; $T_{u_1} = T_{w_i} = \hat{T}_k$ for $i = 1, 2$ if k is odd. For any non-edge $e = xy$ with $u \notin e$, we divide into three cases based on the position of e and the symmetry of x and y.

Case 1. There is an endpoint of e, say x, with degree 3.

Clearly, e is a S_5-saturating edge of T'_k with $S_5 = S_x$ in $T'_k + e$ since $d_{T'_k}(x) = 3$.

In the following we assume $1 \leq d_{T'_k}(x), d_{T'_k}(y) \leq 2$.

Case 2. x and y both belong to T_{u_1} or T_{w_1} or T_{w_2}.

Without loss of generality, we assume $\{x, y\} \subset T_{u_1}$. Let P^1 be a longest path in $T_{u_1} + e$ with an endpoint u_1 and $e \in P^1$, then $|P^1| \geq \lfloor \frac{k}{2} \rfloor$ by Proposition 2.1. Let P^2 is a longest path in T_{u_2} with an endpoint u_2, then P^2 has $\frac{k-1}{2}$ vertices if k is odd or has $\frac{k}{2} - 1$ vertices if k is even by the construction of T'_k. Thus e is a P_k-saturating edge of T'_k with $P_k \subset P^1 u_1 u u_2 P^2$ in $T'_k + e$.

Similarly as above, we can also get that e is a P_k-saturating edge of T'_k if $\{x, y\} \subset T_{w_1}$ or $\{x, y\} \subset T_{w_2}$.

Case 3. x and y belong to different trees in T_{u_1}, T_{w_1} and T_{w_2}.

Without loss of generality, we assume $x \in T_{u_1}$ and $y \in T_{w_1}$. Let P^3 be a longest path in T_{u_1} with x and u_1 as endpoints, P^4 be a longest path in T_{u_1} with an endpoint x and $u_1 \notin P^4$, $P^{3'}$ be a longest path in T_{w_1} with y and w_1 as endpoints, $P^{4'}$ be a longest path in T_{w_1} with an endpoint y and $w_1 \notin P^{4'}$. Then we can know $max\{|V(P^3)|+|V(P^{4'})|, |V(P^{3'})|+|V(P^4)|\} \geq \lfloor \frac{k}{2} \rfloor + 1$ since $|V(P^3)|+|V(P^4)| \geq \lfloor \frac{k}{2} \rfloor + 2$ and $|V(P^{3'})|+|V(P^{4'})| \geq \lfloor \frac{k}{2} \rfloor + 2$ if k is odd, $|V(P^{3'})|+|V(P^{4'})| \geq \lfloor \frac{k}{2} \rfloor + 1$ if k is even. Without loss of generality, we assume $|V(P^{3'})| + |V(P^4)| \geq \lfloor \frac{k}{2} \rfloor + 1$. Let

P^5 be a longest path in T_{w_2} with an endpoint w_2, then P^5 has $\lfloor \frac{k}{2} \rfloor - 1$ vertices if k is odd or has $\frac{k}{2} - 2$ vertices if k is even by the construction of T'_k. Thus $T'_k + e$ has P_k as a subgraph with $P_k \subset P^4 xy P^{3'} w_1 u_2 w_2 P^5$, which means that e is a P_k-saturating edge of T'_k.

Similarly as above, we can also get that e is a P_k-saturating edge of T'_k if $x \in T_{u_1}$ and $y \in T_{w_2}$ or $x \in T_{w_1}$ and $y \in T_{w_2}$. $\square$

Let $\tilde{T}$ be the following tree on 8 vertices. Denote by $T_{k,5}$ the rooted tree with root vertices $\{u, v\}$ obtained by attaching T'_k to each leaf u_i and v_i, $i \in [3]$ of $\tilde{T}$ for $k \geq 9$ and $|T_{k,5}| = c_k$.

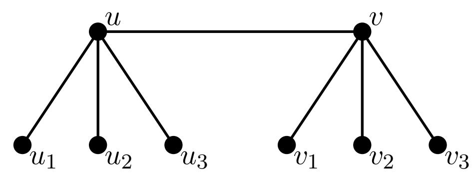

Figure 7 : $\tilde{T}$.

Theorem 2.3. *For $k \geq 9$, $T_{k,5}$ is a $(P_k \cup S_5)$-saturated tree.*

Proof. Firstly $T_{k,5}$ is $(P_k \cup S_5)$-free since $V(P_k) \cap V(S_\ell) \neq \varnothing$ for any subgraphs P_k and S_ℓ in $T_{k,5}$. Let $T_{k,5} - uv = T'_u \cup T'_v$ with $u \in T'_u$ and $v \in T'_v$, we can see that T_{u_i} and T_{v_i} are P_k-free for $i \in [3]$, T'_u and T'_v have P_k as a subgraph. For any non-edge e of a subgraph T'_k in $T_{k,5}$, $T_{k,5} + e$ has $P_k \cup S_5$ as a subgraph by Lemma 2.2 and the structure of $T_{k,5}$. For any non-edge e of $T_{k,5}$ with $u \in e$ or $v \in e$, say $u \in e$, we can get that $T_{k,5} + e$ has $P_k \cup S_5$ as a subgraph with $e \in S_5 \subset S_u$ and $P_k \subset T'_v$. Therefore, for any non-edge $e = xy$ of $T_{k,5}$, it suffices to consider the case that x and y belong to two different subgraphs T'_k of $T_{k,5}$, denoted $T_k^{'1}$ and $T_k^{'2}$.

Case 1. $T_k^{'1}$ and $T_k^{'2}$ both are subgraphs of T'_u or T'_v.

Without loss of generality, we assume $T_k^{'1}, T_k^{'2} \subset T'_u$, $x \in T_k^{'1} = T_{u_1}$ and $y \in T_k^{'2} = T_{u_2}$. If there is an endpoint of e with degree 3, then $T_{k,5} + e$ has $P_k \cup S_5$ as a subgraph with $e \in S_5$ and $P_k \subset T'_v$. If neither endpoint of e has degree 3, we can see that $(T_{u_1} \cup T_{u_2}) + e$ has P_k as a subgraph since $ecc_{T_{u_1}}(x) > \lceil \frac{k}{2} \rceil$ and $ecc_{T_{u_2}}(y) > \lceil \frac{k}{2} \rceil$. Then $T_{k,5} + e$ has $P_k \cup S_5$ as a subgraph with $e \in P_k$ and $S_5 = S_v$.

Case 2. $T_k^{'1} \subset T'_u$ and $T_k^{'2} \subset T'_v$.

Without loss of generality, we assume $x \in T_k^{'1} = T_{u_1}$ and $y \in T_k^{'2} = T_{v_1}$.

If there is an endpoint of e, say x, with degree 3, then $T_{k,5} + e$ has $P_k \cup S_5$ as a subgraph with $e \in S_5 = S_x$ and $P_k \subset T'_v$. If neither endpoint of e has degree 3, then $(T_{u_1} \cup T_{v_1}) + e$ must have a P_k as a subgraph with $u_1 \notin P_k$ or $v_1 \notin P_k$ by Lemma 2.2. Without loss of generality, we assume $u_1 \notin P_k$. Therefore, $T_{k,5} + e$ has $P_k \cup S_5$ as a subgraph with $e \in P_k$ and $S_5 = S_u$. $\square$

Lemma 2.3. *$c_k < a_k$ for $k \geq 14$.*

Proof. By the construction of $T_{k,5}$ and calculation, for $k \geq 9$ and $m = \lfloor \frac{k}{8} \rfloor$,

$$|T_{k,5}| = c_k = \begin{cases} 84 \cdot 2^m - 112, & \text{if } k = 8m, \\ [99 + 6(\lceil \frac{j-1}{2} \rceil + 2\lfloor \frac{j-1}{2} \rfloor)]2^m - 112, & \text{if } k = 8m + j,\ j \in \{1,2,3,4,5,6,7\}. \end{cases}$$

And $|T_k| = a_k = (3+i)2^{\lfloor \frac{k}{2} \rfloor - 1} - 2$, where $k \equiv i \ (mod\ 2)$, $i \in \{0, 1\}$. We can know $c_k < a_k$ for $k \geq 14$ by calculation. □

Let $r \geq \ell - 4$ be an integer and $\ell \geq 5$, we denote $T_{k,r,\ell}$ as the tree obtained by adding r pendant vertices to the lowest level of T_k and there are at least $\ell - 4$ newly-added pendant vertices which have a common neighbour in $T_{k,r,\ell}$. It is not difficult to see that $T_{k,r,\ell}$ is a P_k-saturated tree.

Theorem 2.4. (1). *For $n \geq a_k$ and $k \geq 9$ or $n \geq 2a_k$ and $k = 6, 7, 8$,*

$$sat(n, P_k \cup S_4) \leq n - \lfloor \frac{n}{a_k} \rfloor.$$

(2). *For $k \geq 6$, $\ell \geq 5$ and $n \geq 3(a_k + \ell - 4)$,*

$$sat(n, P_k \cup S_\ell) \leq n - \lfloor \frac{n - 3(\ell - 4)}{a_k} \rfloor.$$

(3). *For $k \geq 6$ and $n \geq max\{3(a_k + 1), c_k\}$,*

$$sat(n, P_k \cup S_5) \leq \begin{cases} n - \frac{n - c_k}{b_k} - 1, & \text{if } n = c_k + ab_k,\ a \geq 0 \text{ and } k \geq 14, \\ n - \lfloor \frac{n-3}{a_k} \rfloor, & \text{otherwise.} \end{cases}$$

Proof. (1). It is not difficult to verify that every minimum P_k-saturated graph on n vertices is also $(P_k \cup S_4)$-saturated for $n \geq a_k$ and $k \geq 9$ or $n \geq 2a_k$ and $k = 6, 7, 8$. Therefore, $sat(n, P_k \cup S_4) \leq sat(n, P_k) = n - \lfloor \frac{n}{a_k} \rfloor$ by Theorem 2.2.

(2). For $k \geq 6$, $\ell \geq 5$ and $n \geq 3(a_k + \ell - 4)$, let G^* be a minimum P_k-saturated graph on n vertices in which there are at least three tree components containing $T_{k,\ell-4,\ell}$, which is $T_{k,r,\ell}$ for $r = \ell - 4$, as a subtree. Thus we can get $m(G^*) = n - \lfloor \frac{n-3(\ell-4)}{a_k} \rfloor$ by Theorem 2.1. It suffices to show that G^* is $(P_k \cup S_\ell)$-saturated. Firstly G^* is $(P_k \cup S_\ell)$-free since G^* is P_k-free. For any non-edge e of G^*, $G^* + e$ has P_k as a subgraph with $e \in P_k$ since G^* is P_k-saturated. Moreover, by the construction of G^*, we can get that there exists at least one subgraph S_ℓ which has no common vertex with P_k in $G^* + e$. Therefore, G^* is $(P_k \cup S_\ell)$-saturated, which means $sat(n, P_k \cup S_\ell) \leq m(G^*) = n - \lfloor \frac{n-3(\ell-4)}{a_k} \rfloor$.

(3). For $a \geq 0$, let $\tilde{G} = T_{k,5} \cup aT_k^5$ on $c_k + ab_k$ vertices. We can verify that $\tilde{G}$ is $(P_k \cup S_5)$-saturated by Lemma 2.1 and Theorem 2.3. Moreover, $m(\tilde{G}) = n - \frac{n-c_k}{b_k} - 1 < n - \lfloor \frac{n-3}{a_k} \rfloor$ for $k \geq 14$ and $n = c_k + ab_k$ by Lemma 2.3 and $b_k < a_k$. Thus we can improve the upper bound of $sat(n, P_k \cup S_5)$ in (2) by $sat(n, P_k \cup S_5) \leq n - \frac{n-c_k}{b_k} - 1$ for $n = c_k + ab_k$ and $k \geq 14$. □

3. The structure of minimum $(P_k \cup S_\ell)$-saturated graph

For $k \geq 6$ and $\ell \geq 5$, let G be a minimum $(P_k \cup S_\ell)$-saturated graph on $n \geq 3(a_k + \ell - 4)$ vertices. We can say $m(G) = n - p$ with $p \geq \lfloor \frac{n-3(\ell-4)}{a_k} \rfloor \geq 3$ by Theorem 2.4. Then there are at least p tree components in G. Moreover, there is exactly one tree component which is either K_1 or P_2 if K_1 or P_2 is a component of G. Denote by $\mathcal{T} = \{T^1, T^2, T^3, \ldots, T^t\}$ the set of tree components of G which are not K_1 or P_2, then $t \geq p - 1 \geq 2$.

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Proposition 3.1. *A graph G is S_ℓ-saturated if and only if $d_G(u) \leq \ell - 2$ for every $u \in V(G)$ and $xy \in E(G)$ for $max\{d_G(x), d_G(y)\} \leq \ell - 3$.*

Proof. The necessity can be given by the definition of S_ℓ-saturated graph. For a graph G, if $d_G(u) \leq \ell - 2$ for every $u \in V(G)$, then G is S_ℓ-free. Moreover, $xy \in E(G)$ for $max\{d_G(x), d_G(y)\} \leq \ell - 3$ is equivalent to the fact that $d_G(x) \geq \ell - 2$ or $d_G(y) \geq \ell - 2$ if xy is a non-edge in G, then $G + xy$ contains S_ℓ as a subgraph. Thus G is a S_ℓ-saturated graph. □

Theorem 3.1. *Let G be a minimum $(P_k \cup S_\ell)$-saturated graph on $n \geq 3(a_k + \ell - 4)$ vertices with $k \geq 6$ and $\ell \geq 5$, $\mathcal{T} = \{T^1, T^2, T^3, \ldots, T^t\}$ $(t \geq 2)$ be the set of tree components of G which are not K_1 or P_2. Then $\mathcal{T}$ satisfies one of the following properties:*

(1). *There exists a tree component, say T^1, which contains S_ℓ and P_k as subgraphs and is $(P_k \cup S_\ell)$-saturated; T^i is S_ℓ-free and P_k-free for $i \neq 1$.*

(2). *There exists a tree component, say T^1, which contains S_ℓ as a subgraph and is P_k-saturated; there exists a tree component, say T^2, which is S_ℓ-free and P_k-saturated; T^i is P_k-saturated for $i \neq 1, 2$.*

(3). *There exists a tree component, say T^1, which is S_ℓ-free and P_k-saturated; T^i is P_k-saturated for $i \neq 1$.*

(4). *T^i is S_ℓ-free and P_k-free for $i \in [t]$.*

(5). *There exists a tree component, say T^1, which contains S_ℓ as a subgraph and is P_k-saturated; T^i is P_k-saturated for $i \neq 1$.*

Proof. Firstly T^i is not S_ℓ-saturated for $i \in [t]$ by Proposition 3.1. For any non-edge e of G, we have $e \in P_k$ or $e \in S_\ell$ in $G + e$ since G is $(P_k \cup S_\ell)$-saturated. We can claim that there has no tree component in $\mathcal{T}$ which is S_ℓ-free and has P_k as a subgraph. Otherwise, let $T^1 \in \mathcal{T}$ be a S_ℓ-free tree with $P_k \subset T^1$. Then G is S_ℓ-free since G is $(P_k \cup S_\ell)$-free. For any two nonadjacent leaves x and y of T^1, we can know that $xy \in P_k$ in $T^1 + xy$ since xy is not a S_ℓ-saturating edge of T^1. Thus $G + xy$ has no $P_k \cup S_\ell$ as a subgraph since $xy \in P_k$ and G is S_ℓ-free, a contradiction. Therefore, for a tree component, say T^1, in $\mathcal{T}$, T^1 is P_k-free and S_ℓ-free or is P_k-free and has S_ℓ as a subgraph or has P_k and S_ℓ as subgraphs.

Case 1. T^1 has P_k and S_ℓ as subgraphs.

In this case, we can claim that T^1 is $(P_k \cup S_\ell)$-saturated. Otherwise, T^1 contains a non-edge e such that $T^1 + e$ has no $P_k \cup S_\ell$ as a subgraph and $G + e$ has $P_k \cup S_\ell$ as a subgraph. Without loss of generality, we assume $e \in P_k$ in $T^1 + e$, then $G - T^1$ contains S_ℓ as a subgraph. Thus G contains $P_k \cup S_\ell$ since $P_k \subset T^1$, which contradicts that G is $(P_k \cup S_\ell)$-free. Moreover, T^i is P_k-free and S_ℓ-free for $i \neq 1$. Thus $\mathcal{T}$ satisfies the property (1).

Case 2. T^1 is P_k-free and S_ℓ-free.

For any two nonadjacent leaves x and y of T^1, $xy \in P_k \subset T^1 + xy$ in $G + xy$. Then there exists a component, say G_1, in $G - T^1$ with $S_\ell \subset G_1$ since T^1 is S_ℓ-free and $G + xy$ has $P_k \cup S_\ell$ as a subgraph. Thus $G - G_1$ is P_k-free.

Subcase 2.1. $G_1 \in \mathcal{T} - T^1$.

We say $G_1 = T^2 \in \mathcal{T} - T^1$ with $S_\ell \subset T^2$. If T^2 has P_k as a subgraph, similarly as the proof of Case 1, we can get that T^2 is $(P_k \cup S_\ell)$-saturated and $\mathcal{T}$ satisfies the property (1). If T^2 is P_k-free, then G is P_k-free. We can claim that T^i is P_k-saturated for $i \in [t]$. If not, there exists a T^i with a S_ℓ-oversaturating edge e which is not a P_k-saturating edge, then $G+e$ has no $P_k \cup S_\ell$ since $e \in S_\ell$ and G is P_k-free, a contradiction. Thus $\mathcal{T}$ satisfies the property (2).

Subcase 2.2. $G_1 \in G - \mathcal{T}$.

Clearly G_1 is not K_1 or P_2 since $S_\ell \subset G_1$, then G_1 is a non-tree component of G. If G_1 is P_k-free, then G is P_k-free. Similarly as the proof of Subcase 2.1, T^i is P_k-saturated for $i \in [t]$. Thus $\mathcal{T}$ satisfies the property (3). If G_1 has P_k as a subgraph, then T^i is P_k-free and S_ℓ-free for $i \in [t]$ since G is $(P_k \cup S_\ell)$-free. Thus $\mathcal{T}$ satisfies the property (4).

Case 3. T^1 is P_k-free and has S_ℓ as a subgraph.

G is P_k-free since $S_\ell \subset T^1$ and G is $(P_k \cup S_\ell)$-free. By a similar reasoning as that in the proof of Subcase 2.1, we can know that T^i is P_k-saturated for $i \in [t]$. Thus $\mathcal{T}$ satisfies the property (5). □

Theorem 3.2. *Let G be a minimum $(P_k \cup S_\ell)$-saturated graph on $n \geq 3(a_k + \ell - 4)$ vertices with $k \geq 8$ and $\ell \geq 6$. If $\mathcal{T}$ satisfies the property (2) or (3) or (5) of Theorem 3.1, then $sat(n, P_k \cup S_\ell) = n - \lfloor \frac{n-3(\ell-4)}{a_k} \rfloor$.*

Proof. If $\mathcal{T}$ satisfies the property (2) or (3) or (5) of Theorem 3.1, then T^i is P_k-saturated for $i \in [t]$. By Theorem 2.1, $T_k \subset T^i$ for $i \in [t]$. We claim that there is no K_1 or P_2 as a tree component of G. If not, without loss of generality, we denote $T_0 = P_2$ as a tree component of G. For any subgraph $T_k \subset T^i \in \mathcal{T}$, $i \in [t]$ and $t \geq 2$, let $x \in V(T_0)$ and y be any vertex from the highest level to the third lowest level of T_k. We consider $G + xy$, then $xy \in S_\ell$ in $G + xy$ since $diam((T^i \cup T_0) + xy) \leq k - 2$. Thus there exists a non-tree component G_1 with $P_k \subset G_1$ of G since $G + xy$ has $P_k \cup S_\ell$ as a subgraph and $\mathcal{T} \cup T_0$ is P_k-free. Moreover, we can get $d_G(y) = \ell - 2 \geq 4$ for $\ell \geq 6$ since G is $(P_k \cup S_\ell)$-free. Therefore, we have $n \geq \sum_{i=1}^{t} |T^i| + |P_2| + |G_1| > (t+1)a_k + 3(\ell - 4)$ for $k \geq 8$ and $\ell \geq 6$ by calculation, then $m(G) \geq n - (t+1) > n - \lfloor \frac{n-3(\ell-4)}{a_k} \rfloor$, which contradicts to $sat(n, P_k \cup S_\ell) \leq n - \lfloor \frac{n-3(\ell-4)}{a_k} \rfloor$ by Theorem 2.4. Thus all tree components of G are P_k-saturated. Moreover, G contains at least three S_ℓ as subgraphs with different central vertices since G is a minimum $(P_k \cup S_\ell)$-saturated graph. Therefore $e(G) = n - p$ with $p \leq \lfloor \frac{n-3(\ell-4)}{a_k} \rfloor$, which means $e(G) \geq n - \lfloor \frac{n-3(\ell-4)}{a_k} \rfloor$. By the result (2) in Theorem 2.4, we have $sat(n, P_k \cup S_\ell) = n - \lfloor \frac{n-3(\ell-4)}{a_k} \rfloor$. □

Let T be a rooted tree which satisfies the property of T^i, $i \neq 1$ in the property (1) and T^i, $i \in [t]$ in the property (5) of Theorem 3.1, which means that T is a P_k-free and S_ℓ-free tree on n vertices such that any non-edge e of T is a P_k-saturating edge or S_ℓ-saturating edge of T. If e is a P_k-saturating edge for any non-edge e of T, then T is a P_k-saturated tree. By Theorem 2.1, we can know $|T| \geq a_k$. Here we consider that T contains a non-edge $e = xy$ which is a S_ℓ-saturating edge but not a P_k-saturating edge of T. Thus $d_T(x) = \ell - 2$ or $d_T(y) = \ell - 2$. Now we consider the vertex with degree 2 in T.

If there exists a vertex u with degree 2 in T, $N_T(u) = \{u_1, u_2\}$ and T has no two adjacent vertices with degree 2, then we have $d_T(u_1) = \ell - 2$ or $d_T(u_2) = \ell - 2$. Otherwise, we consider $T + u_1u_2$ but $T + u_1u_2$ has no P_k or S_ℓ as subgraph since $d_T(u_1), d_T(u_2) \leq \ell - 3$ and $d_T(u) = 2$, a contradiction. Therefore, there are three consecutive vertices with degree sequence $\{\ell - 2, 2, a\}$ and $3 \leq a \leq \ell - 2$ or $\{\ell - 2, 2, 1\}$.

If there exist two adjacent vertices u and v with degree 2 in T, $N_T(u) = \{u_1, v\}$ and $N_T(v) = \{v_1, u\}$, then we can claim $d_T(u_1) = \ell - 2$ and $d_T(v_1) = \ell - 2$. Otherwise, we assume $d_T(u_1) < \ell - 2$ and consider $T + u_1v$, then $u_1v \in P_k$ in $T + u_1v$ since $d_T(u_1), d_T(v) \leq \ell - 3$. Thus we can get that T has P_k as a subgraph by replacing u_1v in P_k of $T + u_1v$ with a $P_3 = u_1uv$, a contradiction. Moreover, we can get that there has no vertex in $N_T(u_1) \cup N_T(u_2)$ with degree 1. Otherwise, we assume $w \in N_T(u_1)$ and $d_T(w) = 1$. We consider $T + wv$ and $wv \in P_k$ in $T + wv$ since $d_T(w), d_T(v) \leq \ell - 3$. Thus we can get that T has P_k as a subgraph by replacing vwu_1 in P_k of $T + wv$ with vuu_1 or replacing $wvuu_1$ in P_k of $T + wv$ with v_1vuu_1, a contradiction. For $2 \leq b, c \leq \ell - 3$, if there exist $w_1 \in N_T(u_1)$ and $w_2 \in N_T(v_1)$ with $d_T(w_1) = b$ and $d_T(w_2) = c$, we consider $T + uw_2$ and $T + vw_1$. Then there have $P_k \subset P^1w_2uvv_1P^2$ in $T + uw_2$ and $P_k \subset P^3w_1vuu_1P^4$ in $T + vw_1$, where P^1 is a longest path with an endpoint w_2 and $v_1 \notin P^1$, P^2 is a longest path with an endpoint v_1 and $\{v, w_2\} \notin P^2$, P^3 is a longest path with an endpoint w_1 and $u_1 \notin P^3$, P^4 is a longest path with an endpoint u_1 and $\{u, w_1\} \notin P^4$. Thus $|P^1|+|P^2|+2 \geq k$ and $|P^3|+|P^4|+2 \geq k$. Since T is P_k-free, we can know $|P^1|+|P^3|+4 \leq k-1$, then $|P^1| < |P^4|-2$. Thus $k \leq |P^1|+|P^2|+2 < |P^4|-2+|P^2|+2 = |P^2|+|P^4| < |P^2|+|P^4|+2$, which means that there has $P_k \subset P^2v_1vuu_1P^4$ in T, a contradiction. Therefore, there are six consecutive vertices with degree sequence $\{\ell-2, \ell-2, 2, 2, \ell-2, b\}$ and $2 \leq b \leq \ell - 2$.

If there exist at least three consecutive vertices u, v and w with degree 2 in T and $N_T(v) = \{u, w\}$, we consider $T + uw$ and $uw \in P_k$ in $T + uw$ since $d_T(u) = d_T(w) = 2$. Thus we can get that T has P_k as a subgraph by replacing uw in P_k of $T + uw$ with a $P_3 = uvw$, a contradiction.

Here we write the number in curly braces as the degree of vertices in the level of figures.

For $k \geq 6$, $\ell \geq 5$ and $3 \leq a \leq \ell - 2$, let $\dot{T}_k^\ell$ be a rooted tree with $\lceil \frac{k}{2} \rceil$ levels and root v. If k is even, the degree sequence of vertices from the highest level to the penultimate level of $\dot{T}_k^\ell$ is $\{\ell-2, 2, a, 2, \ell-2, 2, a, 2, \ell-2, 2\cdots\}$ and $N_{\dot{T}_k^\ell}(v) = \{v_1, v_2, \ldots, v_{\ell-2}\}$. If k is odd, let $\dot{T}_k^\ell$ be the tree obtained from $\dot{T}_{k+1}^\ell$ by deleting the vertices in the lowest level of T_{v_i}, $i = 2, 3, \ldots, \ell-2$. (see Figure 8 for $k = 13, 14$). We can verify that $\dot{T}_k^\ell$ is P_k-free, S_ℓ-free and any non-edge of $\dot{T}_k^\ell$ is P_k-saturating or S_ℓ-saturating in it. Moreover, we have $|\dot{T}_k^\ell| > a_k$ for $\ell \geq 11$ by calculation.

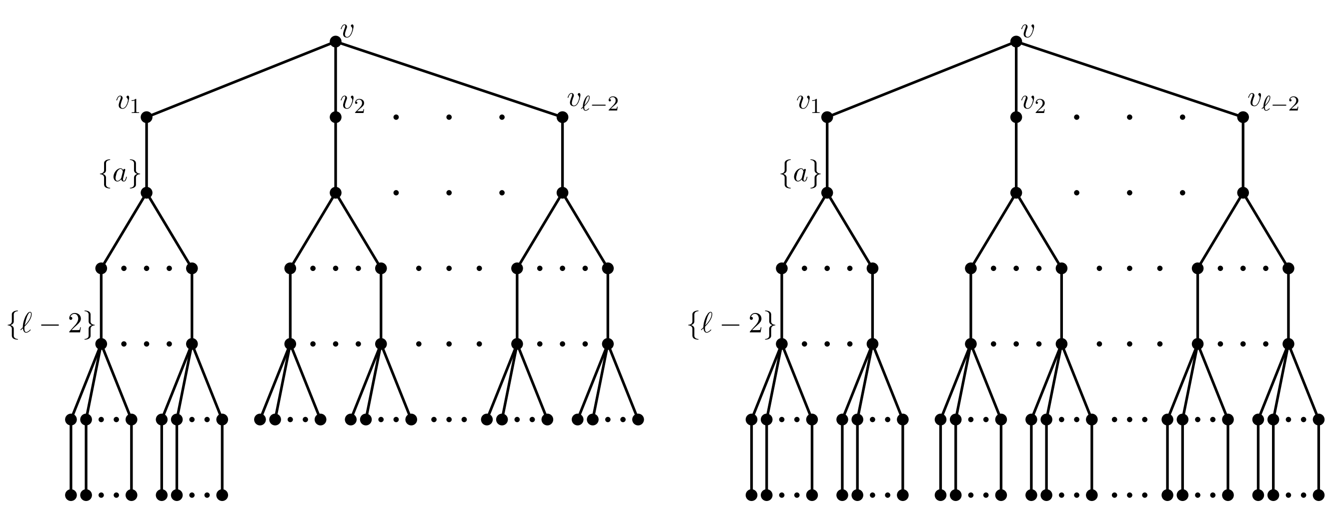

Figure 8 : $\dot{T}_{13}^{\ell}$ (left) and $\dot{T}_{14}^{\ell}$ (right).

For $k \geq 6$, $\ell \geq 5$ and $3 \leq a \leq \ell - 2$, let $\hat{T}_k^{\ell}$ be a rooted tree with $\lceil \frac{k}{2} \rceil$ levels and root v. If k is even, the degree sequence of vertices from the highest level to the penultimate level of $\hat{T}_k^{\ell}$ is $\{\ell - 2, 2, a, a, a \cdots\}$ and $N_{\hat{T}_k^{\ell}}(v) = \{v_1, v_2, \ldots, v_{\ell-2}\}$. If k is odd, let $\hat{T}_k^{\ell}$ be the tree obtained from $\hat{T}_{k+1}^{\ell}$ by deleting the vertices in the lowest level of T_{v_i}, $i = 2, 3, \ldots, \ell - 2$. (see Figure 9 for $k = 10, 11$). We can verify that $\hat{T}_k^{\ell}$ is P_k-free, S_ℓ-free and any non-edge of $\hat{T}_k^{\ell}$ is P_k-saturating or S_ℓ-saturating in it. Moreover, we have $|\hat{T}_k^{\ell}| > a_k$ for $\ell \geq 9$ by calculation.

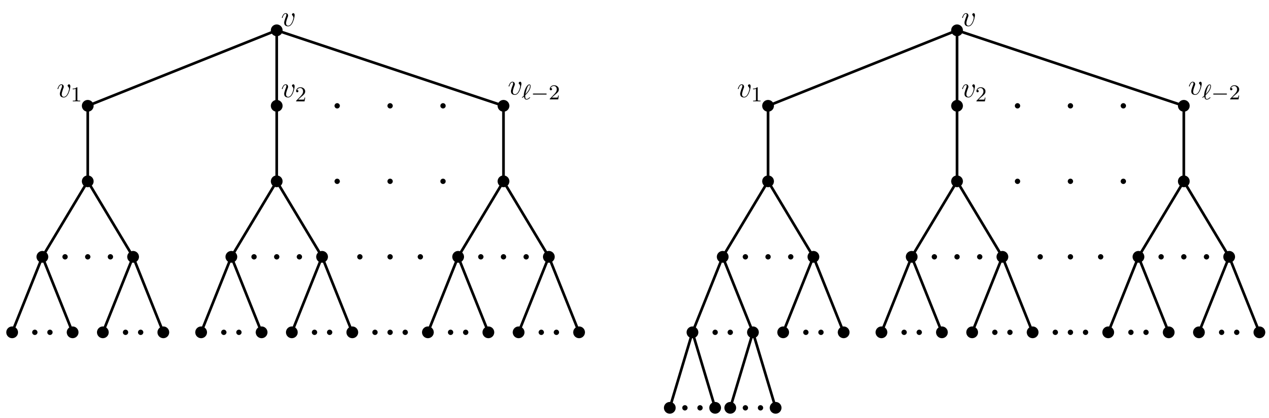

Figure 9 : $\hat{T}_{10}^{\ell}$ (left) and $\hat{T}_{11}^{\ell}$ (right).

For $\ell \geq 5$, $2 \leq a \leq \ell - 2$ and $k = 12 + 8b + c$ with $b \geq 0$, $c \in \{0, 1, 2\}$, let $\tilde{T}_k^{\ell}$ be a rooted tree with $\lceil \frac{k}{2} \rceil$ levels and root v. If k is even (i.e.$c = 0, 2$), the degree sequence of vertices from the highest level to the penultimate level of $\tilde{T}_k^{\ell}$ is $\{\ell - 2, 2, 2, \ell - 2, \ell - 2, 2, 2, \ell - 2 \cdots\}$ and $N_{\tilde{T}_k^{\ell}}(v) = \{v_1, v_2, \ldots, v_{\ell-2}\}$. If k is odd (i.e.$c = 1$), let $\tilde{T}_k^{\ell}$ be the tree obtained from $\tilde{T}_{k+1}^{\ell}$ by deleting the vertices in the lowest level of T_{v_i}, $i = 2, 3, \ldots, \ell - 2$. (see Figure 10 for $k = 12, 14$). We can verify that $\tilde{T}_k^{\ell}$ is P_k-free, S_ℓ-free and any non-edge of $\tilde{T}_k^{\ell}$ is P_k-saturating or S_ℓ-saturating in it. Moreover, we have $|\tilde{T}_k^{\ell}| > a_k$ for $\ell \geq 7$ by calculation.

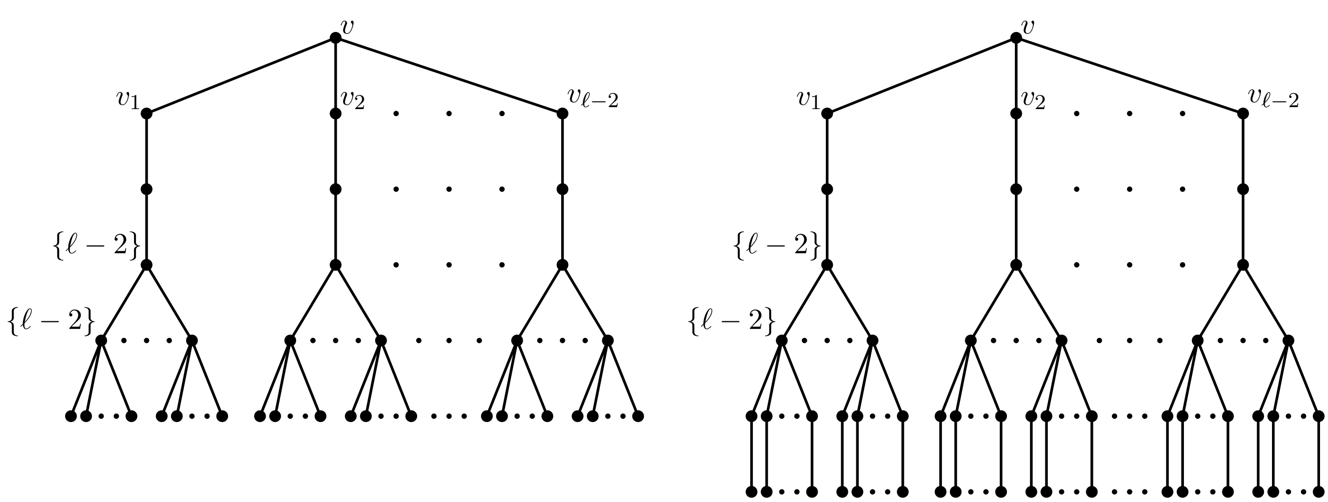

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Figure 10 : $\tilde{T}_{12}^{\ell}$ (left) and $\tilde{T}_{14}^{\ell}$ (right).

For $k \geq 6$ and $\ell \geq 5$, let $\dot{T}_k^{\ell'}$, $\hat{T}_k^{\ell'}$ and $\tilde{T}_k^{\ell'}$ be trees obtained by $\dot{T}_k^{\ell}$, $\hat{T}_k^{\ell}$ and $\tilde{T}_k^{\ell}$ changing the degree of the root vertex to $\ell - 3$, respectively. Let $\dot{T}$ be the following tree on $2\ell - 2$ vertices. Denote by $T_{k,\ell}$ the tree obtained by attaching $\dot{T}_k^{\ell'}$ or $\hat{T}_k^{\ell'}$ or $\tilde{T}_k^{\ell'}$ to each leaf u_i and v_i, $i \in [\ell - 2]$ of $\dot{T}$ for $k, \ell \geq 6$. We can verify that $T_{k,\ell}$ is $(P_k \cup S_\ell)$-saturated and $|T_{k,\ell}| > a_k + 3(\ell - 4)$ by calculation.

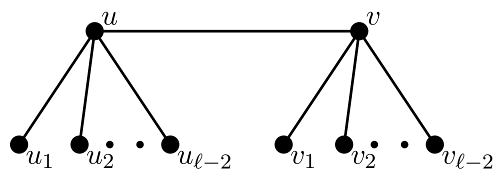

Figure 11 : $\dot{T}$.

Conjecture 3.1. *Let T be a tree in which each non-edge e is a P_k-saturating edge or S_ℓ-saturating edge of T, T' be a $(P_k \cup S_\ell)$-saturated tree. For $k \geq 6$, there exists a positive integer ℓ_0 such that $|T| \geq a_k$ and $|T'| > a_k + 3(\ell - 4)$ for $\ell \geq \ell_0$.*

Conjecture 3.2. *There exists a positive integer ℓ_0 such that $sat(n, P_k \cup S_\ell) = n - \lfloor \frac{n-3(\ell-4)}{a_k} \rfloor$ for $\ell \geq \ell_0$ and $k \geq 6$.*

4. $sat(n, P_k \cup S_4)$ with $k \geq 6$

Let G be a minimum $(P_k \cup S_4)$-saturated graph with $n \geq 2a_k$ vertices and $k \geq 6$, we say $e(G) = n - p$ since $sat(n, P_k \cup S_4) \leq n - \lfloor \frac{n}{a_k} \rfloor$, then there are at least p tree components of G and $p \geq \lfloor \frac{n}{a_k} \rfloor \geq 2$. We can easily verify that $e(G) = n - \lfloor \frac{n}{a_k} \rfloor$ if $p = 2$, since $n \geq 2a_k$ and $e(G) \leq n - \lfloor \frac{n}{a_k} \rfloor$. In the following result we focus on determining the structure of each tree component of G.

Theorem 4.1. *For $n \geq 2a_k$ and $k \geq 6$, let G be a minimum $(P_k \cup S_4)$-saturated graph on n vertices and there are at least three tree components in G. Then T is P_k-saturated for any tree component T of G.*

Proof. It is clear that there is exactly one tree component which is either K_1 or P_2 if K_1 or P_2 is a component of G. Let $\mathcal{T} = \{T^1, T^2, T^3, \ldots, T^t\}$ be the set of tree components of G which are not K_1 or P_2, then $t \geq 2$. Similarly as the proof of Theorem 3.1, we can know that $\mathcal{T}$ satisfies one of the following properties:

(1). There exists a tree component, say T^1, which contains S_4 and P_k as subgraphs and is $(P_k \cup S_4)$-saturated; T^i is P_k-free and S_4-free for $i \neq 1$.

(2). There exists a tree component, say T^1, which contains S_4 as a subgraph and is P_k-saturated; there exists a tree component, say T^2, which is S_4-free and P_k-saturated; T^i is P_k-saturated for $i \neq 1, 2$.

(3). There exists a tree component, say T^1, which is S_4-free and P_k-saturated; T^i is P_k-saturated for $i \neq 1$.

(4). T^i is S_4-free and P_k-free for $i \in [t]$.

(5). There exists a tree component, say T^1, which contains S_4 as a subgraph and is P_k-saturated; T^i is P_k-saturated for $i \neq 1$.

But the properties (2) and (3) cannot happen since a P_k-saturated tree contains S_4 as a subgraph by Theorem 2.1. We can verify that the properties (1) and (4) cannot happen. Otherwise, there exists a P_k-free and S_4-free tree T^2 in G since $t \geq 2$. Then $d_{T^2}(x) \leq 2$ for every $x \in V(T^2)$, which means that T^2 is a P_k-free path. Let x and y be the endpoints of T^2, we consider $G + xy$. But $xy \notin P_k$ and $xy \notin S_4$ in $G + xy$, then $G + xy$ has no $P_k \cup S_4$ as a subgraph, a contradiction. Thus T^i is P_k-saturated for $i \in [t]$. Moreover, there is no K_1 or P_2 as a tree component of G. If not, we denote T_0 as the tree component of G which is K_1 or P_2. Let x be a vertex in T_0 and y be a root vertex of $T_k \subset T^i$ for some $i \in [t]$. We consider $G + xy$, then $xy \in S_\ell$ in $G + xy$ since $diam((T^i \cup T_0) + xy) \leq k - 2$. Thus there exists a component G_1 with $P_k \subset G_1$ of $G - (\mathcal{T} \cup T_0)$ since $G + xy$ has $P_k \cup S_\ell$ as a subgraph and $\mathcal{T} \cup T_0$ is P_k-free. Then G has $P_k \cup S_4$ as a subgraph since $S_4 \subset T_k \subset T^i \in \mathcal{T}$ for $i \in [t]$, which contradicts to that G is $(P_k \cup S_4)$-free. Therefore, T is P_k-saturated for any tree component T of G. □

Theorem 4.2. *$sat(n, P_k \cup S_4) = n - \lfloor \frac{n}{a_k} \rfloor$ with $n \geq 2a_k$ and $k \geq 6$.*

Proof. Let G be a minimum $(P_k \cup S_4)$-saturated graph with $n \geq 2a_k$ vertices and $k \geq 6$, then we can say $e(G) = n - p$ since $sat(n, P_k \cup S_4) \leq n - \lfloor \frac{n}{a_k} \rfloor$ and $e(G) = n - \lfloor \frac{n}{a_k} \rfloor$ if $p = 2$. For $p \geq 3$, we can get that any tree component T of G is P_k-saturated by Theorem 4.1, then $T_k \subset T$ and $|T| \geq a_k$ by Theorem 2.1. Therefore $p \leq \lfloor \frac{n}{a_k} \rfloor$ and $e(G) \geq n - \lfloor \frac{n}{a_k} \rfloor$. Our result follows immediately by Theorem 2.4. □

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References

[1] C.A. Barefoot, L.H. Clark, R.C. Entringer, T.D. Porter, L.A. Székely, Z. Tuza, Cycle-saturated graphs of minimum size, Discrete Math. **150** (1996) 31–48.

[2] J. Balogh, H. Liu, On the number of K_4-saturating edges, J. Combin. Theory Ser. B **109** (2014) 250–257.

[3] T. Bohman, M. Fonoberova, O. Pikhurko, The saturation function of complete partite graphs, J. Comb. **1** (2010) 149–170.

[4] A. Bondy, M.R. Murty, *Graph Theory*, Springer-Verlag, London, 2008.

[5] S. Cao, H. Lei, X. Lian, S. Yao, J. Zhang, Saturation numbers for tP_k with k less than 6, Discrete Appl. Math. **325** (2023) 108–119.

[6] F. Chen, X. Yuan, Some results on the saturation number for unions of cliques, arXiv.2303.11893.

[7] G. Chen, J.R. Faudree, R.J. Faudree, R.J. Gould, M.S. Jacobson, C. Magnant, Results and problems on saturation numbers for linear forests, Bull. Inst. Combin. Appl. **75** (2015) 29–46.

[8] Y. Chen, Minimum C_5-saturated graphs, J. Graph Theory **61** (2009) 111–126.

[9] Y. Chen, Minimum $K_{2,3}$-saturated graphs, J. Graph Theory **76** (2014) 309–322.

[10] G. Chen, R.J. Gould, F. Pfender, B. Wei, Extremal graphs for intersecting cliques, J. Combin. Theory Ser. B **89** (2003) 159–171.

[11] B.L. Currie, J.R. Faudree, R.J. Faudree, J.R. Schmitt, A survey of minimum saturation graphs, Electron. J. Comb. **18** (2021) #DS19.

[12] P. Erdős, A. Hajnal, J. Moon, A problem in graph theory, Amer. Math. Monthly **71** (1964) 1107–1110.

[13] P. Erdős, Z. Füredi, R.J. Gould, D.S. Gunderson, Extremal graphs for intersecting triangles, J. Combin. Theory Ser. B **64** (1995) 89–100.

[14] Q. Fan, C. Wang, Saturation numbers for linear forests $P_5 \cup tP_2$, Graphs Combin. **31** (2015) 2193–2200.

[15] J. Faudree, R.J. Faudree, R.J. Gould, M.S. Jacobson, Saturation numbers for trees, Electron. J. Comb. **16** (2009) 91–109.

[16] J.R. Faudree, R.J. Gould, Saturation numbers for nearly complete graphs, Graphs Combin. **29** (2013) 429–448.

[17] R. Faudree, M. Ferrara, R. Gould, M. Jacobson, tK_p-saturated graphs of minimum size, Discrete Math. **309** (2009) 5870–5876.

[18] M. Frick, J. Singleton, Lower bound for the size of maximal nontraceable graphs, Electron. J. Combin. **12** (2005) 32–40.

[19] Z. Furedi, Y. Kim, Cycle-saturated graphs with minimum number of edges, J. Graph Theory **73** (2013) 203–215.

[20] R. Gould, T. Luczak, J. Schmitt, Constructive upper bounds for cycle-saturated graphs of minimum size, Electron. J. Combin. **13** (2006) 29–47.

[21] J. He, F. Ma, J. Ma, X. Ye, The minimum number of clique-saturating edges, J. Combin. Theory Ser. B **159** (2023) 42–60.

[22] Z. He, M. Lu, Z. Lv, Minimum tP_3-saturation graphs, Discrete Appl. Math. **327** (2023) 148–156.

[23] L. Kászonyi, Z. Tuza, Saturated graphs with minimal number of edges, J. Graph Theory **10** (1986) 203–210.

[24] Y. Lan, Y. Shi, Y. Wang, J. Zhang, The saturation number of C_6, arXiv:2108.03910v1.

[25] Y. Li, K. Xu, Minimizing the number of edges in $(P_k \cup K_3)$-saturated connected graphs, RAIRO-Oper. Res. **57** (2023) 447–458.

[26] X. Lin, W. Jiang, C. Zhang, Y. Yang, On smallest maximally non-Hamiltonian graphs, Ars Combin. **45** (1997) 263–270.

[27] Z. Tuza, C_4-saturated graphs of minimum size, Acta Univ. Carolin. Math. Phys. **30** (1989) 161–167.

[28] J. Wang, W. Yang, The Turán number for spanning linear forests, Discrete Appl. Math. **254** (2019) 291–294.

[29] J. Yan, Saturation numbers for linear forests $P_6 + tP_2$, arXiv:2106.06466v1.

[30] M. Zhang, S. Luo, M. Shigeno, On the number of edges in a minimum C_6-saturated graph, Graphs Combin. **31** (2015) 1085–1106.