Right (b, c)-core inverses in rings

Bing Dong[a], Taohua Jin[a], Huihui Zhu[a,*]

[a]*School of Mathematics, Hefei University of Technology, Hefei 230009, China.*

Abstract

For any a, b, c of a $*$-ring R, the element a is called right (b, c)-core invertible if there exists some $x \in bR$ such that $caxc = c$ and $(cax)^* = cax$. In this paper, several criteria of right (b, c)-core inverses are established. It is shown that a is right (b, c)-core invertible if and only if a is right (b, c)-invertible and c is $\{1, 3\}$-invertible. In addition, the matrix representation of right (b, c)-core inverses is presented. Finally, we present the relations of right (b, c)-core inverses and other generalized inverses. As applications, several known results on right core inverses and right w-core inverses are given as corollaries.

Keywords: (b, c)-core inverses, right (b, c)-inverses, w-core inverses, right w-core inverses, Moore-Penrose inverses
2010 MSC: 15A09, 16W10

1. Introduction

The inverse along an element [9] and the (b, c)-inverse [5] are two important classes of outer generalized inverses, which recover the Drazin inverse [7] and the Moore-Penrose inverse [12]. They are intensively investigated by lots of researchers (see [2, 3, 4, 10]). In 2016, one-sided inverses along an element [15] were introduced. Shortly afterwards, one-sided (b, c)-inverses [6] were given to extend one-sided inverses along an element and (b, c)-inverses.

In 2023, the present author Zhu in [14] seeking new ways to combine (b, c)-inverses and $\{1, 3\}$-inverses to obtain the (b, c)-core inverse in the context of $*$-semigroups, generalizing the core inverse [1], the core-EP inverse [8] and

*Corresponding author

Email addresses: bdmath@163.com (Bing Dong), jthmath@163.com (Taohua Jin), hhzhu@hfut.edu.cn (Huihui Zhu)

the Moore-Penrose inverse, the w-core inverse [18], the right and left w-core inverse [17, 19].

In this paper, we aim to introduce and investigate right (b, c)-core inverses in a $*$-ring. This provides a framework for the theory of generalized inverses.

The paper is organized as follows. In Section 2, for any a, b, c of a $*$-ring R, we define a right (b, c)-core inverse of a, and investigate the corresponding properties. For instance, it is shown that a is right (b, c)-core invertible if and only if a is right (b, c)-invertible and c is $\{1, 3\}$-invertible. Moreover, $a^{\oplus}_{r,(b,c)} = a_r^{(b,c)} c^{(1,3)}$. Then, we characterize right (b, c)-core inverses in terms of properties of the left annihilators and ideals. Further, we present the matrix representations of right (b, c)-core inverses by the Pierce decomposition. In Section 3, we state that several generalized inverses, that is right inverses, right core inverses, right pseudo core inverses, right w-core inverses and Moore–Penrose inverses, are instances of right (b, c)-core inverses. Precisely, for any nonnegative integers m, n satisfying $m + n \geq 1$, we establish the following equivalences in a $*$-ring

(1) a is right invertible if and only if a is right $(1, 1)$-core invertible;

(2) a is right core invertible if and only if a^m is right (a^n, a)-core invertible;

(3) a is right pseudo core invertible if and only if a^m is right (a^n, a^k)-core invertible, for some positive integer k;

(4) a is right w-core invertible if and only if w is right (a, a)-core invertible;

(5) a is Moore-Penrose invertible if and only if a is right (a^*, a^*)-core invertible if and only if a^* is right (a, a)-core invertible.

As applications, we give several characterizations for right core inverses and right w-core inverses and establish the connection between right pseudo core inverses and right w-core inverses. The relation schema of right (b, c)-core inverses and the aforementioned (right) inverses is provided as well.

Let us now recall several notions of generalized inverses.

Let R be an associative ring with unity 1. An element $a \in R$ is called (von Neumann) regular if there exists some $x \in R$ such that $axa = a$. Such an x is called an inner inverse or a $\{1\}$-inverse of a and denoted by a^-. The symbol $a\{1\}$ stands for the set of all inner inverses of a. The set of all regular elements in R is denoted by R^-.

In [15], Zhu et al. extended inverses along an element to one-sided cases. Let $a, d \in R$. An element a is called left invertible along d if there exists some $x \in R$ such that $xad = d$ and $x \in Rd$. Such an element x is called a left inverse of a along d, and is denoted by $a_l^{\| d}$. Dually, an element a is called right invertible along d if there exists some $y \in R$ such that $day = d$ and $y \in dR$. Such an element y is called a right inverse of a along d, and is denoted by $a_r^{\parallel d}$. We use the symbols $R_l^{\parallel d}$ and $R_r^{\parallel d}$ to denote the sets of all left and right invertible elements along d in R, respectively. According to [15, Theorems 2.3 and 2.4], a is left invertible along d if and only if $d \in Rdad$, and a is right invertible along d if and only if $d \in dadR$.

In 2016, Drazin defined one-sided (b, c)-inverses [6]. Let $a, b, c \in R$. We call a left (b, c)-invertible if $b \in Rcab$, or equivalently if there exists $x \in Rc$ such that $xab = b$, in which case, any such x will be called a left (b, c)-inverse of a and denoted by $a_l^{(b,c)}$. Dually, a is right (b, c)-invertible if $c \in cabR$, or equivalently if there exists $y \in bR$ such that $cay = c$, in which case, any such y will be called a right (b, c)-inverse of a and denoted by $a_r^{(b,c)}$. In particular, a is called (b, c)-invertible [5] if it is both left and right (b, c)-invertible. We denote by $R_l^{(b,c)}$, $R_r^{(b,c)}$ and $R^{(b,c)}$ the sets of all left (b, c)-invertible, right (b, c)-invertible and (b, c)-invertible elements in R. It should be pointed out that a is right (d, d)-invertible if and only if it is right invertible along d. Moreover, the right (d, d)-inverse of a is exactly the right inverse of a along d.

A map $* : R \to R$ is an involution of R if it satisfies $(x^*)^* = x$, $(xy)^* = y^*x^*$ and $(x + y)^* = x^* + y^*$ for all $x, y \in R$. Throughout this section, any ring R is assumed to be a unital $*$-ring, that is a ring R with unity 1 and an involution $*$.

An element $a \in R$ is said to be Moore-Penrose invertible [12] if there exists some $x \in R$ such that $axa = a$, $xax = x$, $(ax)^* = ax$ and $(xa)^* = xa$. Such an x is called a Moore-Penrose inverse of a. It is unique if it exists, and is denoted by $a^\dagger$. Generally, any solution x satisfying the equations $axa = a$ and $(ax)^* = ax$ (resp., $(xa)^* = xa$) is called a $\{1, 3\}$-inverse (resp., $\{1, 4\}$-inverse) of a. The symbols $a^{(1,3)}$ and $a^{(1,4)}$ denote a $\{1, 3\}$-inverse and a $\{1, 4\}$-inverse of a, respectively. We denote by $a\{1, 3\}$ and $a\{1, 4\}$ the sets of all $\{1, 3\}$-inverses and $\{1, 4\}$-inverses of a. In general, the sets of all $\{1, 3\}$-invertible, $\{1, 4\}$-invertible and Moore-Penrose invertible elements in R will be denoted by $R^{\{1,3\}}$, $R^{\{1,4\}}$ and $R^\dagger$, respectively. It is known that a is Moore-Penrose invertible if and only if it is both $\{1, 3\}$-invertible and $\{1, 4\}$-invertible. An element $p \in R$ is called a projection if $p^2 = p = p^*$.

An element $a \in R$ is right pseudo core invertible if there exist $x \in R$ and positive integer k such that $axa^k = a^k$, $(ax)^* = ax$ and $ax^2 = x$. Such an x is called a right pseudo core inverse of a and denoted by $a_r^{\textcircled{D}}$. The smallest positive integer k, denoted by $\mathrm{I}(a)$, is called the right pseudo core index of a. In particular, a is called right core invertible when a is right pseudo core invertible with $\mathrm{I}(a) = 1$. In general, $R_r^{\#}$ and $R_r^{\mathbb{D}}$ denote the sets of all right core and right pseudo core invertible elements in R.

The right w-core inverse [19] was introduced in R, which unifies right core inverses, right pseudo core inverses and Moore–Penrose inverses. For any $a, w \in R$, we call a right w-core invertible if there exists some $x \in R$ such that $awxa = a$, $(awx)^* = awx$ and $awx^2 = x$. Any such x is called a right w-core inverse of a, and is denoted by $a_{r,w}^{\#}$. The symbol $R_{r,w}^{\#}$ denotes the set of all right w-core invertible elements in R. It was proved that a is right w-core invertible if and only if w is right invertible along a and a is $\{1,3\}$-invertible, in which case, $a_{r,w}^{\#} = w_r^{\|a} a^{(1,3)}$.

The (b,c)-core inverse was defined in a $*$-monoid M in [14]. For the convenience, we next state this notion in R. Let $a, b, c \in R$. The element a is called (b,c)-core invertible if there exists some $x \in R$ such that $caxc = c$, $xR = bR$ and $Rx = Rc^*$. The (b,c)-core inverse of a is uniquely determined (if it exists) and is denoted by $a_{(b,c)}^{\#}$. As usual, we denote by $R_{(b,c)}^{\#}$ the set of all (b,c)-core invertible elements in R. It was proved in [14] that the (b,c)-core inverse x of a is the unique solution to the system $x \in bR$, $caxc = c$, $(cax)^* = cax$ and $xcab = b$. It follows from [14, Theorem 2.6] that a is (b,c)-core invertible if and only if a is (b,c)-invertible and c is $\{1,3\}$-invertible.

2. Right (b,c)-core inverses

Our main goal in this section is to introduce the right (b,c)-core inverse in a unital $*$-ring R, and to give its several characterizations.

Definition 2.1. *Let $a, b, c \in R$. We call a right (b,c)-core invertible if there exists some $x \in bR$ such that $caxc = c$ and $(cax)^* = cax$. Such an x is called a right (b,c)-core inverse of a.*

By the symbol $a_{r,(b,c)}^{\#}$ we denote a right (b,c)-core inverse of a. An element $a \in R$ could have different right (b,c)-core inverses. For instance, let R be a unital $*$-ring. Take $c = 0 \neq b = 1 \in R$. For any $x \in R$, we have $caxc = c$ and $(cax)^* = cax$. Hence, any $x \in R$ is a right $(1,0)$-core inverse of a. However, the product $caa_{r,(b,c)}^{\#}$ is invariant. Indeed, suppose $z_1, z_2 \in R$ are any two right (b,c)-core inverses of a. It is known that $cx = cy$ for any $x, y \in c\{1,3\}$ (see, e.g., [18, Remark 2.10]). Since $az_1, az_2 \in c\{1,3\}$, we have $caz_1 = caz_2$.

The symbol $R^{\oplus}_{r,(b,c)}$ stands for the set of all right (b,c)-core invertible elements in R.

It is noteworthy to mention that every (b,c)-core invertible element is right (b,c)-core invertible. The converse statement is not valid in general. For instance, let R be the same as that of the previous example. Take $c = 0 \neq b \in R$ and $x \in bR$, then a is right (b,c)-core invertible. Clearly, $xcab = 0 \neq b$, and so that a is not (b,c)-core invertible.

Given any $a \in R$, we write $a^0 = \{x \in R : ax = 0\}$. It is known that (see, e.g., [13]) $Ra \subseteq Rb$ ensures $b^0 \subseteq a^0$ for any $a, b \in R$.

A list of characterizations for right (b,c)-core inverses are given by ideals and annihilators.

Theorem 2.2. *Let $a, b, c \in R$. The following conditions are equivalent*:

(i) $a \in R^{\oplus}_{r,(b,c)}$.

(ii) *There exists some $x \in bR$ such that $caxc = c$, $(cax)^* = cax$ and $xcax = x$.*

(iii) *There exists some $x \in R$ such that $caxc = c$, $xR \subseteq bR$ and $Rx = Rc^*$.*

(iv) *There exists some $x \in R$ such that $caxc = c$, $xR \subseteq bR$ and $x^0 = (c^*)^0$.*

(v) *There exists some $x \in R$ such that $caxc = c$, $xR \subseteq bR$ and $Rx \subseteq Rc^*$.*

(vi) *There exists some $x \in R$ such that $caxc = c$, $xR \subseteq bR$ and $(c^*)^0 \subseteq x^0$.*

(vii) *There exist a projection $p \in R$ and an idempotent $q \in R$ such that $cR \subseteq pR \subseteq caR$, $qR \subseteq bR$ and $Rq \supseteq Rca$.*

In this case, $a^{\oplus}_{r,(b,c)} = q(ca)^- p$ for any $(ca)^- \in (ca)\{1\}$.

Proof. (iii) $\Rightarrow$ (iv) and (v) $\Rightarrow$ (vi) are obvious.

(i) $\Rightarrow$ (ii) Assume $a \in R^{\oplus}_{r,(b,c)}$. Then there exists some $y \in bR$ such that $cayc = c$ and $(cay)^* = cay$. Let $x = ycay$. We get $cax = ca(ycay) = (cayc)ay = cay = (cax)^*$, $caxc = cayc = c$ and $xcax = xcay = (ycay)cay = ycay = x$.

(ii) $\Rightarrow$ (iii) From $caxc = c$ and $(cax)^* = cax$, it follows that $c^* = c^*cax \in Rx$. Also, $xcax = x$ implies $x = x(cax)^* = xx^*a^*c^* \in Rc^*$, as required.

(iv) $\Rightarrow$ (v) Since $c^* = c^*(cax)^*$, we have $1 - (cax)^* \in (c^*)^0 = x^0$, so that $x = x(cax)^* = xx^*a^*c^* \in Rc^*$.

(vi) $\Rightarrow$ (vii) By $c^* = c^*(cax)^*$ and $(c^*)^0 \subseteq x^0$, we obtain $x = x(cax)^*$. This in turn gives $cax = cax(cax)^* = (cax)^*$. Set $p = cax$ and $q = xca$, then $p^2 = p = p^*$ and $q^2 = q$. Therefore, $cR = pcR \subseteq pR \subseteq caR$, $qR \subseteq bR$ and $Rca = Rcaq \subseteq Rq$.

(vii) $\Rightarrow$ (i) Given $Rq \supseteq Rca$, then $ca = caq$. From $cR \subseteq pR \subseteq caR$, it follows that $c = pc$ and $p = caz$ for some $z \in R$. Therefore, $ca = pca = cazca$, so that $ca \in R^-$. Let $x = q(ca)^-p$ for any $(ca)^- \in (ca)\{1\}$. Then

(1) $x = q(ca)^-p \in bR$ by $qR \subseteq bR$.

(2) $cax = caq(ca)^-p = ca(ca)^-caz = caz = p = (cax)^*$.

(3) $caxc = pc = c$.

Thus, $a \in R^{\oplus}_{r,(b,c)}$ and $a^{\oplus}_{r,(b,c)} = q(ca)^-p$ for any $(ca)^- \in (ca)\{1\}$. □

Lemma 2.3. [20, Lemma 2.2] *Let $a \in R$. Then*

(i) $a \in R^{\{1,3\}}$ *if and only if* $a \in Ra^*a$. *In particular, if* $xa^*a = a$ *for some* $x \in R$, *then* x^* *is a* $\{1,3\}$*-inverse of* a.

(ii) $a \in R^{\{1,4\}}$ *if and only if* $a \in aa^*R$. *In particular, if* $aa^*y = a$ *for some* $y \in R$, *then* y^* *is a* $\{1,4\}$*-inverse of* a.

Suppose $a \in R^{\oplus}_{r,(b,c)}$ with a right (b,c)-core inverse x. Then $caxc = c$, we hence deduce that $cax = (cax)^n$ for any positive integer n. It is concluded that $a \in R^{\oplus}_{r,(b,c)}$ implies $x \in bR$, $(cax)^n c = c$ and $((cax)^n)^* = (cax)^n$ for any positive integer n. One may ask whether the converse implication holds. The following theorem gives a positive answer.

Theorem 2.4. *Let $a, b, c \in R$. The following conditions are equivalent*:

(i) $a \in R^{\oplus}_{r,(b,c)}$.

(ii) $c \in R(cab)^*c$.

(iii) $c \in cabR \cap Rc^*c$.

(iv) *There exists some* $x \in bR$ *such that* $(cax)^n c = c$ *and* $((cax)^n)^* = (cax)^n$ *for any positive integer* n.

(v) *There exists some* $x \in bR$ *such that* $(cax)^n c = c$ *and* $((cax)^n)^* = (cax)^n$ *for some positive integer* n.

In this case, $a^{\oplus}_{r,(b,c)} = x(cax)^{n-1}$.

Proof. (i) $\Rightarrow$ (ii) Given $a \in R^{\oplus}_{r,(b,c)}$, then there exists some $x \in bR$ such that $caxc = c$, $(cax)^* = cax$. As a result, $c = caxc = (cax)^*c \in (cabR)^*c = R(cab)^*c$.

(ii) $\Leftrightarrow$ (iii) by [14, Lemma 2.8 (I)].

(iii) $\Rightarrow$ (iv) As $c \in cabR \cap Rc^*c$, then $c = cabt = sc^*c$ for some $t, s \in R$, in which case, $s^* \in c\{1,3\}$ by Lemma 2.3. Let $x = bts^*$. Then $x \in bR$, $cax = cabts^* = cs^* = (cax)^*$ and $caxc = cs^*c = c$. One hence gets $cax =$

$caxc \cdot ax = (cax)^2 = \cdots = (cax)^n$ for any positive integer n. In consequence, $c = caxc = (cax)^n c$ and $(cax)^n = cax = ((cax)^n)^*$.

(iv) $\Rightarrow$ (v) is clear.

(v) $\Rightarrow$ (i) Suppose that there exists some $x \in bR$ such that $(cax)^n c = c$ and $((cax)^n)^* = (cax)^n$ for some positive integer n. Then $y = x(cax)^{n-1}$ is a right (b,c)-core inverse of a. Indeed,

(1) $y = x(cax)^{n-1} \in bR$.

(2) $cayc = (cax)^n c = c$.

(3) $cay = (cax)^n = ((cax)^n)^* = (cay)^*$. □

Let us present a lemma which will be useful in the upcoming results.

Lemma 2.5. [6, Definition 1.2] *Let $a, b, c \in R$. Then*

(i) *a is left (b,c)-invertible if and only if $b \in Rcab$. In this case, $a_l^{(b,c)} = sc$, where $s \in R$ satisfies $b = scab$.*

(ii) *a is right (b,c)-invertible if and only if $c \in cabR$. In this case, $a_r^{(b,c)} = bt$, where $t \in R$ satisfies $c = cabt$.*

Let $a, b, c \in R$. From [14, Theorem 2.6], Zhu showed that a is (b,c)-core invertible if and only if a is (b,c)-invertible and c (ca or cab) is $\{1,3\}$-invertible. An analogous result on right (b,c)-core inverses can be obtained.

Theorem 2.6. *Let $a, b, c \in R$. The following conditions are equivalent*:

(i) $a \in R^{\oplus}_{r,(b,c)}$.

(ii) $a \in R_r^{(b,c)}$ *and* $c \in R^{\{1,3\}}$.

(iii) $a \in R_r^{(b,c)}$ *and* $ca \in R^{\{1,3\}}$.

(iv) $a \in R_r^{(b,c)}$ *and* $cab \in R^{\{1,3\}}$.

In this case, $a^{\oplus}_{r,(b,c)} = a_r^{(b,c)} c^{(1,3)} = a_r^{(b,c)} a (ca)^{(1,3)} = b(cab)^{(1,3)} cab (cab)^{(1,3)}$.

Proof. (i) $\Leftrightarrow$ (ii) directly by Lemmas 2.3, 2.5 and Theorem 2.4 (i) $\Leftrightarrow$ (iii).

(ii) $\Rightarrow$ (iii) As $c \in R^{\{1,3\}}$, one gets $c \in Rc^*c$ by Lemma 2.3. This gives $ca \in Rc^*ca$, which together with $a \in R_r^{(b,c)}$ ensures $ca \in Rc^*ca \subseteq R(cabR)^*ca = R(cab)^*ca \subseteq R(ca)^*ca$. So, $ca \in R^{\{1,3\}}$.

(iii) $\Rightarrow$ (iv) can be proved by a similar way of (ii) $\Rightarrow$ (iii).

(iv) $\Rightarrow$ (ii) Given $cab \in R^{\{1,3\}}$, then $cab \in R(cab)^*cab$ by Lemma 2.3. From $a \in R_r^{(b,c)}$, we get $c \in cabR$ by Lemma 2.5. Then there exists some $t \in R$ such that $c = cabt \in R(cab)^*cabt = R(cab)^*c \subseteq Rc^*c$, so that $c \in R^{\{1,3\}}$.

29 Jul 2024 02:48:52 PDT
240321-ZhuHuihui Version 2 - Submitted to Rocky Mountain J. Math.

We next show that $y = a_r^{(b,c)} c^{(1,3)}$ is a right (b,c)-core inverse of a.

(1) $y = a_r^{(b,c)} c^{(1,3)} \in bR$.

(2) $cay = caa_r^{(b,c)} c^{(1,3)} = cc^{(1,3)} = (cay)^*$.

(3) $cayc = cc^{(1,3)}c = c$.

In addition, it is necessary to prove $a(ca)^{(1,3)} \in c\{1,3\}$. Indeed, $ca(ca)^{(1,3)} = (ca(ca)^{(1,3)})^*$, and $ca(ca)^{(1,3)}c = ca(ca)^{(1,3)}cabt = cabt = c$ by the implication of (iv) $\Rightarrow$ (ii). Analogously, $ab(cab)^{(1,3)} \in c\{1,3\}$. We hence have $c = cab(cab)^{(1,3)}c$, and whence $a_r^{(b,c)} = b(cab)^{(1,3)}c$ by Lemma 2.5.

So, $a^{\oplus}_{r,(b,c)} = a_r^{(b,c)} c^{(1,3)} = a_r^{(b,c)} a(ca)^{(1,3)} = b(cab)^{(1,3)}cab(cab)^{(1,3)}$. □

Suppose $a \in R^{\oplus}_{r,(b,c)}$. Theorem 2.6 guarantees $cab \in R^{\{1,3\}}$, and therefore, $cab \in R^-$. Again as $a \in R^{\oplus}_{r,(b,c)}$, then, by Theorem 2.4, $c \in cabR$ and $c = cabt$ for some $t \in R$. It follows that $c = cab(cab)^- cabt = cab(cab)^- c$ for any $(cab)^- \in (cab)\{1\}$. This implies $a_r^{(b,c)} = b(cab)^- c$ from Lemma 2.5. Hence, another representation of $a^{\oplus}_{r,(b,c)}$ can be presented.

Proposition 2.7. *Let $a, b, c \in R$ with $a \in R^{\oplus}_{r,(b,c)}$. Then $a^{\oplus}_{r,(b,c)} = b(cab)^- cc^{(1,3)}$, for any $(cab)^- \in (cab)\{1\}$ and $c^{(1,3)} \in c\{1,3\}$.*

Remark 2.8. In Theorem 2.4, the right (b,c)-core inverse of a can be represented as bz^*cabz^* provided that $z \in R$ satisfies $c = z(cab)^*c$ by Theorem 2.6. Indeed, since $c = z(cab)^*c = z(ab)^*c^*c \in Rc^*c$, we have $abz^* \in c\{1,3\}$ by Lemma 2.3. Thus, $c = cabz^*c \in cabR$, it follows that $a_r^{(b,c)} = bz^*c$ from Lemma 2.5. So, $a^{\oplus}_{r,(b,c)} = a_r^{(b,c)} c^{(1,3)} = bz^*cabz^*$ by Theorem 2.6.

Characterizations for (b,c)-core inverses are described in terms of properties of the left (right) annihilators and ideals in [14]. It was shown that $a \in R^{\oplus}_{(b,c)}$ if and only if $R = R(cab)^* \oplus {}^0c = Rca \oplus {}^0b$ if and only if $R = R(cab)^* + {}^0c = Rca + {}^0b$. Inspired by this, we consider to derive the characterization for right (b,c)-core inverse of a in R.

Theorem 2.9. *Let $a, b, c \in R$. Then the following statements are equivalent:*

(i) $a \in R^{\oplus}_{r,(b,c)}$.

(ii) $R = R(cab)^* \oplus {}^0c$.

(iii) $R = R(cab)^* + {}^0c$.

PROOF. (i) $\Rightarrow$ (ii) Since $a \in R^{\oplus}_{r,(b,c)}$, we have $c \in R(cab)^*c$ by Theorem 2.4, hence $c = r(cab)^*c = rb^*a^*c^*c$ for some $r \in R$ and $1 - r(cab)^* \in {}^0c$. For any $s \in R$, we have $s = s[(1 - r(cab)^*) + r(cab)^*] = s(1 - r(cab)^*) + sr(cab)^* \in {}^0c + R(cab)^*$, so that $R = {}^0c + R(cab)^*$. Note also that $abr^* \in c\{1, 3\}$. Then for any $z \in R(cab)^* \cap {}^0c$, then $zc = 0$ and there exists some $t \in R$ such that $z = t(cab)^* = t(cc^{(1,3)}cab)^* = t(cabr^*cab)^* = t(cab)^*(cabr^*)^* = zcabr^* = 0$. So, $R = R(cab)^* \oplus {}^0c$.

(ii) $\Rightarrow$ (iii) is trivial.

(iii) $\Rightarrow$ (i) Given $R = R(cab)^* + {}^0c$, then $c \in Rc \subseteq R(cab)^*c$. So, $a \in R^{\oplus}_{r,(b,c)}$ by Theorem 2.4. $\square$

For any $p^2 = p \in R$, any element $a \in R$ can be written as

$$a = pap + pa(1 - p) + (1 - p)ap + (1 - p)a(1 - p)$$

or the matrix form

$$a = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}_p,$$

where $a_1 = pap$, $a_2 = pa(1 - p)$, $a_3 = (1 - p)ap$ and $a_4 = (1 - p)a(1 - p)$. The above decomposition is well known as the Pierce decomposition.

If $p^2 = p = p^*$, then

$$a^* = \begin{bmatrix} a_1^* & a_3^* \\ a_2^* & a_4^* \end{bmatrix}_p.$$

We next give the matrix representations of right (b, c)-core inverses.

Theorem 2.10. *Let $a, b, c \in R$. The following conditions are equivalent*:

(i) $a \in R^{\oplus}_{r,(b,c)}$ *and $x \in R$ is a right (b, c)-core inverse of a.*

(ii) *There exists a projection $p \in R$ such that*

$$a = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}_p, \ b = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}_p, \ c = \begin{bmatrix} c_1 & c_2 \\ 0 & 0 \end{bmatrix}_p \text{ and } x = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}_p, \quad (2.1)$$

where $(c_1a_1+c_2a_3)x_1+(c_1a_2+c_2a_4)x_3 = p$, $(c_1a_1+c_2a_3)x_2+(c_1a_2+c_2a_4)x_4 = 0$ and $\mathcal{R}(x) \subseteq \mathcal{R}(b)$ ($\mathcal{R}(b)$ denotes the column space of b).

(iii) *There exists a projection $q \in R$ such that*

$$a = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}_q, \ b = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}_q, \ c = \begin{bmatrix} 0 & 0 \\ c_3 & c_4 \end{bmatrix}_q \text{ and } x = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}_q,$$

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where $(c_3a_1+c_4a_3)x_1+(c_3a_2+c_4a_4)x_3 = 0$, $(c_3a_1+c_4a_3)x_2+(c_3a_2+c_4a_4)x_4 = 1-q$ *and* $\mathcal{R}(x) \subseteq \mathcal{R}(b)$.

Proof. (i) ⇒ (ii) Suppose $a \in R^{\oplus}_{r,(b,c)}$ with a right (b,c)-core inverse x. Then $x \in bR$, $caxc = c$ and $(cax)^* = cax$. Let $p = cax$. Then $p^2 = p = p^*$. So, a, b, c and x can be represented as (2.1). From $x \in bR$, it follows that $\mathcal{R}(x) \subseteq \mathcal{R}(b)$. By the Pierce decomposition, we have

$$\begin{aligned} & (c_1a_1+c_2a_3)x_1+(c_1a_2+c_2a_4)x_3 \\ = \; & (pcp\cdot pap+pc(1-p)\cdot(1-p)ap)pxp \\ & +(pcp\cdot pa(1-p)+pc(1-p)\cdot(1-p)a(1-p))(1-p)xp \\ = \; & (cpap+c(1-p)ap)pxp+(cpa(1-p)+c(1-p)a(1-p))(1-p)xp \\ = \; & capxp+ca(1-p)xp \\ = \; & caxp = p^2 \\ = \; & p. \end{aligned}$$

The equality $(c_1a_1+c_2a_3)x_2+(c_1a_2+c_2a_4)x_4 = 0$ can be proved similarly.

(ii) ⇒ (i) By $cax = \begin{bmatrix} (c_1a_1+c_2a_3)x_1 & (c_1a_1+c_2a_3)x_2 \\ +(c_1a_2+c_2a_4)x_3 & +(c_1a_2+c_2a_4)x_4 \\ 0 & 0 \end{bmatrix}_p = \begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix}_p =$ p, one can verify $caxc = c$ and $(cax)^* = cax$. Besides, $\mathcal{R}(x) \subseteq \mathcal{R}(b)$ gives $x \in bR$. Consequently, $a \in R^{\oplus}_{r,(b,c)}$ and x is a right (b,c)-core inverse of a.

(i) ⇔ (iii) is analogous to (i) ⇔ (ii) for $q = 1-cax$. □

It should be noted that p and q are invariant in Theorem 3.5, under the choice of x.

It is proved in Theorem 3.1 below that right w-core inverses are instances of right (b,c)-core inverses. As a consequence, we get the matrix representation of right w-core inverses as follows.

Corollary 2.11. *Let* $a, w \in R$. *The following conditions are equivalent*:

(i) $a \in R^{\oplus}_{r,w}$ *and* $x \in R$ *is a right* w*-core inverse of* a.

(ii) *There exists a projection* $p \in R$ *such that*

$$a = \begin{bmatrix} a_1 & a_2 \\ 0 & 0 \end{bmatrix}_p, \; w = \begin{bmatrix} w_1 & w_2 \\ w_3 & w_4 \end{bmatrix}_p \; \textit{and } x = \begin{bmatrix} x_1 & x_2 \\ 0 & 0 \end{bmatrix}_p,$$

where $(a_1w_1 + a_2w_3)x_1 = p$ *and* $(a_1w_1 + a_2w_3)x_2 = 0$.

(iii) *There exists a projection* $q \in R$ *such that*

$$a = \begin{bmatrix} 0 & 0 \\ a_3 & a_4 \end{bmatrix}_q, \ w = \begin{bmatrix} w_1 & w_2 \\ w_3 & w_4 \end{bmatrix}_q \text{ and } x = \begin{bmatrix} 0 & 0 \\ x_3 & x_4 \end{bmatrix}_q,$$

where $(a_3w_2 + a_4w_4)x_3 = 0$ *and* $(a_3w_2 + a_4w_4)x_4 = 1 - q$.

Following [7], an element $a \in R$ is Drazin invertible if there exists some $x \in R$ such that $ax = xa$, $xax = x$ and $a^k = a^{k+1}x$ for some nonnegative integer k. Such an x is called the Drazin inverse of a. It uniquely exists, and is denoted by a^D. The smallest nonnegative integer k is called the Drazin index of a. If the Drazin index of a is 1, then a is called group invertible and the group inverse of a is denoted by $a^{\#}$. R^D and $R^{\#}$ will stand for the sets of all Drazin invertible and group invertible elements in R, respectively.

Let $a \in R^D$ with the Drazin index k. Then $a = c_a + n_a$ is called the core nilpotent decomposition [11] of a, where $c_a = aa^Da$ is the core part of a and $n_a = (1 - aa^D)a$ is the nilpotent part of a. Moreover, $c_a \in R^{\#}$ with $c_a^{\#} = a^D$, $n_a^k = 0$ and $c_an_a = n_ac_a = 0$.

The following theorem shows a similar result for right w-core inverses.

Theorem 2.12. *Let* $a, w \in R$ *with* $a \in R^{\oplus}_{r,w}$. *Then* $aw = a_1 + a_2$, *where*

(i) $a_1 \in R^{\oplus}_r$,

(ii) $a_2^2 = 0$,

(iii) $a_1a_2^* = 0 = a_2a_1$.

In addition, $(aw)^2a^{\oplus}_{r,w} \in R^{\oplus}_r$ *with a right core inverse* $a^{\oplus}_{r,w}$.

Proof. Suppose $a \in R^{\oplus}_{r,w}$ with a right w-core inverse x. Then $awxa = a$, $(awx)^* = awx$ and $awx^2 = x$. Let $a_1 = (aw)^2x$ and $a_2 = aw(1 - awx)$. Then $aw = a_1 + a_2$. It is sufficient to prove (i) as (ii) and (iii) follow directly.

(i) We have

(1) $a_1x = (aw)^2x \cdot x = awx = (a_1x)^*$.

(2) $a_1xa_1 = awxa_1 = awx \cdot (aw)^2x = (aw)^2x = a_1$.

(3) $a_1x^2 = awx^2 = x$.

Hence, $a_1 \in R^{\oplus}_r$ with a right core inverse $a^{\oplus}_{r,w}$. □

From [19] and Proposition 3.2 below, it is known that right core inverses, Moore-Penrose inverses and right pseudo core inverses of a coincide with right 1-core inverses of a, right a^*-core inverses of a and right 1-core inverses

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of a^k, for some positive integer k, respectively. Moreover, $a_r^{\textcircled{\#}}$ is a right 1-core inverse of a, $(a^\dagger)^* a^\dagger$ is a right a^*-core inverses of a and $(a_r^{\textcircled{D}})^k$ is a right 1-core inverses of a^k. We hence have the following corollaries.

Corollary 2.13. *Let $a \in R^\dagger$. Then $aa^* = a_1 + a_2$, where*
(i) $a_1 \in R_r^{\textcircled{\#}}$,
(ii) $a_2^2 = 0$,
(iii) $a_1 a_2^* = 0 = a_2 a_1$.
In addition, $aa^* \in R_r^{\textcircled{\#}}$ with a right core inverse $(a^\dagger)^* a^\dagger$.

Corollary 2.14. *Let $a \in R_r^{\textcircled{D}}$ with $\mathrm{I}(a) = k$. Then $a^k = a_1 + a_2$, where*
(i) $a_1 \in R_r^{\textcircled{\#}}$,
(ii) $a_2^2 = 0$,
(iii) $a_1 a_2^* = 0 = a_2 a_1$.
In addition, $a^{k+1} a_r^{\textcircled{D}} \in R_r^{\textcircled{\#}}$ with a right core inverse $(a_r^{\textcircled{D}})^k$.

3. Connection with several classes of generalized inverses

In this section, we show that right (b,c)-core inverses encompass right inverses, right core inverses, right pseudo core inverses, right w-core inverses and Moore-Penrose inverses by picking different b and c. As shown in Theorems 3.1 and 3.8, for any nonnegative integers m, n satisfying $m + n \geq 1$, the right inverse, the right core inverse, the right pseudo core inverse and the right w-core inverse of a coincide with the right $(1,1)$-core inverse of a, the right (a^n, a)-core inverse of a^m, the right (a^n, a^k)-core inverse of a^m, for some positive integer k, and the right (a,a)-core inverse of w; the Moore-Penrose inverse of a coincides with the right (a^*, a^*)-core inverse of a and that of the right (a,a)-core inverse of a^*.

Theorem 3.1. *Let $a, w \in R$ and let m, n be nonnegative integers such that $m + n \geq 1$. Then*

(i) *a is right invertible if and only if a is right $(1,1)$-core invertible. In this case, $a_r^{-1} = a_{r,(1,1)}^{\textcircled{\#}}$.*

(ii) *a is right pseudo core invertible if and only if a^m is right (a^n, a^k)-core invertible, for some positive integer k. In this case, $a_r^{\textcircled{D}} = a^{k+m-1} (a^m)_{r,(a^n,a^k)}^{\textcircled{\#}}$ and $(a^m)_{r,(a^n,a^k)}^{\textcircled{\#}} = (a_r^{\textcircled{D}})^{k+m}$.*

(iii) *a is right core invertible if and only if a^m is right (a^n, a)-core invertible. In this case, $a_r^{\textcircled{\#}} = a^m (a^m)_{r,(a^n,a)}^{\textcircled{\#}}$ and $(a^m)_{r,(a^n,a)}^{\textcircled{\#}} = (a_r^{\textcircled{\#}})^{m+1}$.*

(iv) *a is right w-core invertible if and only if w is right* (a,a)*-core invertible. In this case,* $a^{\oplus}_{r,w} = w^{\oplus}_{r,(a,a)}$.

Proof. (i) is clear.

(ii) For the "only if" part. Suppose $a \in R^{\mathbb{D}}_r$ with $\mathrm{I}(a) = k$. Then there exists some $x \in R$ such that $axa^k = a^k$, $(ax)^* = ax$ and $ax^2 = x$, whence $ax = a \cdot ax^2 = a^2x^2 = \cdots = a^nx^n$ for arbitrary positive integer n. Let $y = x^{k+m}$. Then

(1) $y = x^{k+m} = ax^2 \cdot x^{k+m-1} = a^nx^{n+1} \cdot x^{k+m-1} = a^nx^{k+m+n} \in a^nR$.

(2) $a^{k+m}y = a^{k+m}x^{k+m} = ax = (a^{k+m}y)^*$.

(3) $a^{k+m}ya^k = axa^k = a^k$.

Hence, $a^m \in R^{\oplus}_{r,(a^n,a^k)}$ and $(a^m)^{\oplus}_{r,(a^n,a^k)} = (a^{\mathbb{D}}_r)^{k+m}$.

For the "if" part. Given $a^m \in R^{\oplus}_{r,(a^n,a^k)}$, we have $(a^m)^{\oplus}_{r,(a^n,a^k)} \in a^nR$, $a^{k+m}(a^m)^{\oplus}_{r,(a^n,a^k)}a^k = a^k$ and $(a^{k+m}(a^m)^{\oplus}_{r,(a^n,a^k)})^* = a^{k+m}(a^m)^{\oplus}_{r,(a^n,a^k)}$, then $(a^m)^{\oplus}_{r,(a^n,a^k)} = a^nz$ for some $z \in R$. Let $x = a^{k+m-1}(a^m)^{\oplus}_{r,(a^n,a^k)}$. Then

(1) $axa^k = a^{k+m}(a^m)^{\oplus}_{r,(a^n,a^k)}a^k = a^k$.

(2) $ax = a^{k+m}(a^m)^{\oplus}_{r,(a^n,a^k)} = (a^{k+m}(a^m)^{\oplus}_{r,(a^n,a^k)})^* = (ax)^*$.

(3) $ax^2 = a^{k+m}(a^m)^{\oplus}_{r,(a^n,a^k)}a^{k+m-1}(a^m)^{\oplus}_{r,(a^n,a^k)} = a^{k+m}(a^m)^{\oplus}_{r,(a^n,a^k)}a^{k+m-1}(a^nz) = (a^{k+m}(a^m)^{\oplus}_{r,(a^n,a^k)}a^k)(a^{m+n-1}z) = a^{k+m-1}(a^nz) = a^{k+m-1}(a^m)^{\oplus}_{r,(a^n,a^k)} = x$.

So, $a \in R^{\mathbb{D}}_r$ and $a^{\mathbb{D}}_r = a^{k+m-1}(a^m)^{\oplus}_{r,(a^n,a^k)}$.

(iii) is clear by (ii).

(iv) By Theorem 2.6 and [19, Theorem 2.5], one has that $a \in R^{\oplus}_{r,w}$ if and only if $w \in R^{\|a}_r$ and $a \in R^{\{1,3\}}$ if and only if $w \in R^{(a,a)}_r$ and $a \in R^{\{1,3\}}$ if and only if $w \in R^{\oplus}_{r,(a,a)}$. Moreover, $a^{\oplus}_{r,w} = w^{\|a}_r a^{(1,3)} = w^{(a,a)}_r a^{(1,3)} = w^{\oplus}_{r,(a,a)}$. □

Let $a \in R$ and let m, n be nonnegative integers such that $m + n \geq 1$. From Theorem 3.1 (iii), we derive that a^k is right core invertible if and only if a^m is right (a^n, a^k)-core invertible, for some positive integer k.

According to the items (ii) and (iv) of Theorem 3.1, we can establish the relation between right pseudo core inverses and right w-core inverses.

Proposition 3.2. *Let* $a \in R$ *and let* m *be a nonnegative integer. Then* a *is right pseudo core invertible if and only if* a^k *is right* a^m*-core invertible, for some positive integer* k*. In this case,* $a^{\mathbb{D}}_r = a^{k+m-1}(a^k)^{\oplus}_{r,a^m}$ *and* $(a^k)^{\oplus}_{r,a^m} = (a^{\mathbb{D}}_r)^{k+m}$.

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It is known that if a is right pseudo core invertible with $\mathrm{I}(a)=1$, then a is right core invertible. As a consequence, the following relation between right core inverses and right w-core inverses is clear. Therein, the cases $m=0$ and $m=1$ were given in [19].

Corollary 3.3. *Let $a \in R$ and let m be a nonnegative integer. Then a is right core invertible if and only if a is right a^m-core invertible. In this case, $a_r^{\oplus} = a^m a_{r,a^m}^{\oplus}$ and $a_{r,a^m}^{\oplus} = (a_r^{\oplus})^{m+1}$.*

As a special case of Theorem 3.1 (iii) and Corollary 3.3, we have the following corollary.

Corollary 3.4. *Let $a \in R$. The following statements are equivalent:*

(i) *$a \in R_r^{\oplus}$.*
(ii) *a is right (a,a)-core invertible.*
(iii) *a is right $(1,a)$-core invertible.*
(iv) *1 is right (a,a)-core invertible.*
(v) *a is right a-core invertible.*
(vi) *a is right 1-core invertible.*
(vii) *a is right (a,a^*)-invertible.*
In this case, $a_r^{\oplus} = aa_{r,(a,a)}^{\oplus} = aa_{r,(1,a)}^{\oplus} = 1_{r,(a,a)}^{\oplus} = aa_{r,a}^{\oplus} = a_{r,1}^{\oplus} = a_r^{(a,a^*)}$.

We remark the fact that any $A \in M_n(\mathbb{C})$ is right pseudo core invertible. Applying Theorem 3.1 (ii) and Proposition 3.2, we get the following result in complex matrices. It should be pointed that [19, Corollary 2.25] is the case $m=1$ of the item (ii) below.

Corollary 3.5. *Let $A \in M_n(\mathbb{C})$ with $\mathrm{I}(A)=k$. Then*

(i) *A^m is right (A^n, A^k)-core invertible, for any nonnegative integers m, n satisfying $m+n \geq 1$. In this case, $(A^{\text{ⓓ}})_r^{k+m}$ is a right (A^n, A^k)-core inverse of A^m.*

(ii) *A^k is right A^m-core invertible, for any nonnegative integer m. In this case, $(A_r^{\text{ⓓ}})^{k+m}$ is a right A^m-core inverse of A^k.*

As shown in Theorem 3.1 (iv), right w-core inverses of a coincides with right (a,a)-core inverses of w. We obtain the following existence criterion of right w-core inverses in rings.

Corollary 3.6. *Let $a, w \in R$. The following conditions are equivalent*:

(i) $a \in R^{\oplus}_{r,w}$.

(ii) *There exists some $x \in aR$ such that $awxa = a$, $(awx)^* = awx$ and $xawx = x$.*

(iii) *There exists some $x \in R$ such that $awxa = a$, $xR \subseteq aR$ and $Rx = Ra^*$.*

(iv) *There exists some $x \in R$ such that $awxa = a$, $xR \subseteq aR$ and $x^0 = (a^*)^0$.*

(v) *There exists some $x \in R$ such that $awxa = a$, $xR \subseteq aR$ and $Rx \subseteq Ra^*$.*

(vi) *There exists some $x \in R$ such that $awxa = a$, $xR \subseteq aR$ and $(a^*)^0 \subseteq x^0$.*

(vii) *There exist a projection $p \in R$ and an idempotent $q \in R$ such that $aR \subseteq pR \subseteq awR$, $qR \subseteq aR$ and $Rq \supseteq Raw$.*

In this case, $a^{\oplus}_{r,w} = q(aw)^- p$ for any $(aw)^- \in (aw)\{1\}$.

As a consequence of Corollary 3.6, we have the following result.

Corollary 3.7. *Let $a \in R$. The following conditions are equivalent*:

(i) $a \in R^{\oplus}_{r}$.

(ii) *There exists some $x \in aR$ such that $axa = a$, $(ax)^* = ax$ and $xax = x$.*

(iii) *There exists some $x \in R$ such that $axa = a$, $xR \subseteq aR$ and $Rx = Ra^*$.*

(iv) *There exists some $x \in R$ such that $axa = a$, $xR \subseteq aR$ and $x^0 = (a^*)^0$.*

(v) *There exists some $x \in R$ such that $axa = a$, $xR \subseteq aR$ and $Rx \subseteq Ra^*$.*

(vi) *There exists some $x \in R$ such that $axa = a$, $xR \subseteq aR$ and $(a^*)^0 \subseteq x^0$.*

(vii) *There exist a projection $p \in R$ and an idempotent $q \in R$ such that $qR \subseteq aR = pR$ and $Rq \supseteq Ra$.*

In this case, $a^{\oplus}_{r} = qa^- p$ for any $a^- \in a\{1\}$.

It was shown in [19, Theorem 2.21] that a is Moore-Penrose invertible if and only if a is right a^*-core invertible if and only if a^* is right a-core invertible. As right (b, c)-core invertible is right (b, c)-invertible, and hence right invertible along an element. This allows us to derive several new existence criteria for the Moore-Penrose inverse by right (b, c)-core inverses.

Theorem 3.8. *Let $a \in R$. The following statements are equivalent*:

(i) $a \in R^{\dagger}$.

(ii) *a is right (a^*, a^*)-core invertible.*

(iii) *a is right* (a^*, a^*)*-invertible.*
(iv) a^* *is right* (a, a)*-core invertible.*
(v) a^* *is right* (a, a)*-invertible.*
(vi) *a is right* a^**-core invertible.*
(vii) a^* *is right* a*-core invertible.*

PROOF. (i) $\Leftrightarrow$ (vi) $\Leftrightarrow$ (vii) by [19, Theorem 2.21].

(i) $\Rightarrow$ (ii) Since $a \in R^{\dagger}$, we have $a^* \in R^{\{1,3\}}$. Again, $a \in R^{\dagger}$ guarantees $a \in R_r^{\|a^*}$ by [15, Corollary 2.21 (iv)], which implies $a \in R_r^{(a^*,a^*)}$. Consequently, $a \in R^{\oplus}_{r,(a^*,a^*)}$ by Theorem 2.6.

(ii) $\Rightarrow$ (iii) by Theorem 2.6 (i) $\Rightarrow$ (ii).

(iii) $\Rightarrow$ (i) Given $a \in R_r^{(a^*,a^*)}$, then $a^* \in a^*aa^*R$, whence $a \in Raa^*a$. One can get $a \in R^{\dagger}$ in terms of [16, Theorem 3.12].

(i) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v) follows dually since $a \in R^{\dagger}$ if and only if $a^* \in R^{\dagger}$. $\square$

Theorem 3.9. *Let* $a, b, c \in R$*. Then a is right* (b, c)*-core invertible if and only if ca is right* (b, c^*)*-invertible. In this case, the right* (b, c)*-core inverse of a coincides with the right* (b, c^*)*-inverse of ca.*

PROOF. Suppose that a is right (b, c)-core invertible with a right (b, c)-core inverse x. Then $x \in bR$, $caxc = c$ and $(cax)^* = cax$. Thus, $c^*cax = c^*(cax)^* = (caxc)^* = c^*$, as required.

Conversely, let $x = (ca)_r^{(b,c^*)}$. Then $x \in bR$ and $c^*cax = c^*$. Consequently, one has $(cax)^* = cax$ and $c = caxc$, i.e., a is right (b, c)-core invertible. Moreover, x is a right (b, c)-core inverse of a. $\square$

Recall that an element $a \in R$ is strongly right (b, c)-invertible if $c \in cabR$ and cab is regular, or equivalently if there exists $x \in R$ such that $xax = x$, $xR \subseteq bR$ and $Rx = Rc$, in which case, any such x will be called a strongly right (b, c)-inverse of a. It is clear that every strongly right (b, c)-invertible element a must be right (b, c)-invertible. Moreover, every strongly right (b, c)-inverse of a is a right (b, c)-inverse of a.

Applying Theorem 3.9, one knows that if ca is strongly right (b, c^*)-invertible, then a is right (b, c)-core invertible. The following theorem shows that the converse also holds.

Theorem 3.10. *Let* $a, b, c \in R$*. Then a is right* (b, c)*-core invertible if and only if ca is strongly right* (b, c^*)*-invertible. In this case, every strongly right* (b, c^*)*-inverse of ca is a right* (b, c)*-core inverse of a.*

PROOF. The "if" part is clear, and every strongly right (b, c^*)-inverse of ca is a right (b, c)-core inverse of a in view of Theorem 3.9.

For the "only if" part. Suppose that a is right (b, c)-core invertible with a right (b, c)-core inverse y. Then $y \in bR$, $cayc = c$ and $(cay)^* = cay$, whence $y = bt$ for some $t \in R$. Consequently, $c^* = c^*cay = c^*cabt \in c^*cabR$, and hence, $c^*cab = c^*cabt \cdot (c^*cabt)^* \cdot ab = c^*cab \cdot t(abt)^* \cdot c^*cab$, which guarantees $c^*cab \in R^-$. So, ca is strongly right (b, c^*)-invertible. □

In terms of Corollary 3.4, Theorems 3.8 and 3.10, we have the following corollaries.

Corollary 3.11. *Let $a \in R$. The following statements are equivalent*:

(i) $a \in R_r^{\text{\textcircled{\#}}}$.

(ii) a^2 *is strongly right* (a, a^*)*-invertible.*

(iii) a^2 *is strongly right* $(1, a^*)$*-invertible.*

(iv) a *is strongly right* (a, a^*)*-invertible.*

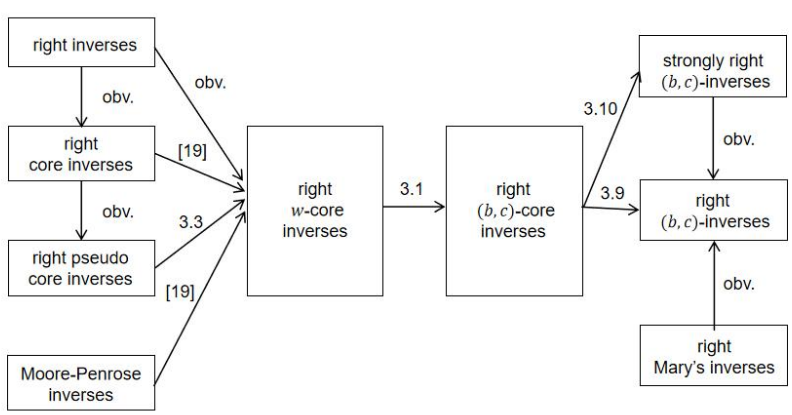
Corollary 3.12. *Let $a \in R$. The following statements are equivalent*:

(i) $a \in R^\dagger$.

(ii) a^*a *is strongly right* (a^*, a)*-invertible.*

(iii) aa^* *is strongly right* (a, a^*)*-invertible.*

At the end of this section, a schema is provided to present the relations between right (b, c)-core inverses and several other (right) generalized inverses.



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ACKNOWLEDGMENTS

The authors are highly grateful to the referee for his/her valuable comments and suggestions which greatly improved this paper. This research is supported by the National Natural Science Foundation of China (No. 11801124).

References

[1] O.M. Baksalary, G. Trenkler, Core inverse of matrices, Linear Multilinear Algebra 58 (2010) 681-697.

[2] J. Benítez, E. Boasso, The inverse along an element in rings, Electron. J. Linear Algebra 31 (2016) 572-592.

[3] J. Benítez, E. Boasso, The inverse along an element in rings with an involution, Banach algebras and C^*-algebras, Linear Multilinear Algebra 65 (2017) 284-299.

[4] D.S. Cvetković-Ilić, Y.M. Wei, Algebraic properties of generalized inverses, Series: Developments in Mathematics, Vol. 52, Springer, 2017.

[5] M.P. Drazin, A class of outer generalized inverses, Linear Algebra Appl. 436 (2012) 1909-1923.

[6] M.P. Drazin, Left and right generalized inverses, Linear Algebra Appl. 510 (2016) 64-78.

[7] M.P. Drazin, Pseudo-inverses in associative rings and semigroups, Amer. Math. Monthly 65 (1958) 506-514.

[8] K. Manjunatha Prasad, K.S. Mohana, Core-EP inverse, Linear Multilinear Algebra 62 (2014) 792-802.

[9] X. Mary, On generalized inverses and Green's relations, Linear Algebra Appl. 434 (2011) 1836-1844.

[10] X. Mary, P. Patrício, Generalized inverses modulo $\mathcal{H}$ in semigroups and rings, Linear Multilinear Algebra 61 (2013) 1130-1135.

[11] P. Patrício, R. Puystjens, Drazin-Moore-Penrose invertibility in rings, Linear Algebra Appl. 389 (2004) 159-173.

[12] R. Penrose, A generalized inverse for matrices, Proc. Camb. Phil. Soc. 51 (1955) 406-413.

[13] D.S. Rakić, N.C. Dinčić, D.S. Djordjević, Group, Moore-Penrose, core and dual core inverse in rings with involution, Linear Algebra Appl. 463 (2014) 115-133.

[14] H.H. Zhu, The (b, c)-core inverse and its dual in semigroups with involution, J. Pure Appl. Algebra 228 (2024) 107526.

[15] H.H. Zhu, J.L. Chen, P. Patrício, Further results on the inverse along an element in semigroups and rings, Linear Multilinear Algebra 64 (2016) 393-403.

[16] H.H. Zhu, J.L. Chen, P. Patrício, X. Mary, Centralizer's applications to the inverse along an element, Appl. Math. Comput. 315 (2017) 27-33.

[17] H.H. Zhu, C.C. Wang, Q-W. Wang, Left w-core inverses in rings with involution, Mediterr. J. Math. 20 (2023) https://doi.org/10.1007/s00009-023-025410-9.

[18] H.H. Zhu, L.Y. Wu, J.L. Chen, A new class of generalized inverses in semigroups and rings with involution, Comm. Algebra 51 (2023) 2098-2113.

[19] H.H. Zhu, L.Y. Wu, D. Mosić, One-sided w-core inverses in rings with involution, Linear Multilinear Algebra 71 (2023) 528-544.

[20] H.H. Zhu, X.X. Zhang, J.L. Chen, Generalized inverses of a factorization in a ring with involution, Linear Algebra Appl. 472 (2015) 142-150.