ROCKY MOUNTAIN JOURNAL OF MATHEMATICS
Vol. , No. , YEAR
https://doi.org/rmj.YEAR..PAGE

HERMITE-HADAMARD TYPE INEQUALITIES FOR THE LEFT RIEMANN-LIOUVILLE FRACTIONAL INTEGRALS WITH VARIABLE ORDER

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ABSTRACT. This study presents several significant results related to the Hermite-Hadamard inequality. We have established new inequalities of the Hermite-Hadamard type and its variant for convex functions. With the help of different approaches of integrals and derivatives, we have presented some integral inequalities for the Riemann-Liouville fractional integral for variable order. We derived two novel equalities to prove new fractional trapezoid and midpoint type inequalities for differentiable convex functions. Furthermore, we have provided the computational analysis of new inequalities with numerical examples for convex functions.

1. Introduction and Preliminaries

Convexity is an essential mathematical concept, especially in geometry and optimisation. Greek philosophers explored convexity, which originated in Egypt and Babylon. Drawing simple geometric shapes like circles and triangles dates back to human civilisation, but its origins are difficult to determine. To the best of my knowledge, in the late 19th century, German mathematician Karl Hermann Amandus Schwarz made a groundbreaking contribution by introducing the convex function [13]. The contributions made by his research on convexity had an enormous impact on the advancement of mathematical theory. A function $F:[\rho,\vartheta]\subset\mathbb{R}\to\mathbb{R}$ is said to be a convex function if the given inequality holds:

$$F(t\rho+(1-t)\vartheta)\leq tF(\rho)+(1-t)F(\vartheta) \tag{1.1}$$

for all $\rho,\vartheta\in\Im$, $t\in[0,1]$. Also, we say that F is concave, if the inequality (1.1) is reversed. Researchers have extensively utilised convexity in economics, engineering, computer science, and other mathematical disciplines [14, 34]. However, the most famous result regarding the convex functions in mathematical inequalities is Hermite-Hadamard (H-H) inequality because of its numerous uses in optimisation theory and the theory of inequalities [27, 23]. These inequalities state that if a function $F:\Im\to\mathbb{R}$ is a convex function in the interval $\Im\supset[\rho,\vartheta]$, then we have

$$F\left(\frac{\rho+\vartheta}{2}\right)\leq\frac{1}{\vartheta-\rho}\int_{\rho}^{\vartheta}F(u)du\leq\frac{F(\rho)+F(\vartheta)}{2}. \tag{1.2}$$

Dragomir and Agarwal [10], derive a specific identity and subsequently utilize it to provide various bounds for the right-hand side of the inequality (1.2).

2020 *Mathematics Subject Classification.*
Key words and phrases. Hermite–Hadamard inequality, Riemann-Liouville fractional integral, Integral inequalities.

Lemma 1.1. *Let $F:\mathfrak{I}^\circ\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on $\mathfrak{I}^\circ$, $\rho,\vartheta\in\mathfrak{I}^\circ$. If $F'\in L[\rho,\vartheta]$, then the following equality holds:*

$$\frac{F(\rho)+F(\vartheta)}{2}-\frac{1}{\vartheta-\rho}\int_\rho^\vartheta F(u)du=\frac{\rho-\vartheta}{2}\int_0^1(1-2t)F'(t\rho+(1-t)\vartheta)dt. \tag{1.3}$$

Kirmaci [21], presents a particular identity and subsequently utilize it to provide various bounds for the left-hand side of the inequality (1.2).

Lemma 1.2. *If $F:\mathfrak{I}^\circ\to\mathbb{R}$ is differentiable on $\mathfrak{I}^\circ$ and $F'\in L[\rho,\vartheta]$, then we obtain*

$$\begin{aligned}&\frac{1}{\vartheta-\rho}\int_\rho^\vartheta F(u)\mathrm{d}u-F\left(\frac{\rho+\vartheta}{2}\right)\\&=(\vartheta-\rho)\left[\int_0^{1/2}tF'(t\rho+(1-t)\vartheta)\mathrm{d}t+\int_{1/2}^1(t-1)F'(t\rho+(1-t)\vartheta)\mathrm{d}t\right],\quad\forall\,\rho,\vartheta\in\mathfrak{I}^\circ.\end{aligned} \tag{1.4}$$

Numerous investigations have been conducted over the past twenty years to find new bounds for the inequality on the left and right sides of (1.2). For more information, please visit [9, 11, 19, 24]. Fractional calculus is a mathematical discipline that extends traditional calculus to include non-integer orders of differentiation and integration, with its origins dating back to the 17th-18th centuries through early contributions by mathematicians like Leibniz and further advancements by Riemann and Liouville [2, 18, 26, 29]. Hezenci et al. have proved Newton's inequalities for differentiable convex functions using Riemann-Liouville fractional integrals. They give a graphical analysis which clarifies the validity of the newly established inequalities [17]. Using different function classes, Budak and Kosem obtained some Milne-type inequalities for Riemann-Liouville fractional integrals [8]. Milne-type inequality for co-ordinated convex functions, Shehzadi et al. [35] have established a novel identity. Also, they presented some new inequalities for Milne-type co-ordinated convex functions. D. Zhao et al. [37] have derived new Bullen-type inequalities for differentiable convex functions with the help of generalised fractional integrals. Hassan et al. [16] have proved an identity using generalised fractional integrals by utilising differentiable functions. Furthermore, they obtained numerous Simpson-type inequalities for the functions whose absolute value derivatives are convex. Iftikhar et al. [20] have derived an identity for local fractional integrals, obtaining new Newton-type inequalities for generalised convex functions and applying inequalities for Simpson's quadrature rules and special means. Tunc [36] has provided definitions for interval-valued left-sided and right-sided fractional integrals. In 2019, Kunt et al. [22] presented a novel approach to establishing new fractional H-H type inequalities exclusively using the left Riemann-Liouville fractional integral. Additionally, they prove two new equalities to derive fractional trapezoid and midpoint type inequalities for differentiable convex functions.

In the following, there are some definitions and mathematical preliminaries which will be extensively used throughout this study.

Definition 1.3 (See [28]). *A function F defined on $\mathfrak{I}$ has a support at $x_0\in\mathfrak{I}$, if there exist an affine function $B(x)=F(x_0)+m(x-x_0)$, such that $B(x)\le F(x)$ for all $x\in\mathfrak{I}$. The graph of the support function B is called a line of support for F at x_0.*

Theorem 1.4 (See [28]). *The function $F:(\rho,\vartheta)\to\mathbb{R}$ is considered convex if and only if there exists at least one line of support for F at every point x_0 within the interval (ρ,ϑ).*

Definition 1.5 (See [30]). *Let $F \in L_1[\rho,\vartheta]$. The Riemann-Liouville fractional integrals $\mathfrak{J}^{\alpha}_{\rho+}F$ and $\mathfrak{J}^{\alpha}_{\vartheta-}F$ of order $\alpha > 0$ are defined by*

$$\mathfrak{J}^{\alpha}_{\rho+}F(x) = \frac{1}{\Gamma(\alpha)}\int_{\rho}^{x}(x-t)^{\alpha-1}F(t)dt, \quad x>\rho$$

and

$$\mathfrak{J}^{\alpha}_{\vartheta-}F(x) = \frac{1}{\Gamma(\alpha)}\int_{x}^{\vartheta}(t-x)^{\alpha-1}F(t)dt, \quad x<\vartheta,$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function and $\mathfrak{J}^{\alpha}_{\rho+}F(x) = \mathfrak{J}^{\alpha}_{\vartheta-}F(x) = F(x)$.

Definition 1.6 (See [15]). *Let $0 < \alpha(x) < 1$ for all $x \in [\rho,\vartheta]$ and $F \in L_1[\rho,\vartheta]$. Then the left and right Riemann-Liouville integrals of variable fractional order $\alpha(x)$ are defined by*

$$J^{\alpha(x)}_{\rho+}F(x) = \frac{1}{\Gamma[\alpha(x)]}\int_{\rho}^{x}(x-t)^{\alpha(x)-1}F(t)dt, \quad \operatorname{Re}\alpha(x)>0$$

and

$$J^{\alpha(x)}_{\vartheta-}F(x) = \frac{1}{\Gamma[\alpha(x)]}\int_{x}^{\vartheta}(t-x)^{\alpha(x)-1}F(t)dt, \quad \operatorname{Re}\alpha(x)>0.$$

The classical Riemann-Liouville fractional integral has a fixed constant order of integration α, while the variable order definition allows the order of integration $\alpha(x)$ to vary as a function of another variable x. In our case, $\alpha(x)$ depends on the integral domain and gives different values at different points, where α and the point of interval vary simultaneously. This flexibility provides a graphical analysis which clarifies the validity of the newly generalized inequalities. Sarikaya et al. (2013) have proved H-H inequalities for Riemann-Liouville fractional integrals using fractional identities [33].

Theorem 1.7 (See [33]). *Let $F : [\rho,\vartheta] \subset \mathbb{R}^+ \to \mathbb{R}$ be a positive function and $F \in L_1[\rho,\vartheta]$. If F is a convex function on $[\rho,\vartheta]$, then the following inequalities for fractional integrals hold:*

$$F\left(\frac{\rho+\vartheta}{2}\right) \le \frac{\Gamma(\alpha+1)}{2(\vartheta-\rho)^{\alpha}}\left[\mathfrak{J}^{\alpha}_{\rho^+}F(\vartheta)+\mathfrak{J}^{\alpha}_{\vartheta^-}F(\rho)\right] \le \frac{F(\rho)+F(\vartheta)}{2} \tag{1.5}$$

with $\alpha > 0$.

In this paper, we derived new fractional H-H type inequalities using the left Riemann-Liouville fractional integral with variable order for convex functions. This dynamic interplay between α and the position within the interval provides increased flexibility and paves the way for graphical analysis, ultimately enhancing our understanding of the applicability of newly introduced generalized inequalities. Through interesting examples, we demonstrate how our generalized H-H inequality with variable order provides more accurate insights into the function's behaviour over specific interval points. In addition, we provide two new identities for differentiable convex functions to derive the fractional trapezoid and midpoint type inequalities.

2. Main results

The H-H inequalities can be expressed in terms of fractional integrals as follows.

Theorem 2.1. *Let $F:[\rho,\vartheta]\to\mathbb{R}$ be a positive function with $0\le\rho<\vartheta$ and $F\in L_1[\rho,\vartheta]$. If F is a convex function on $[\rho,\vartheta]$, then the following inequalities for the left Riemann-Liouville fractional integrals hold:*

$$F\left(\frac{\alpha(\vartheta)\rho+\vartheta}{\alpha(\vartheta)+1}\right)\leqslant\frac{\Gamma(\alpha(\vartheta)+1)}{(\vartheta-\rho)^{\alpha(\vartheta)}}J_{\rho^+}^{\alpha(\vartheta)}F(\vartheta)\leqslant\frac{\alpha(\vartheta)F(\rho)+F(\vartheta)}{\alpha(\vartheta)+1} \tag{2.1}$$

with $\alpha(\vartheta)>0$.

Proof. Assuming that F is a convex function on $[\rho,\vartheta]$, it follows from Theorem 1.4 that at least one line of support exists

$$B(x)=F\left(\frac{\alpha(\vartheta)\rho+\vartheta}{\alpha(\vartheta)+1}\right)+m\left(x-\frac{\alpha(\vartheta)\rho+\vartheta}{\alpha(\vartheta)+1}\right)\le F(x) \tag{2.2}$$

for all $x\in[\rho,\vartheta]$ and $m\in\left[F'_-\left(\frac{\alpha(\vartheta)\rho+\vartheta}{\alpha(\vartheta)+1}\right),F'_+\left(\frac{\alpha(\vartheta)\rho+\vartheta}{\alpha(\vartheta)+1}\right)\right]$. From (2.2), we have

$$B(t\rho+(1-t)\vartheta)=F\left(\frac{\alpha(\vartheta)\rho+\vartheta}{\alpha(\vartheta)+1}\right)+m\left(t\rho+(1-t)\vartheta-\frac{\alpha(\vartheta)\rho+\vartheta}{\alpha(\vartheta)+1}\right)\le F(t\rho+(1-t)\vartheta) \tag{2.3}$$

for all $t\in[0,1]$. By multiplying both sides of (2.3) with the function $\alpha(\vartheta)t^{\alpha(\vartheta)-1}$ and integrating across the interval $(0,1)$ with respect to t, we obtain the following result

$$\begin{aligned}
&\alpha(\vartheta)\int_0^1 t^{\alpha(\vartheta)-1}B(t\rho+(1-t)\vartheta)\mathrm{d}t\\
&=\alpha(\vartheta)\int_0^1 t^{\alpha(\vartheta)-1}\left[F\left(\frac{\alpha(\vartheta)\rho+\vartheta}{\alpha(\vartheta)+1}\right)+m\left(t\rho+(1-t)\vartheta-\frac{\alpha(\vartheta)\rho+\vartheta}{\alpha(\vartheta)+1}\right)\right]\mathrm{d}t\\
&=\alpha(\vartheta)F\left(\frac{\alpha(\vartheta)\rho+\vartheta}{\alpha(\vartheta)+1}\right)\int_0^1 t^{\alpha(\vartheta)-1}\,\mathrm{d}t\\
&\quad+m\left[\int_0^1\alpha(\vartheta)\left[t^{\alpha(\vartheta)}\rho+\left(t^{\alpha(\vartheta)-1}-t^{\alpha(\vartheta)}\right)\vartheta\right]\mathrm{d}t-\frac{\alpha(\vartheta)\rho+\vartheta}{\alpha(\vartheta)+1}\alpha(\vartheta)\int_0^1 t^{\alpha(\vartheta)-1}\,\mathrm{d}t\right]\\
&=F\left(\frac{\alpha(\vartheta)\rho+\vartheta}{\alpha(\vartheta)+1}\right)+m\left[\frac{\alpha(\vartheta)\rho+\vartheta}{\alpha(\vartheta)+1}-\frac{\alpha(\vartheta)\rho+\vartheta}{\alpha(\vartheta)+1}\right]=F\left(\frac{\alpha(\vartheta)\rho+\vartheta}{\alpha(\vartheta)+1}\right)\\
&\le\alpha(\vartheta)\int_0^1 t^{\alpha(\vartheta)-1}F(t\rho+(1-t)\vartheta)\mathrm{d}t=\frac{\alpha(\vartheta)}{(\vartheta-\rho)^{\alpha(\vartheta)}}\int_\rho^\vartheta(\vartheta-t)^{\alpha(\vartheta)-1}F(t)\mathrm{d}t\\
&=\frac{\Gamma(\alpha(\vartheta)+1)}{(\vartheta-\rho)^{\alpha(\vartheta)}}J_{\rho+}^{\alpha(\vartheta)}F(\vartheta).
\end{aligned} \tag{2.4}$$

By utilising the convexity property of function F over the interval $[\rho,\vartheta]$, we can assume that

$$F(t\rho+(1-t)\vartheta)\le tF(\rho)+(1-t)F(\vartheta) \tag{2.5}$$

for all $t \in [0,1]$. By multiplying both sides of (2.5) with the function $\alpha(\vartheta)t^{\alpha(\vartheta)-1}$ and integrating across the interval $(0,1)$ with respect to t, we obtain the following result

(2.6)
$$\alpha(\vartheta)\int_0^1 t^{\alpha(\vartheta)-1}F(t\rho+(1-t)\vartheta)\mathrm{d}t = \frac{\alpha(\vartheta)}{(\vartheta-\rho)^{\alpha(\vartheta)}}\int_a^{\vartheta}(\vartheta-t)^{\alpha(\vartheta)-1}F(t)\mathrm{d}t = \frac{\Gamma(\alpha(\vartheta)+1)}{(\vartheta-\rho)^{\alpha(\vartheta)}}J_{\rho+}^{\alpha(\vartheta)}F(\vartheta)$$
$$\leq \alpha(\vartheta)F(\rho)\int_0^1 t^{\alpha(\vartheta)}\mathrm{d}t + \alpha(\vartheta)F(\vartheta)\int_0^1\left(t^{\alpha(\vartheta)-1}-t^{\alpha(\vartheta)}\right)\mathrm{d}t = \frac{\alpha(\vartheta)F(\rho)+F(\vartheta)}{\alpha(\vartheta)+1}.$$

By using (2.4) and (2.6), we have (2.1). This completes the proof. □

Remark 2.2. *If we choose $\alpha(x)=1$, in Theorem 2.1, then we obtain the inequality (1.2).*

Remark 2.3. *If we choose $\alpha(x)=\beta$ (constant), then we have Theorem* 2.1 *proved by Kunt et al. in* [22].

3. Lemmas

Two identities connected to Lemma 1.1 and Lemma 1.2 will be proven in this section.

Lemma 3.1. *Let $F:\mathfrak{I}^{\circ}\to\mathbb{R}$ be a differentiable mapping on $\mathfrak{I}^{\circ}$, $\rho,\vartheta\in\mathfrak{I}^{\circ}$ with $\vartheta>\rho$. If $F'\in L[\rho,\vartheta]$, then the left Riemann-Liouville fractional integral satisfies the following equality:*

(3.1)
$$\frac{\alpha(\vartheta)F(\rho)+F(\vartheta)}{\alpha(\vartheta)+1}-\frac{\Gamma(\alpha(\vartheta)+1)}{(\vartheta-\rho)^{\alpha(\vartheta)}}J_{\rho+}^{\alpha(\vartheta)}F(\vartheta)=\frac{\vartheta-\rho}{\alpha(\vartheta)+1}\int_0^1\left[1-(\alpha(\vartheta)+1)t^{\alpha(\vartheta)}\right]F'(t\rho+(1-t)\vartheta)\mathrm{d}t$$

with $\alpha(\vartheta)>0$.

Proof. Partial integration on the right side of the equation (3.1) yields

$$\begin{aligned}
&\frac{\vartheta-\rho}{\alpha(\vartheta)+1}\int_0^1\left[1-(\alpha(\vartheta)+1)t^{\alpha(\vartheta)}\right]F'(t\rho+(1-t)\vartheta)\mathrm{d}t\\
&=(\vartheta-\rho)\left[\frac{1}{\alpha(\vartheta)+1}\int_0^1F'(t\rho+(1-t)\vartheta)\mathrm{d}t-\int_0^1t^{\alpha(\vartheta)}F'(t\rho+(1-t)\vartheta)\mathrm{d}t\right]\\
&=(\vartheta-\rho)\left[\frac{1}{\alpha(\vartheta)+1}\left.\frac{F(t\rho+(1-t)\vartheta)}{\rho-\vartheta}\right|_0^1-\left(\left.t^{\alpha(\vartheta)}\frac{F(t\rho+(1-t)\vartheta)}{\rho-\vartheta}\right|_0^1\right.\right.\\
&\quad\left.\left.-\alpha(\vartheta)\int_0^1t^{\alpha(\vartheta)-1}\frac{F(t\rho+(1-t)\vartheta)}{\rho-\vartheta}\,\mathrm{d}t\right)\right]\\
&=\frac{F(\vartheta)-F(\rho)}{\alpha(\vartheta)+1}+F(\rho)-\alpha(\vartheta)\int_0^1t^{\alpha(\vartheta)-1}F(t\rho+(1-t)\vartheta)\mathrm{d}t\\
&=\frac{\alpha(\vartheta)F(\rho)+F(\vartheta)}{\alpha(\vartheta)+1}-\frac{\Gamma(\alpha(\vartheta)+1)}{(\vartheta-\rho)^{\alpha(\vartheta)}}J_{\rho+}^{\alpha(\vartheta)}F(\vartheta).
\end{aligned}$$

This completes the proof. □

Remark 3.2. *If we take $\alpha(x)=1$, in Lemma 3.1, then we get Lemma 1.1.*

Remark 3.3. *If we put $\alpha(x)=\beta$ (constant), then we will obtain Lemma* 3.1 *proved in* [22].

Lemma 3.4. *Let $F:\mathfrak{I}^\circ \to \mathbb{R}$ be a differentiable mapping on $\mathfrak{I}^\circ, \rho, \vartheta \in \mathfrak{I}^\circ$ with $\vartheta > \rho$. If $F' \in L[\rho,\vartheta]$, then the left Riemann-Liouville fractional integral satisfies the following equality:*

$$\begin{aligned}
&\frac{\Gamma(\alpha(\vartheta)+1)}{(\vartheta-\rho)^{\alpha(\vartheta)}} J_{\rho+}^{\alpha(\vartheta)} F(\vartheta) - F\left(\frac{\alpha(\vartheta)\rho+\vartheta}{\alpha(\vartheta)+1}\right) \\
&= (\vartheta-\rho)\left[\int_0^{\frac{\alpha(\vartheta)}{\alpha(\vartheta)+1}} t^{\alpha(\vartheta)} F'(t\rho+(1-t)\vartheta)\mathrm{d}t + \int_{\frac{\alpha(\vartheta)}{\alpha(\vartheta)+1}}^1 \left(t^{\alpha(\vartheta)}-1\right) F'(t\rho+(1-t)\vartheta)\mathrm{d}t\right]
\end{aligned} \tag{3.2}$$

with $\alpha(\vartheta) > 0$.

Proof. Partial integration on the right side of the equation (3.2) yields

$$\begin{aligned}
&(\vartheta-\rho)\left[\int_0^{\frac{\alpha(\vartheta)}{\alpha(\vartheta)+1}} t^{\alpha(\vartheta)} F'(t\rho+(1-t)\vartheta)\mathrm{d}t + \int_{\frac{\alpha(\vartheta)}{\alpha(\vartheta)+1}}^1 \left(t^{\alpha(\vartheta)}-1\right) F'(t\rho+(1-t)\vartheta)\mathrm{d}t\right] \\
&= (\vartheta-\rho)\left[\int_0^1 t^{\alpha(\vartheta)} F'(t\rho+(1-t)\vartheta)\mathrm{d}t - \int_{\frac{\alpha(\vartheta)}{\alpha(\vartheta)+1}}^1 F'(t\rho+(1-t)\vartheta)\mathrm{d}t\right] \\
&= (\vartheta-\rho)\left[\left(t^{\alpha(\vartheta)}\frac{F(t\rho+(1-t)\vartheta)}{\rho-\vartheta}\Big|_0^1 - \alpha(\vartheta)\int_0^1 t^{\alpha(\vartheta)-1}\frac{F(t\rho+(1-t)\vartheta)}{\rho-\vartheta}\,\mathrm{d}t\right)\right. \\
&\qquad \left. - \left(\frac{F(t\rho+(1-t)\vartheta)}{\rho-\vartheta}\Big|_{\frac{\alpha(\vartheta)}{\alpha(\vartheta)+1}}^1\right)\right] \\
&= \left[\left(-F(\rho)+\alpha(\vartheta)\int_0^1 t^{\alpha(\vartheta)-1}F(t\rho+(1-t)\vartheta)\mathrm{d}t\right) + \left(F(\rho) - F\left(\frac{\alpha(\vartheta)\rho+\vartheta}{\alpha(\vartheta)+1}\right)\right)\right] \\
&= \frac{\Gamma(\alpha(\vartheta)+1)}{(\vartheta-\rho)^{\alpha(\vartheta)}} J_{\rho+}^{\alpha(\vartheta)} F(\vartheta) - F\left(\frac{\alpha(\vartheta)\rho+\vartheta}{\alpha(\vartheta)+1}\right).
\end{aligned}$$

This completes the proof. □

Remark 3.5. *If we assign $\alpha(x) = 1$, in Lemma 3.4, then we get Lemma 1.2.*

Remark 3.6. *If we take $\alpha(x) = \beta$ (constant), then we will get Lemma* 3.2 *in* [22].

4. Midpoint and Trapezoid inequalities

In this section, we will derive new inequalities for the left Riemann-Liouville fractional trapezoid and midpoint types. These inequalities will be obtained by utilising Lemma 3.1 and Lemma 3.4.

Theorem 4.1. *Let $F:\mathfrak{I}^\circ \to \mathbb{R}$ be a differentiable mapping on $\mathfrak{I}^\circ$, $\rho,\vartheta \in \mathfrak{I}^\circ$ with $\vartheta > \rho$. If $|F'|$ is convex on $[\rho,\vartheta]$, then the subsequent inequality for the left Riemann-Liouville fractional integral holds:*

$$\begin{aligned}
&\left|\frac{\alpha(\vartheta)F(\rho)+F(\vartheta)}{\alpha(\vartheta)+1} - \frac{\Gamma(\alpha(\vartheta)+1)}{(\vartheta-\rho)^{\alpha(\vartheta)}} J_{\rho+}^{\alpha(\vartheta)} F(\vartheta)\right| \\
&\leq \frac{\vartheta-\rho}{\alpha(\vartheta)+1}\left[A_1(\alpha(\vartheta))\left|F'(\rho)\right| + A_2(\alpha(\vartheta))\left|F'(\vartheta)\right| + A_3(\alpha(\vartheta))\left|F'(\rho)\right| + A_4(\alpha(\vartheta))\left|F'(\vartheta)\right|\right]
\end{aligned}$$

where

$$A_1(\alpha(\vartheta)) = \frac{\alpha(\vartheta)}{2(\alpha(\vartheta)+2)(\alpha(\vartheta)+1)^{\frac{2}{\alpha(\vartheta)}}},\ A_2(\alpha(\vartheta)) = \frac{\alpha(\vartheta)\left(2(\alpha(\vartheta)+2)(\alpha(\vartheta)+1)^{\frac{1}{\alpha(\vartheta)}-1}-1\right)}{2(\alpha(\vartheta)+2)(\alpha(\vartheta)+1)^{\frac{2}{\alpha(\vartheta)}}},$$

$$A_3(\alpha(\vartheta)) = \frac{\alpha(\vartheta)\left(1+(\alpha(\vartheta)+1)^{\frac{2}{\alpha(\vartheta)}}\right)}{2(\alpha(\vartheta)+2)(\alpha(\vartheta)+1)^{\frac{2}{\alpha(\vartheta)}}},$$

$$A_4(\alpha(\vartheta)) = \frac{\alpha(\vartheta)\left(2(\alpha(\vartheta)+2)(\alpha(\vartheta)+1)^{\frac{1}{\alpha(\vartheta)}-1}-1-(\alpha(\vartheta)+1)^{\frac{2}{\alpha(\vartheta)}}\right)}{2(\alpha(\vartheta)+2)(\alpha(\vartheta)+1)^{\frac{2}{\alpha(\vartheta)}}},$$

with $\alpha(\vartheta) > 0$.

Proof. By using Lemma 3.1 and the convexity of $|F'|$, we obtain

$$\left|\frac{\alpha(\vartheta)F(\rho)+F(\vartheta)}{\alpha(\vartheta)+1} - \frac{\Gamma(\alpha(\vartheta)+1)}{(\vartheta-\rho)^{\alpha(\vartheta)}} J_{\rho+}^{\alpha(\vartheta)} F(\vartheta)\right| \leq \frac{\vartheta-\rho}{\alpha(\vartheta)+1}\int_0^1 \left|1-(\alpha(\vartheta)+1)t^{\alpha(\vartheta)}\right| \left|F'(t\rho+(1-t)\vartheta)\right| \mathrm{d}t$$

$$\leq \frac{\vartheta-\rho}{\alpha(\vartheta)+1}\left[\int_0^{\frac{1}{\sqrt[\alpha(\vartheta)]{\alpha(\vartheta)+1}}} \left(1-(\alpha(\vartheta)+1)t^{\alpha(\vartheta)}\right)\left[t\left|F'(\rho)\right|+(1-t)\left|F'(\vartheta)\right|\right]\mathrm{d}t\right.$$

$$\left.+\int_{\frac{1}{\sqrt[\alpha(\vartheta)]{\alpha(\vartheta)}}}^1 \left((\alpha(\vartheta)+1)t^{\alpha(\vartheta)}-1\right)\left[t\left|F'(\rho)\right|+(1-t)\left|F'(\vartheta)\right|\right]\mathrm{d}t\right]$$

$$\leq \frac{\vartheta-\rho}{\alpha(\vartheta)+1}\left[\frac{\alpha(\vartheta)}{2(\alpha(\vartheta)+2)(\alpha(\vartheta)+1)^{\frac{2}{\alpha(\vartheta)}}}\left|F'(\rho)\right| + \frac{\alpha(\vartheta)\left(2(\alpha(\vartheta)+2)(\alpha(\vartheta)+1)^{\frac{1}{\alpha(\vartheta)}-1}-1\right)}{2(\alpha(\vartheta)+2)(\alpha(\vartheta)+1)^{\frac{2}{\alpha(\vartheta)}}}\left|F'(\vartheta)\right|\right.$$

$$+\frac{\alpha(\vartheta)\left(1+(\alpha(\vartheta)+1)^{\frac{2}{\alpha(\vartheta)}}\right)}{2(\alpha(\vartheta)+2)(\alpha(\vartheta)+1)^{\frac{2}{\alpha(\vartheta)}}}\left|F'(\rho)\right|$$

$$\left.+\frac{\alpha(\vartheta)\left(2(\alpha(\vartheta)+2)(\alpha(\vartheta)+1)^{\frac{1}{\alpha(\vartheta)}-1}-1-(\alpha(\vartheta)+1)^{\frac{2}{\alpha(\vartheta)}}\right)}{2(\alpha(\vartheta)+2)(\alpha(\vartheta)+1)^{\frac{2}{\alpha(\vartheta)}}}\left|F'(\vartheta)\right|\right]$$

$$\leq \frac{\vartheta-\rho}{\alpha(\vartheta)+1}\left[A_1(\alpha(\vartheta))\left|F'(\rho)\right|+A_2(\alpha(\vartheta))\left|F'(\vartheta)\right|+A_3(\alpha(\vartheta))\left|F'(\rho)\right|+A_4(\alpha(\vartheta))\left|F'(\vartheta)\right|\right].$$

This completes the proof. □

Remark 4.2. *If we choose* $\alpha(x) = 1$*, in Theorem 4.1, then we obtain Theorem* 2.2 *in* [10].

Remark 4.3. *If we choose* $\alpha(x) = \beta$ *(constant), then we find Theorem* 4.1 *proved in* [22].

Theorem 4.4. *Let $F:\mathfrak{I}^\circ\to\mathbb{R}$ be a differentiable mapping on $\mathfrak{I}^\circ$, $\rho,\vartheta\in\mathfrak{I}^\circ$ with $\vartheta>\rho$. If $|F'|^q$ is convex on $[\rho,\vartheta]$ for $q\geq 1$, then the subsequent inequality for the left Riemann-Liouville fractional integral hold:*

$$\left|\frac{\alpha(\vartheta)F(\rho)+F(\vartheta)}{\alpha(\vartheta)+1}-\frac{\Gamma(\alpha(\vartheta)+1)}{(\vartheta-\rho)^{\alpha(\vartheta)}}J_{\rho+}^{\alpha(\vartheta)}F(\vartheta)\right|\leq\frac{\vartheta-\rho}{\alpha(\vartheta)+1}\left(\frac{2\alpha(\vartheta)}{(\alpha(\vartheta)+1)^{1+\frac{1}{\alpha(\vartheta)}}}\right)^{1-\frac{1}{q}}$$
$$\times\left(A_1(\alpha(\vartheta))\left|F'(\rho)\right|^q+A_2(\alpha(\vartheta))\left|F'(\vartheta)\right|^q+A_3(\alpha(\vartheta))\left|F'(\rho)\right|^q+A_4(\alpha(\vartheta))\left|F'(\vartheta)\right|^q\right)^{\frac{1}{q}}$$

where $A_1(\alpha(\vartheta))-A_4(\alpha(\vartheta))$ are the same as in Theorem 4.1 and $\alpha(\vartheta)>0$.

Proof. By using Lemma 3.1, power mean inequality and the convexity of $|F'|^q$, we obtain

$$\left|\frac{\alpha(\vartheta)F(\rho)+F(\vartheta)}{\alpha(\vartheta)+1}-\frac{\Gamma(\alpha(\vartheta)+1)}{(\vartheta-\rho)^{\alpha(\vartheta)}}J_{\rho+}^{\alpha(\vartheta)}F(\vartheta)\right|\leq\frac{\vartheta-\rho}{\alpha(\vartheta)+1}\int_0^1\left|1-(\alpha(\vartheta)+1)t^{\alpha(\vartheta)}\right|\left|F'(t\rho+(1-t)\vartheta)\right|\mathrm{d}t$$
$$\leq\frac{\vartheta-\rho}{\alpha(\vartheta)+1}\left[\left(\int_0^1\left|1-(\alpha(\vartheta)+1)t^{\alpha(\vartheta)}\right|\mathrm{d}t\right)^{1-\frac{1}{q}}\left(\int_0^1\left|1-(\alpha(\vartheta)+1)t^{\alpha(\vartheta)}\right|\left|F'(t\rho+(1-t)\vartheta)\right|^q\,\mathrm{d}t\right)^{\frac{1}{q}}\right]$$
$$\leq\frac{\vartheta-\rho}{\alpha(\vartheta)+1}\left[\left(\int_0^{\frac{1}{\sqrt[\alpha(\vartheta)]{\alpha(\vartheta)+1}}}\left(1-(\alpha(\vartheta)+1)t^{\alpha(\vartheta)}\right)\mathrm{d}t+\int_{\frac{1}{\sqrt[\alpha(\vartheta)]{\alpha(\vartheta)+1}}}^1\left((\alpha(\vartheta)+)t^{\alpha(\vartheta)}-1\right)\mathrm{d}t\right)^{1-\frac{1}{q}}\right.$$
$$\times\left(\int_0^{\frac{1}{\sqrt[\alpha(\vartheta)]{\alpha(\vartheta)+1}}}\left(1-(\alpha(\vartheta)+1)t^{\alpha(\vartheta)}\right)\left[t\left|F'(\rho)\right|^q+(1-t)\left|F'(\vartheta)\right|^q\right]\mathrm{d}t\right.$$
$$\left.\left.+\int_{\frac{1}{\sqrt[\alpha(\vartheta)]{\alpha(\vartheta)+1}}}^1\left((\alpha(\vartheta)+1)t^{\alpha(\vartheta)}-1\right)\left[t\left|F'(\rho)\right|^q+(1-t)\left|F'(\vartheta)\right|^q\right]\mathrm{d}t\right)^{\frac{1}{q}}\right]$$
$$\leq\frac{\vartheta-\rho}{\alpha(\vartheta)+1}\left(\frac{2\alpha(\vartheta)}{(\alpha(\vartheta)+1)^{1+\frac{1}{\alpha(\vartheta)}}}\right)^{1-\frac{1}{q}}$$
$$\times\left(A_1(\alpha(\vartheta))\left|F'(\rho)\right|^q+A_2(\alpha(\vartheta))\left|F'(\vartheta)\right|^q+A_3(\alpha(\vartheta))\left|F'(\rho)\right|^q+A_4(\alpha(\vartheta))\left|F'(\vartheta)\right|^q\right)^{\frac{1}{q}}.$$

This completes the proof. □

Remark 4.5. *If we choose $\alpha(x)=1$, in Theorem 4.4, then we obtain Theorem 1 in [25].*

Remark 4.6. *If we choose $\alpha(x)=\beta$ (constant), then we have Theorem 4.2 proved in [22].*

Theorem 4.7. *Let $F:\mathfrak{I}^\circ\to\mathbb{R}$ be a differentiable mapping on $\mathfrak{I}^\circ,\rho,\vartheta\in\mathfrak{I}^\circ$ with $\vartheta>\rho$. If $|F'|^q$ is convex on $[\rho,\vartheta]$ for $q>1$, then the subsequent inequality for the left Riemann-Liouville fractional* *integral holds:*

$$\begin{aligned}&\left|\frac{\alpha(\vartheta)F(\rho)+F(\vartheta)}{\alpha(\vartheta)+1}-\frac{\Gamma(\alpha(\vartheta)+1)}{(\vartheta-\rho)^{\alpha(\vartheta)}}J_{\rho+}^{\alpha(\vartheta)}F(\vartheta)\right|\\&\leq\frac{\vartheta-\rho}{\alpha(\vartheta)+1}\left(A_5(\alpha(\vartheta),p)+A_6(\alpha(\vartheta),p)\right)^{\frac{1}{p}}\left(\frac{|F'(\rho)|^q+|F'(\vartheta)|^q}{2}\right)^{\frac{1}{q}}\end{aligned}$$

where

$$\begin{aligned}A_5(\alpha(\vartheta),p)&=\int_0^{\frac{1}{\sqrt[\alpha(\vartheta)]{\alpha(\vartheta)+1}}}\left(1-(\alpha(\vartheta)+1)t^{\alpha(\vartheta)}\right)^p\,\mathrm{d}t,\\A_6(\alpha(\vartheta),p)&=\int_{\frac{1}{\sqrt[\alpha(\vartheta)]{\alpha(\vartheta)+1}}}^1\left((\alpha(\vartheta)+1)t^{\alpha(\vartheta)}-1\right)^p\,\mathrm{d}t,\end{aligned}$$

with $\frac{1}{p}+\frac{1}{q}=1$ *and* $\alpha(\vartheta)>0$.

Proof. By using Lemma 3.1, Hölder inequality and the convexity of $|F'|^q$, we obtain

$$\begin{aligned}&\left|\frac{\alpha(\vartheta)F(\rho)+F(\vartheta)}{\alpha(\vartheta)+1}-\frac{\Gamma(\alpha(\vartheta)+1)}{(\vartheta-\rho)^{\alpha(\vartheta)}}J_{\rho+}^{\alpha(\vartheta)}F(\vartheta)\right|\\&\leq\frac{\vartheta-\rho}{\alpha(\vartheta)+1}\int_0^1\left|1-(\alpha(\vartheta)+1)t^{\alpha(\vartheta)}\right|\left|F'(t\rho+(1-t)\vartheta)\right|\mathrm{d}t\\&\leq\frac{\vartheta-\rho}{\alpha(\vartheta)+1}\left[\left(\int_0^1\left|1-(\alpha(\vartheta)+1)t^{\alpha(\vartheta)}\right|^p\,\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^1\left|F'(t\rho+(1-t)\vartheta)\right|^q\,\mathrm{d}t\right)^{\frac{1}{q}}\right]\\&\leq\frac{\vartheta-\rho}{\alpha(\vartheta)+1}\left(\int_0^{\frac{1}{\sqrt[\alpha(\vartheta)]{\alpha(\vartheta)+1}}}\left(1-(\alpha(\vartheta)+1)t^{\alpha(\vartheta)}\right)^p\,\mathrm{d}t+\int_{\frac{1}{\sqrt{\alpha(\vartheta)+1}}}^1\left((\alpha(\vartheta)+1)t^{\alpha(\vartheta)}-1\right)^p\,\mathrm{d}t\right)^{\frac{1}{p}}\\&\quad\times\left(\int_0^1\left[t\left|F'(\rho)\right|^q+(1-t)\left|F'(\vartheta)\right|^q\right]\mathrm{d}t\right)^{\frac{1}{q}}\\&\leq\frac{\vartheta-\rho}{\alpha(\vartheta)+1}\left[\left(A_5(\alpha(\vartheta),p)+A_6(\alpha(\vartheta),p)\right)^{\frac{1}{p}}\left(\frac{|F'(\rho)|^q+|F'(\vartheta)|^q}{2}\right)^{\frac{1}{q}}\right].\end{aligned}$$

This completes the proof. □

Remark 4.8. *If we assign* $\alpha(x)=1$*, in Theorem 4.7, then we obtain Theorem* 2.3 *in* [10].

Remark 4.9. *If we take* $\alpha(x)=\beta$ *(constant), then we obtain Theorem* 4.3 *in* [22].

Theorem 4.10. *Let* $F:\mathfrak{I}^\circ\to\mathbb{R}$ *be a differentiable mapping on* $\mathfrak{I}^\circ$, $\rho,\vartheta\in\mathfrak{I}^\circ$ *with* $\vartheta>\rho$. *If* $|F'|$ *is convex on* $[\rho,\vartheta]$*, then the subsequent inequality for the left Riemann-Liouville fractional integral holds:*

$$\begin{aligned}&\left|\frac{\Gamma(\alpha(\vartheta)+1)}{(\vartheta-\rho)^{\alpha(\vartheta)}}J_{\rho+}^{\alpha(\vartheta)}F(\vartheta)-F\left(\frac{\alpha(\vartheta)\rho+\vartheta}{\alpha(\vartheta)+1}\right)\right|\\&\leq(\vartheta-\rho)\left[A_7(\alpha(\vartheta))\left|F'(\rho)\right|+A_8(\alpha(\vartheta))\left|F'(\vartheta)\right|+A_9(\alpha(\vartheta))\left|F'(\rho)\right|+A_{10}(\alpha(\vartheta))\left|F'(\vartheta)\right|\right]\end{aligned}$$

where

$$A_7(\alpha(\vartheta)) = \frac{\alpha(\vartheta)\left(\frac{\alpha(\vartheta)}{\alpha(\vartheta)+1}\right)^{\alpha(\vartheta)+1}}{(\alpha(\vartheta)+1)(\alpha(\vartheta)+2)}, \; A_8(\alpha(\vartheta)) = \frac{2\left(\frac{\alpha(\vartheta)}{\alpha(\vartheta)+1}\right)^{\alpha(\vartheta)+1}}{(\alpha(\vartheta)+1)(\alpha(\vartheta)+2)},$$
$$A_9(\alpha(\vartheta)) = \frac{\alpha(\vartheta)\left(2(\alpha(\vartheta))^{\alpha(\vartheta)} + (\alpha(\vartheta)+1)^{\alpha(\vartheta)}\right)}{2(\alpha(\vartheta)+1)^{\alpha(\vartheta)+2}(\alpha(\vartheta)+2)},$$
$$A_{10}(\alpha(\vartheta)) = \frac{4(\alpha(\vartheta))^{\alpha(\vartheta)+1} - \alpha(\vartheta)(\alpha(\vartheta)+1)^{\alpha(\vartheta)}}{2(\alpha(\vartheta)+1)^{\alpha(\vartheta)+2}(\alpha(\vartheta)+2)},$$

with $\alpha(\vartheta) > 0$.

Proof. By using Lemma 3.4 and the convexity of $|F'|$, we obtain

$$\begin{aligned}
&\left|\frac{\Gamma(\alpha(\vartheta)+1)}{(\vartheta-\rho)^{\alpha(\vartheta)}} J_{\rho+}^{\alpha(\vartheta)} F(\vartheta) - F\left(\frac{\alpha(\vartheta)\rho+\vartheta}{\alpha(\vartheta)+1}\right)\right| \\
&\leq (\vartheta-\rho)\left[\int_0^{\frac{\alpha(\vartheta)}{\alpha(\vartheta)+1}} t^{\alpha(\vartheta)} \left|F'(t\rho+(1-t)\vartheta)\right| \mathrm{d}t + \int_{\frac{\alpha(\vartheta)}{\alpha(\vartheta)+1}}^1 \left(1-t^{\alpha(\vartheta)}\right) \left|F'(t\rho+(1-t)\vartheta)\right| \mathrm{d}t\right] \\
&\leq (\vartheta-\rho)\left[\int_0^{\frac{\alpha(\vartheta)}{\alpha(\vartheta)+1}} t^{\alpha(\vartheta)} \left[t\left|F'(\rho)\right| + (1-t)\left|F'(\vartheta)\right|\right] \mathrm{d}t\right. \\
&\quad \left. + \int_{\frac{\alpha(\vartheta)}{\alpha(\vartheta)+1}}^1 \left(1-t^{\alpha(\vartheta)}\right) \left[t\left|F'(\rho)\right| + (1-t)\left|F'(\vartheta)\right|\right] \mathrm{d}t\right] \\
&\leq (\vartheta-\rho)\left[\frac{\alpha(\vartheta)\left(\frac{\alpha(\vartheta)}{\alpha(\vartheta)+1}\right)^{\alpha(\vartheta)+1}}{(\alpha(\vartheta)+1)(\alpha(\vartheta)+2)}\left|F'(\rho)\right| + \frac{2\left(\frac{\alpha(\vartheta)}{\alpha(\vartheta)+1}\right)^{\alpha(\vartheta)+1}}{(\alpha(\vartheta)+1)(\alpha(\vartheta)+2)}\left|F'(\vartheta)\right|\right. \\
&\quad \left. + \frac{\alpha(\vartheta)\left(2\alpha^{\alpha(\vartheta)} + (\alpha(\vartheta)+1)^{\alpha(\vartheta)}\right)}{2(\alpha(\vartheta)+1)^{\alpha(\vartheta)+2}(\alpha(\vartheta)+2)}\left|F'(\rho)\right| + \frac{4\alpha(\vartheta)^{\alpha(\vartheta)+1} - \alpha(\vartheta)(\alpha(\vartheta)+1)^{\alpha(\vartheta)}}{2(\alpha(\vartheta)+1)^{\alpha(\vartheta)+2}(\alpha(\vartheta)+2)}\left|F'(\vartheta)\right|\right].
\end{aligned}$$

This completes the proof. □

Remark 4.11. *If we assign* $\alpha(x) = 1$, *in Theorem 4.10, then we obtain Theorem* 2.2 *in* [21].

Remark 4.12. *If we take* $\alpha(x) = \beta$ *(constant), then we obtain Theorem* 4.4 *in* [22].

Theorem 4.13. *Let* $F : \mathfrak{I}^\circ \to \mathbb{R}$ *be a differentiable mapping on* $\mathfrak{I}^\circ, \rho, \vartheta \in \mathfrak{I}^\circ$ *with* $\vartheta > \rho$. *If* $|F'|^q$ *is convex on* $[\rho, \vartheta]$ *for* $q \geq 1$, *then the subsequent inequality for the left Riemann-Liouville fractional* *integral holds:*

$$\left|\frac{\Gamma(\alpha(\vartheta)+1)}{(\vartheta-\rho)^{\alpha(\vartheta)}}J_{\rho+}^{\alpha(\vartheta)}F(\vartheta)-F\left(\frac{\alpha(\vartheta)\rho+\vartheta}{\alpha(\vartheta)+1}\right)\right|\leq(\vartheta-\rho)\left(\frac{\alpha(\vartheta)^{\alpha(\vartheta)+1}}{(\alpha(\vartheta)+1)^{\alpha(\vartheta)+2}}\right)^{1-\frac{1}{q}}$$
$$\times\left[\left(A_7(\alpha(\vartheta))\left|F'(\rho)\right|^q+A_8(\alpha(\vartheta))\left|F'(\vartheta)\right|^q\right)^{\frac{1}{q}}+\left(A_9(\alpha(\vartheta))\left|F'(\rho)\right|^q+A_{10}(\alpha(\vartheta))\left|F'(\vartheta)\right|^q\right)^{\frac{1}{q}}\right]$$

where $A_7(\alpha(\vartheta))-A_{10}(\alpha(\vartheta))$ are the same as in Theorem 4.10 and $\alpha(\vartheta)>0$.

Proof. By using Lemma 3.4, power mean inequality and the convexity of $|F'|^q$, we obtain

$$\begin{aligned}
&\left|\frac{\Gamma(\alpha(\vartheta)+1)}{(\vartheta-\rho)^{\alpha(\vartheta)}}J_{\rho+}^{\alpha(\vartheta)}F(\vartheta)-F\left(\frac{\alpha(\vartheta)\rho+\vartheta}{\alpha(\vartheta)+1}\right)\right|\\
&\leq(\vartheta-\rho)\left[\int_0^{\frac{\alpha(\vartheta)}{\alpha(\vartheta)+1}}t^{\alpha(\vartheta)}\left|F'(t\rho+(1-t)\vartheta)\right|\mathrm{d}t+\int_{\frac{\alpha(\vartheta)}{\alpha(\vartheta)+1}}^1\left(1-t^{\alpha(\vartheta)}\right)\left|F'(t\rho+(1-t)\vartheta)\right|\mathrm{d}t\right]\\
&\leq(\vartheta-\rho)\left[\left(\int_0^{\frac{\alpha(\vartheta)}{\alpha(\vartheta)+1}}t^{\alpha(\vartheta)}\mathrm{d}t\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{\alpha(\vartheta)}{\alpha(\vartheta)+1}}t^{\alpha(\vartheta)}\left|F'(t\rho+(1-t)\vartheta)\right|^q\,\mathrm{d}t\right)^{\frac{1}{q}}\right.\\
&\quad\left.+\left(\int_{\frac{\alpha(\vartheta)}{\alpha(\vartheta)+1}}^1\left(1-t^{\alpha(\vartheta)}\right)\mathrm{d}t\right)^{1-\frac{1}{q}}\left(\int_{\frac{\alpha(\vartheta)}{\alpha(\vartheta)+1}}^1\left(1-t^{\alpha(\vartheta)}\right)\left|F'(t\rho+(1-t)\vartheta)\right|^q\,\mathrm{d}t\right)^{\frac{1}{q}}\right]\\
&\leq(\vartheta-\rho)\left(\frac{\alpha(\vartheta)^{\alpha(\vartheta)+1}}{(\alpha(\vartheta)+1)^{\alpha(\vartheta)+2}}\right)^{1-\frac{1}{q}}\left[\left(\int_0^{\frac{\alpha(\vartheta)}{\alpha(\vartheta)+1}}t^{\alpha(\vartheta)}\left[t\left|F'(\rho)\right|^q+(1-t)\left|F'(\vartheta)\right|^q\right]\mathrm{d}t\right)^{\frac{1}{q}}\right.\\
&\quad\left.+\left(\int_{\frac{\alpha(\vartheta)}{\alpha(\vartheta)+1}}^1\left(1-t^{\alpha(\vartheta)}\right)\left[t\left|F'(\rho)\right|^q+(1-t)\left|F'(\vartheta)\right|^q\right]\mathrm{d}t\right)^{\frac{1}{q}}\right]\\
&\leq(\vartheta-\rho)\left(\frac{\alpha(\vartheta)^{\alpha(\vartheta)+1}}{(\alpha(\vartheta)+1)^{\alpha(\vartheta)+2}}\right)^{1-\frac{1}{q}}\\
&\quad\times\left[\left(\frac{\alpha(\vartheta)\left(\frac{\alpha(\vartheta)}{\alpha(\vartheta)+1}\right)^{\alpha(\vartheta)+1}}{(\alpha(\vartheta)+1)(\alpha(\vartheta)+2)}\left|F'(\rho)\right|^q+\frac{2\left(\frac{\alpha(\vartheta)}{\alpha(\vartheta)+1}\right)^{\alpha(\vartheta)+1}}{(\alpha(\vartheta)+1)(\alpha(\vartheta)+2)}\left|F'(\vartheta)\right|^q\right)^{\frac{1}{q}}\right.\\
&\quad+\left(\frac{\alpha(\vartheta)\left(2(\alpha(\vartheta))^{\alpha(\vartheta)}+(\alpha(\vartheta)+1)^{\alpha(\vartheta)}\right)}{2(\alpha(\vartheta)+1)^{\alpha(\vartheta)+2}(\alpha(\vartheta)+2)}\left|F'(\rho)\right|^q\right.\\
&\quad\left.\left.+\frac{4(\alpha(\vartheta))^{\alpha(\vartheta)+1}-\alpha(\vartheta)(\alpha(\vartheta)+1)^{\alpha(\vartheta)}}{2(\alpha(\vartheta)+1)^{\alpha(\vartheta)+2}(\alpha(\vartheta)+2)}\left|F'(\vartheta)\right|^q\right)^{\frac{1}{q}}\right].
\end{aligned}$$

This completes the proof. □

Remark 4.14. *In Theorem 4.13, when $\alpha(\vartheta)=1$ is chosen, we find the following midpoint type inequality*

$$\left|\frac{1}{\vartheta-\rho}\int_{\rho}^{\vartheta}F(u)\mathrm{d}u-F\left(\frac{\rho+\vartheta}{2}\right)\right|\leq\frac{\vartheta-\rho}{8}\left[\left(\frac{|F'(\rho)|^{q}+2|F'(\vartheta)|^{q}}{3}\right)^{\frac{1}{q}}+\left(\frac{2|F'(\rho)|^{q}+|F'(\vartheta)|^{q}}{3}\right)^{\frac{1}{q}}\right].$$

Remark 4.15. *If we take $\alpha(x)=\beta$ (constant), then we will get Theorem 4.5 proved in [22].*

Theorem 4.16. *Let $F:\mathfrak{I}^{\circ}\to\mathbb{R}$ be a differentiable mapping on $\mathfrak{I}^{\circ},\rho,\vartheta\in\mathfrak{I}^{\circ}$ with $\rho<\vartheta$. If $|F'|^{q}$ is convex on $[\rho,\vartheta]$ for $q>1$, then the subsequent inequality for the left Riemann-Liouville fractional integral holds:*

$$\begin{aligned}&\left|\frac{\Gamma(\alpha(\vartheta)+1)}{(\vartheta-\rho)^{\alpha(\vartheta)}}J_{\rho+}^{\alpha(\vartheta)}F(\vartheta)-F\left(\frac{\alpha(\vartheta)\rho+\vartheta}{\alpha(\vartheta)+1}\right)\right|\\&\leq(\vartheta-\rho)\left[A_{11}^{\frac{1}{p}}(\alpha(\vartheta),p)\left(\frac{\alpha(\vartheta)^{2}}{2(\alpha(\vartheta)+1)^{2}}\left|F'(\rho)\right|^{q}+\frac{\alpha(\vartheta)^{2}+2\alpha(\vartheta)}{2(\alpha(\vartheta)+1)^{2}}\left|F'(\vartheta)\right|^{q}\right)^{\frac{1}{q}}\right.\\&\quad\left.+A_{12}^{\frac{1}{p}}(\alpha(\vartheta),p)\left(\frac{2\alpha(\vartheta)+1}{2(\alpha(\vartheta)+1)^{2}}\left|F'(\rho)\right|^{q}+\frac{1}{2(\alpha(\vartheta)+1)^{2}}\left|F'(\vartheta)\right|^{q}\right)^{\frac{1}{q}}\right]\end{aligned}$$

where

$$A_{11}(\alpha(\vartheta),p)=\int_{0}^{\frac{\alpha(\vartheta)}{\alpha(\vartheta)+1}}t^{\alpha(\vartheta)p}\,\mathrm{d}t,\quad A_{12}(\alpha(\vartheta),p)=\int_{\frac{\alpha(\vartheta)}{\alpha(\vartheta)+1}}^{1}(1-t^{\alpha}(\vartheta))^{p}\,\mathrm{d}t,$$

with $\frac{1}{p}+\frac{1}{q}=1$ and $\alpha(\vartheta)>0$.

Proof. By using Lemma 3.4, Hölder inequality and the convexity of $|F'|^{q}$, we obtain

$$\begin{aligned}&\left|\frac{\Gamma(\alpha(\vartheta)+1)}{(\vartheta-\rho)^{\alpha(\vartheta)}}J_{\rho+}^{\alpha(\vartheta)}F(\vartheta)-F\left(\frac{\alpha(\vartheta)\rho+\vartheta}{\alpha(\vartheta)+1}\right)\right|\\&\leq(\vartheta-\rho)\left[\int_{0}^{\frac{\alpha(\vartheta)}{\alpha(\vartheta)+1}}t^{\alpha(\vartheta)}\left|F'(t\rho+(1-t)\vartheta)\right|\mathrm{d}t+\int_{\frac{\alpha(\vartheta)}{\alpha(\vartheta)+1}}^{1}\left(1-t^{\alpha(\vartheta)}\right)\left|F'(t\rho+(1-t)\vartheta)\right|\mathrm{d}t\right]\\&\leq(\vartheta-\rho)\left[\left(\int_{0}^{\frac{\alpha(\vartheta)}{\alpha(\vartheta)+1}}t^{\alpha(\vartheta)p}\,\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_{0}^{\frac{\alpha(\vartheta)}{\alpha(\vartheta)+1}}\left|F'(ta+(1-t)\vartheta)\right|^{q}\,\mathrm{d}t\right)^{\frac{1}{q}}\right.\\&\quad\left.+\left(\int_{\frac{\alpha(\vartheta)}{\alpha(\vartheta)+1}}^{1}\left(1-t^{\alpha(\vartheta)}\right)^{p}\,\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_{\frac{\alpha(\vartheta)}{\alpha(\vartheta)+1}}^{1}\left|F'(ta+(1-t)\vartheta)\right|^{q}\,\mathrm{d}t\right)^{\frac{1}{q}}\right]\end{aligned}$$

$$\begin{aligned}
&\leq (\vartheta-\rho)\left[\left(\int_0^{\frac{\alpha(\vartheta)}{\alpha(\vartheta)+1}} t^{\alpha(\vartheta)p}\,\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^{\frac{\alpha(\vartheta)}{\alpha(\vartheta)+1}}\left[t\left|F'(\rho)\right|^q+(1-t)\left|F'(\vartheta)\right|^q\right]\mathrm{d}t\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\int_{\frac{\alpha(\vartheta)}{\alpha(\vartheta)+1}}^1\left(1-t^{\alpha(\vartheta)}\right)^p\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_{\frac{\alpha(\vartheta)}{\alpha(\vartheta)+1}}^1\left[t\left|F'(\rho)\right|^q+(1-t)\left|F'(\vartheta)\right|^q\right]\mathrm{d}t\right)^{\frac{1}{q}}\right]\\
&\leq (\vartheta-\rho)\left[A_{11}^{\frac{1}{p}}(\alpha(\vartheta),p)\left(\frac{\alpha(\vartheta)^2}{2(\alpha(\vartheta)+1)^2}\left|F'(\rho)\right|^q+\frac{\alpha(\vartheta)^2+2\alpha(\vartheta)}{2(\alpha(\vartheta)+1)^2}\left|F'(\vartheta)\right|^q\right)^{\frac{1}{q}}\right.\\
&\left.+A_{12}^{\frac{1}{p}}(\alpha(\vartheta),p)\left(\frac{2\alpha(\vartheta)+1}{2(\alpha(\vartheta)+1)^2}\left|F'(\rho)\right|^q+\frac{1}{2(\alpha(\vartheta)+1)^2}\left|F'(\vartheta)\right|^q\right)^{\frac{1}{q}}\right].
\end{aligned}$$

This completes the proof. □

Remark 4.17. *In Theorem 4.16, when $\alpha(\vartheta)=1$ is chosen, we get Theorem* 2.3 *in* [21].

Remark 4.18. *If we assign $\alpha(x)=\beta$ (constant), then we will get Theorem* 4.6 *proved in* [22].

5. Numerical Examples

Example 5.1. *We define a convex mapping $F(x)=x^2$. Then, from inequality (2.1) for $\alpha(x)=\sin x, \rho=0$ and $\vartheta=1$, we have*

$$F\left(\frac{0+1}{\sin(1)+1}\right)\approx\frac{294897}{1000000}$$
$$\frac{\Gamma(\sin(1)+1)}{(1-0)^{\sin(1)}}J_{0^+}^{\sin(1)}F(1)\approx\frac{382227}{1000000}$$

and

$$\frac{\sin(1)F(0)+F(1)}{\sin(1)+1}\approx\frac{135761}{250000}.$$

It is obvious that

$$\frac{294897}{1000000}<\frac{382227}{1000000}<\frac{135761}{250000}.$$

This exemplifies that the inequality (2.1) is valid for convex functions.

Example 5.2. *We define a convex mapping $F(x)=x^2$. Then, from inequality (2.1) for $\alpha(x)=\cos x, \rho=0$ and $\vartheta=1$, we have*

$$F\left(\frac{0+1}{\cos(1)+1}\right)\approx\frac{421491}{1000000}$$
$$\frac{\Gamma(\cos(1)+1)}{(1-0)^{\cos(1)}}J_{0^+}^{\cos(1)}F(1)\approx\frac{511139}{1000000}$$

and

$$\frac{\cos(1)F(0)+F(1)}{\cos(1)+1} \approx 1\frac{649223}{1000000}.$$

It is obvious that

$$\frac{421491}{1000000} < \frac{511139}{1000000} < 1\frac{649223}{1000000}.$$

This is another example of application of the inequality (2.1), which is valid for convex functions.

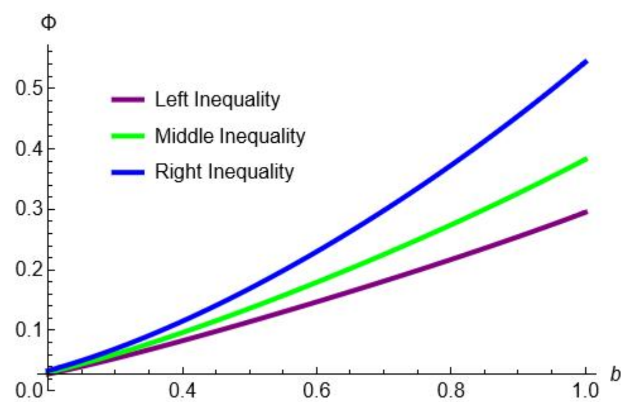

FIGURE 1. In 2D-Plot when ρ is fixed and ϑ lies between 0 and 1.

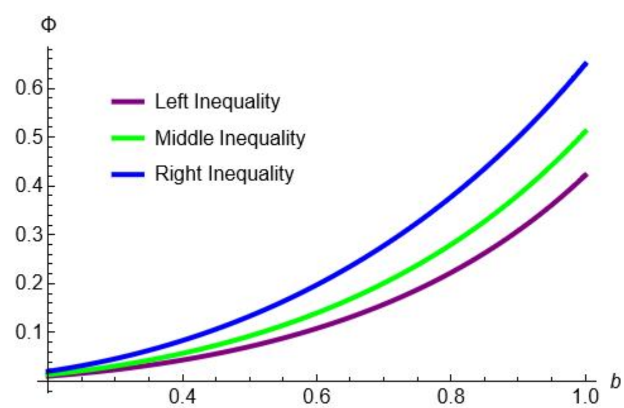

FIGURE 2. In 2D-Plot when ρ is fixed and ϑ lies between 0 and 1.

Remark 5.3. *The inequalities stated in Theorem 2.1 are exemplified by both Examples 5.1 and 5.2. A comparative analysis further emphasises this finding, as seen in figures 1 and 2.*

Conclusion

This study has added new results to H-H inequalities and looked at how they can be used by looking at convex functions and fractional integrals with variable order. We derived new H-H type inequalities by studying convex functions for fractional integrals and also found new fractional trapezoid and midpoint type inequalities with variable order. We gave numerical examples and a graphical analysis of the novel inequalities. These inequalities will be helpful for researchers who are working in the field of optimization theory and mathematical inequalities.

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