NON-RECURSIVE CANONICAL BASIS COMPUTATIONS FOR LOW RANK KASHIWARA CRYSTALS OF AFFINE TYPE $A$

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Abstract. We give an improved algorithm for the action of the divided power of a Chevalley basis element of an affine Lie algebra of type $A$ on canonical basis elements satisfying an easily checked uniformity condition and compare calculation times for our algorithm against the standard algorithm. For symmetric Kashiwara crystals of affine type $A$ and rank $e = 2$, and for the canonical basis elements that we call external, corresponding to weights on the outer skin of the Kashiwara crystal, we construct the canonical basis elements in a non-recursive manner. In particular, for a symmetric crystal with $\Lambda = a\Lambda_0 + a\Lambda_1$, we give formulae for the canonical basis elements for all the $e$-regular multipartitions with defects either $k(a - k)$ or $k(a - k) + 2a$, for $0 \leq k \leq a$.

1. INTRODUCTION

The highest weight representations of the enveloping algebra of an affine Lie algebra have been intensively studied through the Kashiwara crystal $B(\Lambda)$ [[10], §2, §3] corresponding to the chosen highest weight $\Lambda$. For Lie algebras of affine type $A$, there are three different representations for the basis elements, through multipartitions, through Littelmann paths [13] and through canonical bases. In an earlier paper [17], we studied the passage from multipartitions to Littelmann paths, and in this paper we consider the passage from multipartitions to canonical basis elements. In both cases, the use of computer algebra was critical in making calculations and formulating conjectures.

The Kashiwara crystal, long studied for its importance to the representation theory of the quantum enveloping algebra, is also important because of the categorification theory of Chuang and Rouquier [5]. Under categorification, the canonical basis elements that we will study correspond to simple modules in blocks of cyclotomic Hecke algebras.

Our original conjectures were obtained by experimentation, using programs in SageMath [19] or programs privately available to interested researchers. In generating the highest weight representation of a particular dominant integral weight, the particular object from which we start is the object `CrystalOfLSPaths(CartanType)`, whose authors are Mark Shimozono and Anne Schilling. This program generates the Littelmann paths.

In the case of affine type $A$, there are two additional representations of the basis elements of the crystal, one by multipartitions and one by canonical basis elements. Of these three representations, the multipartition is the most compact, the Littelmann path is second and the canonical basis is the most verbose. We choose the Littelmann path model, which is available for all supported Cartan types, as our primary representation.

Our own program [19] made use of an implementation by Travis Scrimshaw of an algorithm of Matthew Fayers [7], extending the algorithm in [14], for constructing the

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canonical bases and a variant of Kleshchev’s algorithm [12] for constructing $e$-regular multipartitions recursively. Our program, CanonicalBasisfromPaths(CartanType, HighestWeight) proceeds recursively degree by degree. For each basis element $w$, we keep track of all the different paths which can be used to reach $w$. Each basis element corresponds to a unique multipartition, called an $e$-regular multipartition, and by using the signature method [12], we can calculate the new multipartition. From the point of view of information theory, there is obviously a great deal of redundancy in the representation by canonical basis, from which the $e$-regular multipartition can be obtained immediately. When we write the canonical basis element in the natural basis of the Fock space by multiplying with canonical basis, from which the $e$-regular multipartition can be obtained immediately.

2. DEFINITIONS AND NOTATION

Let $\mathfrak{g}$ be the affine Lie algebra $A_1^{(1)}$, untwisted of affine type $A$, with a Cartan matrix given by

$$C = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$$

Let $Q$ be the $\mathbb{Z}$-lattice generated by the simple roots, $\alpha_0, \alpha_1$. Let $Q_+$ be the subset of $Q$ in which all coefficients are non-negative. We define an order on the set of weights by setting $\psi \geq \psi'$ if $\psi - \psi' \in Q_+$ and say the $\psi$ is higher than $\psi'$.

The weight space $P$ of the affine Lie algebra has two different bases. One is given by the fundamental weights and null root, $\Lambda_0, \Lambda_1, \delta$, and one is given by $\Lambda_0, \alpha_0, \alpha_1$. We will usually use the first basis for our weights.

**Definition 2.1.** The projection of a weight $\psi$ onto the subspace generated by the fundamental weights will be called the hub of the weight, given for $e = 2$ by the formula

$$h(\psi) = [\langle h_0, \psi \rangle, \langle h_1, \psi \rangle],$$

where the $\{h_0, h_1\}$ are dual to the fundamental weights in a dual basis. The hubs as originally defined by Fayers in [6] were the negatives of the hubs used here.

The Cartan matrix of $\mathfrak{g}$ is symmetric, with a symmetric product on the weight space, where the values we will need are generated by $(\alpha_i \mid \alpha_i) = 2$, $(\Lambda_i \mid \alpha_i) = 1$, $(\alpha_0 \mid \alpha_1) = (\alpha_1 \mid \alpha_0) = 2$ and $(\Lambda_i \mid \alpha_j) = 0$ for $i \neq j$, for $i, j \in \{0, 1\}$.

A weight $\Lambda = c_0 \Lambda_0 + c_1 \Lambda_1$ will be called a dominant integral weight if the $c_i$ are non-negative integers. A highest weight module $V(\Lambda)$ will be generated from an element $u_\Lambda$ of weight $\Lambda$ by action of the Chevalley generators $f_i$, which always lower the weight. Let $P(\Lambda)$ be the set of weights of $V(\Lambda)$ as in Kac [[9], Ch. 11]. Every weight $\psi$ in $P(\Lambda)$ has the form $\Lambda - \alpha$, for $\alpha \in Q_+$. The vector of nonnegative integers giving the coefficients of $\alpha$ is called the content of $\psi$ and the sum of the coefficients is called the degree.

In order to operate on the set $P(\Lambda)$ of weights of $V(\Lambda)$, we use operators $\tilde{e}_i$ and $\tilde{f}_i$. For a weight $\psi$ in $P(\Lambda)$, we define $\tilde{f}_i \cdot \psi = \psi - \alpha_i$ if the new weight lies in $P(\Lambda)$ and 0 otherwise, while $\tilde{e}_i$ is defined using $\tilde{e}_i \cdot \psi$ equal to $\psi + \alpha_i$ or 0.

**Definition 2.2.** For any weight $\psi$, the set of weights $\ldots, \tilde{e}_i^2(\psi), \tilde{e}_i(\psi), \psi, \tilde{f}_i(\psi), \tilde{f}_i^2(\psi), \ldots$ will be called the $i$-string of $\psi$.

A highest weight module $V(\Lambda)$ is integrable, which implies that each of these $i$-strings is finite. The structure of the $i$-strings is treated in some detail in [[8], §3] and particularly in Remark 3.1 and Example 2. The treatment in the published version [4] is less detailed.
We follow [12, §3.3] in defining the defect by

\[ \text{def}(\psi) = \frac{1}{2}((\Lambda | \Lambda) - (\psi | \psi)) = (\Lambda | \alpha) - \frac{1}{2}(\alpha | \alpha). \]

The defect is non-negative and is, in fact, an integer for the untwisted affine Lie algebras of affine type \( A \) treated in this paper. The weights of defect 0 are those lying in the Weyl group orbit of \( \Lambda \) and play an important role in the theory. Since the symmetric product is invariant under the action of the Weyl group, so is the defect. Let \( W \) denote the Weyl group, generated by reflections \( s_0, s_1 \). The reflection \( s_i \) reflects each \( i \)-string, so the defects are symmetric along an \( i \)-string. For the affine Lie algebras, the Weyl group is infinite, being the semidirect product of a finite Weyl group corresponding to a finite Lie group which we get by crossing out the first row and column of the Cartan matrix, acting on the root lattice for the finite Lie group, which is an infinite abelian subgroup [9].

A partition \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell) \) is a sequence of integers with \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell \) of length \( \ell(\lambda) = t \). A multipartition \( \lambda = (\lambda^1, \lambda^2, \ldots, \lambda^r) \) is a sequence of partitions. The dominance order on multipartitions is given by \( \mu \succeq \lambda \) if, for all integers \( k \) with \( 1 \leq k \leq r \) and \( j \leq \ell(\mu^k) \),

\[ \sum_{\ell=1}^{k-1} | \mu^{\ell} | + \sum_{i=1}^{j} \mu^{k}_{\ell} \geq \sum_{\ell=1}^{k-1} | \lambda^{\ell} | + \sum_{i=1}^{j} \lambda^{k}_{\ell}. \]

Our quantum enveloping algebra will be \( \mathcal{U} = U_q(\hat{\mathfrak{sl}}(2)) \), where we are using Lusztig’s \( v \) in place of the more common quantum parameter \( q \) because we are using balanced quantum integers \( [n]_v = v^{n-1} + v^{n-3} + \cdots + v^{-(n-3)} + v^{-(n-1)} \). The quantum factorial is \( [n]_v! = [n]_v \cdot [n-1]_v \cdot \cdots \cdot [1]_v \). The underlying ring of the enveloping algebra is \( \mathbb{Q}(v) \), and the generators are \( e_i, f_i, v^{h_i} \) for \( i \in I = \mathbb{Z}/2\mathbb{Z} \) and a central element \( v^d \).

Choose a sequence \( s = (k_1, \ldots, k_r) \), called a multicharge, such that \( \Lambda = \Lambda_{k_1} + \cdots + \Lambda_{k_r} \). In affine type \( A \), the number \( r \) of terms in the sum is called the level.

**Definition 2.3.** The Fock space \( \mathcal{F}^s \) for multicharge \( s \) is a vector space over the field \( \mathbb{Q}(v) \) with a natural basis corresponding to multipartitions consisting of \( r \) partitions, where \( |\mu| \) is the natural basis element corresponding to the multipartition \( \mu \), and the empty multipartition is represented by \( | \) \). The multicharge determines residues attached to the nodes, as follows. For a multipartition \( \lambda \) with Young diagram \( Y(\lambda) \), the node \((t,u)\) in partition \( \ell \) is given residue

\[ k_\ell + u - t \mod 2. \]

Using these residues, we can define an action of the quantum enveloping algebra \( \mathcal{U} = U_q(\hat{\mathfrak{sl}}(2)) \). An addable \( i \)-node \( n \) is a node of residue \( i \) outside \( Y(\lambda) \) such that if added it would give a multipartition, which we denote by \( \lambda^n \). A removable \( i \)-node \( m \) inside a multipartition \( \mu \) is a node of residue \( i \) at the end of a row or column which, if removed, would give a multipartition, which we denote by \( \mu_m \). The quantum enveloping algebra acts on the Fock space by determining actions for \( e_i, f_i, v^{h_i} \) and \( v^d \) where \( i \in I = \mathbb{Z}/2\mathbb{Z} \), as follows:

- For an addable node, let us now define \( N(n, i) = \# \{ \text{addable } i \text{-nodes above } n \} - \# \{ \text{removable } i \text{-nodes above } n \} \) and set

\[ f_i(|\lambda|) = \sum_n v^{N(n,i)} |\lambda^n|. \]
The operator we get by dividing the Fock space element \( f \) giving two natural basis elements. The coefficients have a common factor equal to the highest exponent of \( v \) is the highest exponent of \( v \) with highest exponent equal to the defect, namely 3.

Letting \( N(i) = \# \{ \text{addable } i \text{-nodes} \} - \# \{ \text{removable } i \text{-nodes} \} \), we let \( v^{N(i)} \) act on an element \( |\mu\rangle \) of the natural basis by multiplication by \( v^{N(i)}|\mu\rangle \).

Letting \( N_0 \) be the number of 0-nodes in \( \mu \), \( v^0|\mu\rangle = v^{N_0}|\mu\rangle \).

**Example 1.** Let \( e = 2 \), \( s = (0, 0, 1) \). Starting with \(|\rangle \) we calculate some actions by \( f_i \).

- \( f_0(|\rangle) = |[(1, \emptyset; \emptyset)] + v|[(0, 1); \emptyset]\rangle \),
- \( f_1 f_0(|\rangle) = |[(2, \emptyset; \emptyset)] + v^2|[(1, \emptyset; (1))] + v|[(0, 2); \emptyset]\rangle + v^2|[(\emptyset, (1); \emptyset)] + v^3|[(\emptyset, (1); (1))]\rangle \),
- \( f_2 f_0(|\rangle) = (v^{-1} + v)|[(2, 1); \emptyset]\rangle + v^2|[(2, 1); (1)] + v|[(0, 2); \emptyset]\rangle + v^2|[(\emptyset, (1); (1))]\rangle \),
- \( f_2 f_0(|\rangle) = (v^{-2} + 1 + v^2(v^{-1} + v)|[(2, 1); \emptyset]\rangle + v|[(0, 2); (1)]\rangle) \).

The Fock space element \( f_0(|\rangle) \) has two natural basis elements, each with three addable 1-nodes and no removable 1-nodes. The corresponding weight, \( \Lambda - \alpha_0 \), has defect 1, which is the highest exponent of \( v \). After acting by \( f_1 \), we get defect 3 and six natural basis elements. Each of the six is obtained from one of the original multipartitions by choosing one of the addable nodes and adding it.

Acting again by \( f_1 \), we again get defect 3 and six natural basis elements, each obtained by adding two addable nodes to one of the original basis elements. If we pull out a common factor \( v^{-1} + v = [2]_v \), the remaining Fock space element has coefficients which are polynomial in \( v \) with highest exponent equal to the defect, namely 3.

The result of acting by \( f_1^3 \) is to fill in all the last of the original three addable 1-nodes, giving two natural basis elements. The coefficients have a common factor equal to the quantum factorial \( [3]_v \), and when we pull out the common factor, we get polynomial coefficients with highest exponent equal to 1, the defect. This is intended to illustrate the previous definition but also to motivate our next definition.

**Definition 2.4.** The operator we get by dividing \( e_i^k \) or \( f_i^k \) by the quantum factorial \([k]_v \) is called the divided power and will be represented by \( e_i^{(k)} \) and \( f_i^{(k)} \).

We define \( F^2 \) to be the subalgebra of \( F^1 \) generated from \(|\rangle \) by the divided powers \( f_i^{(k)} \), with coefficients in the algebra \( A \) of Laurent polynomials in \( v \) with integral coefficients.

We now define two operators on the multipartitions. We will give this definition for general \( e \), and return to the case \( e = 2 \) when we return to the Fock space. Our exposition will generally follow that in \([12], \S 3\), except of a duality issue which we will explain later. For any given residue \( i \), we denote the addable \( i \)-nodes, as defined in Def. 2.3, by a “+”, and the \( i \)-removable nodes by a “−”. We then write from left to right all the pluses and minuses from the bottom to the top, remove any cases of “−−”, and call the remaining sequence of plus and minus signs the signature. The first removable \( i \)-node from the left, if such exists, is called \( i \)-good, and the first addable \( i \)-node from the right, if such exists, is called \( i \)-cogood. In Example 1, \([(1, \emptyset; \emptyset)] \) has 1-signature \( +++ \). Adding the 1-cogood node gives a new multipartition \([(2, \emptyset; \emptyset)] \) with 1-signature \( ++− \).

We define an operation of \( \tilde{e}_i \) which is then the removal of the \( i \)-good node if it exists and otherwise gives 0, and an operation \( \tilde{f}_i \), which is the addition of the \( i \)-cogood node if it exists and otherwise gives 0.
We now define a subset of all the multipartitions of level \( r \), and describe a recursive procedure for calculating them, starting from the empty multipartition of level \( r \). The set of all multipartitions obtained by acting on elements of the recursively defined set by various \( \tilde{f}_i \), will be called the \( e \)-regular multipartitions. The \( e \)-regular multipartitions are in one-to-one correspondence with basis elements of a highest weight module \( V(\Lambda) \) [12], and the operations \( \tilde{e}_i \) and \( \tilde{f}_i \) correspond to the action of the Chevalley basis elements \( e_i \) and \( f_i \) of the Lie algebra \( \mathfrak{g} \) on the module \( V(\Lambda) \).

If \( r = 1 \), the procedure will give all partitions which are \( e \)-regular in the sense that they do not have \( e \) identical rows. Since this is a condition which is evident from the structure of the partition, in the case \( r = 1 \) there is no need for a recursive construction. When \( r > 1 \), there are multipartitions in which every individual partition is \( e \)-regular but the multipartition as a whole cannot be obtained by this recursive procedure. For example, when \( e = 2 \), and the multicharge is \((0, 0)\), the multipartition \([\emptyset, (1)]\) is not \( e \)-regular although neither of the individual partition has two identical rows.

There is an analogous construction, usually preferred by Brundan and Kleshchev [12, §3.4], which produces multipartitions in which every partition is \( e \)-restricted, which means that there are no \( e \) consecutive columns which are equal.

In [2], we introduced for the spin representations of the symmetric group a graph \( \hat{P}(\Lambda) \) with vertices \( P(\Lambda) \) and an edge labelled by residue \( i \) between two weights when there are two \( e \)-regular multipartitions with those weights which can be obtained one from the other by \( \tilde{e}_i \) and \( \tilde{f}_i \). We called it the block-reduced crystal graph. The same definition can be given for \( P(\Lambda) \) for any affine Lie algebra, where the weights are connected by an edge if they lie on the same \( i \)-string as in Def. 2.2.

### 3. CANONICAL BASIS ELEMENTS

There is an involution of the quantum enveloping algebra called the bar-involution which fixes \( e_i, f_i \), sends \( v^h_i \) to \( v^{-h_i} \), \( v^d \) to \( v^{-d} \), and interchanges \( v \) and \( v^{-1} \). We return to the case of \( e = 2 \). For each 2-regular partition \( \mu \), there is an element \( G(\mu) \) of the Fock space \( F_A(\Lambda) \) that is invariant under the bar involution, and is congruent to the standard basis element of \( \mu \) modulo \( v \). These are called the canonical basis elements. An algorithm for constructing the \( G(\mu) \) recursively in the case of partitions was given originally by Lascoux, Leclerc and Thibon in [14], and later extended to multipartitions by Fayers in [7]. As an intermediate step it calculates a bar-invariant Fock space element \( A(\mu) \) called the auxiliary vector and converts this to \( G(\mu) \) by linear algebra.

For \( e = 2 \), Mathas [15] completely determined the \( e \)-regular multipartitions. Ariki, Kleiman and Tsuchioka [1] did the same for the \( r = 2 \), using the Littelmann path model. Less in known about the possibility of non-recursive construction of the canonical basis elements. In [3], for \( e = 2 \), we managed to show that for multipartitions we called strongly residue-homogeneous, we can move directly back and forth between the multipartitions and the Littelmann paths. Many, but not all, of these strongly residue homogeneous multipartitions are external in the sense that \( \tilde{e}_i \) and \( \tilde{f}_i \) is zero. We would like to find some such procedure for external canonical basis elements.

We introduce some notation which will allow us to describe families of multipartitions:

- \((n)\) is a row of of length \( n \),
- \( T_n \) is triangular with \( n \) rows if \( n > 0 \) or \( \emptyset \) otherwise,
- \( \lambda \lor \mu \) is the partition obtained by taking the rows in \( \lambda \) followed by the rows in \( \mu \), presuming that this a well-formed partition.
As mentioned before, we will usually follow Mathas in [15] in assuming that all partitions with the same corner residue will lie in an interval in the multipartition, and we will also take them in increasing order, \( \Lambda = a_0 \Lambda_0 + a_1 \Lambda_1 + \cdots + a_{e-1} \Lambda_{e-1} \). In the case \( e = 2 \), the notation \( \Lambda = a \Lambda_0 + b \Lambda_1 \) will indicate that \( k_1, k_2, \ldots, k_a \) is 0 and the first \( a \) partitions will be called \( 0 \)-corner partitions, while \( k_{a+1}, \ldots, k_{a+b} = 1 \) and the corresponding partitions will be called \( 1 \)-corner partitions. For \( e = 2 \), we also put a semicolon between the \( 0 \)-corner partitions and the \( 1 \)-corner partitions. If \( i \) is a residue in the set \( \{0, 1\} \), we let \( i' = 1 - i \) be the opposite residue.

In [6], Fayers describes two involutions on the multipartitions. These involutions may change our preferred order. We give the definitions for the case of general \( e \), since one of the important distinctions does not show up in the case \( e = 2 \), and this might be misleading.

**Definition 3.1.** If \( \lambda = (\lambda^1, \ldots, \lambda^e) \) is a multipartition of rank \( e \) and level \( r \), then the conjugate \( \lambda' \) of \( \lambda \) is given by \( \lambda' = (\lambda'^e, \ldots, \lambda'^1) \), where \( \lambda'^i \) is the transposed partition of \( \lambda^i \), corresponding to reflection of the Young diagram in the main diagonal.

**Definition 3.2.** If \( \lambda = (\lambda^1, \ldots, \lambda^e) \) is an \( e \)-regular multipartition of rank \( e \) and level \( r \) for \( \Lambda = \Lambda_{k_1} + \cdots + \Lambda_{k_e} \), then the diamond \( \lambda^o \) of \( \lambda \) is an \( e \)-multipartition in the crystal for \( \tilde{\Lambda} = \Lambda_{-k_1} + \cdots + \Lambda_{-k_e} \). If \( \mu \) is obtained from the empty multipartition of level \( r \) by acting by a sequence of operators \( \tilde{f}_{i_1}, \tilde{f}_{i_2}, \ldots, \tilde{f}_{i_t} \) then the corresponding sequences of operators \( \tilde{f}_{-i_1}, \tilde{f}_{-i_2}, \ldots, \tilde{f}_{-i_t} \) will give a non-zero \( e \)-regular multipartitions for \( \tilde{\Lambda} \), and this \( e \)-regular multipartions will be designated \( \mu^o \). The sequence of residues \( (i_1, i_2, \ldots, i_t) \) will be called a path of \( \mu \) and its negative will be a path of \( \mu^o \).

**Example 2.** For \( e = 2 \), let \( \Lambda = 2 \Lambda_0 + \Lambda_1 \) be a dominant integral weight, and let \( s = (0, 0, 1) \) be a multicharge defining a Fock space \( \mathcal{F}_A^s \).

In Example 1 we calculated the result of acting by \( f_0, f_1 \) and \( f_1 \) from \( \mathcal{U} \) on the highest weight element | \( \) of \( \mathcal{F}_A^s \). Let us find the 2-regular multipartition obtained by acting by \( \tilde{f}_0, \tilde{f}_1, \tilde{f}_1 \), corresponding to a path \((0, 1, 1)\). Adding the first 0-cogood node gives us a multipartition \( \mu_1 = [(1), 0; 0] \). Adding one 1-cogood node brings us to \( \mu_2 = [(2), 0; 0] \). Adding a second 1-cogood node brings us to \( \mu_3 = [(2, 1), 0; 0] \). The final step gives us \( \mu_4 = [(2, 1), 0; (1)] \) and then \( \tilde{f}_1(\mu_4) = 0 \).

We first illustrate the conjugate, by calculating \( \mu_2^o = [(0, 0), (1^2)] \), first reversing the order of the partitions and then taking the conjugate of each individual partition.

Now we consider \( \mu_3 = [(2, 1), 0; 0] \) with path \((0, 1, 1)\) and calculate \( \mu_3^o \). The new multicharge is \( \hat{s} = (1, 0, 0) \). When \( e = 2 \) we have the same dominant integral weight, \( \hat{\Lambda} = \Lambda \) because \( 0 = -0 \) and \( 1 = -1 \), the order of the summands being irrelevant in the abelian weight space, and the same path, \((0, 1, 1)\). We think of this path as a path in the reduced crystal. We now draw the reduced crystal to six degrees, with multipartitions according to the two different multicharges, and edges of residue 0 having positive slope, edges of residue 1 having negative slope. In particular, \( \mu_3^o = [(1); (2), 0] \).

![Figure 1: Reduced crystal for \( e = 2 \) for \( \Lambda = 2 \Lambda_0 + \Lambda_1, n \leq 6 \), path \((0, 1, 1)\).](image)
The vertex reached by the path \( \hat{f}_0 \) followed by \( \hat{f}_1 \) corresponds to a weight \( \Lambda - \alpha_0 - \alpha_1 = \Lambda - \delta \). Its hub is the same as the hub of \( \Lambda \), which is \([2, 1]\). There are two multipartitions with this weight, \( \mu_2 = [(2), \emptyset; \emptyset] \) reached by path \((0, 1)\), but if we instead took the path \((1, 0)\) leading to the same vertex, we would have gotten a multipartition \([(1), \emptyset; (1)]\). We now draw the same block reduced crystal with the weights labelled by hubs and by defect, which will represented as a superscript.

\[
\begin{array}{c}
\text{\( n = 2 \)}
\end{array}
\]

\[
\begin{array}{c}
\text{\( n = 4 \)}
\end{array}
\]

\[
\begin{array}{c}
\text{\( n = 6 \)}
\end{array}
\]

\[
\begin{array}{c}
\text{\( n = 8 \)}
\end{array}
\]

Figure 2: Reduced crystal for \( e = 2 \) for \( \Lambda = 2\Lambda_0 + \Lambda_1 \), \( n \leq 8 \), with hubs and defects.

The sum of the coordinates of every hub equals the level \( r \) and every \( i \)-string of length \( \ell \) has an \( i \)-coordinate of the hub equal to \( \ell \) at the top end, and an \( i \)-coordinate equal to \(-\ell \) at the bottom end, with matching defects, when we draw the crystal as in Fig. 2.

**Definition 3.3.** For an \( e \)-regular multipartition \( \mu \), we can write the canonical basis element \( G(\mu) \) with respect to the natural basis. If \(|\lambda\rangle \) is a natural basis element with non-zero coefficient, then \( \mu \) and \( \lambda \) have the same content. Furthermore, in the dominance relation defined above, \( \mu \geq \lambda \). The coefficient of \(|\lambda\rangle \) in \( G(\mu) \) as a function of \( v \) will be denoted by \( d_{\mu \lambda}(v) \).

In Theorem 2.1 of [6], Fayers proves that if \( \text{def}(\mu) \) is the defect of an \( e \)-regular multipartition \( \mu \), then

\[
\hat{d}_{X^{\mu^c}}(v) = v^{\text{def}(\mu)} d_{\lambda \mu}(v^{-1}).
\]

If \( \mu \) is an \( e \)-regular multipartition and we denote by \( G(\mu) \) the corresponding canonical basis element, then [6, Thm. 2.1] implies that if \( \text{def}(\mu) \) is the defect of the block containing \( \mu \), then if we represent the canonical basis element \( G(\mu) \) in the natural basis, there is a unique natural basis element with coefficient \( v^{\text{def}(\mu)} \) and it is given by \( (\mu^c)' \). This already gives us some information about low defects, without any other restrictions:

- **Defect 0:** The multipartitions of defect 0 are precisely those for which \( \mu = (\mu^c)' \). For \( e = 2 \), we already know from [15] that the defect 0 multipartitions for \( \Lambda = a\Lambda_0 + b\Lambda_1 \) consist of \( a \) triangular partitions of size \( n \) and \( b \) triangular partitions of size \( n \pm 1 \). The diamond operation reverses the order, and the prime operation reverses the order back and transposes all the partitions, which is not noticed because they are triangular and thus invariant under transpose.

- **Defect 1:** The canonical basis is \( G(\mu) = |\mu\rangle + v(|\mu^c\rangle)' \).

One more word about \( (\mu^c)' \), a result of the strong duality of the canonical basis element. We chose \( \mu \) to be \( e \)-regular. Then \( (\mu^c)' \) is \( e \)-restricted, that is to say, it is the partition we would get from the same path through the crystal for \( \Lambda \) if we were always calculating our signatures from the bottom to the top instead of from the top to the bottom as we do.
4. NON-RECURSIVE CONSTRUCTIONS

From the results of \([4, \text{Prop. 2.6}]\), there is a fundamental region of \(P(\Lambda)\) under the action of the normal translation subgroup of the Weyl group, and the defects which can occur in the basis graph \(P(\Lambda)\) are all congruent to these defects modulo \(r\), since subtracting the null root \(\delta\) adds the level \(r\) to the defect. In the case of rank \(e = 2\), let \(\Lambda = a\Lambda_0 + b\Lambda_1\), and \(u\) representing the empty multipartition. Then since the null root \(\delta = \alpha_0 + \alpha_1\), the defects which can occur, modulo \(r\), all occur as defects of \(f^k_0(u), 0 \leq k \leq a\) or of \(f^k_1(u), 0 \leq k \leq b\). In Fig. 1 in Example 2 the first of these are the multipartitions labelling the 0-string of vertices along the straight line going out from the empty partitions to the left, which we call the top row on the left, and the second set label the vertices of the 1-string which is the top row on the right. The defects for the weights of these multipartitions are given by the following lemma. For each such weight there is a unique multipartition, and we will say that this multipartition is derived from the top row.

**Lemma 4.1.** In a crystal with \(\Lambda = a\Lambda_0 + b\Lambda_1\), the defect of \(\psi = \Lambda - k\alpha_0\) for \(0 \leq k \leq a\) is \(k(a - k)\) and of \(\psi = \Lambda - \ell\alpha_1\) for \(0 \leq \ell \leq b\) is \(\ell(b - \ell)\).

**Proof.** We simply compute the defect explicitly for the case of \(\alpha_0\), the other case being symmetric:

\[
def(\lambda) = (\Lambda|k\alpha_0) - \frac{1}{2}(k\alpha_0|k\alpha_0) = ka - \frac{1}{2}k^2(\alpha_0|\alpha_0) = ka - k^2
\]

□

**Definition 4.1.** The *shape* of a canonical basis element is the number of multipartitions, counting repetitions, for each power of \(\psi\) between 0 and the defect. A canonical basis element of defect \(d\) whose shape is \((1, 1, \ldots, 1)\) with \(d + 1\) entries will be called *svelte*.

**Example 3.** Let \(e = 2\), \(s = (0, 0, 1)\), \(r = 3\). We continue Examples 1 and 2.

- \(G(\mu_1) = \left\langle (1), 0; \emptyset \right\rangle + v\left\langle (0), 1; \emptyset \right\rangle\) has defect 1 and shape \((1, 1)\), so it is svelte.
- \(G(\mu_2) = \left\langle (2), 0; \emptyset \right\rangle + v\left\langle (1^2), 0; \emptyset \right\rangle + v^2\left\langle (1^2), 0; (1) \right\rangle + v\left\langle (0), (2); \emptyset \right\rangle + v^2\left\langle (0), (1^2); \emptyset \right\rangle + v^3\left\langle (0, (1); (1)) \right\rangle\) has defect 3, since its weight is obtained from a weight of defect 0 by subtracting a copy of the null root \(\delta\), and its shape is \((1, 2, 2, 1)\).
- \(G(\mu_3) = \left\langle (2, 1), 0; \emptyset \right\rangle + v\left\langle (2), 1; (1) \right\rangle + v^2\left\langle (1^2), 0; (1) \right\rangle + v\left\langle (0), (2, 1); \emptyset \right\rangle + v^2\left\langle (0), (2); (1) \right\rangle + v^3\left\langle (0, (1^2); (1)) \right\rangle\) also has defect 3, since defects are preserved by the action of the Weyl group, and reflecting the 1-string of 2-regular multipartitions \(\mu_1, \mu_2, \mu_3, \mu_4\) interchanges \(\mu_2\) with \(\mu_3\). The shape is also \((1, 2, 2, 1)\).
- \(G(\mu_4) = \left\langle (2, 1), (0); (1) \right\rangle + v\left\langle (0), (2, 1); (1) \right\rangle\), has defect 1 and shape \((1, 1)\).

4.1. A faster algorithm for canonical basis calculation. The definitions, lemmas and algorithms in this subsection are valid for any rank \(e\), but the examples are given for the case \(e = 2\). For any set \(X\) in a universe \(I\) we define the characteristic function \(\chi_X\) by

\[
\chi_X(y) = \begin{cases} 
1 & y \in X \\
0 & y \notin X 
\end{cases}
\]

In Def. 2.3 we described the action of the Chevalley basis element \(f_i\) on an element \(|\lambda\rangle\). In the course of Def. 2.3 we defined the addable and removable nodes of the multipartition \(\lambda\).
Definition 4.2. Fix some $i, 0 \leq i < e$. We order the $i$-addable nodes of a multipartition $\lambda$ in descending order in the Young diagram as $N_\lambda = (n_1, \ldots, n_e)$, starting from the first row of the first partition with an $i$-addable node, if such exists. Now let $\hat{N}_\lambda$ be the set of $i$-addable nodes in the sequence $N_\lambda$, so that $|\hat{N}_\lambda| = c$.

1. For $X \subset \hat{N}_\lambda$ we have a characteristic sequence $\chi_X(\lambda) = (\chi_X(n_1), \ldots, \chi_X(n_e))$.
2. For any sequence $S = (s_j | s_j \in \{0, 1\}_{j=1}^e)$, we apply $S$ to $\lambda$ by adding the nodes $n_j$ to $\lambda$ if $s_j = 1$. If $\nu$ is the result of applying a sequence $S$ to $\lambda$ and $\ell = \sum_{j=1}^e s_j$, then the degree of $\nu$ is $\ell$ greater than the degree of $\lambda$ and the Fock space element $|\nu\rangle$ is contained in the support of $f^\ell_\lambda |\lambda\rangle$.
3. If $\lambda$ is $e$-regular, then we can form a uniquely determined sequence of $e$-regular multipartitions by adding the $i$-cogood node, then the $i$-cogood node of the resulting multipartition, and so forth until one reaches the end of the $i$-string. If $\nu$ is an $e$-regular multipartition obtained by adding $\ell$ $i$-cogood nodes, and $X \subset \hat{N}_\lambda$ is the subset of nodes added to get $\nu$, the characteristic sequence $\hat{S} = \chi_X(\lambda)$ will be called special.
4. If $|X| = \ell$ and $\pi$ is a permutation of $\ell$, then if we add nodes to the multipartition $\lambda$ in the order $\pi(1), \pi(2), \ldots, \pi(\ell)$, we will say that we added the nodes according to $\pi$.

We give an example to illustrate the first item in this definition.

Example 4. As in Example 2, we let $e = 2$, and take the dominant integral weight $\Lambda = 2\Lambda_0 + 1\Lambda_1$, with multicharge $s = (0, 0, 1)$. The 2-regular multipartition $\mu = [(2, 1, \emptyset; \emptyset)]$ is the result of applying first a 0-addable choice sequence $(1, 0)$ add the 0-node, and then a 1-addable choice sequence $(1, 1, 0)$ to add the two 1-nodes which appear in the Young diagram of $\mu$. The canonical basis element corresponding to $\mu$ can be found in Example 1 by factoring out $[2]_e!$. We take $\lambda$ to be $[(1^2), \emptyset; (1)]$, a multipartition which is not 2-regular, whose natural basis element $|\lambda\rangle$ occurs in $G(\mu)$, and we calculate the sequences of addable nodes $N_\mu$ and $N_\lambda$. We denote the addable 0-nodes in the Young diagram by a +. Nodes will represented in the notation $(i, j, k)$ where $i$ is the row, $j$ is the column and $k$ is the partition.

$$
\mu:\begin{align*}
0 & 1 + \\
1 & + \\
\emptyset & \\
\end{align*}
$$

$$
\lambda:\begin{align*}
0 & + \\
1 & + \\
\emptyset & 1 + \\
\end{align*}
$$

$N_\mu = ((1, 3, 1), (2, 2, 1), (3, 1, 1), (1, 1, 2))$

$N_\lambda = ((3, 1, 1), (1, 1, 2), (1, 2, 3), (2, 1, 3))$

Since both multipartitions have four addable nodes, there are $\binom{4}{1} = 6$ characteristic sequences $S$. Since $\mu$ is 2-regular, it determines a special characteristic sequence $\hat{S} = (1, 1, 0, 0)$. The characteristic sequence $(0, 1, 0, 1)$ for $\mu$ would mean adding the second node in the list $N_\mu$, which is $(2, 2, 1)$ and the fourth node in the list, which is $(1, 1, 2)$. Adding the first of these nodes changes the first partition $(2, 1)$ in $\mu$ into $(2, 2)$ and adding
the node $(1, 1, 2)$ changes the $\emptyset$ into $(1)$ to give a multipartition $[(2^2), (1); \emptyset]$, while the characteristic sequence $(0, 1, 0, 1)$ for $\lambda$ would give a multipartition $[(1^2), (1); (1^2)]$.

What we are hoping to avoid with a refined algorithm is the current situation that in order to go down an $i$-string from one basis element of defect $d$ to another, we have to construct the large canonical basis elements in between, which blow up in an exponential manner, as happened in Example 1.

**Definition 4.3.** For a sequence $S$ chosen from a two element ordered set $\{0, 1\}$, the number of inversions, $\text{Inv}(S)$, is the sum of the number of elements 0 appearing before each element 1. We will use the function for characteristic sequences, Def. 4.2

**Example 5.** We now start a new example in which the crystal will be symmetric, $\Lambda = 3\Lambda_0 + 3\Lambda_1$ and will be used in the next subsection to illustrate the duality in our calculations.

Consider a path $(0, 0)$ in the crystal. There are three addable 0-nodes in the empty partitions, which we represent by a sequence $(0, 0, 0)$. We represent a choice of two of these by putting 1’s in place of the corresponding 0 to get the characteristic sequence. The 2-regular multipartition obtained by acting twice by $f_0$ as required by our path is $\lambda_2 = [(1), (1), \emptyset; \emptyset, \emptyset, \emptyset]$ with special characteristic sequence $\bar{S} = (1, 1, 0)$. The canonical basis element corresponding to $\lambda_2$ is

$$G(\lambda_2) = [(1), (1), \emptyset; \emptyset, \emptyset, \emptyset] + v[(1), (1); \emptyset, \emptyset, \emptyset] + v^2[\emptyset, (1), (1); \emptyset, \emptyset, \emptyset].$$

We now calculate the inversions of the 3 characteristic sequences:

1. $\text{Inv}((1, 1, 0)) = 0$, because no 0 appears before a 1.
2. $\text{Inv}((1, 0, 1)) = 1$, because the zero appears before the second 1, and
3. $\text{Inv}((0, 1, 1)) = 2$, because the zero appears before both copies of 1.

These inversion numbers give the exponent of $v$ in the canonical basis for the natural basis element corresponding to that characteristic sequence.

In all the results below, we will frequently use Theorem 6.16 in Mathas [16]. The result there is stated for partitions rather than multipartitions, and Mathas is working with $e$-restricted rather than $e$-regular multipartitions. To take care of these differences, we will give a slightly different proof compatible with our set-up. However, except when the differences are important, we will simply quote Mathas.

**Lemma 4.2.** If a multipartition $\mu$ has at least $\ell$ $i$-addable nodes, and if $\lambda$ is the result of adding $\ell$ $i$-addable nodes with characteristic sequence $S$ choosing among addable nodes, then $\tilde{f}^e_i(\mu)$ contains $\lambda$ with coefficient $v_{\text{Inv}(S)[\ell]!}.$

**Proof.** For each permutation $\pi$ in the symmetric group $S_\ell$, define

$$a(\pi) := \sum_{j=1}^{\ell} \#\{i : i < j, \pi(i) > \pi(j)\} - \#\{i : i < j, \pi(i) < \pi(j)\}$$

This is closely related to the inversion number of permutations, defined by

$$\text{Inv}(\pi) := \#\{(i, j) : i < j, \pi(i) > \pi(j)\}.$$ 

Indeed, since there are $j - 1$ natural numbers $i$ with $i < j$, clearly for fixed $j$

$$\#\{i : i < j, \pi(i) > \pi(j)\} - \#\{i : i < j, \pi(i) < \pi(j)\} = 2\#\{i : i < j, \pi(i) > \pi(j)\} - (j - 1).$$
and therefore

\[ a(\pi) = 2 \text{Inv}(\pi) - \binom{\ell}{2}. \]

If the characteristic sequence \( S \) contained only copies of 1, we would nearly be finished, but since it may also contain copies of 0, we also have to consider the inversion number \( \text{Inv}(S) \), from Definition 4.3. We now claim that a partition \( \lambda \) obtained by adding the addable \( i \)-nodes whose positions correspond to the copies of 1 in \( S \) according to the permutation \( \pi \) as in Def. 4.2 (2) will have coefficient \( v^{\text{Inv}(S)} \mu^{a(\pi)} \).

We let \( S_0, S_1, \ldots, S_\ell \) be the characteristic sequences as we add copies of 1 according to the permutation \( \pi \). Assuming the claim true for \( \ell - 1 \) with permutation \( \bar{\pi} \), we want to prove it for \( \ell \) and permutation \( \pi \). Assume that the last number 1 we insert in \( S_{\ell-1} \) is in the coordinate \( t \), and that there are \( s \) copies of 1 before it in \( S_{\ell-1} \). By the induction hypothesis, the previous multipartition \( \bar{\lambda} \) had coefficient \( v^{\text{Inv}(S_{\ell-1})+a(\bar{\pi})} \). To add this new node, we must calculate the number of addable and removable \( i \)-nodes before position \( t \), which is to say, the number of zeros minus the number of ones. There are \( t-1 \) components to the vector before \( t \), and of these, \( s \) are ones, so we add \( t-1-2s \) to the exponent of \( v \). Now \( \text{Inv}(S_{\ell}) - \text{Inv}(S_{\ell-1}) = t-1-s \), the number of zeros in front of the new 1, minus \( t-1-s \), the number of copies of 1 after \( t \) which will be missing one zero, altogether \( t-\ell \). The difference \( a(\pi) - a(\bar{\pi}) \) will be \( (\ell - 1 - s) - s \), since we add 1 for each \( i \) with \( \pi(i) > s \) and subtract 1 for each of the \( i \) with \( \pi(i) \leq s \). Altogether,

\[ \text{Inv}(S) + a(\pi) - \text{Inv}(S_{\ell-1}) - a(\bar{\pi}) = (t-\ell) + (\ell - 1 - 2s) = t - 1 - 2s \]

as needed. After adding the \( \ell \) nodes which produce \( \lambda \) in all possible orders, we thus get \( \lambda \), multiplied by \( v^{\text{Inv}(S)} \sum_{\pi \in S_{\ell}} v^{a(\pi)} \).

It follows, from MacMahon’s formula (see, for example, [18]) for the inversion number generating functions that

\[
\sum_{\pi \in S_{\ell}} v^{a(\pi)} = v^{-\left(\binom{\ell}{2}\right)} \sum_{\pi \in S_{\ell}} v^{2 \text{Inv}(\pi)} = v^{-\left(\binom{\ell}{2}\right)} \prod_{j=1}^{\ell} \frac{v^{2j} - 1}{v^2 - 1} = \prod_{j=1}^{\ell} \frac{v^j - v^{-j}}{v^j - v^{-1}} = \prod_{j=1}^{\ell} [j]_v = [\ell]_v!.
\]

The algorithm has been implemented by a short program “quickf” and a different program “quickerf”. In the algorithm “quickf” to calculate \( f^{(p)}_i \), on a natural basis element with \( p \)-addable nodes and no removable nodes, one runs \( f_i \) \( p \) times but suppresses the power of \( v \) in the definition of \( f_i \). At the end, instead of dividing by the quantum factorial, one divides by the ordinary integer factorial. This does blow up in the middle, but the arithmetic for integer coefficients is faster and one need not calculate the power of \( v \), so at least for \( p > 3 \) the algorithm provides a significant decrease in central processing unit (CPU) computation time for canonical basis elements with only addable nodes and no removable nodes. For \( p > 10 \) the time is halved by quickf.

For “quickerf” the programming was trickier but large canonical basis elements for large defects in the middle of the \( i \)-string are eliminated. One again needs that every natural basis element has \( p \)-addable nodes and no removable nodes, but then one simply adds all the addable nodes to each natural basis element in the canonical basis element. For \( p > 10 \), the time is reduced by a factor of 3000 for “quickerf”. We give here a comparison of computation times for multipartitions with this property for the case of \( \Lambda = 2\Lambda_0 + \Lambda_1 \). The column “d” is the defect and the column labeled by “Part.” gives the number of natural basis elements needed to represent the canonical basis element of
the 2-regular multipartition $\mu$. The computation times are roughly linear in “Part.” The CPU computation times are in seconds and indicate that the process is a heavy one, since calculation time for a single function are normally in milliseconds.

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>d</th>
<th>Part.</th>
<th>i</th>
<th>p</th>
<th>quickerf</th>
<th>quickf</th>
<th>f</th>
</tr>
</thead>
<tbody>
<tr>
<td>(2, 1), (2, 1); (1)</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>8</td>
<td>0.0005 s</td>
<td>0.10 s</td>
<td>0.18 s</td>
</tr>
<tr>
<td>(3, 2, 1), (3, 2, 1); (2, 1)</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>11</td>
<td>0.008 s</td>
<td>1.20 s</td>
<td>2.39 s</td>
</tr>
<tr>
<td>(6, 1), (2, 1); (1)</td>
<td>6</td>
<td>41</td>
<td>0</td>
<td>8</td>
<td>0.02 s</td>
<td>4.30 s</td>
<td>7.83 s</td>
</tr>
<tr>
<td>(7, 2, 1), (3, 2, 1); (2, 1)</td>
<td>6</td>
<td>41</td>
<td>1</td>
<td>11</td>
<td>0.03 s</td>
<td>50.63 s</td>
<td>100.12 s</td>
</tr>
</tbody>
</table>

The SageMath code of “quickf” and “quickerf” are included in an appendix.

4.2. Symmetric crystals. Theorem 2.1 in [6], giving the relationship between $\mu$ and $\mu^0$, is sometimes awkward to use, particularly in computer algebra computations, because it involves constructing two distinct crystals and comparing them. However, in the rank 2 symmetric case, $\Lambda = a\Lambda_0 + a\Lambda_1$, we can do everything within the confines of a single crystal, and thus gain considerable information about the coefficients in the canonical basis. Our question is this: To what extent can we determine the canonical basis element from the multipartition and the block-reduced crystal $\hat{P}(\Lambda)$ without resorting to recursive calculations?

Our strategy for giving a non-recursive construction of canonical basis elements is to find a uniform notation for all the multipartitions occuring in the canonical basis elements. For compactness, we will write the sequences of residues giving that path through the reduced crystal using exponential notation, and will call such a sequence residue-collected path, so that, for example, $(0, 0, 1, 1, 1, 0)$ would be written in residue-collected form as $(0^2, 1^3, 0)$. Consider paths $(0^{(1)}, 1^{(2)}, 0^{(3)}, \ldots)$ or $(1^{(1)}, 0^{(2)}, 1^{(3)}, \ldots)$ in $P(\Lambda)$ starting with residue $i = 0, 1$, and let $(d_1, d_2, \ldots)$ be the defects of the weights at the ends of each fixed-residue string.

**Definition 4.4.** We say that the path is stabilizing at $t$ if the defects rise to a fixed defect $d$ in place $t$ and afterwards are all $d$, so that in fact from $t$ forward, the actions are actions of Weyl group elements reflecting the strings.

**Example 6.** Consider a residue collected path $(0, 1^2, 0^4, 1^7)$ in the block reduced crystal for $\Lambda = 2\Lambda_0 + \Lambda_1$ computed in Example 2. The sequence of defects is $(1, 3, 3, 3)$, as we can see by reflecting $0$-strings and $1$-strings according to the action of the generators $s_0$ and $s_1$ of the Weyl groups, so we easily construct a infinite path which is 2-stabilizing because the defect are constant at 3 from the second entry in the string of defects.

For a stabilizing path as in Def. 4.4, let $c_1 = a$ be the number of $i$-addable nodes in the highest weight vector $u_\Lambda$ and let $S_1$ be a characteristic sequence of length $c_1$ choosing $k^{(1)}$ nodes addable nodes out of $c_1$. Given such a choice, we have a new number $c_2 = 2k^{(1)} + a$ of $i'$-addable nodes for the second residue, and make a new choice of $S_2$ among them. In general, we let the $S_\ell$ be characteristic sequences of subsets of size $k^{(\ell)}$ in $[1, 2, \ldots, c_\ell]$, where $c_\ell$ may depend on the previous characteristic sequences $S_j$, $j < \ell$. Let $\mathcal{S}(c, k)$ be the set of all choice sequences of length $c$ with $k$ 1-entries.

The method is simplest to apply when the number $c_i$ of addable nodes is independent of the various choices $S_1, S_2, \ldots, S_{i-1}$ made previously, but we will show in Example 7 below that this is not always true. What affects the value of $c_\ell$ the most is the distribution of $S_\ell$ between 0- and 1-corner nodes, so we let $S_\ell^0$ be the part of the sequence $S_\ell$ lying in 0-corner nodes, and let $S_\ell^1$ be the subsequence of $S_\ell$ lying in the 1-corner nodes, so that
we obtain \( S_\ell \) by concatenation, \( S_\ell = S_\ell^0 \cup S_\ell^1 \). At each stage, we let \( \hat{c}_\ell \) be the number of addable nodes in the \( e \)-regular multipartition \( \mu_\ell \) obtained after \( \ell - 1 \) steps. We define a sequence of length \( \hat{c}_\ell \) which we denote by \( \hat{S}_\ell \). An entry will be 1 if the corresponding addable node was added to \( \mu_{\ell-1} \) to give \( \mu_\ell \). We call this the choice sequence of the \( e \)-regular multipartition, which will usually have all the 1-entries at the beginning, unless there is subsequence \(-+\) occurring in the calculation of the signature as described after Def. 2.4.

We must also define new families of partitions depending on \( n \) for \( n \geq 1 \). We let

\[
U_1^n = (n + 1) \lor T_{n-2}; U_2^n = T_{n-1} \lor (1^2)
\]

be families built from (2) and (1\(^2\)) by alternately adding all \( i \)-addable nodes. Note that \( U_1^n \) is the transpose of \( U_1^n \).

What do we mean by “non-recursive”? We will rely heavily on the block-reduced crystal graph \( \hat{P}(\Lambda) \), defined for spin representations of the symmetric group in [2], Def. 2.6] and generalized in [[8], §2,§3], which is most easily computed recursively. The block-reduced crystal graph does have a non-recursive construction, given in [4], Thm. 2.7]. Suppose we are given a weight \( \psi \) of defect \( d \). By the criterion in [4], we can determine if it lies in \( P(\Lambda) \). Assuming it does, we can try to act by reflection \( s_t \) to bring this weight to a weight of lower degree with the same defect, find the \( e \)-regular multipartitions and their canonical basis elements for this weight, and then use Lemma 4.2 to transfer the \( e \)-regular multipartitions and canonical basis element back to \( \psi \).

The following conjecture summarizes the results of numerous computer calculations of canonical basis elements. Recall that a path stabilizing at \( t \) was defined in Def. 4.4.

**Conjecture:** Let \( e = 2 \) and \( \Lambda = a\Lambda_0 + a\Lambda_1 \). Let \( \mu \) be an \( e \)-regular multipartition reached by a path \( p \) of length \( q \) stabilizing at \( t, p = ((i)\ell^{(1)}), (i')\ell^{(2)}, \ldots, (i)\ell^{(q)} \). Let \( t' \) be the first index greater than or equal to \( t \) for which we get an external weight space, if such \( t' \) exist, and otherwise let it be \( q \). Set \( n = q - t \). Then there is a number \( m \) with \( t \leq m \leq t' \) for which we can define families of multipartitions \( \pi^n(S_1, S_2, \ldots, S_m) \) depending on the characteristic sequences \( S_\ell \) such that

\[
G(\mu) = \sum_{S_1 \in S(c_1,k^{(1)})} \cdots \sum_{S_m \in S(c_m,k^{(m)})} v^{\text{Inv}(S_1)+\cdots+\text{Inv}(S_m)} \pi^n(S_1, S_2, \ldots, S_m)
\]

where the \( \pi^n \) are multipartitions determined entirely by the choices of addable nodes given by the \( S_\ell \). If \( q > t' \), all the canonical basis element from \( t' \) up have the same shape.

In this paper, we will verify the conjecture for a number of cases for which \( t' \) and \( t \) are small. If \( S_\ell \) has a single 1 in the position \( j_\ell \), then \( \text{Inv}(S_\ell) = j_\ell - 1 \). If \( S_\ell \) is all copies of 1, with a single 0 in position \( j_\ell \), then \( \text{Inv}(S_\ell) = c_\ell - j_\ell \), where \( c_\ell \) is the length of \( S_\ell \). In either case, we denote by \( u(j_\ell) \) the index \( u \) of the partition in which the addable node represented by \( j_\ell \) lies.

**Example 7.** Let us take \( a = 3 \), as in Fig. 3 below, and consider the path \((0,1,0)\), this being a case where the value of \( c_3 \) depends on the value of \( S_2 \). We have \( c_1 = 3, k^{(1)} = 1, \) and \( \hat{S}_1 = (1,0,0,0) \) so the \( e \)-regular multipartition after one step on the path is \( \mu_1 = [(1), \emptyset, \emptyset; \emptyset, \emptyset, \emptyset] \) that appears in the canonical basis with coefficient 1. If we choose a different choice sequence \( S_1 = (0,0,1) \), with the 1 in position \( j_1 \), we get a multipartition \([\emptyset, \emptyset, (1); \emptyset, \emptyset, \emptyset] \), which occurs in the canonical basis element \( G(\mu_1) \) with coefficient \( v^{\text{Inv}(S_1)} = v^2 \). Altogether, \( G(\mu_1) = [(1), \emptyset, \emptyset; \emptyset, \emptyset, \emptyset] + v[\emptyset, (1), \emptyset; \emptyset, \emptyset, \emptyset] + v^2[\emptyset, \emptyset, (1); \emptyset, \emptyset, \emptyset] \).
A different characteristic sequence \( k \) We are continuing Example 5 with Example 8.

For which the addable nodes are contained in \( 1 \) 1-node is added to a 1-corner partition, then \( c \) partition. After the first step, there can be several addable nodes in a single position \( t \). In particular, we let \( u \) let \( t \) be the range of superscripts indicating the various partitions in \( k \) by adding nodes \( u \) of a single residue. We now introduce the notation which will allow us to describe the multipartitions occurring in the canonical basis of the 2-regular \( k \) nodes of a single residue. We now introduce the notation which will allow us to describe the multipartitions occurring in the canonical basis of the 2-regular multipartitions at the top of the crystal, reached by adding nodes \( k \) which can be changed to (1), and one 1-corner partition (1) with two addable 0-nodes. On the other hand, if \( S_0^0 = (1, 0) \) or \( (0, 1) \), giving \( U_1^1 = (2) \) or \( U_2^1 = (1^2) \) respectively in position \( j_1 \), then \( c_3 = 3 \) and there is one addable 0-node for each of the three 0-corner partitions.

At the second stage, we have \( c_2 = 2k^{(1)} + a = 5, k^{(2)} = 1 \), and \( \tilde{S}_2 = (1, 0, 0, 0, 0) \), giving an \( e \)-regular multipartition \( \mu_2 = [(2), \emptyset, \emptyset, \emptyset, \emptyset] \). Now we pick a choice sequence \( S_2 = S_2^0 \cup S_2^1 \), with \( S_2^0 \) of length 2 and \( S_2^1 \) of length 3. The choice sequence \( S_2 = (0, 0, 0, 1, 0) \), following \( S_1 \) above, would give a multipartition \( \lambda_2 = [\emptyset, \emptyset, (1) ; \emptyset, (1), \emptyset] \), which would have a coefficient \( v^{\text{inv}(S_1)+\text{inv}(S_2)} = v^5 \).

Finally, at the third stage, \( c_3 \) depends on the structure of \( S_2 \). If \( S_2^0 = (0, 0) \), so that the 1-node is added to a 1-corner partition, then \( c_3 = 4 \), there being two 0-corner copies of \( \emptyset \) which can be changed to (1), and one 1-corner partition (1) with two addable 0-nodes. On the other hand, if \( S_2^0 = (1, 0) \) or \( (0, 1) \), giving \( U_1^1 = (2) \) or \( U_2^1 = (1^2) \) respectively in position \( j_1 \), then \( c_3 = 3 \) and there is one addable 0-node for each of the three 0-corner partitions.

Using the notation \( t \) and \( t' \) given in the conjecture, we start with the case where \( t = t' = 1 \), that is to say, canonical basis elements at the top of the crystal, reached by adding nodes \( k \) of a single residue. We now introduce the notation which will allow us to describe the multipartitions occurring in the canonical basis of the 2-regular multipartitions at the top of the crystal, which by Lemma 4.1 have defect \( k(a - k) \). We let \( u \) with \( 1 \leq u \leq 2a = r \) be the range of superscripts indicating the various partitions in the multipartition, and we recall that \( S_1 \) is the subsequence of the characteristic sequence \( S_1 \) for which the addable nodes are contained in \( i \)-corner partitions. If the addable nodes are indexed by an integer \( j \), we let \( u(j) \) indicate the index of the partition containing that addable node. After the first step, there can be several addable nodes in a single partition.

\[
\tau^u_i(S_1) = \begin{cases} 
T_{n+1}, & (S^i_1)_{u-ia} = 1, 1 \leq u - ia \leq a, \\
T_{n-1}, & (S^i_1)_{u-ia} = 0, 1 \leq u - ia \leq a, \\
T_n, & a + 1 \leq u + ia \leq 2a, 
\end{cases}
\]

**Example 8.** We are continuing Example 5 with \( a = 3 \). There are three addable 1-nodes, and we set \( k = 2 \), with special characteristic sequence \( \tilde{S}_1 = (1, 1, 0) \) as Def. 4.2. Consider a different characteristic sequence \( S_1 = (1, 0, 1) \). Then

\[
\tau^0(S_1) = [\emptyset, \emptyset, (1), \emptyset, (1)]
\]
\[ \tau^1(S_1) = [(1), (1), (1); (2, 1), \emptyset, (2, 1)] \]
\[ \tau^2(S_1) = [(2, 1), (2, 1), (2, 1); (3, 2, 1), (1), (3, 2, 1)] \]

**Lemma 4.3.** For a dominant integral weight \( \Lambda = a\Lambda_0 + a\Lambda_1 \) of an affine Lie algebra of type \( A \), the 2-regular multipartition \( f_i^{(k)}(u_\Lambda) \) for an integer \( k \) is \( \tau^0(S_1) \), with \( 0 \leq k \leq a \), and the canonical basis element is

\[ G(\tau^0(S_1)) = \sum_{S_1 \in \mathcal{S}(a,k)} v^{\text{Inv}(S_1)} \tau^0_1(S_1). \]

The shape of the canonical basis element \( G(\tau^0(S_1)) \) is given by \( s(a, k, 0), \ldots, s(a, k, k(a-k)) \) where \( s(a, k, \ell) \) is a recursive function which is 0 except for \( 0 \leq \ell \leq k(a-k) \), and satisfies the following recursion scheme:

\[ s(1, 0, 0) = s(1, 1, 0) = 1, \]
\[ s(a, k, \ell) = s(a - 1, k - 1, \ell) + s(a - 1, k, \ell - k). \]

**Proof.** The 2-regular multipartition \( \mu = \tau^0(S_1) \) consists of \( k \) partitions (1) at the beginning of the \( i \)-corner multipartitions, of which there are \( a \). The only addable \( i \)-nodes in \( u_\Lambda \) are the corners of those \( a \) partitions, so by the formula for the actions of the divided power in the Fock space, there are \( \binom{a}{\ell} \) different multipartitions which can occur in the canonical basis element \( G(\mu) \) and they will all occur. Let \( S_1 \) be the characteristic sequence of any such choice of \( k \) partitions from \( a \) copies of a 0-corner \( \emptyset \). By Mathas [16, Thm. 6.16], when we act on the highest weight vector \( u_\Lambda \) by \( f_i^{(k)} \), we get each \( \tau^0(S_1) \) multiplied by \([k]_r!\). After dividing by the factorial, as we showed in the Lemma 4.2, the power of \( v \) which occurs as coefficient of \( \tau^0(S_1) \) is \( \text{Inv}(S) \).

The function \( s(a, k, \ell) \) which gives the shape is then a function counting all the multipartitions with coefficient \( v^\ell \). For \( a = 1 \), we have \( 0 \leq k \leq 1 \), so \( k = 0, 1 \). This means that \( 0 \leq \ell \leq 1(1-1) = 0 \), so \( \ell = 0 \). We have only the defect 0 multipartitions and so get \( s(1, 1, 0) = s(1, 1, 0) = 1 \). Thereafter, the number of multipartitions with coefficient \( v^\ell \) is the sum of those starting with 1, for which the power is determined by the remaining \( a-1 \) elements of the sequence, and those starting with 0, for which the initial 0 adds \( k \) to the power of \( v \) determined by the remainder of the sequence, giving the desired recursion formula.

For every defect, there is a degree after which all weight spaces with this defect occur at the end of strings in the block-reduced crystal graph [17], and once this happens, almost all addable nodes have the same residue. For the images of the multipartitions in the top rows under the action of the Weyl group, this is true from the very beginning.

**Corollary 4.3.1.** The shape function in closed form:

- for \( k = 1 \), \( s(a, 1, \ell) = 1 \) for \( 0 \leq \ell \leq a - 1 \).
- for \( k = 2 \), \( s(a, 2, \ell) = \left[ a - \frac{(a - 2)}{2} \right], 0 \leq \ell \leq 2(a - 2) \).
- for \( k = 3 \), \( s(a, 3, \ell) = \sum_{t=1}^{\left[ 1 + \frac{a}{2} \right]} \left[ a - \frac{(a - 2)}{2} \right] \).

**Proof.** For \( k = 1 \), the canonical basis element is svelte, since the multipartition multiplying \( v^\ell \) will be that obtained from the highest weight vector by adding an \( i \)-node to partition \( \ell + 1 \).
For $k = 2$, we can separate $s(a, 2, \ell) = s(a - 1, 1, \ell) + s(a - 2, \ell - 2)$. If we continue to separate the term with $k = 2$, each time adding $s(a - t, 1, \ell - 2(t - 1))$ as long as $\ell - 2(t - 1) \geq 0$, which means $t \leq \ell + 1$, we get a sum of elements which are all 0 or 1, and we must count the number which are 1:

$$\sum_{t=1}^{[\frac{\ell+1}{2}]} s(a - t, 1, \ell - 2(t - 1)).$$

Case 1: If $\ell \leq \frac{d}{2} = a - 2$, then $s(a - t, 1, \ell - 2(t - 1)) = 1$ from $t = 1$ as long as $\ell - 2(t - 1) \geq 0$, which is to say, until $t = \lfloor \frac{\ell+1}{2} \rfloor$, altogether $\lfloor \frac{\ell+1}{2} \rfloor$ copies of 1. However, since $\ell \leq a - 2$, we have $|\ell - (a - 2)| = a - 2 - \ell$ and thus

$$\left\lfloor \frac{a - |\ell - (a - 2)|}{2} \right\rfloor = \left\lfloor \frac{(\ell + 2)}{2} \right\rfloor,$$

as desired.

Case 2: $\ell > a - 2$, then for $t = 1$ we have $s(a - t, 1, \ell - 2(t - 1)) = 0$. The first value of $t$ for which we get the value 1 is when $\ell - 2(t - 1) \leq a - t - 1$, which is equivalent to $\ell - a + 3 \leq t$. The total number of copies of 1 in the sum is then $\lfloor \frac{(\ell+2)}{2} \rfloor - (\ell - a + 3) + 1$. This equals $\lfloor \frac{a - |\ell - (a - 2)|}{2} \rfloor$, as desired.

For $k = 3$, by the same arguments we used above, the sum goes from $t = 1$, which could give a value 0, as long as $\ell - 3(t - 1) \geq 0$, giving

$$s(a, 3, \ell) = \sum_{t=1}^{[\frac{\ell+1}{3}]} s(a - t, 2, \ell - 3(t - 1)).$$

Substituting from the result for case $k = 2$, we get the desired formula.

Note that for $k = 1, 2$ the closed form is symmetric around $\frac{d}{2}$, where $d$ is the defect $k(a - k)$, and we presume this to be true in general. To prove that would probably require reformulating the recursion in terms of a symmetric parameter $|\ell - \frac{d}{2}|$.

**Lemma 4.4.** A multipartition derived from the top rows under the action of a reduced word in the Weyl group generators of length $n$ is

$$\tau_i^n(\tilde{S}_1).$$

The canonical basis element of the images of a multipartition from the top rows under the action of a reduced word in the Weyl group generators of length $n$ is

$$G(\tau_i^n(\tilde{S}_1)) = \sum_{S_1 \in S(a, k)} v^{\text{inv}(S_1)} \tau_i^n(S_1).$$

The shape is given by the same recursive formula $s(a, k, \ell)$ given in the previous lemma.

**Proof.** We are doing explicitly the case residue $i = 0$, the case $i = 1$ being dual. We now apply Theorem 6.16 from [16] to each of the multipartitions $\tau^0(S_1)$ in $G(\tau^0(\tilde{S}_1))$. The hub is $[a - 2k, a + 2k]$, so there are $a + 2k$ addable 1-nodes, two for each partition (1) in the 0-corner part, and $a$ for all the 1-corner nodes. The result of adding all these nodes is exactly $\tau^1(S)$, and by the theorem quoted above, that is the result of acting on $\tau^0(S)$ by the divided power $j_i^{(2k+a)}$. The result is a canonical basis element of exactly the same shape as before.
We now continue by induction, assuming that we have a canonical basis element of the desired shape, and calculating that the number of addable nodes in $\tau^n(S)$ must be $k(n + 2) + (a - k)n + a(n + 1)$, and the result of adding them all will be $\tau^{n+1}(S)$. After applying the Mathas result [16, Theorem 6.16] again, we get the desired canonical basis element. The case $i = 1$ is dual.

\[ \square \]

In Figure 3 above, we drew the symmetric block reduced crystal for $a = 3$, where the vertices are labelled by the hub with the defect as superscript. Note that the hubs and defects on the two sides are symmetric. The label of any vertex in the interior can be obtained by going down the lattice, adding $r = 6$ to the defect and leaving the hub the same.

**Lemma 4.5.** For a symmetric crystal for $\Lambda = a\Lambda_0 + a\Lambda_1$ with $e = 2$, any weight space which has content $(k, 1)$ or $(1, k)$ for $1 \leq k \leq a$ has dimension 2 if $k = 1$ and 3 if $k \geq 2$. For each path $p$ through the crystal, there is an integer $m$ such that:

$$G(\pi^0(\tilde{S}_1, \ldots, \tilde{S}_m)) = \sum_{S_1 \in \mathcal{S}(c_1, k^{(1)})} \cdots \sum_{S_m \in \mathcal{S}(c_m, k^m)} t^{\text{Inv}(S_1) + \cdots + \text{Inv}(S_m)} \pi^0(S_1, \ldots, S_m)$$

where

- $p = (0^k, 1^t)$: $m = 2$, $t = 2, c_1 = a, k^{(1)} = k, S_1 \in \mathcal{S}(a, k), c_2 = a + 2k, k^{(2)} = 1, S_2 \in \mathcal{S}(2k + a, 1)$ with the single 1 in position $j_2$, and $\pi^0(S_1, S_2)$ is identical to $\tau^0(S_1)$, except for the following partitions:

  \[ \pi^0(S_1, S_2)^u = \begin{cases} 
  (2), & 1 \leq u \leq a, u = u(j_2), (S_2^0)_u = 1, j_2 \equiv 1 \mod 2, \\
  (1^2), & 1 \leq u \leq a, u = u(j_2), (S_2^0)_u = 1, j_2 \equiv 0 \mod 2, \\
  (1), & j_2 > 2k, u = u(j_2).
\]

The case $p = (1^k, 0^t)$ is dual.

- $p = (1, 0^k)$: $m = 2$, $t = 2, c_1 = a, k^{(1)} = 1, S_1 \in \mathcal{S}(a, 1)$ with the single 1 in position $j_1$, $c_2 = a + 2, k^{(2)} = k, S_2 \in \mathcal{S}(a + 2, k)$, and $\pi^0(S_1, S_2)$ is identical to $\tau^0(S_1)$, except for the following partitions:

  \[ \pi^0(S_1, S_2)^u = \begin{cases} 
  (1), & u \leq a, (S_2^0)_u = 1, \\
  (2), & u = u(j_1), S_2^0 = (1, 0), \\
  (1^2), & u = u(j_1), S_2^1 = (0, 1), \\
  (T_2), & u = u(j_1), S_2^1 = (1, 1).
\]

The case $p = (0, 1^k)$ is dual.

- $p = (0, 1, 0^{k-1})$ $m = 2$, $t = 3, c_1 = a, k^{(1)} = 1, S_1 \in \mathcal{S}(a, 1)$ with the single 1 in position $j_1$, $c_2 = a + 2, k^{(2)} = 1, S_2 \in \mathcal{S}(a + 2, 1)$ with a single 1 in position $j_2$, $k^{(3)} = k - 1$. Then $\pi^0(S_1, S_2, S_3)$ is identical to $\tau^0(S_1)$, except for a few special partitions depending on the values of $j_2$: 

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If \( j_2 \leq 2, c_3 = a, S_3 = S_3^0 \in \mathcal{S}(a, k - 1), \)
\[
\pi^0(S_1, S_2, S_3)^a = \begin{cases} 
(1), & (S_3^0)_u = 1, u \neq j_1, \\
(3), & (S_3^0)_u = (1, 0), u = j_1, (S_3)_{j_1} = 1, \\
(2), & (S_3^0) = (1, 0), u = j_1, (S_3)_{j_1} = 0, \\
(1^2), & (S_3^0) = (0, 1), u = j_1, (S_3)_{j_1} = 1.
\end{cases}
\]

If \( j_2 > 2, c_3 = a + 1, S_3 = \mathcal{S}(a + 1, k - 1), \)
\[
\pi^0(S_1, S_2, S_3)^a = \begin{cases} 
(1), & (S_3^0)_u = 1, u < j_1 \\
(1), & (S_3^0)_{u-1} = 1, j_1 < u \leq a \\
(2), & S_3 = (1, 0), u = u(j_2), \\
(1^2), & S_3^1 = (0, 1), u = u(j_2), \\
T_2, & S_3^1 = (1, 1), u = u(j_2).
\end{cases}
\]

The case \( p = (1, 0, 1^{k-1}) \) is dual.

**Proof.** Before dividing into cases, we review the notation. If \( j_k \) is the index of an addable node in a list of all addable nodes, then \( u(j_k) \) is the index of the partition in which this node is located.

All \( e \)-regular partitions for content \((k, 1)\) are obtained by straightforward application of the signature method, with no removable nodes involved. The paths \((0^i, 1, 0^{k-i})\) give the same \( e \)-regular partition for any \( 1 \leq j \leq k - 1 \), so we will assume \( j = 1 \).

- Now we do the canonical basis elements, starting with path \( p = (0^k, 1) \). This first case is easier, because it can be obtained by a single application of \( f_1 \) from the canonical basis element constructed in Lemma 4.3. The multipartition \( \tau^0(S_1) \) has \( k \) copies of \((1)\) in the 0-corner partitions and every other partition \( \emptyset \). We now add the \( j_2 \)th 1-node, and there are three possibilities. If \( j_2 \) is odd and \( j_2 \leq 2k \), then we add to the side of the copy of \((1)\) in position \( u = u(j_2) \) to get \((2)\). If \( j_2 \) is even and \( j_2 \leq 2k \), then we add to the bottom of the copy of \((1)\) in position \( u = u(j_2) \) to get \((1^2)\). If \( j_2 > 2k \), then we add a new 1-corner copy of \((1)\) in position \( u \) with \( u = u(j_2) = a + (j_2 - 2k) \). The multipartition \( \tau^0(S_1) \) had a coefficient Inv\((S_1)\), and now we multiply that by \( v \) to the power \( j_2 - 1 = \text{Inv}(S_2) \), the number of addable 1-nodes above the one we just added. There are no removable 1-nodes because this is the first 1-node that we are adding.

- To calculate the case \( p = (1, 0^k) \) we start with the canonical basis element of \( f_1 u_A \), which by Corollary 4.3.1 must be svelte, with the multipartitions dependent on a choice \( j_1 \) of a number between 1 and \( a \) for the position of the partition \((1)\) among the 0-corner partitions. Each such multipartition is multiplied by \( v^{j_1 - 1} \), because there are \( j_1 - 1 \) addable 1-nodes before it. There are now \( a + 2 \) addable 0-nodes, and we let \( S_2 \) represent the choice of \( k \) nodes from among them. Each of the resulting multipartitions is multiplied by an additional factor of \( v^\text{Inv}(S_2) \), as described in Lemma 4.3.

- Finally, we have to calculate a longer path than the two before it, \((0, 1, 0^{k-1})\). We start from a svelte canonical basis element \( G(\tau^0(S_1)) \). Each multipartition in the canonical basis has no removable 1 nodes and \( a + 2 \) addable 1-nodes, these being the original 1-corner partitions and two new addable nodes from the 0-corner.
Proposition 4.1. All external weight spaces with defect \((k - 1)(a - k + 1) + 2a\) for \(1 \leq k \leq a\) have canonical basis elements for \(n \geq 1\) depending on the path as follows:

\[
G(\pi^n(\tilde{S}_1, \ldots, \tilde{S}_m)) = \sum_{S_1 \in \mathcal{S}(c_1, k^{(1)})} \cdots \sum_{S_m \in \mathcal{S}(c_r, k^{(r)})} v^{\text{Inv}(S_1) + \cdots + \text{Inv}(S_m)} \pi^n(S_1, \ldots, S_m),
\]

- \(p = (0^k, 1^{2k+a-1}, \ldots)\) : \(t = 2, c_1 = a, k^{(1)} = k, S_1 \in \mathcal{S}(a, k), c_2 = 2k + a, k^{(2)} = 2k + a - 1, S_2 \in \mathcal{S}(2k + a, 2k + a - 1)\) with the single 0 in position \(j_2\), and \(\pi^n(S_1, S_2)\) is identical to \(\pi^n(S_1)\) for \(n \geq 1\), except for the following partitions:

\[
\pi^n(S_1, S_2)^n = \begin{cases} 
U_1^n, & u \leq a, u = u(j_2), j_2 \equiv 0 \mod 2, \\
U_2^n, & u \leq a, u = u(j_2), j_2 \equiv 1 \mod 2, \\
T_{n-2}, & u > a, u = u(j_2).
\end{cases}
\]

The case \(p = (1^k, 0^{2k+a-1}, \ldots)\) is dual.
- \(p = (1, 0^k, 1^{2k+a-2}, \ldots)\) : \(t = 2, c_1 = a, k^{(1)} = 1, S_1 \in \mathcal{S}(a, 1)\) with the single 1 in position \(j_1\), \(c_2 = a + 2, k^{(2)} = k, S_2 \in \mathcal{S}(a + 2, k)\), and \(\pi^n(S_1, S_2)\) is identical to \(\pi^n(S_2^c)\) for \(n \geq 1\), except for the following partitions:
If $S_2^1 = (0, 0), (1, 0), (0, 1)$, we need an additional characteristic sequence $S_3$ of length $2k + a - 1$ with a single 0 in position $j_3$:

$$\pi^n(S_1, S_2, S_3)^u = \begin{cases} 
T_{n+1}, & u \leq a, u \neq u(j_3), (S_2^0)_u = 1, \\
U_1^n, & u \leq a, u = u(j_3), j_3 \equiv 0 \mod 2, \\
U_2^n, & u \leq a, u = u(j_3), j_3 \equiv 1 \mod 2, \\
T_{n-2}, & u > a, u \neq u(j_1), u = u(j_3), \\
U_1^n, & u > a, u = u(j_3), \\
U_2^n, & u > a, u \neq u(j_3), S_2^1 = (1, 0), \\
U_2^{n+1}, & u > a, u \neq u(j_3), S_2^1 = (0, 1), \\
U_2^n, & u > a, u = u(j_3), S_2^3 = (0, 1), \\
T_{n-2}, & u > a, u = u(j_3), u \neq u(j_2).
\end{cases}$$

If $S_2^1 = (1, 1)$, then there is no need for a third characteristic sequence.

$$\pi^n(S_1, S_2)^u = \begin{cases} 
T_{n+1}, & u \leq a, (S_2^0)_u = 1, \\
T_{n+2}, & u > a, u = u(j_1), \\
T_n & u > a, u \neq u(j_1).
\end{cases}$$

The case $p = (0, 1^k, 0^{2k+a-2}, \ldots)$ is dual.

• $p = (0, 1^k, 1^{2k+a-2}, \ldots)$ $t = 3, c_1 = a, k^{(1)} = 1, S_1 \in S(a, 1)$ with the single 1 in position $j_1$, $c_2 = a + 2, k^{(2)} = 1, S_2 \in S(a + 2, 1)$ with a single 1 in position $j_2$. The structure and length of $S_3$ depends on the value of $j_2$. In all cases where it is needed, $S_4$ is of length $2k + a - 1$, and has a single 0 in position $j_4$. Then $\pi^n(S_1, S_2, S_3, S_4)$ is identical to $\pi^n(S_1)$ for $n \geq 1$, except for a few special partitions depending on the values of $j_2, S_3$ and $S_4$:

If $j_2 \leq 2$, then for $(S_3^0)_{j_1} = 1$,

$$\pi^n(S_1, S_2, S_3)^u = \begin{cases} 
T_{n+1}, & u \neq j_1, (S_3^0)_u = 1, \\
U_1^{n+1}, & u = j_1, (S_3^0) = (1, 0), \\
U_2^{n+1}, & u = j_1, (S_3^0) = (0, 1).
\end{cases}$$

and for $(S_3^0)_{j_1} = 0$, we have

$$\pi^n(S_1, S_2, S_3, S_4)^u = \begin{cases} 
T_{n+1}, & u \neq j_1, (S_3^0)_u = 1, u \neq u(j_4), \\
U_1^n, & u \neq j_1, (S_3^0)_u = 1, u = u(j_4), j_4 \equiv 0 + \left\lfloor \frac{u(j_1)}{u(j_1)} \right\rfloor \mod 2, \\
U_2^n, & u \neq j_1, (S_3^0)_u = 1, u = u(j_4), j_4 \equiv 1 + \left\lfloor \frac{u(j_1)}{u(j_1)} \right\rfloor \mod 2, \\
T_{n-1}, & u = j_1, u(j_4) \neq j_1, \\
U_1^n, & u = j_1, u(j_4) = j_1, S_4^0 = (1, 0), \\
U_2^n, & u = j_1, u(j_4) = j_1, S_4^0 = (0, 1), \\
T_{n-2}, & u > a, u = u(j_4).
\end{cases}$$

If $j_2 > 2$, we set $\pi^n(j_1, S_2, j_3)^u$ equal to $\pi^n(S_2^0)^u$ except for the following cases:
Proof. We treat each case separately.

• \( S_3^1 = (0, 0) \),

\[
\pi^n(S_1, S_2, S_3)^u = \begin{cases} 
T_{n+1}, & (u = j_1) \lor (S_3^0)_u = 1, u \neq u(j_4), \\
U_1^n, & u \leq a, u = u(j_4), j_4 \equiv 0 \mod 2, \\
U_2^n, & u \leq a, u = u(j_4), j_4 \equiv 1 \mod 2, \\
T_n, & u = u(j_2) > a, \\
T_n, & u > a, u \neq u(j_4), \\
T_{n-2}, & u > a, u = u(j_4).
\end{cases}
\]

if \( S_3^1 = (1, 0), (0, 1) \),

\[
\pi^n(S_1, S_2, S_3, S_4)^u = \begin{cases} 
T_{n+1}, & (u = j_1) \lor (S_3^0)_u = 1, u \neq u(j_4), \\
T_{n-1}, & u \neq j_1 \land (S_3^0)_u = 0, \\
U_1^n, & u = u(j_4), j_4 \equiv 1 \mod 2, \\
U_2^n, & u = u(j_4), j_4 \equiv 0 \mod 2, \\
U_1^{n+1}, & u = u(j_2), S_3^1 = (1, 0), u \neq u(j_4), \\
U_1^n, & u = u(j_2), S_3^1 = (1, 0), u = u(j_4), \\
U_2^{n+1}, & u = u(j_2), S_3^1 = (0, 1), u \neq u(j_4), \\
U_2^n, & u = u(j_2), S_3^1 = (0, 1), u = u(j_4), \\
T_n, & u > a, u \neq u(j_4), u \neq u(j_2), \\
T_{n-2}, & u > a, u = u(j_4).
\end{cases}
\]

if \( S_3^1 = (1, 1), k \geq 3 \),

\[
\pi^n(S_1, S_2, S_3)^u = \begin{cases} 
T_{n+1}, & u = j_1 \lor (S_3^0)_u = 1, \\
T_{n-1}, & u \neq j_1 \lor (S_3^0)_u = 0, \\
T_{n+2}, & u = u(j_2), \\
T_n, & u > a, u \neq u(j_2).
\end{cases}
\]

\text{The case } p = (1, 0, 1^{k-1}) \text{ is dual.}

Proof. We treat each case separately.

• \( p = (0^k, 1^{2k+a-1}, \ldots) \) : In this case, we want to start the numbering at \( n = 1 \). The hub of \( \tau_n(\tilde{S}_1) \) is \([a-2k, a+2k] \). The 1-string is of length \( 2k+a \) and ends at \( \tau_n(\tilde{S}_1) \), whose canonical basis element is given in Lemma 4.4. Then the operation by \( e_i \) on the canonical basis element gives multipartitions which have been modified by removing the \( j_2 \)th removable node. If \( 1 \leq j_2 \leq 2k \), then if \( j_2 \) is odd, we replace one copy of the triangular partition \( T_2 \) by \((1^2)\) because we remove the first removable node, whereas if \( j_2 \) is even, we replace one \( T_2 \) by \((2)\), having removed the second 1-removable node. On the other hand, if \( j > 2k \), then we remove the \((j_2 - 2k)\)th of the 1-corner partitions, leaving \( \emptyset \) in that spot. This gives exactly the values of \( \tau_1(S_1) \), except for one partition, which is described in the proposition, its location depending on the value of \( j_2 \), so the case \( n = 1 \) is solved for this path.

The hub of this weight space is \([a + 2k, a - 2k] + (a + 2k - 1)(-2, 2] = [-a - 2k + 2, 3a + 2k - 2] \). The 1-string above this 1-string is shorter by two vertices, so there is no point above this weight space in the 0-direction. Thus our weight space is external and lies at the beginning of a 0-string of length \( 3a+2k-2 \). In the
original $\tau^1_0(S_1)$ there were exactly $3a + 2k$ addable 0-nodes, one for each of $a - k$
0-corner $\emptyset$, three for each 0 corner $T_2$, and two for each 1-corner (1). In each of
the three different cases for $j_2$, the effect of removing the 1-node was to reduce
the number of possible 0-nodes by 2. If we left out the first 1-node in a $T_2$, then
we can only add a single 0-node at the bottom, giving $(1^3)$. If we omitted the
bottom 1-node of a $T_2$, then we can only add a single 0-node at the end, giving
(3). If we omitted a copy of (1) among the 1-corner nodes, then we cannot add
any 0-corner nodes at the spot. Altogether, as we go down the 0-string, we add
all the 0-nodes to every multipartition, so there are no choices, and the shape of
the multipartition at the end of the string is exactly as it was at the beginning,
and we get the formula in the proposition for $n = 2$ and all the multipartitions
appearing in the canonical basis have only 1-addable nodes. We now continue by
induction, since for each $n$, all the partitions occurring in the formulae for the
multipartitions for have only addable nodes, in this case, of the parity opposite
to that of $n$. Since the vertex is external, the number of addable nodes equals
the length of the string, so adding all addable nodes gets one to the vertex of the
same defect at the other end of the string. There are never any choices and the
shape of the canonical basis element is preserved.

The dual case $(1^k, 0^{2k+a-1}, \ldots)$ is very similar.

* $p = (1, 0^k, 1^{2k+a-2}, \ldots)$: Here $1 \leq j_1 \leq a$ chooses one of the 1-corner partitions,
in position $u(j_1) = a + j_1$. Then $S_2$ distributes $k$ 0-nodes, of which none, one, or
two can be places on a 1-corner partition, so that the total number of possibilities is
$\binom{a+2}{k}$. We now need to determine the number of addable 1-nodes and identify
the multipartitions. In the case $n = 1$, which is after adding one 1-node, $k$ 0-
nodes, and $2k + a - 2$ 1-nodes, most of the 0-corner partitions are $T_2$ or $\emptyset$, and
most of the 1-corner partitions are (1). The various special cases depend on $S_3^1$
and on $j_3$, and we will go over them now for the case $n = 1$, letting $k'$ be the
number of 1-entries in $S_3^0$:

- $S_3^1 = (0, 0)$: If all of the $k$ 0-nodes are in the 0-corner section, then there are
  $2k$ addable 1-nodes in the 0-corner section, and $a - 1$ addable 1-nodes in the
  1 corner section, as we replace $\emptyset$ with (1) giving $2k + a - 1$ altogether so a
  choice sequence $S_3$ is necessary. Thus in applying $J_{1}^{2k+a-2}$ to the canonical
  basis element, the characteristic sequence $S_3$ is all copies of 1, except for a
  0 in position $j_3$ where $1 \leq j_3 \leq 2k + a - 1$. In this case $k' = k$. If $j_3 \leq 2k$
  then there are $k - 1$ copies of $T_2$, and one copy of $1^2$ or (2) depending on the
  parity of $j_3$.

- $S_3^1 = (1, 0), (0, 1)$. If there are $k - 1$ 0-nodes in the 0-corner section, this
gives $2(k - 1)$ addable 1-nodes. There are still $a - 1$ copies of $\emptyset$ to be filled,
but in addition we now have either (2) or (1$^2$) at the previously chosen 1-
corner partition, and this can be converted to (3) or (1$^3$) respectively, giving
an additional addable 1 node, so that we have $2k + a - 1$ altogether, as
before, and again a characteristic sequence is necessary. If $j_3 > 2k'$ but
corresponds to not filling one of the partitions $\emptyset$, then the partition in the
1-corner partition numbered by $j_1$ is (1), (2), $1^2$, or $T_2$ after the 0-nodes are
filled in, and becomes (1), (3), (1$^3$), or $T_3$ after all the 1 nodes are filled in.

- $S_3^1 = (1, 1)$ Finally, we come to the case where $k \geq 2$ and there are only $k - 2$
0-nodes in the 0-corner partitions, giving $2(k - 2)$ addable 1-nodes there. As
before we have the \( a - 1 \) copies of \( \emptyset \) to be filled, but now there is also a 1-corner copy of \( T_2 \), to which 3 different 1 nodes can be added, altogether \( 2k + a - 2 \) nodes, the total number we need to add, so here we do not need a characteristic sequence \( S_4 \), and we continue the results in this case from Lemma 4.5

As before, the induction results from the weight space being external and from noting that all the partitions have only addable nodes and no removable nodes.

- \( p = (0, 1, 0^{k-1}, 1^{2k+a-2}, \ldots) \): We have \( u(j_1) = j_1 \).

  If \( j_2 \leq 2 \), then we are adding a 1-entry to the 0-corner (1) in position \( u(j_1) \). If \( S_2^0 = (1, 0) \), then we get (2), and if \( S_2^0 = (0, 1) \), then we get \((1^2)\). Next we need to add \( k - 1 \) 0-nodes, which must all go into 0-corner partitions, so that \( S_3 = S_3^0 \) with \( k - 1 \) entries equal to 1 and is of length \( c_3 = a \). We now need to distinguish two cases:

  \( (S_3^0)_{j_1} = 1 \): In this case, adding a 0-node to (2) gives (3), or adding a 0-node to (1^2) gives (1^3). Both of these have 2 addable 1-nodes. In addition, we have added \( k - 2 \) 0-nodes to copies of \( \emptyset \), giving an additional \( 2(k - 2) \) addable 1-nodes. Putting these together with the \( a \) addable 1-nodes in the 1-corner partitions, we have \( 2k + a - 2 \) addable 1-nodes, just the number we need to add, so there is no need for a characteristic sequence \( S_4 \).

  \( (S_3^0)_{j_1} = 0 \): In this case, we added \( k - 1 \) 0-nodes in place of \( \emptyset \), each giving 2 addable 1-nodes, but we also have another addable 1-node in the partitions in position \( u = u(j_1) \), giving, together with the \( a \) addable 1-nodes in the 1-corner partitions, a total of \( 2k + a - 1 \) addable one nodes, which is one too many. Therefore, we need a characteristic sequence \( S_3 \) of length \( c_3 = 2k + a - 1 \), which will choose \( 2k + a - 2 \) addable nodes. Letting \( j_4 \) be the position of the single 0-entry in \( S_4 \), we consider the effect of adding all addable 1-nodes to the existing partition, and combine that with considering the various possible positions in which \( j_4 \) can lie. In the 0-corner partitions, if we add two 1-nodes to a (1), we get \( T_2 \), while if we omit one of them, we get (2) or \((1^2)\). If \( u < u(j_1) \), then \( j_4 \equiv 0 \mod 2 \) means that we omit the bottom node, giving (2) and if \( j_4 \equiv 1 \mod 2 \) we omit the side node, giving \((1^2)\). If \( j_4 \) corresponds to \( u = j_1 \), then we remain with what we had, dependant on \( S_2^0 \). If \( u > u(j_1) \), then we again get (2) or \((1^2)\), but we have to shift down by one, because there was only one possibility for \( u(j_4) = u(j_1) = j_1 \), so the required parities of \( j_4 \) are reversed. In order to compensate for this shift, we added \( [\frac{u - j_1}{u(j_1)}] \), which is equal to 1 when \( u > j_1 \). If \( u > a \), then all the partitions are \( T_n \) as in \( \tau^{n}(S_1) \), and if \( j_4 \) is in this section, then for \( u = u(j_4) \) we get \( T_{n-2} \).

  If \( j_2 > 2 \), there are three cases, depending on the value of \( S_3^0 \), which is of length 2.

  If \( S_3^0 = (0, 0) \), then all the 0-nodes are in the 0-corner partitions. Each has two addable nodes, and when we add the 1-nodes, we get \( T_2 \) if \( u = j_1 \) or \((S_3^0)_u = 1 \). Together with \( a - 1 \) addable 1-corner 1-nodes, we get \( 2k + a - 1 \), so we need a characteristic sequence \( S_4 \). When \( u = j_1 \) or \((S_3^0)_u = 1 \) but \( u = u(j_4) \), then we get (2) or \((1^2)\), depending on the parity of \( j_4 \). As for the 1-corner nodes, they are all filled with (1) except possibly when \( u(j_4) > a \), in which case we get \( \emptyset \). Since the weight space is external, we continue to \( u > 1 \) by filling all nodes.

  If \( S_3^0 = (1, 0) \) or \((0, 1) \), we again need \( S_4 \). The possibilities for 0-corner partitions are the same as in the previous case, but now for \( u > a \) and \( u = u(j_2) \) we have a few
new possibilities. Adding the 0-nodes gives (2) or \((1^2)\). Then in the continuation, this becomes (3) or \((1^3)\) if we don’t have \(u = u(j_4)\) and stays as it was if \(u = u(j_4)\).

If \(S^3_1 = (1, 1)\), then there is no need for \(S_1\) and no partitions which are not equal to their own transpose. For \(n = 1\) we get \(T_3\), and for larger \(n\) we get \(T_{n+2}\) at that spot.

Finally, the induction. The action by \(e_0\) gives 0, so going down the string from one end to the other involves adding all the addable nodes with no choices. This sends \(T_m\) to \(T_{m+1}\), sends \(U^{n}_i\) to \(U^{n+1}_i\) for \(i = 1, 2\).

**Corollary 4.5.1.** Let us define \(\mu^T\) to be the multipartition obtained from \(\mu\) by transposing each individual partition, but without reversing the order as in \(\mu'\). For every multipartition \(\mu\) occurring in a canonical basis element as in the Proposition 4.1, the multipartition \(\mu^T\) also occurs.

**Proof.** The partitions \(T_n\) are all transpose to themselves, and the only other partitions which occur are the \(U^{n}_i, i = 1, 2\), which always occur in pairs, so that if one exists in the canonical basis element, the other occurs in the same position.

We give one more lemma of a different flavor, being concerned not with defects showing up near the top of the block reduced crystal, but with defects appearing above a defect 0 block in an \(i\)-string.

**Lemma 4.6.** For \(e = 2\), and a symmetric crystal, consider an \(i\)-string of length \(s\) ending with a vertex with weight \(\psi\) of defect 0. A defect 0 weight corresponds to a unique 2-regular multipartition \(\lambda\). The canonical basis element of the multipartition \(\hat{e}_i(\lambda)\) is svelte.

**Proof.** We begin with a defect 0 multipartition which is obtained from the highest weight vector by action of the Weyl group.

A symmetric crystal, in addition to being symmetric, also looks rather similar to a spruce tree, with lower branches much longer that upper branches, as in Fig. 3. Then we start on the right side and act by the elements of the Weyl group, the hubs of vertices of defect 0 are \([3a, -a], [-3a, 5a], [7a, -5a], \ldots\) with the coordinates reversed on the left side. At the top, the hub before \([3a, -a]\) is \([3a - 2, 2 + a]\). Furthermore, since the hubs going down always drop by two in length and start at the same place, the vertex \(\mu\) one above the defect 0 vertex is also the beginning of an \(i\)-string. We are trying to show that the canonical basis element of \(\mu\) is svelte. The only multipartitions which can occur in \(G(\mu)\) are those which can produce the multipartition \(\lambda\), whose removable nodes are all of residue \(i\), so it must be obtained by removing one removable node of residue \(i\).

The length of the string was \((2c + 1)a\), so there are \((2c + 1)a\) nodes which were added and can be removed, giving \((2c + 1)a\) candidate multipartitions. Since, by an argument similar to that in Lemma 4.1, the defect of \(\mu\) is \((2c + 1)a - 1\), we have to show that they all occur. Now, in order to get the correct canonical basis element for \(\lambda\), we simply apply \(\hat{e}_i\) to \(\lambda\). The formula for \(\hat{e}_i\) has us remove the removable nodes one-by-one, multiplying each time by \(v\) to a power which is the number of removable nodes below the one we are removing, minus the number of addable nodes. Since there are no addable nodes, and the number of removable nodes is the same as the length of the string, we get a different power of \(v\) for each of \((2c + 1)a\) removable nodes, running from 0 for the bottom node to \((2c + 1)a - 1\) for the top removable node. This gives a svelte canonical basis element.
5. BLOCKS OF SMALL DEFECT

The defect 0 case is as already described, even in the non-symmetric case, so we will begin with defect 1. There can be a block of defect 1 only if it occurs in the first string going out from the highest weight element, by results in [4], §3. Furthermore, since the defects rise towards the center of the string in a parabolic fashion as described in that paper, the only possible values of $a$ for which the crystal can contain a block of defect 1 are $a = 2$, for which the defects in the highest string are $0 - 1 - 0$. In this case, we can simply determine all blocks of defect 1 by using the action of the Weyl group. The sequence of multipartitions with path beginning at zero consists entirely of triangular partitions, the first of side $n$, the second of side $n - 2$, and the last two of sides $n - 1$.

From the structure of the block reduced crystal graph, the defects which can occur in a symmetric crystal are the defects appearing in this first row, modulo $2a$. Thus

- for $a = 1$, the defects are all even numbers,
- for $a = 2$, the defects are congruent to 0 or 1, modulo 4,
- for $a = 3$, the defects are congruent to 0 or 2, modulo 6,
- for $a = 4$, the defects are congruent to 0, 3 or 4, modulo 8.

We now turn to the case of defect 2. This can occur only when $a = 1$ where it lies on a string with defects $0 - 2 - 2 - 0$ and the block of defect 2 in internal, or for $a = 3$. When $a > 2$, there must be a $k$ with $k(a - k) = 2$, and this happens only when $a = 3$ and $k = 1, 2$. The multipartitions with defect 2 have a very distinctive form, and we can calculate all of them. The first example in defect 2 is in degree 2 and there are two 2-regular multipartitions, both svelte. In degree 3, there are two blocks of defect 2, each of which has one canonical basis element which is svelte, corresponding to the path $(1, 0, 0)$ or $(0, 1, 1)$ as in Lemma 4.6 and two which are not, corresponding to the alternating paths $(0, 1, 0)$ and $(1, 0, 1)$. This gives an example to show that the action of the Weyl group on internal vertices of a string need not preserve the shape of the canonical basis element. We now list all multipartitions of defect 2.

- $a = 1$. In this case the 2-regular multipartitions are in one of two dual forms:
  1. For $\mu = [T_{n+1}, T_{n-2}]$, or $\mu^2 = [T_n, U_1^n = (n + 1) \lor T_{n-2}],$
  2. For $\mu = [U_1^n = (n + 1) \lor T_{n-2}, T_{n-2}]$, or $\mu^2 = [U_1^n = (n + 1) \lor T_{n-2}, T_n].$

In the first case, there are three monomials in all the $G(\mu)$, so they are all svelte. The coefficient for $v$ in $\mu$ was given by taking the transpose of the non-regular partition.

   In the second case, $G(\mu)$ is not svelte. There are two multipartitions multiplied by $v$, being given by the transposes of $\mu$ and $(\mu^2)'$ as in the corollary to Proposition 4.1. For example, in degree 3, we have
   
   $$G([(3), \emptyset]) = [(3), \emptyset] + v[(1^3), \emptyset] + v[(1), (2)] + v^2[(1), (1^2)].$$

   This can be checked easily for the cases of lowest degree. Thereafter, we appeal to Prop. 4.1, and note that in the form given, adding all addable nodes preserves the property of being transpose.

- $a = 3$. As in the case $a = 1$, we have two dual forms:
  1. For $\mu = [T_{n+1}, T_{n-1}, T_{n-1}, T_n, T_n, T_n]$, or $\mu^2 = [T_n, T_n, T_n, T_{n+1}, T_{n-1}, T_{n-1}],$
  2. For $\mu = [T_{n+1}, T_{n+1}, T_{n-1}, T_n, T_n, T_n].$
or \( \mu^2 = [T_n, T_n, T_{n+1}, T_{n+1}, T_{n-1}] \).

as follows from Lemma 4.3.

\[
[(1), \emptyset, \emptyset, \emptyset, \emptyset, \emptyset] + v[(1), \emptyset, \emptyset, \emptyset, \emptyset] + v^2[(0, (1), \emptyset, \emptyset, \emptyset, \emptyset)].
\]

To produce the middle terms, we move the larger triangle down, and this is preserved under adding all addable nodes, so by Lemma 4.3 we get the desired structure of all the canonical basis elements.

6. Appendix: SageMath code

After Lemma 4.2 we gave two examples of implementation of the algorithm for computing canonical basis elements given in that lemma. The code given here is for the interactive version of SageMath, and specifies the second example. The code below is for “quickf”. After each colon, the next block needs to be indented.

```python
F=FockSpace(2,(0,0,1));
hw=F.highest_weight_vector();
CB=hw.f(0,1,0,1,(0,2),(1,5));CB
P=CB.parent();
def quickf(ret,i,p):
    if (len(P.removable(la,i))==0 for la,c in ret)and (len(P.addable(la,i))==p for la,c in ret):
        for j in range(p):
            ret = P.sum_of_terms( (la.add_cell(*x), c) for la,c in ret for x in P.addable(la, i) )
        ret = ret / factorial(p)
    else:
        for in range(p):
            ret = ret.f(i)
        ret = ret / q_factorial(p, q)
    return ret
```

Here is the code which replaces “quickf” to produce “quickerf”. Again, the indentation should be as standard in Python.

```python
def jump(la,i):
    nu = la
    for x in P.addable(la, i):
        nu = nu.add_cell(*x)
    return nu
def quickerf(ret,i,p):
    if (len(P.removable(la,i))==0 for la,c in ret)and (len(P.addable(la,i))==p for la,c in ret):
        ret = P.sum_of_terms((jump(la,i),c) for la,c in ret for x in P.addable(la, i) )
    else:
        for in range(p):
            ret = ret.f(i)
        ret = ret / q_factorial(p, q)
    return ret
```

REFERENCES


