Hölder Continuity for Vectorial Local Minimizers of Variational Integrals

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Abstract. This note deals with vectorial local minimizers of some integral functionals related to nonlinear elasticity theory. Under some structural assumptions, we derive that each component of the local minimizers lies in the De Giorgi class, thus the minimizers are locally Hölder continuous. We emphasize that no convexity is assumed. On the other hand, a special dependence on minors, taken from the Jacobian matrix, plays an important role in the proof.

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1 Introduction.

Throughout this paper we let Ω be a bounded domain in the three-dimensional Euclidean space \( \mathbb{R}^3 \). Consider the energy functional \( F \) defined for every map \( u \in W_{loc}^{1,p}(\Omega; \mathbb{R}^3) \), \( p > 1 \), and for every measurable subset \( E \) of \( \Omega \) by

\[
F(u; E) = \int_E f(x, u(x), Du(x)) \, dx
\]

with the integrand \( f : \Omega \times \mathbb{R}^3 \times \mathbb{R}^{3 \times 3} \to \mathbb{R} \) be of the form

\[
f(x, u, Du) = \sum_{\alpha=1}^{3} [F^\alpha(x, Du^\alpha) + G^\alpha(x, (\text{adj}_2 Du)^\alpha)] + H(x, \det Du) + F_0 u,
\]

where \( u = (u^1, u^2, u^3)^t : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3 \) is a vector-valued map defined in \( \Omega \), \( Du \) is the \( 3 \times 3 \) Jacobian matrix of its partial derivatives, i.e.,

\[
Du = \begin{pmatrix}
D_{u^1} \\
D_{u^2} \\
D_{u^3}
\end{pmatrix} = \begin{pmatrix}
D_{1u^1} & D_{2u^1} & D_{3u^1} \\
D_{1u^2} & D_{2u^2} & D_{3u^2} \\
D_{1u^3} & D_{2u^3} & D_{3u^3}
\end{pmatrix}, \quad D_{\beta u^\alpha} = \frac{\partial u^\alpha}{\partial x_\beta}, \quad \alpha, \beta \in \{1, 2, 3\},
\]

\( \text{adj}_2 Du = ((\text{adj}_2 Du)^\gamma_i) \in \mathbb{R}^{3 \times 3} \) denotes the adjugate matrix of order 2,

\[
\text{adj}_2 Du = \begin{pmatrix}
(\text{adj}_2 Du)^1_i \\
(\text{adj}_2 Du)^2_i \\
(\text{adj}_2 Du)^3_i
\end{pmatrix} = \begin{pmatrix}
(\text{adj}_2 Du)^1_1 & (\text{adj}_2 Du)^1_2 & (\text{adj}_2 Du)^1_3 \\
(\text{adj}_2 Du)^2_1 & (\text{adj}_2 Du)^2_2 & (\text{adj}_2 Du)^2_3 \\
(\text{adj}_2 Du)^3_1 & (\text{adj}_2 Du)^3_2 & (\text{adj}_2 Du)^3_3
\end{pmatrix},
\]

whose components are

\[
(\text{adj}_2 Du)^\gamma_i = (-1)^{\gamma+i} \det \begin{pmatrix}
D_{k u^\alpha} & D_{l u^\alpha} \\
D_{k u^\beta} & D_{l u^\beta}
\end{pmatrix}, \quad \gamma, i \in \{1, 2, 3\},
\]

where \( \alpha, \beta \in \{1, 2, 3\} \setminus \{\gamma\}, \alpha < \beta \) and \( k, l \in \{1, 2, 3\} \setminus \{i\}, \ k < l \), \( \det Du \) is the determinant of the matrix \( Du \) and \( F_0 \) is a three-dimensional vector function.

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We assume that there exist exponents
\[ 1 < p < 3, \quad (1.3) \]
\[ 0 < q < \frac{p}{2}, \quad (1.4) \]
\[ 0 < r < \frac{p}{3}, \quad (1.5) \]
constants
\[ k_1, k_3 > 0, \quad k_2 \geq 0, \quad (1.6) \]
and functions
\[ a(x), b(x), c(x) \in L^p_{\text{loc}}(\Omega), \quad \sigma > 1, \quad (1.7) \]
\[ F_0(x) \in L^{\tau}_{\text{loc}}(\Omega; \mathbb{R}^3), \quad \tau > p', \quad (1.8) \]
such that for \( \alpha \in \{1, 2, 3\} \), almost all \( x \in \Omega \), all \( \lambda \in \mathbb{R}^3 \) and all \( t \in \mathbb{R} \),
\[ k_1 |\lambda|^p - k_2 \leq F^\alpha(x, \lambda) \leq k_3 |\lambda|^p + a(x), \quad (1.9) \]
\[ -k_1 |\lambda|^q - k_2 \leq G^\alpha(x, \lambda) \leq k_3 |\lambda|^q + b(x), \quad (1.10) \]
\[ -k_1 |t|^r - k_2 \leq H(x, t) \leq k_3 |t|^r + c(x). \quad (1.11) \]
A vectorial function \( u \in W^{1,p}_{\text{loc}}(\Omega; \mathbb{R}^3) \) is a local minimizer of \( F \) if
\[ F(u; \text{supp}\varphi) \leq F(u + \varphi; \text{supp}\varphi) \quad (1.12) \]
for all \( \varphi \in W^{1,p}(\Omega; \mathbb{R}^3) \) with compact support.

Regularity results for vector-valued minimizers \( u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m \) \((m, n \geq 2)\) of variational integrals (1.1) with \( f : \Omega \times \mathbb{R}^m \times \mathbb{R}^{n \times m} \to \mathbb{R} \) are very rich. Partial regularity results (i.e., the regularity of solutions up to a set \( \Omega_0 \) and the study of the properties of the singular set) can be found in [5, 15, 16]. For everywhere regularity results, we refer the reader to [17], where the everywhere continuity is proved in the case \( n = m = 2 \), [7] where local boundedness is proved in the case \( n = m = 3 \). Global pointwise bounds can be found in [10, 19, 29–32].

The integral functional (1.1) is closely related to nonlinear elasticity theory, see, for example, [1–3, 5, 8, 14, 38]. A remarkable development in this field is the recent paper [7] by Cupini, Leonetti and Mascolo, in which a local boundedness result for local minimizers of some polyconvex functionals is obtained. A related result can be found in [4]. Some global regularity results are obtained in [18].

We note that the functionals considered in [4, 7] are polyconvex. For the definition of a polyconvex functional, we refer the reader to [1], or Page 270 in [7]. In [1] Ball pointed out that the convexity of \( f \) with respect to \( Du \) is unrealistic in the vectorial case. Indeed, it conflicts with the natural requirement that the elastic energy is frame-indifferent. The convexity must be replaced by different and more general assumptions, such as quasiconvexity and polyconvexity, already introduced by [37] in an abstract setting.

The difference between the present paper and the above mentioned ones is that we do not impose any convexity assumptions on the energy density. We show, under the structural assumptions (1.3)-(1.11), if \( \sigma, \tau \) are big enough and \( q, r \) are small enough, then the first component \( u^1 \) lies in the De Giorgi class, so it is locally Hölder continuous. We can argue similarly for the other components \( u^\alpha, \alpha = 2, 3 \), thus the minimizers are locally Hölder continuous. We note that this method has been used in some papers, among them, we refer [6], where local Hölder continuity of vectorial local minimizers of special classes of integrals functionals with rank-one and polyconvex integrands are
considered.

Our main result in this note is the following

**Theorem 1.** Let \( f \) be as in (1.2) with the conditions (1.3)-(1.11). Assume \( \sigma, \tau \) are big enough and \( q, r \) are small enough such that

\[
\max \left\{ \frac{1}{\sigma}, \frac{q}{p - q}, \frac{2r}{p - r}, \frac{p'}{\tau} \right\} < \frac{p}{3}.
\]

(1.13)

Then all the local minimizers \( u \in W_{loc}^{1,p}(\Omega; \mathbb{R}^3) \) of \( F \) are locally Hölder continuous.

We emphasize that regularity is not expected, in general, in vectorial framework, see [13, 21, 23–26, 28, 33, 36, 39–41]; see also the surveys [27, 34, 35].

Let us end this introduction by mentioning [11, 12] where higher integrability is studied.

## 2 Proof of the Main Theorem.

We first recall the definition of De Giorgi classes \( DG^+, DG^- \) and \( DG \), see [20].

**Definition 1.** Let \( \Omega \subset \mathbb{R}^n \) be a bounded open set and \( v : \Omega \to \mathbb{R} \). We say that \( v \in W_{loc}^{1,p}(\Omega) \) belongs to the De Giorgi class \( DG^+(\Omega, p, y, y_*, \delta) \), \( p > 1 \), \( y \) and \( \delta > 0 \), \( y_* \geq 0 \) if

\[
\int_{B_\rho(x_0)} |D(v - k)_+|^p \, dx \leq \int_{B_{R_\delta}(x_0)} \left( \frac{(v - k)_+}{R_\delta - \rho} \right)^p \, dx + y_* \left( \{v > k\} \cap B_{R_\delta}(x_0) \right)^{1 - \frac{p}{p'}} + p^p
\]

for all \( k \in \mathbb{R} \) and all pair of concentric balls \( B_\rho(x_0) \subset B_{R_\delta}(x_0) \subset \subset \Omega \), where \( |E| \) is the \( n \)-dimensional Lebesgue measure of \( E \).

The De Giorgi class \( DG^-(\Omega, p, y, y_*, \delta) \) is defined similarly with \( (v - k)_+ \) replaced by \( (v - k)_- \). Define \( DG(\Omega, p, y, y_*, \delta) = DG^+(\Omega, p, y, y_*, \delta) \cap DG^-(\Omega, p, y, y_*, \delta) \).

(2.1) is a Caccioppoli type inequality on super-/sub-level sets and contains several information on the smoothness of the function \( v \). As a matter of fact, they are locally bounded and locally Hölder continuous in \( \Omega \), see Theorems 2.1 and 3.1 in [9] and Chapter 7 in [22].

**Lemma 1.** Let \( v \in DG(\Omega, p, y, y_*, \delta) \) and \( \tau \in (0, 1) \). There exists a constant \( C > 1 \), depending only upon the data and independent of \( v \), such that for every pair of balls \( B_{\tau \rho}(x_0) \subset B_\rho(x_0) \subset \subset \Omega \),

\[
\|v\|_{L^\infty(B_{\tau \rho}(x_0))} \leq \max \left\{ y_* \rho^{\alpha_0}; \frac{C}{(1 - \tau)^{\frac{1}{2}} \left( \int_{B_\rho(x_0)} |v|^p \, dx \right)^{\frac{1}{p}}} \right\},
\]

moreover, there exists \( \tilde{\alpha} \in (0, 1) \) depending only upon the data and independent of \( v \), such that

\[
\text{osc}(v, B_\rho(x_0)) \leq C \max \left\{ y_* \rho^{\tilde{\alpha}_0}; \left( \frac{\rho}{R} \right)^{\tilde{\alpha}_0} \text{osc}(v, B_R(x_0)) \right\},
\]

where \( \text{osc}(v, B_\rho(x_0)) = \text{ess sup}_{B_\rho(x_0)} v - \text{ess inf}_{B_\rho(x_0)} v \). Therefore, \( v \in C_{loc}^{0, \tilde{\alpha}_0}(\Omega) \) with \( \tilde{\alpha}_0 := \tilde{\alpha} \wedge (n\delta) \).

This lemma comes from Lemma 6.1 of [22], which has an important application to the hole filling technique.
Lemma 2. Let \( h : [r, R_0] \to \mathbb{R} \) be a non-negative bounded function and \( 0 < \theta < 1 \), \( A, B \geq 0 \) and \( \beta > 0 \). Assume that
\[
h(s) \leq \theta h(t) + \frac{A}{(t-s)^\beta} + B,
\]
for all \( r \leq s < t \leq R_0 \). Then there exists \( c = c(\theta, \beta) > 0 \) such that
\[
h(r) \leq \frac{cA}{(R_0 - r)^\beta} + cB.
\]

Proof of Theorem 1. Let \( x_0 \in \Omega \subset \mathbb{R}^3 \) be arbitrarily fixed and \( B_{R_0}(x_0) \subset \Omega \) with \( |B_{R_0}(x_0)| < 1 \). Let \( 0 < \rho < R_0 \) and \( s, t \) be such that \( \rho \leq s < t \leq R_0 \).

Consider a cut-off function \( \eta \in C_0^\infty(B_t) \) (we shall often write \( B_t \) for \( B_t(x_0) \) if no confusion can arise) satisfying the following assumptions:
\[
0 \leq \eta \leq 1, \quad \eta \equiv 1 \text{ in } B_s(x_0), \quad |D\eta| \leq \frac{2}{t-s}.
\]

Let \( u = (u^1, u^2, u^3)^t \in W^{1,p}_\text{loc}(\Omega; \mathbb{R}^3) \) be a local minimizer of \( F \). Fixing \( k \in \mathbb{R} \), define \( w = (w^1, w^2, w^3)^t \in W^{1,p}_\text{loc}(\Omega; \mathbb{R}^3) \) by
\[
w^1 = \max\{u^1 - k; 0\}, \quad w^2 = 0, \quad w^3 = 0.
\]

Take
\[
\varphi = -\eta^p w. \tag{2.2}
\]

It is easy to see that \( \text{supp}\varphi = \{u^1 > k\} \cap \{\eta > 0\} \) and for almost every \( x \in \Omega \setminus \text{supp}\varphi \) we have \( \varphi = 0 \), then
\[
f(x, u + \varphi, Du + D\varphi) = f(x, u, Du) \text{ almost everywhere in } \Omega \setminus \text{supp}\varphi.
\]

For almost every \( x \in \text{supp}\varphi \) denote
\[
A = \begin{pmatrix} p\eta^{-1}(k - u^1)D\eta \\ Du^2 \\ Du^3 \end{pmatrix} \tag{2.3}
\]

We notice that
\[
Du + D\varphi = \begin{pmatrix} (1 - \eta^p)Du^1 + p\eta^{p-1}(k - u^1)D\eta \\ Du^2 \\ Du^3 \end{pmatrix} = (1 - \eta^p)Du + \eta^p A. \tag{2.4}
\]

Let
\[
A^1_{k,t} = \{x \in B_t(x_0) : u^1(x) > k\},
\]
then (1.12) implies
\[
\int_{A^1_{k,t} \cap \{\eta > 0\}} f(x, u, Du)dx \leq \int_{A^1_{k,t} \cap \{\eta > 0\}} f(x, u + \varphi, Du + D\varphi)dx. \tag{2.5}
\]

Taking into account that a.e. in \( \Omega \),
\[
F^2(x, Du^2) = F^2(x, Du^2 + D\varphi^2),
\]
\[
F^3(x, Du^3) = F^3(x, Du^3 + D\varphi^3).
\]

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\[ G^1(x,(\text{adj}_2Du)^1) = G^1(x,(\text{adj}_2(Du+D\varphi))^1), \]

and the particular structure of \( f \) in (1.2), we obtain from (2.5) that

\[
\int_{A_{k,t}^1} \left\{ F^1(x,Du^1) + \sum_{\alpha=2}^{3} G^\alpha(x, (\text{adj}_2 Du)^\alpha) + H(x, \det Du) \right\} \, dx \\
\leq \int_{A_{k,t}^1} \left\{ F^1(x,Du^1 + D\varphi^1) + \sum_{\alpha=2}^{3} G^\alpha(x, (\text{adj}_2 (Du + D\varphi))^\alpha) \\
+ H(x, \det (Du + D\varphi)) + F_0\varphi \right\} \, dx. \tag{2.6} \]

Our goal next is to estimate each term in the above inequality. By the growth assumption (1.9),

\[
\int_{A_{k,t}^1} F^1(x,Du^1) \, dx \geq k_1 \int_{A_{k,t}^1} |Du^1|^p \, dx - k_2 |A_{k,t}^1|, \tag{2.7} \]

The equality (2.4) together with (1.9) allows us to estimate

\[
\int_{A_{k,t}^1} F^1(x,Du^1 + D\varphi^1) \, dx \\
\leq k_3 \int_{A_{k,t}^1} |(1-\eta^p)Du^1 + pm^{p-1}(k-u^1)D\eta|^p \, dx + \int_{A_{k,t}^1} a(x) \, dx \\
\leq 2^{p-1} k_3 \int_{A_{k,t}^1 \setminus A_{k,s}^1} |Du^1|^p \, dx + 2^{p-1} k_3 p^p \int_{A_{k,t}^1} \left( \frac{u^1-k}{t-s} \right)^p \, dx \\
+ \|a\|_{L^p(B_{R_0}(x_0))} |A_{k,t}^1|^{1-\frac{1}{\beta}}. \tag{2.8} \]

Define the norm \( |\xi| \) for a matrix \( \xi = \begin{pmatrix} \xi_i^j \end{pmatrix} \in \mathbb{R}^{3\times 3} \) by

\[
|\xi| = \sum_{i=1}^{3} |\xi_i^i| = \sum_{i,j=1}^{3} |\xi_i^j|. \]

Thus for every matrix \( \xi \in \mathbb{R}^{3\times 3}, \alpha \in \{1, 2, 3\}, \beta, \gamma = \{1, 2, 3\} \setminus \{\alpha\}, \beta < \gamma, \) one has

\[
|(\text{adj}_2 \xi)^\alpha| \leq |\xi^\beta||\xi^\gamma|. \]

We use these estimates, (1.10), Young inequality and Hölder inequality in order to derive

\[
\int_{A_{k,t}^1} \sum_{\alpha=2}^{3} G^\alpha(x, (\text{adj}_2 Du)^\alpha) \, dx \\
\geq -k_1 \int_{A_{k,t}^1} \sum_{\alpha=2}^{3} |(\text{adj}_2 Du)^\alpha|^q \, dx - 2k_2 |A_{k,t}^1| \\
\geq -k_1 \int_{A_{k,t}^1} |Du^1|^q (|Du^2|^q + |Du^3|^q) \, dx - 2k_2 |A_{k,t}^1| \tag{2.9} \]
\[
\geq -2k_1 \varepsilon \int_{A_{k,t}^1} |Du^1|^p \, dx - 2k_1 C_\varepsilon \int_{A_{k,t}^1} |Du|_{F^q}^p \, dx - 2k_2 |A_{k,t}^1| \\
\geq -2k_1 \varepsilon \int_{A_{k,t}^1} |Du^1|^p \, dx - 2k_1 C_\varepsilon \int_{A_{k,t}^1} |Du|_{F^q}^p \, dx - 2k_2 |A_{k,t}^1| \\
\geq -2k_1 \varepsilon \int_{A_{k,t}^1} |Du^1|^p \, dx - 2k_1 C_\varepsilon \int_{A_{k,t}^1} |Du|_{F^q}^p \, dx - 2k_2 |A_{k,t}^1|.
and

\[
\int_{A_{k,t}^1} \sum_{\alpha=2}^3 G^\alpha(x, (\text{adj}_2(Du + D\varphi))^{\alpha})\,dx \\
\leq k_3 \int_{A_{k,t}^1} \sum_{\alpha=2}^3 (\text{adj}_2(Du + D\varphi))^{\alpha}\,dx + \|b\|_{L^s(B_R(x_0))} |A_{k,t}^1|^{1-\frac{1}{s}} \\
\leq k_3 \int_{A_{k,t}^1} |Du^1 + D\varphi^1|^q((|Du^2|^q + |Du^3|^q))\,dx + \|b\|_{L^s(B_R(x_0))} |A_{k,t}^1|^{1-\frac{1}{s}} \\
\leq 2k_3 \varepsilon \int_{A_{k,t}^1} |(1 - \eta^p)Du^1 + np^{p-1}(k - u^1)Du^1|^p\,dx + 2k_3 C_\varepsilon \int_{A_{k,t}^1} |Du|^\frac{pq}{p-q}\,dx \\
+ \|b\|_{L^s(B_R(x_0))} |A_{k,t}^1|^{1-\frac{1}{s}} \\
\leq 2^p k_3 \varepsilon \int_{A_{k,t}^1 \setminus A_{k,t}^1} |Du^1|^p\,dx + 4^p k_3 \varepsilon \int_{A_{k,t}^1} \left(\frac{u^1 - k}{t - s}\right)^p\,dx \\
+ 2k_3 C_\varepsilon \left(\int_{A_{k,t}^1} |Du|^p\,dx\right)^\frac{2}{p-q} |A_{k,t}^1|^{\frac{p-2q}{p-q}} + \|b\|_{L^s(B_R(x_0))} |A_{k,t}^1|^{1-\frac{1}{s}} \\
\tag{2.10}
\end{align}

For every matrix \( \xi \in \mathbb{R}^{3 \times 3} \), we have \( |\det \xi| \leq |\xi_1^1||\xi_2^2||\xi_3^3| \leq |\xi|^3 \). By (1.11), using Young inequality and Hölder inequality again, one has

\[
\int_{A_{k,t}^1} H(x, \det Du)\,dx \\
\geq -k_1 \int_{A_{k,t}^1} |\det Du|^r\,dx - k_2 |A_{k,t}^1| \\
\geq -k_1 \int_{A_{k,t}^1} |Du^1|^r(|Du^2||Du^3|)^r\,dx - k_2 |A_{k,t}^1| \\
\geq -k_1 \varepsilon \int_{A_{k,t}^1} |Du^1|^p\,dx - k_1 C_\varepsilon \int_{A_{k,t}^1} (|Du^2||Du^3|)\frac{2r}{r}\,dx - k_2 |A_{k,t}^1| \\
\geq -k_1 \varepsilon \int_{A_{k,t}^1} |Du^1|^p\,dx - k_1 C_\varepsilon \left(\int_{A_{k,t}^1} (|Du^2||Du^3|)\frac{2r}{r}\,dx\right)^\frac{r}{2r} |A_{k,t}^1|^{\frac{2r-2r}{2r}} - k_2 |A_{k,t}^1| \\
\geq -k_1 \varepsilon \int_{A_{k,t}^1} |Du^1|^p\,dx - k_1 C_\varepsilon \|Du\|_{L^p(B_R(x_0))} |A_{k,t}^1|^{\frac{2r}{2r-r}} - k_2 |A_{k,t}^1| \\
\tag{2.11}
\end{align}

and

\[
\int_{A_{k,t}^1} H(x, \det(Du + D\varphi))\,dx \\
\leq k_3 \int_{A_{k,t}^1} |\det(Du + D\varphi)|^r\,dx + \|c\|_{L^s(B_R(x_0))} |A_{k,t}^1|^{1-\frac{1}{s}} \\
\leq k_3 \int_{A_{k,t}^1} |(1 - \eta^p)Du^1 + np^{p-1}(k - u^1)Du^1|^r |Du|^{2r}\,dx + \|c\|_{L^s(\Omega)} |A_{k,t}^1|^{1-\frac{1}{s}} 
\]
≤ k_3 \varepsilon \int_{A_{k,t}^1} |(1 - \eta^p)Du^1 + pm^{p-1}(k - u^1)Du|^p dx \\
+ k_3 C_\varepsilon \int_{A_{k,t}^1} |Du|^\frac{2\nu}{p+\nu} dx + ||c||_{L^\sigma(\Omega)} |A_{k,t}^1|^{1-\frac{1}{\sigma}}
\leq 2^{p-1} k_3 \varepsilon \int_{A_{k,t}^1 \setminus A_{k,s}^1} |Du|^p dx + 2^{p-1} k_3 \varepsilon \int_{A_{k,t}^1} \left(\frac{u^1 - k}{t - s}\right)^p dx \\
+ k_3 C_\varepsilon \int_{A_{k,t}^1 \setminus A_{k,s}^1} |Du|^\frac{2\nu}{p+\nu} dx + |A_{k,t}^1|^{p-3\nu} + ||c||_{L^\sigma(\Omega)} |A_{k,t}^1|^{1-\frac{1}{\sigma}}.
(2.12)

Since |B_{R_0}(x_0)| = \omega_3 R_0^3 < 1, then R_0 < 1, from which, and considering the fact s < t ≤ R_0, we have t - s < 1. We use this fact and Young inequality, then

\begin{align*}
\int_{A_{k,t}^1} F_0 dx & \leq \int_{A_{k,t}^1} |F_0|(u^1 - k) dx \\
& \leq \frac{1}{p} \int_{A_{k,t}^1} (u^1 - k)^p dx + \frac{1}{p'} \int_{A_{k,t}^1} |F_0|^p dx \\
& \leq \int_{A_{k,t}^1} \left(\frac{u^1 - k}{t - s}\right)^p dx + ||F_0||_{L^p(B_{R_0}(x_0))} |A_{k,t}^1|^{1-\frac{p'}{p}}.
(2.13)
\end{align*}

Substituting (2.7)-(2.13) into (2.6), taking \( \varepsilon = \frac{1}{4} \), we arrive at

\begin{align*}
& \frac{k_1}{4} \int_{A_{k,s}^1} |Du|^p dx \leq \frac{k_1}{4} \int_{A_{k,t}^1} |Du|^p dx \\
& \leq c_1 \int_{A_{k,s}^1} |Du|^p dx + c_2 \int_{A_{k,t}^1} \left(\frac{u^1 - k}{t - s}\right)^p dx \\
& + c_3 \left(|A_{k,s}^1| + |A_{k,t}^1|^{1-\frac{1}{\sigma}} + |A_{k,t}^1|^{p-2\nu} + |A_{k,t}^1|^{p-3\nu} + |A_{k,t}^1|^{p-3\nu} + |A_{k,t}^1|^{1-\frac{p'}{p}}\right),
\end{align*}

where \( c_1, c_2, c_3 \) are constants depending on the quantities \( p, q, r, k_1, k_2, k_3, ||a||_{L^\sigma(\Omega)}, ||b||_{L^\rho(B_{R_0}(x_0))}, ||c||_{L^\sigma(\Omega)}(B_{R_0}(x_0)), ||Du||_{L^p(B_{R_0}(x_0))} \) and \( ||F_0||_{L^p(B_{R_0}(x_0))} \). Next we use the technique of hole-filling, i.e., adding to both sides

\[ c_1 \int_{A_{k,s}^1} |Du|^p dx, \]

we arrive at

\begin{align*}
& \int_{A_{k,s}^1} |Du|^p dx \\
& \leq \frac{c_1}{c_1 + k_1/4} \int_{A_{k,t}^1} |Du|^p dx + \frac{c_2}{c_1 + k_1/4} \int_{A_{k,t}^1} \left(\frac{u^1 - k}{t - s}\right)^p dx \\
& + \frac{c_3}{c_1 + k_1/4} \left(|A_{k,s}^1| + |A_{k,t}^1|^{1-\frac{1}{\sigma}} + |A_{k,t}^1|^{p-2\nu} + |A_{k,t}^1|^{p-3\nu} + |A_{k,t}^1|^{1-\frac{p'}{p}}\right),
\end{align*}
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