ON SEQUENCE SPACES DEFINED BY ARITHMETIC FUNCTION AND HAUSDORFF MEASURE OF NON-COMPACTNESS

TAJA YAYING AND NIPEN SAIKIA

ABSTRACT. In this study, we construct an infinite matrix \( \mathcal{S} = (s_{nk}) \) defined by

\[ s_{nk} = \begin{cases} \frac{k}{S(n)}, & k \mid n, \\ 0, & k \nmid n \end{cases} \]

for all \( n, k = 1, 2, 3, \ldots \), where \( S(n) \) stands for the sum of the positive divisors of \( n \), and introduce sequence spaces \( \ell_p(\mathcal{S}), c_0(\mathcal{S}), c(\mathcal{S}) \) and \( \ell_\infty(\mathcal{S}) \) by employing the matrix \( \mathcal{S} \), where \( 0 < p < \infty \). We construct Schauder bases and compute \( \alpha-, \beta- \) and \( \gamma- \) duals of the newly constructed spaces. We state and prove characterization theorems related to matrix transformation from the spaces \( \ell_p(\mathcal{S}), c_0(\mathcal{S}), c(\mathcal{S}) \) and \( \ell_\infty(\mathcal{S}) \) to the spaces \( \ell_\infty, c, c_0 \) and \( \ell_1 \). Finally, we determine essential conditions for compactness of a matrix operator from the sequence space \( X \in \{ \ell_p(\mathcal{S}), c_0(\mathcal{S}), c(\mathcal{S}), \ell_\infty(\mathcal{S}) \} \) to anyone of the sequence spaces \( \ell_\infty, c, c_0 \) or \( \ell_1 \).

1. Introduction, Preliminaries and Notations

Let \( \omega \) be the set of all real- or complex-valued sequences. Any linear subset of \( \omega \) is called a sequence space. The sets \( \ell_p, \ell_\infty, c_0 \) and \( c \) of absolutely \( p \)-summable, bounded, null and convergent sequences, respectively, are some of the examples of classical sequence spaces, where \( 0 < p < \infty \). Throughout the text, we assume that \( p^{-1} + q^{-1} = 1 \) and \( 0 < p < \infty \), if not stated. Also \( cs = \{ f = (f_k) \in \omega : \lim_{n \to \infty} |\sum_{k=1}^n f_k| \text{ exists} \} \) and \( bs = \{ f = (f_k) \in \omega : \sup_n |\sum_{k=1}^n f_k| < \infty \} \). A Banach space \( X \) is a \( BK \)-space if each projection \( f \mapsto f_k \) on the \( k \)th coordinate is continuous. It is known that the sequence spaces \( \ell_\infty, c, c_0 \) and \( \ell_p \), \( 1 < p < \infty \), are \( BK \)-spaces equipped with the usual norms \( \| f \|_{\ell_\infty} = \sup_k |f_k| \) and \( \| f \|_{\ell_p} = \left( \sum_k |f_k|^p \right)^{1/p} \), respectively. Also, the sequence space \( \ell_p \), for \( 0 < p \leq 1 \), is a complete \( p \)-normed space under the \( p \)-norm \( \| f \|_{\ell_p} = \sum_k |f_k|^p \). Here and henceforth, for our convenience, we use the notations \( \sum_k \) and \( \sup_k \) instead of \( \sum_{k=0}^\infty \) and \( \sup_{k \in \mathbb{N}} \), respectively, where \( \mathbb{N} \) is the set of all non-negative integers.

Assume \( X \) and \( Y \) to be any two sequence spaces and \( H = (h_{nk}) \) is an infinite matrix of real or complex entries. Let \( H_n \) denote the \( n \)th row of the matrix \( H \). Define the sequence \( g = (g_n) \) by \( g_n = (Hf)_n = \sum_k H_{nk}f_k \) for any sequence \( f = (f_k) \in \omega \), where we presumed that infinite sum exists. Then, the sequence \( g \) is termed as the \( H \)-transform of the sequence \( f \). Further, the matrix \( H \) defines a Compact operator.
mapping from $X$ to $Y$ if the sequence $g \in Y$ for all $f \in X$. The notation $(X, Y)$ denotes the family of all matrices that map from $X$ to $Y$. The set $X_H := \{ f = (f_k) \in \omega : Hf \in X \}$ is a sequence space and is known as the domain of $H$ in $X$. Also, $H$ is known as a triangle if and only if $h_{nm} \neq 0$ and $h_{nk} = 0$ for all $n < k$. It is well known that if $X$ is a BK-space and $H$ is a triangle, then $X_H$ is also a BK-space equipped with the norm $\|f\|_H = \|Hf\|_X$.

1.1. Arithmetic functions. Let $n$ be any positive integer. Then, by the notations `$k \mid n$’ and `$k \nmid n$’, we mean $k$ is a divisor of $n$ and $k$ is not a divisor of $n$, respectively. An arithmetic function is a function from $\mathbb{N} \rightarrow \mathbb{R}$. Some of the well known arithmetic functions are

\[
S(n) = \sum_{k \mid n} k,
\]

\[
\mu(n) = \begin{cases} 
1, & n = 1, \\
(-1)^k, & n = p_1 p_2 \cdots p_k, \\
0, & n \text{ has a square prime factor},
\end{cases} \text{(Möbius function)}
\]

\[
\varphi(n) = n \sum_{k \mid n} \mu(k), \text{ (Euler’s totient function)}
\]

\[
J(n) = n \sum_{k \mid n} \mu(k), \text{ (Jordan’s totient function)}
\]

where $p_k$ denotes successive prime numbers. The arithmetic function $S(n)$ exhibits the following interesting properties:

(a) $S$ is a multiplicative function i.e. $S(mn) = S(m)S(n)$ whenever $m$ and $n$ are coprime.

(b) $(S(n), n)$ is a Möbius pair i.e.

\[
S(n) = \sum_{k \mid n} k \text{ if and only if } n = \sum_{k \mid n} \mu\left(\frac{n}{k}\right) S(k).
\]

(c) If $n = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$, then $S(n) = \frac{p_1^{r_1+1} - 1}{p_1 - 1} \cdot \frac{p_2^{r_2+1} - 1}{p_2 - 1} \cdots \frac{p_k^{r_k+1} - 1}{p_k - 1}$. In particular $S(n) = n + 1$ if $n$ is a prime number.

We refer to [28] for detailed studies concerning arithmetic functions.

1.2. Literature and motivation of the study. We provide a brief literature concerning sequence spaces derived by using arithmetic functions.

Ruckle [30] studied the sequence spaces $AC$ and $AS$ as the set of all sequences that are arithmetic convergent and arithmetic summable (cf. [30] for terminologies). The author mentioned that the sequences $f_n(k) = 1$ if $n \mid k$ and 0 otherwise, are arithmetic convergent but not convergent in the ordinary sense, intact periodic. Yaying and Hazarika further extended these sequence spaces to the multiplier sequence spaces [33] and lacunary statistical sequence spaces [34]. For relevant studies concerning theory of sequence spaces (including domains of triangular matrices in classical sequence spaces) and summability, the readers may refer to the monographs [6, 25], and the research papers
We are motivated by the aforementioned studies to construct arithmetic matrix operator $\mathcal{S}$ and study its domain in the spaces $\ell_p, c_0, c$ and $\ell_\infty$. We determine essential conditions for a matrix operator to be compact on the newly constructed matrix domains.

2. Sequence spaces $\ell_p(\mathcal{S}), c_0(\mathcal{S}), c(\mathcal{S})$ and $\ell_\infty(\mathcal{S})$

Define the matrix $\mathcal{S} = (s_{nk})_{n,k\in\mathbb{N}}$ by

$$s_{nk} = \begin{cases} \frac{k}{S(n)} & , \quad k \mid n, \\ 0 & , \quad k \nmid n, \end{cases}$$
which can also be expressed explicitly as

\[
\mathcal{S} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
1 & 2 & 0 & 0 & 0 & 0 & \cdots \\
1 & 3 & 0 & 0 & 0 & 0 & \cdots \\
1 & 4 & 0 & 4 & 0 & 0 & \cdots \\
1 & 5 & 0 & 0 & 5 & 0 & \cdots \\
1 & 6 & 0 & 0 & 0 & 6 & \cdots \\
1 & 7 & 0 & 0 & 0 & 0 & 7 & \cdots \\
1 & 8 & 0 & 0 & 0 & 0 & 0 & 8 & \cdots \\
1 & 9 & 0 & 0 & 0 & 0 & 0 & 0 & 9 & \cdots \\
1 & 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10 & \cdots \\
1 & 11 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 11 & \cdots \\
1 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{bmatrix}.
\]

It is noted that \( \mathcal{S} \) is a triangle and so its inverse exists and is unique. By using the property (1.1), the inverse \( \mathcal{S}^{-1} := \mathcal{T} = (t_{nk}) \) of the matrix \( \mathcal{S} \) is defined by

\[
t_{nk} = \begin{cases}
\frac{\mu\left(\frac{n}{k}\right) S(k)}{n}, & k \mid n, \\
0, & k \nmid n.
\end{cases}
\]

The sequence \( g = (g_n) \) is defined as the \( \mathcal{S} \)-transform of a sequence \( f = (f_k) \in \omega \), that is,

\[
g_n = (\mathcal{S}f)_n = \sum_{k\mid n} \frac{k}{S(n)} f_k
\]

for all \( n \in \mathbb{N} \). The equality (2.1) can also be expressed in terms of the sequence \( g \) as follows:

\[
f_n = (\mathcal{T}g)_n = \sum_{k\mid n} \frac{\mu\left(\frac{n}{k}\right) S(k)}{n} g_k
\]

for all \( n \in \mathbb{N} \). In what follows, the sequences \( f \) and \( g \) are connected by (2.1) or equivalently by (2.2).

Now, we define the sequence spaces \( \ell_p(\mathcal{S}) \), \( c_0(\mathcal{S}) \), \( c(\mathcal{S}) \) and \( \ell_\infty(\mathcal{S}) \), as follows:

\[
\ell_p(\mathcal{S}) := \{ f = (f_k) \in \omega : g = \mathcal{S}f \in \ell_p \},
\]

\[
c_0(\mathcal{S}) := \{ f = (f_k) \in \omega : g = \mathcal{S}f \in c_0 \},
\]

\[
c(\mathcal{S}) := \{ f = (f_k) \in \omega : g = \mathcal{S}f \in c \},
\]

\[
\ell_\infty(\mathcal{S}) := \{ f = (f_k) \in \omega : g = \mathcal{S}f \in \ell_\infty \}.
\]

It is trivial that the sequence spaces \( X(\mathcal{S}) \) can be redefined as \( X(\mathcal{S}) = X_\mathcal{S} \), where \( X \) represents any one of the sequence spaces \( \ell_p, c_0, c \) or \( \ell_\infty \). In other words, \( X(\mathcal{S}) \) is the domain of \( \mathcal{S} \) in \( X \).

The following result is given without proof being a routine exercise.

**Theorem 2.1.** We have the following statements:
(a) Let $X$ be anyone of the spaces $c_0(\mathcal{G})$, $c(\mathcal{G})$ or $\ell_\infty(\mathcal{G})$. Then, $X$ is a BK-space endowed with the supremum norm defined by
\[ \|f\|_X = \|\mathcal{G}f\|_{\ell_\infty} = \sup_n \left| \sum_{k \mid n} \frac{k}{S(n)} f_k \right|. \]

(b) For $0 < p \leq 1$, $\ell_p(\mathcal{G})$ is a complete $p$-normed space under the $p$-norm.
\[ \|f\|_{\ell_p(\mathcal{G})} = \left( \sum_n \left| \sum_{k \mid n} \frac{k}{S(n)} f_k \right|^p \right)^{1/p} < \infty. \]

(c) For $1 < p < \infty$, the space $\ell_p(\mathcal{G})$ is BK-space endowed with the norm
\[ \|f\|_{\ell_p(\mathcal{G})} = \left[ \sum_n \left| \sum_{k \mid n} \frac{k}{S(n)} f_k \right|^p \right]^{1/p} < \infty. \]

**Theorem 2.2.** Let $X$ be anyone of the sequence spaces $\ell_p$, $c_0$, $c$ or $\ell_\infty$. Then, $X(\mathcal{G}) \cong X$.

**Proof.** Let the sequences $f = (f_k)$ and $g = (g_k)$ be connected by the relation (2.1). It is known that the matrix $\mathcal{G}$ is a triangle and so is invertible. This immediately leads to the fact that the mapping $T$ defined by
\[ T : X(\mathcal{G}) \rightarrow X \]
\[ f \mapsto T f = y = \mathcal{G} f. \]

is a norm (or $p$-norm) preserving linear bijection. Therefore $X(\mathcal{G})$ is linearly isomorphic to the space $X$, that is, $\ell_p(\mathcal{G}) \cong \ell_p$, $c_0(\mathcal{G}) \cong c_0$, $c(\mathcal{G}) \cong c$ and $\ell_\infty(\mathcal{G}) \cong \ell_\infty$. □

**Theorem 2.3.** Assume that $1 \leq p < \infty$. Then, $\ell_p(\mathcal{G})$ is not a Hilbert space for $p \neq 2$.

**Proof.** Consider the sequences $x = (x_k)$ and $y = (y_k)$ defined by
\[ x_n = \begin{cases} \frac{1}{n} \left( \mu(n) + 3 \mu\left(\frac{n}{2}\right) \right) & \text{if } n \text{ is even} \\ \frac{\mu(n)}{n} & \text{if } n \text{ is odd,} \end{cases} \]
\[ y_n = \begin{cases} \frac{1}{n} \left( \mu(n) - 3 \mu\left(\frac{n}{2}\right) \right) & \text{if } n \text{ is even} \\ \frac{\mu(n)}{n} & \text{if } n \text{ is odd,} \end{cases} \]

for all $n \in \mathbb{N}$. Then, $\mathcal{G} x = (1, 1, 0, 0, \ldots) \in \ell_p$ and $\mathcal{G} y = (1, -1, 0, 0, \ldots) \in \ell_p$. Moreover
\[ \|x + y\|_{\ell_p(\mathcal{G})}^2 + \|x - y\|_{\ell_p(\mathcal{G})}^2 \neq 2 \left( \|x\|_{\ell_p(\mathcal{G})}^2 + \|y\|_{\ell_p(\mathcal{G})}^2 \right). \]

Thus parallelogram law does not hold. This completes the proof. □

Recalling the definition of the matrix $\mathcal{G} = (s_{nk})$, we emphasize that $\sum_{k=0}^n s_{nk} = 1$ for each $n \in \mathbb{N}$. That is $\sup_{n \in \mathbb{N}} \sum_{k=0}^n s_{nk} < \infty$ and $\lim_{n \rightarrow \infty} \sum_{k} s_{nk} < \infty$. Moreover $\lim_{n \rightarrow \infty} s_{nk} = 0$ for each $k \in \mathbb{N}$. This implies that $\mathcal{G}$ is a regular matrix. This immediately allows us to write the following result:

**Theorem 2.4.** The inclusion $X \subset X(\mathcal{G})$, where $X$ denotes anyone of the sequence spaces $\ell_p$, $c_0$, $c$ or $\ell_\infty$.

**Proof.** The result follows from the aforementioned statements. □
Theorem 2.5. The inclusion $\ell_p(\mathcal{S}) \subset c_0(\mathcal{S}) \subset c(\mathcal{S}) \subset \ell_\infty(\mathcal{S})$ strictly holds.

Proof. It is known that the matrix $\mathcal{S}$ is regular and the inclusion $\ell_p \subset c_0 \subset c \subset \ell_\infty$ holds. Therefore, the inclusion part is obvious.

Since the inclusion $\ell_p \subset c_0$ is strict, we consider a sequence $y = (y_k) \in c_0 \setminus \ell_p$. Now we define a sequence $x = (x_k)$ by $x_k = \sum_{j \mid k} \mu(\frac{j}{k})S(j) y_j$ for all $k \in \mathbb{N}$. Then, $\mathcal{S}x = y \in c_0 \setminus \ell_p$. This implies the fact that $x \in c_0(\mathcal{S}) \setminus \ell_p(\mathcal{S})$.

In the similar manner strictness of the inclusion $c_0(\mathcal{S}) \subset c(\mathcal{S}) \subset \ell_\infty(\mathcal{S})$ can be established. □

Theorem 2.6. The inclusion $\ell_p(\mathcal{S}) \subset \ell_q(\mathcal{S})$ strictly holds, where $1 \leq p < q < \infty$.

Proof. This is obtained by the similar technique used in the proof of Theorem 2.5. □

Now, we construct Schauder basis for the sequence spaces $\ell_p(\mathcal{S})$, $c_0(\mathcal{S})$ and $c(\mathcal{S})$.

A sequence $f = (f_k)$ of a normed space $(X, \|\cdot\|)$ is called a Schauder basis if there exists a unique sequence of scalars $(\alpha_k)$ for every $z \in X$ such that $\|z - \sum_{k=0}^{n} \alpha_k f_k\| \to 0$, as $n \to \infty$.

We refer to Theorem 2.3 of Jarrah and Malkowsky [20] which states that the domain $X_A$ of a triangle $A$ in a normed space $X$ has a basis if and only if $X$ has a basis. As an immediate consequence of this fact, we conclude that the inverse image of the basis of spaces $\ell_p$, $c_0$ or $c$ form the basis of the spaces $\ell_p(\mathcal{S})$, $c_0(\mathcal{S})$ or $c(\mathcal{S})$, respectively. This allows us to state the following result:

Corollary 2.7. Define the sequence $b^{(k)} = (b^{(k)}_n)$, $k \in \mathbb{N}$, by

\[
(2.3) \quad b^{(k)}_n = \begin{cases} \frac{1}{n} \mu(\frac{n}{k})S(k), & k \mid n, \\ 0, & k \nmid n. \end{cases}
\]

Then

(a) the sequence $b^{(k)}$ is a Schauder basis for $\ell_p(\mathcal{S})$ and $c_0(\mathcal{S})$, and every $f \in \ell_p(\mathcal{S})$ or $f \in c_0(\mathcal{S})$ is uniquely represented by $f = \sum_k g_k b^{(k)}$, where $g_k = (\mathcal{S}f)_k$ for each $k \in \mathbb{N}$.

(b) the set $\{e, b^{(k)}\}$ is a Schauder basis for $c(\mathcal{S})$ and every $f \in c(\mathcal{S})$ is uniquely represented by $f = le + \sum_k (g_k - l) b^{(k)}$, where $g_k \to l$, as $k \to \infty$ and $e = (1, 1, 1, \ldots)$.

(c) the space $\ell_\infty(\mathcal{S})$ has no Schauder basis.

Corollary 2.8. The sequence spaces $\ell_p(\mathcal{S})$, $c_0(\mathcal{S})$ and $c(\mathcal{S})$ are separable.

3. Alpha-, Beta- and Gamma-duals

In this section, we compute $\alpha$, $\beta$- and $\gamma$-duals of the spaces $\ell_p(\mathcal{S})$, $c_0(\mathcal{S})$, $c(\mathcal{S})$ and $\ell_\infty(\mathcal{S})$.

For $X, Y \in \omega$, the set $M(X, Y)$ defined by

$$M(X, Y) := \{d = (d_k) \in \omega : df = (d_kf_k) \in Y \text{ for all } f = (f_k) \in X\}$$
The following lemmas are necessary for our examinations. Throughout \( \mathcal{N} = \{ N \subset \mathbb{N} : N \text{ is finite} \} \).

**Lemma 3.1.** [31] The following results are well known:

(i) \( H = (h_{nk}) \in (c_0, \ell_1) = (c, \ell_1) = (\ell_\infty, \ell_1) \) if and only if

\[
\sup_{N \in \mathcal{N}} \sum_{k=0}^{\infty} \sum_{n \in N} |h_{nk}| < \infty.
\]

(ii) \( H = (h_{nk}) \in (c_0, c) \) if and only if

\[
\exists \alpha_k \in \mathbb{C} \ni h_{nk} = \alpha_k \text{ for each } k \in \mathbb{N},
\]

(iii) \( H = (h_{nk}) \in (c, c) \) if and only if (3.2) and (3.3) hold, and

\[
\sup_{n \in \mathbb{N}} \sum_{k=0}^{\infty} |h_{nk}| < \infty.
\]

(iv) \( H = (h_{nk}) \in (\ell_\infty, c) \) if and only if (3.2) holds, and

\[
\lim_{n \to \infty} \sum_{k=0}^{\infty} |h_{nk}| = \sum_{k=0}^{\infty} \lim_{n \to \infty} |h_{nk}|.
\]

(v) \( H = (h_{nk}) \in (c_0, \ell_\infty) = (c, \ell_\infty) = (\ell_\infty, \ell_\infty) \) if and only if (3.3) holds.

(vi) Assume that \( 1 < p < \infty \). Then, \( H = (h_{nk}) \in (\ell_p, \ell_\infty) \) if and only if

\[
\sup_{n \in \mathbb{N}} \sum_{k=0}^{\infty} |h_{nk}|^q < \infty.
\]

(vii) Assume that \( 1 < p < \infty \). Then, \( H = (h_{nk}) \in (\ell_p, c) \) if and only if (3.2) and (3.6) hold.

**Lemma 3.2.** The following results are well known:

(i) [15, Theorem 5.1.0] with \( p_k = p \) for all \( k \) \( H = (h_{nk}) \in (\ell_p, \ell_1) \) if and only if

\[
\sup_{N \in \mathcal{N}} \sup_{k \in \mathbb{N}} \left| \sum_{n \in N} h_{nk} \right|^p < \infty, \quad (0 < p \leq 1).
\]

(ii) [23, Theorem 1 (i)] with \( p_k = p \) for all \( k \) \( H = (h_{nk}) \in (\ell_p, \ell_\infty) \) if and only if

\[
\sup_{n, k \in \mathbb{N}} |h_{nk}|^p < \infty, \quad (0 < p \leq 1).
\]
(iii) [23, Corollary for Theorem 1 with \( p_k = p \) for all \( k \)] \( H = (h_{nk}) \in (\ell_p, c) \) if and only if (3.2) and (3.9) hold.

**Theorem 3.3.** Consider the following sets

\[
D_1 := \left\{ d = (d_k) \in \omega : \sup_{N \in \mathbb{N}, k \in \mathbb{N}} \left| \sum_{n \in N, k \in \mathbb{N}} \frac{1}{n} \mu \left( \frac{n}{k} \right) S(k) d_n \right| ^p < \infty \right\}, \quad (0 < p \leq 1),
\]

\[
D_2 := \left\{ d = (d_k) \in \omega : \sup_{N \in \mathbb{N}, k \in \mathbb{N}} \left| \sum_{n \in N, k \in \mathbb{N}} \frac{1}{n} \mu \left( \frac{n}{k} \right) S(k) d_n \right| ^q < \infty \right\}, \quad (1 < p < \infty),
\]

\[
D_3 := \left\{ d = (d_k) \in \omega : \sup_{N \in \mathbb{N}, k \in \mathbb{N}} \left| \sum_{n \in N, k \in \mathbb{N}} \frac{1}{n} \mu \left( \frac{n}{k} \right) S(k) d_n \right| < \infty \right\}.
\]

Then,

(i) \([\ell_p(\mathcal{G})]^\alpha = \left\{ D_1 \right\}, \quad 0 < p \leq 1,
\]

(ii) \([c_0(\mathcal{G})]^\alpha = [c(\mathcal{G})]^\alpha = [\ell_{\infty}(\mathcal{G})]^\alpha = D_3.

**Proof.** For any arbitrary sequence \( d = (d_k) \in \omega \), consider the matrix \( \Theta = (\theta_{nk}) \) defined for all \( n, k \in \mathbb{N} \) by

\[
(3.10) \quad \theta_{nk} = \begin{cases} 
\frac{1}{n} \mu \left( \frac{n}{k} \right) S(k) d_n, & k \mid n, \\
0, & k \nmid n. 
\end{cases}
\]

Let \( d = (d_k) \in \omega \). Then, we get the following equality:

\[
d_n f_n = \sum_{k \in \mathbb{N}} \frac{1}{n} \mu \left( \frac{n}{k} \right) S(k) g_k d_n = (\Theta g)_n, n \in \mathbb{N},
\]

where the sequence \( g = (g_k) \) is the \( \mathcal{G} \)-transform of the sequence \( f = (f_k) \). We observe that \( df = (d_n f_n) \in \ell_1 \) whenever \( f \in \ell_p(\mathcal{G}) \) if and only if \( \Theta g \in \ell_1 \) whenever \( g \in \ell_p \) which yields the fact that \( d = (d_k) \in [\ell_p(\mathcal{G})]^\alpha \) if and only if \( \Theta \in (\ell_p, \ell_1) \). Thus by employing Part (i) of Lemma 3.2, we deduce that

\[
[\ell_p(\mathcal{G})]^\alpha = \left\{ D_1 \right\}, \quad 0 < p \leq 1,
\]

\[
D_2, \quad 1 < p < \infty.
\]

The proof of the Part (ii) is given in the similar way by using Part (i) of Lemma 3.1 instead of Part (i) of Lemma 3.2 in the aforementioned statements. Hence, to avoid unnecessary repetition of the similar statements we omit details. \(\square\)
Theorem 3.4. Consider the following sets

\[ D_4 := \left\{ d = (d_k) \in \omega : \lim_{n \to \infty} \sum_{l=1}^{n} \frac{1}{l} \mu\left(\frac{l}{k}\right) S(k) d_l \text{ exists} \right\}, \]

\[ D_5 := \left\{ d = (d_k) \in \omega : \sup_{n,k} \left| \sum_{l=k}^{n} \frac{1}{l} \mu\left(\frac{l}{k}\right) S(k) d_l \right|^p < \infty \right\}, \quad (0 < p \leq 1), \]

\[ D_6 := \left\{ d = (d_k) \in \omega : \sum_{n,k} \left| \sum_{l=k}^{n} \frac{1}{l} \mu\left(\frac{l}{k}\right) S(k) d_l \right|^q < \infty \right\}, \]

\[ D_7 := \left\{ d = (d_k) \in \omega : \lim_{n \to \infty} \sum_{l=k}^{n} \frac{1}{l} \mu\left(\frac{l}{k}\right) S(k) d_l \text{ exists} \right\}, \]

\[ D_8 := \left\{ d = (d_k) \in \omega : \lim_{n \to \infty} \left| \sum_{l=k}^{n} \frac{1}{l} \mu\left(\frac{l}{k}\right) S(k) d_l \right| \text{ exists} \right\}. \]

Then,

(i) \( [\ell_p(\mathcal{G})]^\beta = \begin{cases} D_4 \cap D_5, & 0 < p \leq 1, \\ D_4 \cap D_6, & 1 < p < \infty. \end{cases} \)

(ii) \( [c_0(\mathcal{G})]^\beta = D_3 \cap D_6 \text{ with } q = 1. \)

(iii) \( [c(\mathcal{G})]^\beta = D_3 \cap D_6 \cap D_7 \text{ with } q = 1. \)

(iv) \( [\ell_\infty(\mathcal{G})]^\beta = D_3 \cap D_8. \)

Proof. For any arbitrary sequence \( d = (d_k) \in \omega \), define the matrix \( \Psi = (\psi_{nk}) \) for all \( n,k \in \mathbb{N} \) by

\[ \psi_{nk} = \begin{cases} \sum_{l=k}^{n} \frac{1}{l} \mu\left(\frac{l}{k}\right) S(k) d_l, & 1 \leq k \leq n, \\ 0, & k > n. \end{cases} \]

Then, we have the following equality:

\[ \sum_{k=1}^{n} d_k f_k = \sum_{k=1}^{n} \sum_{l=k}^{n} \frac{1}{l} \mu\left(\frac{l}{k}\right) S(l) g_l d_k = \sum_{k=1}^{n} \sum_{l=k}^{n} \frac{1}{l} \mu\left(\frac{l}{k}\right) S(k) d_l g_k = (\Psi g)_n, \]

where the sequence \( g = (g_k) \) is the \( \mathcal{G} \)-transform of the sequence \( f = (f_k) \). We observe that \( \left( \sum_{k=1}^{n} d_k f_k \right) \) converges whenever \( f \in \ell_p(\mathcal{G}) \) if and only if \( \Psi g \in c \) whenever \( g \in \ell_p \). This yields the fact that \( d = (d_k) \in [\ell_p(\mathcal{G})]^\beta \) if and only if \( \Psi \in (\ell_p, c) \). Thus by engaging Part (vii) of Lemma 3.1 and Part (iii)
of Lemma 3.2, we conclude that

$$[l_p(\mathcal{S})]^\alpha = \begin{cases} D_4 \cap D_5 & 0 < p \leq 1, \\ D_4 \cap D_6 & 1 < p < \infty. \end{cases}$$

In the similar way the $\beta$-dual of the sequence spaces $c_0(\mathcal{S})$, $c(\mathcal{S})$ and $\ell_\infty(\mathcal{S})$ can be determined by engaging Parts (ii), (iii) and (iv) of Lemma 3.1, respectively, instead of Part (vii) of Lemma 3.1 and Part (iii) of Lemma 3.2 in the aforementioned statements. We omit details so as to avoid repetition of the similar statements.

**Theorem 3.5.** We have

(i) $l_p(\mathcal{S})]^\gamma = \begin{cases} D_5 & 0 < p \leq 1, \\ D_6 & 1 < p < \infty. \end{cases}$

(ii) $[c_0(\mathcal{S})]^\gamma = [c(\mathcal{S})]^\gamma = [\ell_\infty(\mathcal{S})]^\gamma = D_6$ with $q = 1$.

**Proof.** This is similar to the proof of Theorem 3.4 except that Part (ii) of Lemma 3.2 and Part (vi) of Lemma 3.1 are engaged instead of Part (iii) of Lemma 3.2 and Part (vii) of Lemma 3.1, respectively, in proving the $\gamma$-dual of the sequence space $l_p(\mathcal{S})$. For computing the $\gamma$-dual of the sequence spaces $c_0(\mathcal{S})$, $c(\mathcal{S})$ and $\ell_\infty(\mathcal{S})$, we engage Part (v) of Lemma 3.1 instead of Parts (ii), (iii) and (iv) of Lemma 3.1, respectively. We omit details to avoid repetition of the similar statements.

### 4. Matrix Transformations

In this section, we characterize the classes of matrix transformations from the space $X(\mathcal{S})$ to any arbitrary sequence space $Y$, where $X$ is any one of the sequence spaces $\ell_p$, $c_0$, $c$ and $\ell_\infty$. As a direct consequence of the main result, we give characterizations of certain classes of infinite matrices from $X(\mathcal{S})$ to $\ell_1$, $c_0$, $c$ and $\ell_\infty$.

The following result is the base for our examinations which is an immediate consequence of Theorem 4.1 in [22].

**Theorem 4.1.** Let $X$ be any one of the spaces $\ell_p$, $c_0$, $c$ or $\ell_\infty$. Then, $H = (h_{nk}) \in (X(\mathcal{S}), Y)$ if and only if $Z^{(n)} = (z^{(n)}_{lk}) \in (X, c)$ for all $n \in \mathbb{N}$ and $Z = (z_{lk}) \in (X, Y)$, where

$$z^{(n)}_{lk} = \begin{cases} \sum_{v=k}^{l} \frac{\mu(v)}{v} S(k) h_{nv} & 1 \leq k \leq l, \\ 0 & k > l, \end{cases}$$

and $z_{nk} = \sum_{v=k}^{\infty} \frac{\mu(v)}{v} S(k) h_{nv}$

for all $k, l, n = 1, 2, 3, \cdots$.

**Proof.** Assume that $H \in (X(\mathcal{S}), Y)$ and $f \in X(\mathcal{S})$. Then, following the similar technique as given in the proof of Theorem 3.4, we deduce the following equality

$$\sum_{k=1}^{l} h_{nk} f_k = \sum_{k=1}^{l} \sum_{v|k} \frac{1}{v} \mu(k) S(v) g_v h_{nk} = \sum_{k=1}^{l} \sum_{v|k} \frac{1}{v} \mu(k) S(k) h_{nv} g_k = \sum_{k=1}^{l} z^{(n)}_{lk} g_k$$

(4.1)
The following result is an immediate consequence of Lemma 4.3 that allows us to characterize certain
matrix classes from the sequence space $X \in \{ \ell_1, c_0, c, \ell_\infty \}$ to the sequence space $Y \in \{ \ell_p(\mathfrak{S}), c_0(\mathfrak{S}), c(\mathfrak{S}), \ell_\infty(\mathfrak{S}) \}$.

**Lemma 4.4.** Assume that $1 < p < \infty$. Then, the necessary and sufficient condition that a matrix
$H = (h_{nk}) \in (X,Y)$, where $X \in \{ \ell_1(\mathfrak{S}), \ell_p(\mathfrak{S}), c_0(\mathfrak{S}), c(\mathfrak{S}), \ell_\infty(\mathfrak{S}) \}$ and $Y \in \{ \ell_1, c_0, c, \ell_\infty \}$ can be
read from Table 1, where

<table>
<thead>
<tr>
<th>A. (4.2) and (4.3)</th>
<th>B. (4.2) and (4.4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>C. (4.2) and (4.4) with $q = 1$</td>
<td>D. (4.2), (4.4) with $q = 1$ and (4.5)</td>
</tr>
<tr>
<td>E. (4.2) and (4.6)</td>
<td>F. (3.1) with $z_{nk}$ instead of $h_{nk}$</td>
</tr>
<tr>
<td>G. (3.2) with $\alpha_k = 0$ for all $k$ and $z_{nk}$ instead of $h_{nk}$</td>
<td>H. (3.2) with $z_{nk}$ instead of $h_{nk}$</td>
</tr>
<tr>
<td>I. (3.3) with $z_{nk}$ instead of $h_{nk}$</td>
<td>J. (3.4) with $\beta_k = 0$ for all $k$ and $z_{nk}$ instead of $h_{nk}$</td>
</tr>
<tr>
<td>K. (3.4) with $z_{nk}$ instead of $h_{nk}$</td>
<td>L. (3.5) with $z_{nk}$ instead of $h_{nk}$</td>
</tr>
<tr>
<td>M. (3.6) with $z_{nk}$ instead of $h_{nk}$</td>
<td>N. (3.7) with $p = 1$ and $z_{nk}$ instead of $h_{nk}$</td>
</tr>
<tr>
<td>O. (3.8) with $z_{nk}$ instead of $h_{nk}$</td>
<td>P. (3.9) with $p = 1$ and $z_{nk}$ instead of $h_{nk}$</td>
</tr>
</tbody>
</table>
We need the following list of conditions, identified by $A'$, $B'$, and $C'$, respectively, for our next result:

(4.7) \[ A' := \sup_n \sum_k |\bar{s}_{nk}|^p < \infty; \]

(4.8) \[ B' := \sup_n \left( \sum_{k \in K} |\bar{s}_{nk}|^p \right) < \infty; \]

(4.9) \[ C' := \lim_{n \to \infty} \sum_k |\bar{s}_{nk}| = 0. \]

**Theorem 4.4.** Let $X \in \{\ell_1, c_0, c, \ell_\infty\}$, $Y \in \{\ell_p, c_0, c, \ell_\infty\}$ and $H$ be an infinite matrix. Define the matrix $\tilde{H} = (\tilde{s}_{nk})$ by

\[ \tilde{s}_{nk} = \sum_{v \in S(n)}^v h_{nk}, \quad n, k = 1, 2, 3, \ldots. \]

Then, $H \in (X, Y(\tilde{H}))$ if and only if $\tilde{H} \in (X, Y)$.

**Proof.** It is evident from Lemma 4.3. \[ \square \]

We need the following list of conditions, identified by $A'$, $B'$, and $C'$, respectively, for our next result:

(4.7) \[ A' := \sup_n \sum_k |\bar{s}_{nk}|^p < \infty; \]

(4.8) \[ B' := \sup_n \left( \sum_{k \in K} |\bar{s}_{nk}|^p \right) < \infty; \]

(4.9) \[ C' := \lim_{n \to \infty} \sum_k |\bar{s}_{nk}| = 0. \]

**Lemma 4.5.** Assume that $1 < p < \infty$. Then, the necessary and sufficient condition that a matrix $H = (h_{nk}) \in (X, Y)$, where $X \in \{\ell_1, c_0, c, \ell_\infty\}$ and $Y \in \{\ell_p, c_0, c, \ell_\infty\}$ can be read from Table 2, where the conditions $A$-L and $P$ listed as in Lemma 4.2 are taken into consideration except that the statement ‘with $z_{nk}$ instead of $h_{nk}$’ is replaced by ‘with $\bar{s}_{nk}$ instead of $h_{nk}$’.

<table>
<thead>
<tr>
<th>From \ To</th>
<th>$\ell_1$</th>
<th>$c_0$</th>
<th>$c$</th>
<th>$\ell_\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\ell_1(\mathcal{S})$</td>
<td>A &amp; N</td>
<td>A, G &amp; P</td>
<td>A, H &amp; P</td>
<td>A &amp; P</td>
</tr>
<tr>
<td>$\ell_p(\mathcal{S})$</td>
<td>B &amp; O</td>
<td>B, G &amp; M</td>
<td>B, H &amp; M</td>
<td>B &amp; M</td>
</tr>
<tr>
<td>$c_0(\mathcal{S})$</td>
<td>C &amp; F</td>
<td>C, G &amp; I</td>
<td>C, H &amp; I</td>
<td>C &amp; I</td>
</tr>
<tr>
<td>$c(\mathcal{S})$</td>
<td>D &amp; F</td>
<td>D, G, I &amp; J</td>
<td>D, H, I &amp; K</td>
<td>D &amp; I</td>
</tr>
<tr>
<td>$\ell_\infty(\mathcal{S})$</td>
<td>E &amp; F</td>
<td>E &amp; J</td>
<td>E, H &amp; L</td>
<td>E &amp; I</td>
</tr>
</tbody>
</table>

**Table 2.** Characterization of the matrix class $(X, Y)$, where $X \in \{\ell_1, c_0, c, \ell_\infty\}$ and $Y \in \{\ell_p(\mathcal{S}), c_0(\mathcal{S}), c(\mathcal{S}), \ell_\infty(\mathcal{S})\}$
Now, we characterize matrix classes from the sequence space $X \in \{ \ell_1(\mathbb{S}), \ell_p(\mathbb{S}), c_0(\mathbb{S}), c(\mathbb{S}), \ell_{\infty}(\mathbb{S}) \}$ to some of the recently studied sequence spaces in the literature as another important consequence of Lemma 4.3.

**Corollary 4.6.** Take $M = \alpha$, $0 < \gamma < 1$, (cf. [13]) and define the matrix $C := \alpha H = (c_{nk})$ by

$$
c_{nk} = \frac{\gamma^n}{n!} h_{vk}, \quad n, k = 1, 2, 3, \ldots.
$$

Then, the characterization of the matrix classes $(X, \mathcal{X}_0^\alpha(\gamma))$, $(X, \mathcal{X}_c^\alpha(\gamma))$ and $(X, \mathcal{X}_\infty^\alpha(\gamma))$ can be determined from Lemma 4.2 by replacing the statement ‘with $z_{nk}$ by $h_{nk}$’ with ‘with $c_{nk}$ by $h_{nk}$’ in Table 1, where $X \in \{ \ell_1(\mathbb{S}), \ell_p(\mathbb{S}), c_0(\mathbb{S}), c(\mathbb{S}), \ell_{\infty}(\mathbb{S}) \}$ and $\mathcal{X}_0^\alpha$, $\mathcal{X}_c^\alpha$ and $\mathcal{X}_\infty^\alpha$ are $\gamma$-Cesàro sequence spaces studied by Demiriz and Şahin [13] and Yaying et al. [39].

**Corollary 4.7.** Take $M = \Phi$ (cf. [17]) and define the matrix $\Phi := \Phi H = (\phi_{nk})$ by

$$
\phi_{nk} = \frac{\phi(n)}{n} h_{vk}, \quad n, k = 1, 2, 3, \ldots.
$$

Then, the characterization of the matrix classes $(X, \mathcal{X}_0^\Phi)$, $(X, \mathcal{X}_c^\Phi)$ and $(X, \ell_{\infty}^\Phi)$ can be determined from Lemma 4.2 by replacing the statement ‘with $z_{nk}$ by $h_{nk}$’ with ‘with $\phi_{nk}$ by $h_{nk}$’ in Table 1, where $X \in \{ \ell_1(\mathbb{S}), \ell_p(\mathbb{S}), c_0(\mathbb{S}), c(\mathbb{S}), \ell_{\infty}(\mathbb{S}) \}$ and $\mathcal{X}_0^\Phi$, $\mathcal{X}_c^\Phi$ and $\ell_{\infty}^\Phi$ are sequence spaces defined by Euler totient function $\phi$ studied by İlkhan et al. [16, 17].

5. Compactness of operators via $H_{mc}$

Throughout this section, $H_{mc}$ is the abbreviation for Hausdorff measure of non-compactness. We use the following notations in the sequel:

- $\mathcal{B}(X) := \text{Unit ball in } X$;
- $\mathcal{B}(f, r) := \text{Unit ball with centre } f \text{ and radius } r$;
- $B(X, Y) := \{ T : T : X \to Y \text{ is bounded and linear}; X \text{ and } Y \text{ are Banach spaces} \}$;
- $D(X) := \text{Domain of } X$;

\begin{equation}
\|v\|_{X}^\mathcal{B} := \sup_{f \in \mathcal{B}(X)} \left| \sum_{k} v_k f_k \right|, \quad v = (v_k) \in \omega;
\end{equation}

- $\sigma := \{ A \subset \omega : A \text{ is finite and ends in zeros} \}$;
- $\mathcal{N} := \{ B : B \subset \mathbb{N} \}$;
- $\mathcal{N}_r := \{ B \subset \mathcal{N} : \text{elements of } B \text{ are greater } r \}$,

where we presumed that the series on the right hand side of (5.1) exists. It is noted that $v \in X^\beta$. We note that $\|\cdot\| = \sup_{f \in \mathcal{B}(X)} \|\Theta f\|$ is norm on the Banach space $B(X, Y)$.

An operator $\Theta$ is called compact if $D(X) = X$ and the sequence $(\Theta f_k)$ has a convergent subsequence in $Y$, for every bounded sequence $f = (f_k)$ in $X$. 

We recall certain well known results in the literature that are necessary for our investigation:

Lemma 5.1. Assume that $1 < p < \infty$. Then, $\ell_1^p = \ell_\infty$, $\ell_p^0 = \ell_q$ and $\ell_0^p = \ell_c = \ell_1$. Further, $\|f\|_{\ell_1^p} = \|f\|_{\ell_\infty}$, $\|f\|_{\ell_p^0} = \|f\|_{\ell_q}$ and $\|f\|_{\ell_0^p} = \|f\|_{\ell_1}$, where $X \in \{\ell_\infty, c, c_0\}$.

Lemma 5.2. [32, Theorem 4.2.8] Let $X$ and $Y$ be any two BK sequence spaces. Then, $(X, Y) \subset B(X, Y)$, i.e. every $H \in (X, Y)$ defines a linear operator $\Theta_H \in B(X, Y)$, where $\Theta_H f = H f$ for all $f \in X$.

Lemma 5.3. [24, Theorem 1.23] Let $X \supset \sigma$ be a BK-space and $H \in (X, Y)$. Then

$$\|\Theta_H\| = \|H\|_{(X,Y^\prime\prime)} = \sup_{n \in \mathbb{N}} \|H_n\|_{X} < \infty.$$ 

Lemma 5.4. [24, Theorem 2.15] Let $Q \subset X = \ell_p$ or $c_0$ be bounded, and $T_r : X \rightarrow X$ is defined by $T_r(f_1, f_2, f_3, \cdots) = (f_1, f_2, f_3, \cdots, f_r, 0, 0, \ldots) \forall f = (f_k) \in X$. Then

$$\chi(Q) = \limsup_{r \rightarrow \infty, f \in Q} \|(I - T_r) f\|,$$

where $I$ is the identity operator on $X$.

Lemma 5.5. [27, Theorem 3.7] Let $X \supset \sigma$ be a BK-space. Then following statements are true:

(a) Assume that $H \in (X, c_0)$. Then $\|\Theta_H\|_X = \limsup_{n \rightarrow \infty} \|H_n\|_X^p$ and, as $n \rightarrow \infty$,

$$\Theta_H \text{ is compact } \iff \|H_n\|_X^p = 0.$$ 

(b) Assume that $X$ has AK and $H \in (X, c)$. Then

$$\frac{1}{2} \limsup_{n \rightarrow \infty} \|H_n - h\|_X^p \leq \|\Theta_H\|_X \leq \limsup_{n \rightarrow \infty} \|H_n - h\|_X^p$$

and, as $n \rightarrow \infty$,

$$\Theta_H \text{ is compact } \iff \|H_n - h\|_X^p = 0,$$

where $h = (h_k)$ with $h_k = \lim_{n \rightarrow \infty} h_{nk} \forall k \in \mathbb{N}$.

(c) Assume that $H \in (X, \ell_\infty)$. Then, $0 \leq \|\Theta_H\|_X \leq \limsup_{n \rightarrow \infty} \|H_n\|_X^p$ and $\Theta_H$ is compact if $\|H_n\|_X^p = 0$ as $n \rightarrow \infty$. 

Hmnc of a bounded set $Q$ in a metric space $X$ is defined by

$$\chi(Q) = \inf \left\{ \varepsilon > 0 : Q \subset \bigcup_{k=1}^{n} B(f_k, r_k), f_k \in X, r_k < \varepsilon \ (k = 1, 2, \ldots, n), n \in \mathbb{N} \right\}.$$ 

An important application of Hmnc is that it helps in determining the compactness of an operator between BK-spaces. Hmnc of the operator $\Theta$ is defined by $\|\Theta\|_X = \chi(\Theta(B(X)))$. An operator $\Theta : X \rightarrow Y$ is compact if and only if $\|\Theta\|_X = 0$. We refer to [24, 27, 35, 36] for studies concerning compactness of a matrix operator between well-known BK-spaces via Hmnc.
Lemma 5.6. [27, Theorem 3.11] Assume that $X \supset \sigma$ be $a$ BK-space and $H \in (X, \ell_1)$. Then

$$\lim_{r \to \infty} \sup_{N \in A_r} \left\| \sum_{n \in N} H_n \right\|_X^{\dagger} \leq \left\| \Theta H \right\|_X \leq 4 \cdot \lim_{r \to \infty} \sup_{N \in A_r} \left\| \sum_{n \in N} H_n \right\|_X^{\dagger}$$

and, as $r \to \infty$,

$$\Theta H \text{ is compact } \iff \sup_{N \in A_r} \left\| \sum_{n \in N} H_n \right\|_X^{\dagger} = 0.$$ 

For all $n, k = 1, 2, 3, \ldots$, we define

$$z_{nk} = \sum_{v=k}^{\infty} \frac{\mu(v) S(k)}{v} h_{nv}.$$ 

Then, we write the following result which is significant in deriving our findings:

Lemma 5.7. Let $X \in \mathcal{O}$ and $H \in (X(\mathcal{G}), Y)$. Then, $Z = (z_{nk}) \in (X, Y)$ and $Hf \in Zg$ $\forall f \in X(\mathcal{G})$, where $X \in \{\ell_1, \ell_p, c, \ell_\infty\}$.

Proof. It is straightforward from Theorem 4.1. \hfill \Box

Theorem 5.8. Let $1 < p < \infty$. Then, following statements are true:

(a) Assume that $H \in (\ell_p(\mathcal{G}), c_0)$. Then

$$\left\| \Theta H \right\|_X = \lim_{n \to \infty} \sup \left( \sum_{k} \left| z_{nk} \right|^q \right)^{1/q}.$$ 

(b) Assume that $H \in (\ell_p(\mathcal{G}), c)$. Then

$$\frac{1}{2} \lim_{n \to \infty} \sup \left( \sum_{k} \left| z_{nk} - z_{k} \right|^q \right)^{1/q} \leq \left\| \Theta H \right\|_X \leq \lim_{n \to \infty} \sup \left( \sum_{k} \left| z_{nk} - z_{k} \right|^q \right)^{1/q},$$

where $z = (z_{k})$ and $z_k = \lim_{n \to \infty} z_{nk}$ for each $k \in \mathbb{N}$.

(c) Assume that $H \in (\ell_p(\mathcal{G}), \ell_\infty)$. Then

$$0 \leq \left\| \Theta H \right\|_X \leq \lim_{n \to \infty} \sup \left( \sum_{k} \left| z_{nk} \right|^q \right)^{1/q}.$$ 

(d) Assume that $H \in (\ell_p(\mathcal{G}), \ell_1)$. Then

$$\lim_{r \to \infty} \left\| H \right\|_{(\ell_p(\mathcal{G}), \ell_1)} \leq \left\| \Theta H \right\|_X \leq 4 \lim_{r \to \infty} \left\| H \right\|_{(\ell_p(\mathcal{G}), \ell_1)},$$

where $\left\| H \right\|_{(\ell_p(\mathcal{G}), \ell_1)} = \sup_{N \in A_r} \left( \sum_{k} \left| \sum_{n \in N} z_{nk} \right|^q \right)^{1/q}, r \in \mathbb{N}$.
Proof. (a) Assume that $H \in (\ell_p(\mathcal{S}), c_0)$. We have

$$
\|H_n\|_{\ell_p(\mathcal{S})} = \|Z_n\|_{\ell_p} = \|Z_n\|_{\ell_q} = \left(\sum_k |z_{nk}|^q\right)^{1/q}
$$

for each $n \in \mathbb{N}$. In the view of Part (a), Lemma 5.5, we conclude that

$$
\|\Theta H\|_X = \limsup_{n \to \infty} \left(\sum_k |z_{nk}|^q\right)^{1/q}
$$

(b) Note that

$$
\|Z_n - z_k\|_{\ell_p} = \|Z_n - z_k\|_{\ell_q} = \left(\sum_k |z_{nk} - z_k|^q\right)^{1/q}
$$

for each $n \in \mathbb{N}$. Assume that $H \in (\ell_p(\mathcal{S}), c)$. Then, in the light of Lemma 5.7, we get that $H \in (\ell_p, c)$. By engaging Part (b) of Lemma 5.5, we deduce that

$$
\frac{1}{2} \limsup_{n \to \infty} \|Z_n - z\|_{\ell_p} \leq \|\Theta H\|_X \leq \limsup_{n \to \infty} \|Z_n - z\|_{\ell_p}
$$

which on employing (5.2) yields

$$
\frac{1}{2} \limsup_{n \to \infty} \left(\sum_k |z_{nk} - z_k|^q\right)^{1/q} \leq \|\Theta H\|_X \leq \limsup_{n \to \infty} \left(\sum_k |z_{nk} - z_k|^q\right)^{1/q}
$$

as desired.

(c) We use Part (c) of Lemma 5.5 instead of Part (a) of Lemma 5.5 in the proof of Part (a). Rest of the statements being similar are discarded.

(d) Note that

$$
\left\|\sum_{n \in \mathbb{N}} Z_n\right\|_{\ell_p} = \left\|\sum_{n \in \mathbb{N}} Z_n\right\|_{\ell_q} = \left(\sum_k \left\|\sum_{n \in \mathbb{N}} z_{nk}\right\|_{\ell_q}^q\right)^{1/q}
$$

Assume that $H \in (\ell_p(\mathcal{S}), \ell_1)$. Then, by applying Lemma 5.7, we get that $Z \in (\ell_p, \ell_1)$. Engaging Lemma 5.6, we get

$$
\limsup_{r \to \infty} \left\|\sum_{n \in N} Z_n\right\|_{\ell_p} \leq \|\Theta H\|_X \leq 4 \cdot \limsup_{r \to \infty} \left\|\sum_{n \in N} Z_n\right\|_{\ell_p}
$$

which further reduces on using (5.3) to

$$
\lim_{r \to \infty} \|H\|_{(\ell_p(\mathcal{S}), \ell_1)} \leq \|\Theta H\|_X \leq 4 \lim_{r \to \infty} \|H\|_{(\ell_p(\mathcal{S}), \ell_1)},
$$

as desired.

□

Theorem 5.9. The following statements are true:
(a) Assume that $H ∈ (ℓ_1(Θ), c_0)$. Then

$$||Θ_H||_X = \limsup_{n→∞} \left( \sup_k |z_{nk}| \right).$$

(b) Assume that $H ∈ (ℓ_1(Θ), c)$. Then

$$\frac{1}{2} \limsup_{n→∞} \left( \sup_k |z_{nk} - z_k| \right) ≤ ||Θ_H||_X ≤ \limsup_{n→∞} \left( \sup_k |z_{nk} - z_k| \right),$$

where $z = (z_k)$ and $z_k = \lim_{n→∞} z_{nk}$ for each $k ∈ N$.

(c) Assume that $H ∈ (ℓ_1(Θ), ℓ_∞)$. Then

$$0 ≤ ||Θ_H||_X ≤ \limsup_{n→∞} \left( \sup_k |z_{nk}| \right).$$

(d) Assume that $H ∈ (ℓ_1(Θ), ℓ_1)$. Then

$$||Θ_H||_X = \lim_{r→∞} \left( \sup_k \sum_{n>r} |z_{nk}| \right), \quad r ∈ N.$$ 

Proof. One can obtain this in the similar way as in the proof of Theorem 5.8. Hence the proof is discarded to avoid repetative statements. □

**Theorem 5.10.** The following statements are true:

(a) Assume that $H ∈ (ℓ_∞(Θ), c_0)$ or $H ∈ (c_0(Θ), c_0)$. Then

$$||Θ_H||_X = \limsup_{n→∞} \left( \sum_k |z_{nk}| \right).$$

(b) Assume that $H ∈ (ℓ_∞(Θ), c)$ or $H ∈ (c_0(Θ), c)$. Then

$$\frac{1}{2} \limsup_{n→∞} \left( \sum_k |z_{nk} - z_k| \right) ≤ ||Θ_H||_X ≤ \limsup_{n→∞} \left( \sum_k |z_{nk} - z_k| \right),$$

where $z = (z_k)$ and $z_k = \lim_{n→∞} z_{nk}$ for each $k ∈ N$.

(c) Assume that $H ∈ (ℓ_∞(Θ), ℓ_∞)$ or $H ∈ (c_0(Θ), ℓ_∞)$. Then

$$0 ≤ ||Θ_H||_X ≤ \limsup_{n→∞} \left( \sum_k |z_{nk}| \right).$$

(d) Assume that $H ∈ (ℓ_∞(Θ), ℓ_1)$ or $H ∈ (c_0(Θ), ℓ_1)$. Then

$$\lim_{r→∞} ||H||_{(ℓ_∞(Θ), ℓ_1)} ≤ ||Θ_H||_X ≤ 4 \lim_{r→∞} ||H||_{(ℓ_∞(Θ), ℓ_1)},$$

where $||H||_{(ℓ_∞(Θ), ℓ_1)} = \sup_{N ∈ N} \left( \sum_k |\sum_{n∈N} z_{nk}| \right), \quad r ∈ N$.

Proof. One can obtain this in the similar way as in the proof of Theorem 5.8. Hence the proof is discarded to avoid repetative statements. □
Now we list the following conditions indicated as \( C1, C2, C3 \) and \( C4 \):

\[
C1 := \lim_{n \to \infty} \left( \sum_k |z_{nk} - z_k|^q \right)^{1/q} = 0;
\]

\[
C2 := \lim_{r \to \infty} \sup_{N \in \mathbb{N}} \left( \sum_k \left| \sum_{n \in \mathbb{N}} z_{nk} \right|^q \right)^{1/q} = 0;
\]

\[
C3 := \lim_{n \to \infty} \left( \sup_k |z_{nk} - z_k| \right) = 0;
\]

\[
C4 := \lim_{r \to \infty} \left( \sup_k \sum_{n=r}^\infty |z_{nk}| \right) = 0.
\]

The following result is an immediate consequence of Lemmas 5.5 and 5.6, and Theorems 5.8, 5.9 and 5.10:

**Corollary 5.11.** The necessary and sufficient conditions that the operator \( \Theta_H \) is compact for \( H \in (X,Y) \) can be read from Table 3, where \( X \in \{ \ell_p(S), \ell_1(S), c_0(S), \ell_\infty(S) \} \) and \( Y \in \{ c_0, c, \ell_\infty, \ell_1 \} \). In the cases where \( Y = \ell_\infty \), the condition is only necessary but not sufficient.

<table>
<thead>
<tr>
<th>From \ To \</th>
<th>( c_0 ) with ( z_k = 0 \ \forall \ k ) ( q = 1 )</th>
<th>( c ) with ( q = 1 )</th>
<th>( \ell_\infty ) with ( z_k = 0 \ \forall \ k ) ( q = 1 )</th>
<th>( \ell_1 ) with ( z_k = 0 \ \forall \ k ) ( q = 1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \ell_p(S) )</td>
<td>C1</td>
<td>C1</td>
<td>C1 with ( z_k = 0 \ \forall \ k )</td>
<td>C2</td>
</tr>
<tr>
<td>( \ell_1(S) )</td>
<td>C3 with ( z_k = 0 \ \forall \ k ) ( q = 1 )</td>
<td>C3 with ( q = 1 )</td>
<td>C1 with ( z_k = 0 \ \forall \ k ) ( q = 1 )</td>
<td>C4</td>
</tr>
<tr>
<td>( c_0(S) )</td>
<td>C1 with ( z_k = 0 \ \forall \ k ) ( q = 1 )</td>
<td>C1 with ( q = 1 )</td>
<td>C1 with ( z_k = 0 \ \forall \ k ) ( q = 1 )</td>
<td>C2 with ( q = 1 )</td>
</tr>
<tr>
<td>( \ell_\infty(S) )</td>
<td>C1 with ( z_k = 0 \ \forall \ k ) ( q = 1 )</td>
<td>C1 with ( q = 1 )</td>
<td>C1 with ( z_k = 0 \ \forall \ k ) ( q = 1 )</td>
<td>C2 with ( q = 1 )</td>
</tr>
</tbody>
</table>

**TABLE 3.** Necessary and sufficient condition for compactness from \( X \in \{ \ell_p(S), \ell_1(S), c_0(S), \ell_\infty(S) \} \) to \( Y \in \{ c_0, c, \ell_\infty, \ell_1 \} \).

**Conflict of Interest**

The corresponding author, on behalf of all the authors, states that there is no conflict of interest.

**Acknowledgment**

The authors are thankful to the anonymous reviewer for making helpful comments and suggesting necessary changes which have improved the readability of the paper to a great extent.
References


[26] M. Mursaleen, F. Başar, B. Altay, On the Euler sequence spaces which include the spaces \( \ell_p \) and \( \ell_\infty \), II, Nonlinear Anal. 65 (3) (2006), 707-717.

DEPARTMENT OF MATHEMATICS, DERANATUNG GOVERNMENT COLLEGE, ITANAGAR 791113, INDIA
E-mail address: tajayaying20@gmail.com

DEPARTMENT OF MATHEMATICS, RAJIV GANDHI UNIVERSITY, DOIMUKH-791112, INDIA
E-mail address: nipennak@yahoo.com