SYMMETRY OF MAXIMALS FOR FRACTIONAL IDEALS OF CURVES

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ABSTRACT. The purpose of this paper is to extend the symmetry of maximals of the ring of a germ of reducible plane curve proved by Delgado to a relation between the relative maximals of a fractional ideal and the absolute maximals of its dual for any admissible ring. In particular, it includes the case of germs of reduced reducible curve of any codimension. We then apply this symmetry to characterize the elements in the set of values of a fractional ideal from some of its projections and the irreducible absolute maximals of the dual ideal.

1. Introduction. Let \((C, 0)\) be the germ of a curve with reduced ring \(\mathcal{O}_C\) and \(p\) irreducible components. We associate with any fractional ideal \(I\) the set of the values of the non zero divisors \(g \in I\), which is a subset of \(\mathbb{Z}^p\). The set of values of certain ideals are related to properties of the curve. For example, for plane curves, the set of values of \(\mathcal{O}_C\), which is called the semigroup, characterizes the topological class of the curve (see [16] and [15]). The set of values of the modules of Kähler differentials appears in the analytic classification of plane curves considered in [9] and [10]. The theorems of Kunz for irreducible curves (see [12]) and Delgado for reducible curves (see [6]) show that a curve \(C\) is Gorenstein if and only if the semigroup satisfies a symmetry property. We extend the latter symmetry to any fractional ideal of a plane curve and a Gorenstein curve in respectively [13] and [14], and then it is extended to any fractional ideal of more general rings called admissible rings in [11]. This symmetry recently gave rise to other characterizations of Gorenstein curves (see [7]).
It is therefore natural to study how to compute explicitly set of values of fractional ideals of curves. The explicit computation of the semigroup of a plane curve is considered for example in [16], [2] and [5]. An algorithm for the semigroup of a space irreducible curve is given in [3]. A standard basis of a fractional ideal for an irreducible curve can be computed with the algorithm [8, Theorem 2.4] (see definition 4.7 for the notion of standard basis), and the set of valuations of $I$ can then be deduced from this standard basis. The set of values of the Jacobian ideal and the set of values of its dual, namely the module of logarithmic residues, are studied in [13] and [14] (see definition 2.6 for the notion of dual). We suggest in [14, §4.3.3] an algorithm for the computation of the values of the module of logarithmic residues for plane curves with exactly two branches which uses the algorithm [8, Theorem 2.4]. However, this procedure cannot be extended to curves with three or more branches.

The computation of the semigroup of a plane curve given in [5] is based on a symmetry between two particular kinds of elements of the semigroup, which are called relative maximals and absolute maximals (see definition 2.16). The proof of this result relies on the symmetry property of the semigroup of a plane curve we mentioned before, and on an induction on the number of components of $C$.

The main result of this paper is a generalization of the symmetry between relative and absolute maximals to any fractional ideal of admissible ring as follows:

**Theorem 1.1.** Let $R$ be an admissible local ring and let $I \subseteq \text{Frac}(R)$ be a fractional ideal. Let $\alpha \in \mathbb{Z}$ and $\beta = \gamma - \alpha - 1$, where $\gamma$ is the conductor of $R$ (see definition 2.8). Then $\beta$ is an absolute maximal of $I^\vee$ if and only if $\alpha$ is a relative maximal of $I$.

In addition, the proof we suggest here gives an alternative proof of Delgado’s result since we do not use an induction on the number of branches. We then use this symmetry to investigate the computation of the set of values of a fractional ideal. Our main result has been recently reobtained in the particular case of Gorenstein curves in [7, Corollary 27] by other methods.
Let us describe the content of this paper.

In section 2, we recall several properties of the set of values of a fractional ideal which will be used in the next sections.

Section 3 is devoted to the proof of the main theorem 1.1, which uses the symmetry theorem of [14] and [11] (see theorem 2.15) and properties of the set of values of fractional ideals. The proof in the basic case of Gorenstein curves relies on [14], and the proof in the general case, as presented here using [11], is very similar.

In section 4, we investigate the computation of the set of values of a fractional ideal using induction on the number of branches as it is done in [5] for the semigroup of the curve. The generation theorem [5, Theorem 1.5] can be generalized to any fractional ideal, and is not specific to plane curves (see theorem 4.1). This theorem gives a characterization of the set of values of $I$ from some projections of val($I$) and the relative maximals of $I$. Thanks to theorem 1.1, to determine the set of the relative maximals of $I$ is equivalent to determine the set of the absolute maximals of $I^{\cap}$. In subsection 4.2, we study the set of the absolute maximals of an ideal in the case of germs of analytic curves.

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2. Notations and preliminary results. We recall in this section definitions and properties from [5], [6], [14] and [11] which will be used in the rest of this paper.

2.1. Setup. Let $C$ be the germ of a reduced complex analytic curve, with $p$ irreducible components $C_1, \ldots, C_p$. We denote by $\mathcal{O}_C$ the reduced ring of $C$. The ring $\mathcal{O}_{C_i}$ of the branch $C_i$ is a one-dimensional integral domain, so that its normalization $\tilde{\mathcal{O}}_{C_i}$ is isomorphic to $\mathbb{C}\{t_i\}$ (see for example [4, Corollary 4.4.10]). The total ring of fractions of
\( \mathcal{O}_C \) satisfies (see \[4\] for example):

\[
\text{Frac}(\mathcal{O}_C) = \text{Frac}(\mathcal{O}_\tilde{C}) = \bigoplus_{i=1}^p \text{Frac}(\mathcal{C}\{t_i\}).
\]

**Definition 2.1.** Let \( g \in \text{Frac}(\mathcal{O}_C) \). We define the valuation of \( g \) along the branch \( C_i \) as the order on \( t_i \) of the image of \( g \) by the map \( \text{Frac}(\mathcal{O}_C) \to \text{Frac}(\mathcal{C}\{t_i\}) \). We denote the valuation of \( g \) along \( C_i \) by \( \text{val}_i(g) \in \mathbb{Z} \cup \{\infty\} \), with the convention \( \text{val}_i(0) = \infty \).

We then define the value of \( g \) by \( \text{val}(g) := (\text{val}_1(g), \ldots, \text{val}_p(g)) \in (\mathbb{Z} \cup \{\infty\})^p \).

The previous definition can be extended to more general rings introduced in \[11\], which are called admissible rings. We recall here the definition, and we refer to \[11\] for more details. We will only consider the local case. Properties of a semilocal ring can be deduced from the local case thanks to \[11, \text{Theorem } 3.2.2\].

We denote by \( |\cdot| \) the cardinality of a set.

**Definition 2.2.** Let \( (R, m) \) be a one dimensional Noetherian local Cohen-Macaulay ring. The ring \( R \) is called admissible if the following conditions are satisfied:

1. the completion \( \hat{R} \) of \( R \) is reduced,
2. the integral closure \( \tilde{R} \) of \( R \) in \( \text{Frac}(R) \) satisfies \( \tilde{R}/n = R/n \cap R \) for any maximal ideal \( n \) of \( R \),
3. we have \( |R/m| \geq |\mathcal{V}| \), where \( \mathcal{V} \) is the set of discrete valuation rings of \( \text{Frac}(R) \) over \( R \).

A value map \( \text{val} : \text{Frac}(R) \to (\mathbb{Z} \cup \{\infty\})^{|\mathcal{V}|} \) can be defined using the set of discrete valuation rings (see \[11, \text{Definition } 3.1.2\]).

In particular, the ring \( \mathcal{O}_C \) of the germ of a reduced curve is admissible, and the value map is the one defined in definition 2.1.

We fix \( R \) a local admissible ring, we set \( p = |\mathcal{V}| \), and \( \text{val} : \text{Frac}(R) \to (\mathbb{Z} \cup \{\infty\})^p \) the value map.
Definition 2.3. Let \( I \subset \text{Frac}(R) \) be an \( R \)-module. We call \( I \) a fractional ideal if there exists a non zero divisor \( h \in R \) such that \( hI \subseteq R \) and if \( I \) contains a non zero divisor of \( \text{Frac}(R) \). We set:

\[
\text{val}(I) := \{ \text{val}(g) \mid g \in I \text{ non zero divisor} \} \subseteq \mathbb{Z}^p.
\]

One can notice that this set is an ideal over the semigroup \( \text{val}(R) \): if \( a \in \text{val}(I) \) and \( b \in \text{val}(R) \), then \( a + b \in \text{val}(I) \).

For \( I, J \) ideals in \( \text{Frac}(R) \), we set \( (I : J) = \{ a \in R \mid aJ \subseteq I \} \).

Definition 2.4. Let \( K \subset \text{Frac}(R) \) be a fractional ideal. We say that \( K \) is a canonical ideal if for all fractional ideals \( I \subseteq \text{Frac}(R) \), we have

\[
(K : (K : I)) = I.
\]

The ring \( R \) is called Gorenstein if \( R \) is a canonical ideal.

Proposition 2.5 ([11, Corollary 5.1.7]). There exists a unique canonical ideal \( K^0 \) up to multiplication by an invertible element of \( \tilde{R} \) such that

\[
R \subseteq K^0 \subseteq \tilde{R}.
\]

In particular, an admissible ring \( R \) is Gorenstein if and only if \( K^0 = R \).

Definition 2.6. Let \( I \subset \text{Frac}(R) \) be a fractional ideal. The dual of \( I \) is:

\[
I^\vee := (K^0 : I).
\]

In particular, from the definition of \( K^0 \), we have \( (I^\vee)^\vee = I \).

Remark 2.7. We also have \( I^\vee \cong \text{Hom}_R (I, K^0) \) (see for example [4, Proof of Lemma 1.5.14]).

Definition 2.8. The conductor ideal of \( R \) is \( C_R = \tilde{R}^\vee \). In particular, there exists \( \gamma \in \mathbb{N}^p \) such that \( \text{val}(C_R) = \gamma + N^p \). We call \( \gamma \) the conductor of the ring \( R \).
2.2. Properties of the set of values of fractional ideals. Let $I \subseteq \text{Frac}(R)$ be a fractional ideal. From the definition of a fractional ideal, one can notice that there exists $\lambda \in \mathbb{Z}^{p}$ such that

$$(1) \quad \text{val}(I) \subseteq \lambda + \mathbb{N}^{p}.$$  

By [11, Proposition 3.1.9], the set of values of any fractional ideal $I \subseteq \text{Frac}(R)$ is a good semigroup ideal, which means that we have the following properties.

By convention, we will always denote by $(\alpha_1, \ldots, \alpha_p) \in \mathbb{Z}^{p}$ the coordinates of an element $\alpha \in \mathbb{Z}^{p}$.

**Lemma 2.9** ([11, Proposition 3.1.9 (b)]). Let $I \subseteq \text{Frac}(R)$ be a fractional ideal. There exists $\nu \in \mathbb{Z}^{p}$ such that:

$$\nu + \mathbb{N}^{p} \subseteq \text{val}(I).$$

We consider the product order on $\mathbb{Z}^{p}$ defined by:

$$(\alpha_1, \ldots, \alpha_p) \leq (\beta_1, \ldots, \beta_p) \iff \forall i \in \{1, \ldots, p\}, \alpha_i \leq \beta_i.$$  

In particular, for $\alpha, \beta \in \mathbb{Z}^{p}$, $\inf(\alpha, \beta) = (\min(\alpha_1, \beta_1), \ldots, \min(\alpha_p, \beta_p))$.

**Proposition 2.10** (see [11, Proposition 3.1.9 (c)]). Let $I$ be a fractional ideal and $\alpha, \beta \in \text{val}(I)$. Then $\inf(\alpha, \beta) \in \text{val}(I)$.

**Proposition 2.11** (see [11, Proposition 3.1.9 (d)]). Let $I$ be a fractional ideal and $\alpha, \beta \in \text{val}(I)$. Let us assume that $\alpha \neq \beta$ and that there exists $i \in \{1, \ldots, p\}$ such that $\alpha_i = \beta_i$. Then there exists $\eta \in \text{val}(I)$ such that:

1. For all $j \in \{1, \ldots, p\}$, $\eta_j \geq \min(\alpha_j, \beta_j)$,
2. $\eta_i > \alpha_i$,
3. For all $j \in \{1, \ldots, p\}$ such that $\alpha_j \neq \beta_j$, $\eta_j = \min(\alpha_j, \beta_j)$.

**Remark 2.12.** Proposition 2.11 will often be used in the following, and if $\alpha, \beta, i$ satisfy the assumptions of proposition 2.11, we will say that we apply proposition 2.11 to the triple $(\alpha, \beta, i)$.

The following lemma is a direct consequence of the definition of $I^{\vee}$.
**Lemma 2.13.** Let $\alpha \in \text{val}(I)$ and $\beta \in \text{val}(I^\vee)$. Then $\alpha + \beta \in \text{val}(K^0)$.

Let us recall the symmetry theorem that will be used in the proof of our main theorem 1.1. We first need the following notations.

**Notation 2.14.** Let $\alpha = (\alpha_1, \ldots, \alpha_p) \in \mathbb{Z}^p$ and $E \subseteq \mathbb{Z}^p$ an arbitrary subset of $\mathbb{Z}^p$.

- Let $i \in \{1, \ldots, p\}$. We set:
  \[ \Delta_i(\alpha, E) = \{ v \in E | v_i = \alpha_i \text{ and } v_j > \alpha_j \text{ for all } j \neq i \} . \]
  We then define $\Delta(\alpha, E) = \bigcup_{i=1}^{p} \Delta_i(\alpha, E)$.
- For $J \subseteq \{1, \ldots, p\}$ we set:
  \[ \Delta_J(\alpha, E) = \{ v \in E | v_j = \alpha_j \text{ for all } j \in J \text{ and } v_j > \alpha_j \text{ for all } j \not\in J \} . \]

For a fractional ideal $I \subseteq \text{Frac}(R)$, we denote for all $J \subseteq \{1, \ldots, p\}$, $\Delta_J(\alpha, I) = \Delta_J(\alpha, \text{val}(I))$ and $\Delta(\alpha, I) = \Delta(\alpha, \text{val}(I))$.

The following symmetry theorem is proved in [14] for Gorenstein curves, and is extended to admissible rings in [11]:

**Theorem 2.15 (\cite{14, Theorem 1.2}, \cite{11, Theorem 5.3.4, Lemma 5.2.8}).** Let $R$ be an admissible ring with canonical ideal $K^0$ as in proposition 2.5. Let $I \subseteq \text{Frac}(R)$ be a fractional ideal. Then, for all $v \in \mathbb{Z}^p$:

\[(2) \quad v \in \text{val}(I^\vee) \iff \Delta(\gamma - v - 1, I) = \emptyset.\]

The previous theorem generalizes [6, Theorem 2.8] which characterizes Gorenstein curves by the symmetry of the semigroup.

**2.3. Absolute and relative maximals.** The following definitions and properties are generalizations to fractional ideals of the ones given in [5].

Let $I \subseteq \text{Frac}(R)$ be a fractional ideal.

**Definition 2.16.** Let $\alpha \in \text{val}(I)$.

1. If $\Delta(\alpha, I) = \emptyset$, we call $\alpha$ a maximal of $I$. 
(2) If for all $J \subseteq \{1, \ldots, p\}$, $J \neq \{1, \ldots, p\}$ and $J \neq \emptyset$ we have $\Delta_J(\alpha, I) = \emptyset$ then we call $\alpha$ an absolute maximal of $I$.

(3) If $\Delta(\alpha, I) = \emptyset$ and for all $J \subseteq \{1, \ldots, p\}$ such that $|J| \geq 2$ we have $\Delta_J(\alpha, I) \neq \emptyset$ then we call $\alpha$ a relative maximal of $I$.

**Remark 2.17.** The three notions of maximals, absolute maximals and relative maximals coincide in the case $p = 2$. If $p = 1$, the set of maximals of any fractional ideal is empty. From now on, we assume that $p \geq 2$.

**Remark 2.18.** If $\alpha$ is a maximal of $I$, then by definition $\alpha \in \text{val}(I)$ and by theorem 2.15, $\gamma - \alpha - 1 \in \text{val}(I')$.

**Remark 2.19.** Let $\lambda, \nu \in \mathbb{Z}^p$ be such that $\nu + \mathbb{N}^p \subseteq \text{val}(I) \subseteq \lambda + \mathbb{N}^p$. Let $\alpha$ be a maximal of $I$. It follows from the fact that $\alpha \in \text{val}(I)$ that $\alpha \geq \lambda$, and since $\Delta(\alpha, I) = \emptyset$, we also have $\alpha < \nu$. Therefore, the set of the maximals of $I$ is contained in $\{v \in \mathbb{Z}^p | \lambda \leq v < \nu\}$, so that it is a finite set.

The following lemma is a generalization of [5, Lemma 1.3] to any fractional ideal. The proof is essentially the same as for the ring $\mathcal{O}_C$ of a plane curve.

**Lemma 2.20.** Let $\alpha \in \mathbb{Z}^p$ be such that there exists $i \in \{1, \ldots, p\}$ satisfying:

1. $\Delta_i(\alpha, I) = \emptyset$,
2. for all $j \neq i$, $\Delta_{i,j}(\alpha, I) \neq \emptyset$.

Then $\alpha$ is a relative maximal of $I$.

**Proof.** For all $j \neq i$, let $\alpha^j \in \Delta_{i,j}(\alpha, I)$. Then $\alpha = \inf((\alpha^j)_{j \neq i})$ so that by proposition 2.10, we have $\alpha \in \text{val}(I)$.

Let us assume that there exists $k \in \{1, \ldots, p\}$ such that $\Delta_k(\alpha, I) \neq \emptyset$. Let $\eta \in \Delta_k(\alpha, I)$. Since $\eta_k = \alpha^k_k$ and $\eta \neq \alpha^k$, by proposition 2.11 applied to the triple $(\eta, \alpha^k, k)$, there exists $\mu \in \text{val}(I)$ such that $\mu_k > \alpha^k_k$, $\mu_i = \alpha^k_i = \alpha_i$ and for all $\ell \notin \{i, k\}$, $\mu_\ell \geq \min(\alpha^k_\ell, \eta_\ell)$. Since $\alpha^k_i > \alpha_i$ and $\eta_k > \alpha_k$, we have $\mu \in \Delta_i(\alpha, I)$, which is impossible. Therefore, for all $k \in \{1, \ldots, p\}$, $\Delta_k(\alpha, I) = \emptyset$. 

Let us prove that for all \( J = \{k, \ell\} \subseteq \{1, \ldots, p\} \setminus \{i\} \), \( \Delta_J(\alpha, I) \neq \emptyset \). Since \( \alpha_k = \alpha_\ell = \alpha_i \), and \( \alpha_k \neq \alpha_\ell \), by proposition 2.11 applied to the triple \((\alpha_k, \alpha_\ell, i)\), there exists \( \eta \in \text{val}(I) \) such that \( \eta_i > \alpha_i \), \( \eta_k = \alpha_k \), \( \eta_\ell = \alpha_\ell \), and for all \( j \notin \{i, k, \ell\} \), \( \eta_j \geq \min(\alpha^k_j, \alpha^\ell_j) > \alpha_j \). Therefore, \( \eta \in \Delta_{k,\ell}(\alpha, I) \).

Let us consider now \( J \subseteq \{1, \ldots, p\} \) with \( |J| \geq 2 \). We set \( J = J_1 \cup J_2 \cup \cdots \cup J_k \) with for all \( j \in \{1, \ldots, k\} \), \( |J_j| = 2 \). For all \( j \in \{1, \ldots, k\} \), let \( \eta^j \in \Delta_{J_j}(\alpha, I) \). Then by proposition 2.10, \( \inf(\eta^1, \ldots, \eta^k) \in \Delta_J(\alpha, I) \).

Hence the result. \( \square \)

The following lemma is a generalization of [5, Lemma 2.8].

**Lemma 2.21.** Let \( \alpha \in \text{val}(I) \). Let us assume\(^1\) that for all \( J \subseteq \{1, \ldots, p\} \) such that \( |J| \geq 2 \), we have \( \Delta_J(\alpha, I) \neq \emptyset \). Let \( \beta = \gamma - \alpha - 1 \). Then for all \( A \subseteq \{1, \ldots, p\} \) with \( A \neq \{1, \ldots, p\} \) and \( A \neq \emptyset \), we have \( \Delta_A(\beta, I') = \emptyset \).

If in addition \( \beta \in \text{val}(I') \), then \( \beta \) is an absolute maximal of \( I' \) and \( \alpha \) is a relative maximal of \( I \).

**Proof.** We set \( \beta = \gamma - \alpha - 1 \). Let us assume that there exists \( \emptyset \neq A \subseteq \{1, \ldots, p\} \), \( A \neq \{1, \ldots, p\} \), such that \( \Delta_A(\beta, I') \neq \emptyset \). Let \( \eta \in \Delta_A(\beta, I') \). We set \( J = A^c \cup \{i\} \) with \( A^c \) the complement of \( A \) in \( \{1, \ldots, p\} \) and \( i \in A \). Let \( \mu \in \Delta_J(\alpha, I) \). Then \( \eta + \mu \in \Delta_J(\gamma - 1, K^0) \). However, by [11, Lemma 5.2.2], \( \gamma \) is also the conductor of \( K^0 \), and by [11, Lemma 4.1.10], we have \( \Delta(\gamma - 1, K^0) = \emptyset \). Hence the result.

If we assume in addition that \( \beta \in \text{val}(I') \), then \( \beta \) is an absolute maximal of \( I' \), and by theorem 2.15, \( \Delta(\alpha, I) = \emptyset \) so that \( \alpha \) is a relative maximal of \( I \). \( \square \)

As an immediate corollary we have:

**Corollary 2.22.** If \( \alpha \) is a relative maximal of \( I \), then \( \beta = \gamma - \alpha - 1 \) is an absolute maximal of \( I' \).

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\(^1\)One can notice that \( \alpha \) may not be a relative maximal because we do not assume that \( \Delta(\alpha, I) = \emptyset \).
3. Symmetry of the maximals. The purpose of this section is to prove that the converse implication of corollary 2.22 is satisfied, which will give theorem 1.1.

Theorem 1.1 is a generalization to any admissible ring and any fractional ideal of the theorem of symmetry between relative and absolute maximals of the ring of a plane curve proved in [5]. We recall that if \( p = 1 \), then there is no maximal element in the sense of definition 2.16, and if \( p = 2 \), the statement can be rephrased as "\( \beta \) is a maximal of \( I' \) if and only if \( \alpha \) is a maximal of \( I \) " , which is a direct consequence of theorem 2.15.

The proof given in [5] uses an induction on the number of branches. We suggest here a different proof which does not use an induction on the number of branches.

From now on, we assume that \( p \geq 3 \).

Proof of theorem 1.1.

Let us assume that:

- \( \beta \) is an absolute maximal of \( I' \)
- \( \alpha = \gamma - \beta - 1 \) is not a relative maximal of \( I \).

By remark 2.18 we have \( \beta \in \text{val}(I') \) and \( \alpha \in \text{val}(I) \).

Lemma 3.1. There exists \( j \in \{2, \ldots, p\} \) such that

\[ E_{1,j} := \{ v \in \mathbb{Z}^p | v_1 = \beta_1, v_j = \beta_j \text{ and } v_i < \beta_i \text{ for all } i \neq \{1, j\} \} \]

satisfies the following property:

\[ \forall v \in E_{1,j}, \; \Delta(v, I') \neq \emptyset. \]  

Proof. Since \( \beta \in \text{val}(I') \), by theorem 2.15, we have \( \Delta(\alpha, I) = \emptyset \). In particular, \( \Delta_1(\alpha, I) = \emptyset \). Therefore, by lemma 2.20, since \( \alpha \) is not a relative maximal of \( I \), there exists \( j \neq 1 \) such that \( \Delta_{1,j}(\alpha, I) = \emptyset \). Let \( v \in E_{1,j} \), and let \( w = \gamma - v - 1 \). In particular, \( w_1 = \alpha_1, w_j = \alpha_j \) and for all \( \ell \notin \{1, j\}, w_\ell > \alpha_\ell \). Since \( \Delta_{1,j}(\alpha, I) = \emptyset \), then \( w \notin \text{val}(I) \). Therefore, by theorem 2.15, we have \( \Delta(v, I') \neq \emptyset \). \( \square \)

By renumbering the branches, we assume that the index \( j \) satisfying lemma 3.1 is \( j = 2 \), and we set \( \mathcal{E} = E_{1,2} \).
We will prove thanks to the following lemmas the existence of an element $\eta$ which is a maximal element for the product order in $\mathcal{E}$ and satisfies the property $\Delta_1(\eta, I') \neq \emptyset$ and $\Delta_2(\eta, I') \neq \emptyset$ (see lemma 3.5).

**Lemma 3.2.** Let $v \in \mathcal{E}$. Then $\Delta_1(v, I') \neq \emptyset$ if and only if $\Delta_2(v, I') \neq \emptyset$.

*Proof.* Let $v \in \mathcal{E}$. Let us assume that $\Delta_1(v, I') \neq \emptyset$. There exists $w \in \text{val}(I')$ such that $w_1 = v_1 = \beta_1$, $w_2 > v_2 = \beta_2$ and for all $i \geq 3$, $w_i > v_i$. Since $\beta \in \text{val}(I')$, and $w_1 = \beta_1$, by the proposition 2.11 applied to the triple $(\beta, w, 1)$, there exists $u' \in \text{val}(I')$ such that $w'_1 > \beta_1 = w_1$, $w'_2 = \beta_2$, and for all $i \geq 3$, $w'_i \geq \min(w_i, \beta_i)$. Since $\beta_1 = v_1$, $\beta_2 = v_2$ and for all $i \geq 3$, $\min(w_i, \beta_i) > v_i$, we have $u' \in \Delta_2(v, I')$. □

**Lemma 3.3.** Let $\beta' = (\beta_1, \beta_2, \beta_3 - 1, \ldots, \beta_p - 1)$. In particular, $\beta'$ is an element in $\mathcal{E}$ which is maximal for the product order. Then $\Delta_1(\beta', I') = \Delta_2(\beta', I') = \emptyset$.

*Proof.* Let us assume that there exists $v \in \Delta_1(\beta', I')$. Let $J = \{i \in \{1, \ldots, p\} \mid v_i = \beta_i\}$. Then $v \in \Delta_J(\beta', I')$. In addition, $J \neq \emptyset$ since $1 \in J$, and $J \neq \{1, \ldots, p\}$ since $v_2 > \beta_2$. It contradicts the fact that $\beta$ is an absolute maximal. Hence $\Delta_1(\beta', I') = \emptyset$ and by lemma 3.2, $\Delta_2(\beta', I') = \emptyset$. □

**Lemma 3.4.** Let $\lambda \in \mathbb{Z}^p$ be such that $\text{val}(I') \subseteq \lambda + \mathbb{N}^p$. Let $\lambda' = (\beta_1, \beta_2, \lambda_3 - 1, \ldots, \lambda_p - 1) \in \mathcal{E}$. Then $\Delta_1(\lambda', I') \neq \emptyset$ and $\Delta_2(\lambda', I') \neq \emptyset$.

*Proof.* Since $\lambda' \in \mathcal{E}$, we have $\Delta(\lambda', I') \neq \emptyset$ by lemma 3.1. Since $\text{val}(I') \subseteq \lambda + \mathbb{N}^p$, we have for all $i \geq 3$, $\Delta_i(\lambda', I') = \emptyset$. Therefore, using lemma 3.2, we have $\Delta_1(\lambda', I') \neq \emptyset$ and $\Delta_2(\lambda', I') \neq \emptyset$. □

The following lemma is a consequence of lemmas 3.4 and 3.2, and of the existence of a maximal element for the product order in $\mathcal{E}$.

**Lemma 3.5.** There exists $\eta \in \mathcal{E}$ such that:

1. $\Delta_1(\eta, I') \neq \emptyset$ and $\Delta_2(\eta, I') \neq \emptyset$. 

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*Proof.* Let us assume that there exists $\eta \in \mathcal{E}$ such that $\Delta_1(\eta, I') \neq \emptyset$ and $\Delta_2(\eta, I') \neq \emptyset$. Then $\eta$ is an element in $\mathcal{E}$ which is maximal for the product order. Hence $\Delta_1(\eta, I') = \Delta_2(\eta, I') = \emptyset$. □
(2) for all $\eta' \in E$ such that $\eta' \triangleright \eta$ and such that there exists $i_0 \geq 3$ with $\eta'_{i_0} > \eta_{i_0}$, we have $\Delta_1(\eta', I') = \Delta_2(\eta', I') = \emptyset$.

In other words, $\eta$ is maximal for the product order in the subset of $E$ composed of the elements satisfying the first condition (1).

Notation 3.6. We fix an element $\eta \in E$ satisfying lemma 3.5. We set:

- $K = \{i \in \{3, \ldots, p\} \mid \eta_i \leq \beta_i - 2\}$,
- $J_1 = \{i \in \{3, \ldots, p\} \mid \eta_i = \beta_i - 1\}$.

In particular, $K \cup J_1 = \{3, \ldots, p\}$, and $K \neq \emptyset$ by lemma 3.3.

Lemma 3.7. Let $\mu \in \Delta_1(\eta, I')$. Then:

1. $\mu_1 = \beta_1$, $\mu_2 > \beta_2$ and for all $j \in J_1$, $\mu_j \geq \beta_j$,
2. for all $i \in K$, $\mu_i = \eta_i + 1$.

Proof. The first property comes from the definitions of $\Delta_1(\eta, I')$ and $J_1$. We have for all $i \in K$, $\mu_i \geq \eta_i + 1$. Let us assume that there exists $i_0 \in K$ such that $\mu_{i_0} > \eta_{i_0} + 1$. Let $\eta' \in E$ be such that for all $i \neq i_0$, $\eta'_i = \eta_i$ and $\eta'_{i_0} = \eta_{i_0} + 1 < \beta_{i_0} - 1$. Then $\mu \in \Delta_1(\eta', I')$, which is impossible from the definition of $\eta$. $\square$

The following proposition is the initialization of the induction of proposition 3.9.

Proposition 3.8. Let $v^1 \in E$ be such that for all $i \in K$, $v^1_i = \eta_i + 1$ and for $j \in J_1$, $v^1_j = \beta_j - 1$. We set

$$J_2 = \{j \in \{1, \ldots, p\} \mid \Delta_j(v^1, I') \neq \emptyset\}.$$  

Then $J_2 \subseteq J_1$. In addition, $J_2 \neq \emptyset$.

Proof. It follows from the definition of $\eta$ that we have $\Delta_1(v^1, I') = \Delta_2(v^1, I') = \emptyset$. Let us assume that there exists $i_0 \in K$ such that $\Delta_{i_0}(v^1, I') \neq \emptyset$. Let $w \in \Delta_{i_0}(v^1, I')$. Let $\mu \in \Delta_1(\eta, I')$ as in lemma 3.7. Since $\mu_{i_0} = w_{i_0} = \eta_{i_0} + 1$ and for all $i \in K \setminus \{i_0\}$, $\mu_i = \eta_i + 1 < w_i$, by proposition 2.11 applied to the triple $(\mu, w, i_0)$, there exists $w' \in \text{val}(I')$ such that $w'_i = \mu_i = \beta_i$, $w'_i \geq \min(w_{2, 2'}, \mu_2) > \beta_2$, for all $i \in K \setminus i_0$, $w'_i = \mu_i = \eta_i + 1$, $w'_{i_0} > \eta_{i_0} + 1$ and for all $j \in J_1$, $w_j \geq \min(\mu_j, w_j) > \beta_j$. Let $\eta' \in E$ be such that for $i \neq i_0$, $\eta'_{i} = \eta_i$ and
Let $\eta_{i_0} = \eta_{i_0} + 1$. Then $w' \in \Delta_1(v_q, I')$, which leads to a contradiction with the definition of $\eta$. Therefore, $J_2 \subseteq J_1$. In addition, since $v^1 \in E$, we have $J_2 \neq \emptyset$ by lemma 3.1.

**Proposition 3.9.** Let $q \geq 2$. Let us assume that there exist a sequence of sets

$$J_q \subseteq J_{q-1} \subseteq \cdots \subseteq J_2 \subseteq J_1$$

and a sequence of elements of $E$

$$v^1, v^2, \ldots, v^{q-1}$$

such that, if $q \geq 3$, for all $\ell \in \{2, \ldots, q-1\}$, we have:

1. For all $i \notin J_q$, $v^\ell_i = v^{\ell-1}_i$,
2. For all $j \in J_q$, $v^\ell_j = v^{\ell-1}_j - 1$,
3. $J_{\ell+1} = \{ j \in \{1, \ldots, p\} \mid |\Delta_j(v^\ell, I') \neq \emptyset \}$.

Let $v^q \in E$ be such that for all $i \notin J_q$, $v^q_i = v^{q-1}_i$ and for all $j \in J_q$, $v^q_j = v^{q-1}_j - 1$. Let $J_{q+1} = \{ j \in \{1, \ldots, p\} \mid |\Delta_j(v^q, I') \neq \emptyset \}$. Then $J_{q+1} \subseteq J_q$ and $J_{q+1} \neq \emptyset$.

**Remark 3.10.** One can notice that for all $\ell \in \{1, \ldots, q-1\}$, for all $m \in \{1, \ldots, \ell-1\}$ and for all $j \in J_m \setminus J_{m+1}$, $v^\ell_j = \beta_j - m$ and for all $j \in J_{\ell+1}$, $v^\ell_j = \beta_j - \ell$.

**Proof.** Let us assume that there exists $i_0 \in \{1, \ldots, p\} \setminus J_q$ such that $\Delta_{i_0}(v^q, I') \neq \emptyset$. If $w \in \Delta_{i_0}(v^q, I')$, then:

- $w_{i_0} = v^q_{i_0} = v^{q-1}_{i_0}$,
- for all $i \notin J_q \cup i_0$, $w_i > v^q_i = v^{q-1}_i$,
- for all $i \in J_q$, $w_i \geq v^q_i + 1 = v^{q-1}_i$.

Since $i_0 \notin J_q$, from the definition of $J_q$ we have $\Delta_{i_0}(v^{q-1}, I') = \emptyset$. Therefore, there exists $j_0 \in J_q$ such that $w_{j_0} = v^{q-1}_{j_0} = v^q_{j_0} + 1$.

We set

$$s = \inf_{w \in \Delta_{i_0}(v^q, I')} \left( \text{Card} \left( \{ j \in J_q \mid w_j = v^q_j - 1 \} \right) \right).$$

We then have $s \geq 1$. 

**□**
Let us choose an element $w \in \Delta_{\omega}(v^q, I')$ such that the cardinality of $\{j \in J_q | w_j = v_j^{q-1}\}$ is $s$.

Let $j_0 \in \{j \in J_q | w_j = v_j^{q-1}\}$. Since $j_0 \in J_q$, there exists $w' \in \Delta_{j_0}(v^{q-1}, I')$. We thus have:

- $w'_{j_0} = w_{j_0} = v_{j_0}^{q-1} = v_{j_0}^q + 1$,
- $w'_{i_0} > w_{i_0} = v_{i_0}^{q-1}$,
- for all $i \notin J_q \cup \{i_0\}$, $w'_i > v_i^{q-1} = v_i^q$,
- for all $j \in J_q \setminus \{j_0\}$, $w'_j > v_j^{q-1} > v_j^q$.

By proposition 2.11 applied to the triple $(w, w', j_0)$, there exists $u \in \text{val}(I')$ such that for all $i \notin J_q \cup \{i_0\}$, $u_i > v_i^{q-1} = v_i^q$, $u_{i_0} = v_{i_0}^{q-1} = v_{i_0}^q$, $u_{j_0} > v_{j_0}^{q-1} > v_{j_0}^q$ and for all $j \in J_q \setminus \{j_0\}$, $u_j \geq \min(w_j, w'_j) > v_j^q = v_j^{q-1} - 1$. In addition, for all $j \in J_q \setminus \{j_0\}$, if $w_j > v_j^{q-1}$, then $u_j > v_j^{q-1}$. Therefore, $u \in \Delta_{i_0}(v^q, I')$ and $\text{Card}(\{j \in J_q | u_j = v_j^{q-1}\}) = s - 1$, which contradicts the minimality of $s$. Hence the result: $J_{q+1} \subseteq J_q$.

By lemma 3.1, for all $\ell \in \{1, \ldots, q\}$, since $v^\ell \in \mathcal{E}$, we have $\Delta(v^\ell, I') \neq \emptyset$, thus $J_{\ell+1} \neq \emptyset$. \hfill \square

We can now finish the proof of theorem 1.1.

By proposition 3.9, for all $\ell \in \mathbb{N}$, $v^\ell \in \mathcal{E}$ so that $\Delta(v^\ell, I') \neq \emptyset$ and we have $J_{\ell+1} \neq \emptyset$. There exists $q \in \mathbb{N}$ such that for all $j \in J_1$, $\beta_j - q < \lambda_j$ where $\lambda$ is defined in lemma 3.4. Since for all $j \in J_q$, $v_j^q = \beta_j - q < \lambda_j$, $\Delta_j(v^q, I') = \emptyset$, which is impossible, and which finishes the proof of the missing implication of theorem 1.1: if $\beta$ is an absolute maximal of $I$ then $\alpha$ is a relative maximal of $I$. \hfill \square

Remark 3.11. Since we only used combinatorial properties of the set of values in the proof of theorem 1.1, this theorem can also be stated in the context of good semigroup ideals as considered in [11].

4. Description of the set of values of a fractional ideal.

4.1. The generation theorem. The purpose of this subsection is to generalize the generation theorem [5, Theorem 1.5]. If $J \subseteq$
{1, \ldots, p}$, we denote by $pr_J : \mathbb{Z}^p \to \mathbb{Z}^{\lvert J \rvert}$ the surjective map defined by $pr_J(v_1, \ldots, v_p) = (v_j)_{j \in J}$.

It follows from remark 2.19 that the set of the relative maximals of any fractional ideal is finite.

**Theorem 4.1.** Let $R$ be a local admissible ring, and $I \subseteq \text{Frac}(R)$ be a fractional ideal.

Let $RM(I) = \{ \alpha^1, \ldots, \alpha^q \}$ be the set of the relative maximals of $I$. Let $v \in \mathbb{Z}^p$ be such that for all $J \subseteq \{1, \ldots, p\}$ with $\lvert J \rvert = p - 1$, we have $pr_J(v) \in pr_J(\text{val}(I))$. Then $v \in \text{val}(I)$ if and only if for all $i \in \{1, \ldots, q\}$, $v \notin \Delta(\alpha^i, \mathbb{Z}^p)$.

**Proof.** We extend the proof of [5, Theorem 1.5] as follows. The first implication is immediate, since for all $i \in \{1, \ldots, p\}$, $\Delta(\alpha^i, I) = \emptyset$.

Let us denote for $i \in \{1, \ldots, p\}$, $J \subseteq \{1, \ldots, p\}$ and $\alpha \in \mathbb{Z}^p$:

$$\Delta^i_J(\alpha, I) = \{ w \in \text{val}(I) | \forall j \in J, w_j = \alpha_j \text{ and } \forall r \leq i, w_r \geq \alpha_r \text{ and } \forall s > i \text{ with } s \notin J, w_s > \alpha_s \}.$$ 

Let $v$ be such that for all $J \subseteq \{1, \ldots, p\}$ with $\lvert J \rvert = p - 1$, we have $pr_J(v) \in pr_J(\text{val}(I))$, and such that for all $i \in \{1, \ldots, q\}$, $v \notin \Delta(\alpha^i, \mathbb{Z}^p)$. Let us assume that $v \notin \text{val}(I)$. Then, there exists $j \in \{1, \ldots, p\}$ such that $\Delta^j_I(v, I) = \emptyset$. Indeed, if for all $j \in \{1, \ldots, p\}$, $\emptyset = \Delta^j_I(v, I) \ni w^j$, then $v = \inf(w^1, \ldots, w^p) \in \text{val}(I)$, which contradicts our assumption. By renumbering the branches, we can assume that $\Delta^j_I(v, I) = \emptyset$.

Let $i \in \{1, \ldots, p\}$ be the smallest integer such that there exists $v_{i+1}', \ldots, v_p' \in \mathbb{Z}$, $\eta_{i+1}', \ldots, \eta^p' \in \text{val}(I)$ such that by denoting $v^{(i+1)'} = (v_1, \ldots, v_i, v_{i+1}', \ldots, v_p')$ we have:

- for all $k \in \{i + 1, \ldots, p\}$, $v_k' < v_k$,
- for all $k \in \{i + 1, \ldots, p\}$, $\eta^k \in \Delta^i_{i,k}(v^{(i+1)'}, I)$,
- $\Delta^i_I(v^{(i+1)'}, I) = \emptyset$.

One can notice that this condition is always satisfied for $i = p$ since we then have $v = v^{(p+1)'}$ so that $\Delta^p_I(v^{(p+1)'}, I) = \emptyset$.
Let us assume that $i > 1$. We recall that there exists $\lambda \in \mathbb{N}^p$ such that $\text{val}(I) \subseteq \lambda + \mathbb{N}^p$. We set $v^* = (v_1, \ldots, v_{i-1}, \lambda, v'_{i+1}, \ldots, v'_p)$.

By assumption, we have
\[(v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_p) \in \text{pr}_{\{1, \ldots, i-1, i+1, \ldots, p\}}(I)\]
so that $\Delta^1_1(v^*, I) \neq \emptyset$.

In addition, since $\Delta^1_1(v^{(i+1)'}, I) = \emptyset$, if $(v_1, w_2, \ldots, w_p) \in \Delta^1_1(v^*, I)$ then $w_i < v_i$. Let $i'$ be the maximal integer such that there exists $\eta^i = (v_1, w_2, \ldots, w_{i-1}, v'_i, w_{i+1}, \ldots, w_p) \in \Delta^1_1(v^*, I)$. In particular, $v'_i < v_i$. We set $v^{(i)'} = (v_1, \ldots, v_{i-1}, v'_i, v'_{i+1}, \ldots, v'_p)$. Then $\eta^i \in \Delta^{i-1}_1(v^{(i)'}, I)$, and for all $k \in \{i+1, \ldots, p\}$, $\eta^k \in \Delta^{i-1}_{i,k}(v^{(i)'}, I)$. In addition, since $v'_i$ is maximal, $\Delta^{i-1}_1(v^{(i)'}, I) = \emptyset$, so that $i$ was not minimal.

Therefore, $i = 1$, which means that there exists $v'_2, \ldots, v'_p \in \mathbb{N}$ such that for all $k \in \{2, \ldots, p\}$, $v'_k < v_k$, there exists $\eta^k \in \Delta_{1,k}(v^{(2)'}, I)$, and $\Delta_{1}(v^{(2)'}, I) = \emptyset$. By lemma 2.20, it means that $v^{(2)'}$ is a relative maximal of $I$. In addition, $v \in \Delta_1(v^{(2)'}, \mathbb{Z}^p)$, which contradicts the hypothesis that for all relative maximal $\alpha$, $v \notin \Delta(\alpha, \mathbb{Z}^p)$. Therefore, $v \in \text{val}(I)$.

For an irreducible curve, the computation of the set of values of an ideal is described in [8]. An algorithm for two branches is suggested in [14]. If the set of relative maximals is known, the previous theorem can be used to compute the set of values of an ideal of a curve with three branches, and by induction for an arbitrary number of branches, provided that at each step the relative maximals are known.

The question is therefore to compute the relative maximals of $I$. Thanks to theorem 1.1, it is equivalent to the computation of the absolute maximals of $I^{(p)}$. In the following, we prove that we only need to know the set of irreducible absolute maximals which is defined below and generalizes [5, Remark 3.11].

**Proposition 4.2.** Let $R$ be a local admissible ring and let $I \subseteq \text{Frac}(R)$ be a fractional ideal. Let $v \in \text{val}(I)$ be an absolute maximal. Let us assume that $v = \alpha + \beta$ with $\alpha \in \text{val}(R)$ and $\beta \in \text{val}(I)$. Then $\alpha$ is an absolute maximal of $R$ and $\beta$ is an absolute maximal of $I$. 
Proof. Let us assume that $\beta$ is not an absolute maximal of $I$. Then there exists $J \subseteq \{1, \ldots, p\}$ with $J \neq \emptyset$ and $J \neq \{1, \ldots, p\}$ such that there exists $w \in \Delta_J(\beta, I)$. Then, $\alpha + w \in \text{val}(I)$ since $\alpha \in \text{val}(R)$ and $w \in \text{val}(I)$. Thus $\alpha + w \in \Delta_J(v, I)$, which contradicts the fact that $v$ is an absolute maximal of $I$. A similar argument can be used to prove that $\alpha$ is an absolute maximal of $R$. □

Definition 4.3. Let $v \in \text{val}(I)$ be an absolute maximal. We call $v$ an irreducible absolute maximal if it cannot be written as $v = \alpha + \beta$ with $\alpha \in \text{val}(R) \setminus \{0\}$ and $\beta \in \text{val}(I)$. More generally, if $v \in \text{val}(I)$, we say that $v$ is irreducible if for all $\alpha \in \text{val}(R)$ and for all $\beta \in \text{val}(I)$, the condition $v = \alpha + \beta$ implies $\alpha = 0$.

The following proposition is a direct consequence of the definition:

Proposition 4.4. Let $G = \{g_1, \ldots, g_q\}$ be the set of irreducible absolute maximals of $R$ and $\{\alpha_1, \ldots, \alpha_r\}$ be the set of irreducible absolute maximals of $I$. Then the set of absolute maximals of $I$ is contained in the set $\bigcup_{i=1}^r (\mathbb{N}g_1 + \mathbb{N}g_2 + \cdots + \mathbb{N}g_q + \alpha_i)$.

To apply the generation theorem 4.1 to a fractional ideal $I$, we need the absolute maximals of $I^\vee$.

Notation 4.5. Let $\{g_1, \ldots, g_q\}$ be the set of irreducible absolute maximals of $R$, and $\{\beta_1, \ldots, \beta_s\}$ be the set of irreducible absolute maximals of the ideal $I^\vee$. Let $\nu_{I^\vee} \in \mathbb{Z}^p$ be such that $\nu_{I^\vee} + \mathbb{N}^p \subseteq \text{val}(I^\vee)$. We set

$$F = \left\{ \sum_{j=1}^q \lambda_j g_j + \beta_i \mid i \in \{1, \ldots, s\} \text{ and } \lambda_j \in \mathbb{N} \text{ for all } j \in \{1, \ldots, q\} \right\} \subseteq \text{val}(I^\vee).$$

and

$$\left\{ \sum_{j=1}^q \lambda_j g_j + \beta_i \leq \nu_{I^\vee} - 1 \right\} \subseteq \text{val}(I^\vee).$$

Since $\nu_{I^\vee} + \mathbb{N}^p \subseteq \text{val}(I^\vee)$, one can notice that if $\beta$ is an absolute maximal of $I^\vee$, then $\beta \leq \nu_{I^\vee} - 1$.

We also set $F' = \{\gamma - u - 1 \mid u \in F\}$, where $\gamma$ is the conductor of $R$. 
We therefore have the following result:

**Proposition 4.6.** Let $v \in \mathbb{Z}^p$ be such that for all $J \subseteq \{1, \ldots, p\}$ with $|J| = p - 1$, we have $\text{pr}_J(v) \in \text{pr}_J(\text{val}(I))$. Then:

$$v \in \text{val}(I) \iff \forall w \in F', v \notin \Delta(w, \mathbb{Z}^p).$$

**Proof.** Let us assume that $v \in \text{val}(I)$ and that there exists $w \in F'$ such that $v \in \Delta(w, \mathbb{Z}^p)$. For example, we may assume that $v \in \Delta^1(w, \mathbb{Z}^p)$. Since $v \in \text{val}(I)$, by theorem 2.15, $\Delta(\gamma - v - 1, I) = \emptyset$. We have $(\gamma - v - 1)1 = \gamma_1 - w_1 - 1$ and for all $j \geq 2$, $(\gamma - v - 1)_j < \gamma_j - w_j - 1$. In addition, $\gamma - w - 1 \in \text{val}(I')$ by definition of $F$ and $F'$. Thus $\gamma - w - 1 \in \Delta(\gamma - v - 1, I')$, which is a contradiction.

The other implication is a consequence of theorem 4.1. Indeed, the set of absolute maximals of $I'$ is contained in $F$, and by theorem 1.1, the set of relative maximals of $I$ is contained in $F'$.

\[\Box\]

### 4.2. Irreducible absolute maximals of an ideal.

The remaining problem would be to determine the absolute maximals of $I'$. When $C$ is a plane curve, the set of irreducible absolute maximals of $\mathcal{O}_C$ coincide with the values of maximal contact which are finite, see [5] for more details.

We identify here a subset of the set of irreducible absolute maximals of a fractional ideal $I$ of a reduced reducible germ of curve $C \subseteq (\mathbb{C}^m, 0)$. Let $p$ be the number of irreducible components of $C$. For $\alpha \in \mathbb{Z}^p$ we will denote by $t^\alpha$ the element $(t_1^{\alpha_1}, \ldots, t_p^{\alpha_p})$ of $\mathcal{O}_C = \bigoplus_{i=1}^p \mathbb{C} \{t_i\}$.

The set of values of a fractional ideal of an irreducible curve can be deduced from a standard basis. An algorithm computing the standard basis of an ideal is given in [8].

**Definition 4.7 ([8, Definition 2.1]).** Let $\mathcal{G} = \{g_1, \ldots, g_s\} \subseteq \mathcal{O}_C$.

- A $\mathcal{G}$-product is an element of the form $\prod_{i=1}^s g_i^{\alpha_i}$ where for all $i \in \{1, \ldots, s\}$, $\alpha_i \in \mathbb{N}$.
- The set $\mathcal{G}$ is called a standard basis of $\mathcal{O}_C$ if for all $f \in \mathcal{O}_C$, there exists a $\mathcal{G}$-product $g$ such that $\text{val}(g) = \text{val}(f)$. In other words,

$$\text{val}(g_1)\mathbb{N} + \cdots + \text{val}(g_s)\mathbb{N} = \text{val}(\mathcal{O}_C).$$
• Let \( \mathcal{H} \subseteq I \). The couple \((\mathcal{H}, \mathcal{G})\) is called a standard basis of \( I \) if \( \mathcal{G} \) is a standard basis of \( \mathcal{O}_C \) and if for all \( f \in I \), there exist \( h \in \mathcal{H} \) and a \( \mathcal{G} \)-product \( g \) such that \( \text{val}(f) = \text{val}(g) + \text{val}(h) \).

• Let \((\mathcal{H}, \mathcal{G})\) be a standard basis of \( I \). We say that \( \mathcal{H} \) is minimal if for all \( h \neq h' \in \mathcal{H} \) we have \( \text{val}(h') \notin \text{val}(\mathcal{O}_C) + \text{val}(h) \).

**Remark 4.8.** Let \((\mathcal{H}, \mathcal{G})\) be a standard basis of \( I \). If there exists \( h \neq h' \in \mathcal{H} \) such that \( \text{val}(h') \in \text{val}(\mathcal{O}_C) + \text{val}(h) \), then there is a \( \mathcal{G} \)-product \( g \) such that \( \text{val}(h') = \text{val}(g) + \text{val}(h) \). Then \((\mathcal{H} \setminus \{ h' \}, \mathcal{G})\) is also a standard basis of \( I \). By iterating this process, one can deduce from \((\mathcal{H}, \mathcal{G})\) a standard basis \((\mathcal{H}', \mathcal{G}')\) where \( \mathcal{H}' \) is minimal.

**Notation 4.9.** Let for \( i \in \{1, \ldots, p\} \), \( \pi_i : \text{Frac}(\mathcal{O}_C) \to \text{Frac}(\mathcal{O}_{C_i}) \) be the natural surjection. We set \( I_i = \pi_i(I) \subseteq \text{Frac}(\mathcal{O}_{C_i}) \).

For all \( i \in \{1, \ldots, p\} \), let us consider a standard basis \((\mathcal{H}^i, \mathcal{G}^i)\) of \( I_i \) where \( \mathcal{H}^i \) is minimal, and let us write \( \mathcal{H}^i = \{ H^i_0, \ldots, H^i_{s_i} \} \) and \( \mathcal{G}^i = \{ g^i_0, \ldots, g^i_{s_i} \} \). For all \( j \in \{0, \ldots, s_i\} \), let \( h^i_{j,i} = \text{val}_i(H^i_j) \in \mathbb{Z} \).

Let us fix \( \nu \in \text{val}(I) \) such that \( \nu + \mathbb{N}^p \subseteq \text{val}(I) \). We do not assume that \( \nu \) is the conductor of \( I \).

**Notation 4.10.** For all \( i \in \{1, \ldots, p\} \) and for all \( j \in \{0, \ldots, s_i\} \), we set

\[
E^i_j(\nu) = \{ v \in \text{val}(I) | v_i = h^i_{j,i} \text{ and } v_\ell \leq v_i \text{ for all } \ell \neq i \}.
\]

**Lemma 4.11.** Let \( \alpha \in E^i_j(\nu) \). Then \( \alpha \) is an irreducible element.

**Proof.** Let us assume that \( \alpha = a + h \) with \( a \in \text{val}(\mathcal{O}_C) \) and \( h \in \text{val}(I) \). In particular, \( h^i_{j,i} = a_i + h_i \) with \( a_i \in \text{val}_i(\mathcal{O}_{C_i}) \) and \( h_i \in \text{val}_i(I_i) \). Since \( \mathcal{H}^i \) is minimal we have \( a_i = 0 \) and \( h_i = h^i_{j,i} \). Therefore, since \( \mathcal{O}_C \) is a local ring, one has \( a = 0 \) (see \([6, (1.1.1)]\)). Therefore, \( \alpha \) is irreducible. \( \square \)

**Proposition 4.12.** Let \( \alpha \in E^i_j(\nu) \). Then \( \alpha \) is an irreducible absolute maximal of \( I \) if and only if for all \( \ell \in \{1, \ldots, p\}, \alpha_\ell \neq v_\ell \) and \( \alpha \) is an element of \( E^i_j(\nu) \) which is maximal for the product order of \( \mathbb{Z}^p \).
Proof. Let $\alpha \in E_j^i(\nu)$. If $\alpha$ is not a maximal element of $E_j^i(\nu)$ for the product order, then there exists $\alpha' \in E_j^i(\nu)$ such that $\alpha' \geq \alpha$, $\alpha' \neq \alpha$ and $\alpha'_i = \alpha_i$, so that $\alpha' \in \Delta_J(\alpha, I)$ for a subset $J \subseteq \{1, \ldots, p\}$ with $J \neq \emptyset$ and $J \neq \{1, \ldots, p\}$. Thus, $\alpha$ is not an absolute maximal of $I$.

Let us assume that there exists $\ell \in \{1, \ldots, p\}$ such that $\alpha_\ell = \nu_\ell$. We define $\alpha' \in \mathbb{Z}_+$ such that for all $q \neq \ell$, $\alpha'_q = \alpha_q$ and $\alpha'_\ell = \alpha_\ell + 1$. Then, since $\nu' \in \mathbb{Z}_+$, by [14, Corollary 3.12] we have $\alpha' \in \text{val}(I)$ so that $\alpha' \in \Delta_{\{1, \ldots, p\}\setminus \{\ell\}}(\alpha, I)$ and $\alpha$ is not an absolute maximal of $I$.

Let us prove the converse implication. Let $\alpha \in E_j^i(\nu)$ be such that for all $\ell \in \{1, \ldots, p\}$, $\alpha_\ell \neq \nu_\ell$ and $\alpha$ is an element of $E_j^i(\nu)$ maximal for the product order. Let us prove that $\alpha$ is an irreducible absolute maximal of $I$. By lemma 4.11, $\alpha$ is irreducible. Let us assume that $\alpha$ is not an absolute maximal of $I$. There exists $J \subseteq \{1, \ldots, p\}$, $J \neq \emptyset$ and $J \neq \{1, \ldots, p\}$ such that $\Delta_J(\alpha, I) \neq \emptyset$. Let $w \in \Delta_J(\alpha, I)$.

If $i \in J$, then by proposition 2.10, $w' := \inf(\nu, w) \in E_j^i(\nu)$, $w' \geq \alpha$, and $w' \neq \alpha$, which contradicts the fact that $\alpha$ is maximal for the product order in $E_j^i(\nu)$.

Thus, $i \notin J$. We then have $w_i > \alpha_i$, and there exists $\ell \in \{1, \ldots, p\}$, $\ell \neq i$, such that $\alpha_\ell = w_\ell$. Let us apply proposition 2.11 to the triple $(\alpha, w, \ell)$. There exists $v \in \text{val}(I)$ such that $v_i = \alpha_i$, $v_\ell > \alpha_\ell$, and for all $q \notin \{i, \ell\}$, $v_q \geq \min(\alpha_q, w_q) \geq \alpha_q$. Therefore, by proposition 2.10, $v' := \inf(\nu, v) \in E_j^i(\nu)$, $v' \geq \alpha$ and $v'_\ell > \alpha_\ell$, which contradicts the fact that $\alpha$ is a maximal element for the product order in $E_j^i(\nu)$. Hence the result.

Lemma 4.13. The set of the elements $\alpha \in E_j^i(\nu)$ such that for all $\ell \in \{1, \ldots, p\}$, $\alpha_\ell \neq \nu_\ell$ and $\alpha$ is an element of $E_j^i(\nu)$ maximal for the product order does not depend on the choice of an element $\nu$ satisfying $\nu + \mathbb{N}^p \subseteq \text{val}(I)$.

Proof. Let $\nu, \nu'$ be such that $\nu + \mathbb{N}^p \subseteq \text{val}(I)$ and $\nu' + \mathbb{N}^p \subseteq \text{val}(I')$. Let $\alpha \in E_j^i(\nu)$ be such that for all $\ell \in \{1, \ldots, p\}$, $\alpha_\ell \neq \nu_\ell$ and $\alpha$ is a maximal element for the product order in $E_j^i(\nu)$. Let us prove that $\alpha \in E_j^i(\nu')$. Let us set $\alpha' = \inf(\alpha, \nu')$. Let us prove that for all $\ell \in \{1, \ldots, p\}$, $\alpha'_\ell \neq \nu'_\ell$. Let us assume that there exists $L \subseteq \{1, \ldots, p\}$, $L \neq \emptyset$, such that for all $\ell \in L$, $\alpha'_\ell = \nu'_\ell$. In particular, it means that
for all $\ell \in L$, $\nu'_\ell < \nu_\ell$. Let $f \in I$ be such that $\text{val}(f) = \alpha'$. Since $t^{\nu_\ell} g' \subseteq I$, one can choose an element $g \in I$ such that the restriction $g|_{C_q}$ of $g$ to $C_q$ for $q \notin L$ is zero, and for all $\ell \in L$, $g|_{C_\ell} = f|_{C_\ell} + t^{\nu_\ell}_{\ell}$, so that for $q \notin L$, $\text{val}_q(f-g) = \alpha_q$ and for $\ell \in L$ $\text{val}_\ell(f-g) = \nu_\ell$.

Then $\text{val}(f-g) \in E_j(\nu)$ and $\alpha \leq \text{val}(f-g)$, which contradicts the fact that $\alpha$ is an element of $E_j(\nu)$ which is maximal for the product order.

Therefore, for all $q \in \{1, \ldots, p\}$, $\alpha_q < \nu'_q$. In particular, $\alpha \in E_j(\nu')$.

It remains to prove that $\alpha$ is maximal for the product order in $E_j(\nu')$.

Let us assume the contrary. Then there exists $v \in E_j(\nu')$ such that $v \geq \alpha$ and $v \neq \alpha$. Let $v' = \inf(v, \nu)$. Then $v' \in E_j(\nu)$, $v' \geq \alpha$ and since for all $j, \alpha_j \neq \nu_j$, $v' \neq \alpha$, it contradicts the maximality for the product order of $\alpha$ in $E_j(\nu')$. □

Remark 4.14. We determine in proposition 4.12 the irreducible absolute maximals which have at least one coordinate equal to the valuation of an element in a minimal standard basis of the ideal along the corresponding branch. It is not obvious that this property is satisfied for all irreducible absolute maximals of $I$. The computation of the irreducible absolute maximals in [5] relies on the Hamburger-Noether expansion of plane curves and elements of maximal contact which are particular to the semigroup of a plane curve.

If we do not restrict ourselves to semigroups arising from curves, it is possible to find irreducible absolute maximals for which all the coordinates are not irreducible. Indeed, if we consider the semigroup defined in [1, Example 2.16, Figure 3], which is an example of good semigroup which is not the semigroup of values of any ring, then $(16, 20)$ is an irreducible absolute maximal of the semigroup, but 16 and 20 are not irreducible when we consider the projection respectively along the first and second coordinate.

REFERENCES


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