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G-STABLE RANK OF SYMMETRIC TENSORS AND LOG CANONICAL THRESHOLD

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ABSTRACT. Shitov recently gave counterexamples over the real and complex field to Comon's conjecture that the symmetric tensor rank and tensor rank of a symmetric tensor are the same. In this paper we show that an analog of Comon's conjecture for the G-stable rank introduced by Derksen is true: the symmetric G-stable rank and G-stable rank of a symmetric tensor are the same over perfect fields. We also show that the log-canonical threshold of a complex singularity is bounded by the G-stable rank of the defining ideal.

1. Introduction

An order d tensor is a vector in a tensor product of d vector spaces. The are several generalizations of the rank of a matrix to tensors of order ≥ 3, for example the tensor rank, border rank, sub-rank, slice rank and G-stable rank. A simple tensor in $V_1 \otimes V_2 \otimes \cdots \otimes V_d$ is a tensor of the form $v_1 \otimes v_2 \otimes \cdots \otimes v_d$, where $v_i \in V_i$. The tensor rank of $T \in V_1 \otimes V_2 \otimes \cdots \otimes V_d$ is the smallest number of simple tensors that sum up to T. The G-stable rank of a tensor was introduced by Derksen in [2]. The slice rank and G-stable rank have been used to find bounds for the cap set problem (see [2], [3], [5], [14]).

If $V_1 = V_2 = \cdots = V_d = V$ then there is a natural action of the symmetric group S_d on $V^{\otimes d} = V \otimes V \otimes \cdots \otimes V$. A tensor invariant under this action is called a symmetric tensor of order d. The Waring rank or symmetric rank of a symmetric tensor $T \in V^{\otimes d}$ is the smallest number d such that T can be written as a sum of d tensors of the form $v^{\otimes d} = v \otimes v \otimes \cdots \otimes v$. It is clear that the tensor rank is less than or equal to the symmetric rank. It was conjectured by Comon [1] that the symmetric rank and tensor rank of a symmetric tensor are equal. Sufficient conditions were given in [4] under which Comon's conjecture is true. Comon's conjecture was proved when the rank of a tensor is less than its order [15] [16]. However, Shitov gave counterexamples over the real [12] and complex field [13]. It is also proved in [10] that there is an order 6 real tensor whose rank and symmetric rank differ. In this paper, we study the notion of G-stable rank of a tensor. The G-stable rank of a tensor is defined in terms of geometric invariant theory and the notion of stability for algebraic group actions on tensors. It is also natural to define a symmetric G-stable rank for a symmetric tensor. One main result of this paper is that the symmetric G-stable rank and G-stable rank of a symmetric tensor are the same over perfect fields.

In algebraic geometry and singularity theory, the log canonical threshold is an important invariant of singularities. We will show that the symmetric G-stable rank and the log canonical threshold are closely related. We extend the notion of G-stable rank to ideals in a coordinate ring of a smooth complex irreducible affine variety. In this context, we show that the log canonical threshold is less than or equal to the G-stable rank. In the case of monomial ideals in the polynomial ring we show equality.

1.1. ***Stability of tensors.*** Let K be a perfect field and G be a reductive algebraic group over K. Suppose $\rho : G \to \mathrm{GL}(W)$ is a rational representation of G. Let $\mathbb{O}_v, \overline{\mathbb{O}}_v, G_v$ denote the orbit, orbit-closure and stabilizer of v respectively. We have the following notions of stability:

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Definition 1. We say $v \in W$ is:

- G-unstable, if $0 \in \overline{\mathbb{O}}_v$;
- G-semistable, if $0 \notin \overline{\mathbb{O}}_v$;
- G-polystable, if $v \neq 0$ and $\mathbb{O}_v$ is closed;
- G-stable, if v is G-polystable and $\dim G_v = \dim(\text{kernel of } \rho)$.

We often omit the prefix G if G is clear in the context. The subset of G-unstable points is called the *null cone*. A 1-parameter subgroup of an algebraic group G is a homomorphism of algebraic groups $\lambda : \mathbb{G}_m \to G$, where $\mathbb{G}_m$ is the multiplicative group. For any integer m, we define the multiple of λ by m, denoted by $m \cdot \lambda$, which is also a 1-parameter subgroup with $(m \cdot \lambda)(t) = (\lambda(t))^m$. A 1-parameter subgroup is *indivisible* if it is not a multiple of any other 1-parameter subgroup with factor $m \geq 2$. It follows from the *Hilbert-Mumford* criterion that v is G-unstable iff there is a 1-parameter subgroup $\lambda : \mathbb{G}_m \to G$, such that

$$\lim_{t\to 0} \lambda(t) \cdot v = 0.$$

Example 2. Let $W = K^n$ be an n dimensional vector space. Fix a basis of W and let $p \in W$ be a point with coordinates $p = (x_1, x_2, \cdots, x_n)$. Consider the action of $\mathbb{G}_m$ on p by

$$t \cdot (x_1, x_2, \cdots, x_n) = (t^{-1}x_1, tx_2, \cdots, tx_n)$$

We note that any 1-parameter subgroup of $\mathbb{G}_m$ is of the form $\lambda_k(t) = t^k$ for some integer $k \in \mathbb{Z}$. If $x_1 \neq 0$ and $(x_2, x_3, \cdots, x_n) \neq 0$, then

$$\lim_{t\to 0} \lambda_k(t) \cdot (x_1, x_2, \cdots, x_n) = \lim_{t\to 0} (t^{-k}x_1, t^k x_2, \cdots, t^k x_n)$$

does not exist for all $k \neq 0$, hence p is $\mathbb{G}_m$-semistable. One can also check that in this case the orbit $\mathbb{O}_p$ is closed, therefore p is also $\mathbb{G}_m$-stable.

If $x_1 = 0$, by taking $k = 1$, we have $\lim_{t\to 0} \lambda_1(t) \cdot (x_1, x_2, \cdots, x_n) = \lim_{t\to 0}(0, tx_2, \cdots, tx_n) = 0$. Similarly, if $(x_2, x_3, \cdots, x_n) = 0$, by taking $k = -1$, we have $\lim_{t\to 0} \lambda_{-1}(t) \cdot (x_1, x_2, \cdots, x_n) = \lim_{t\to 0}(tx_1, 0, \cdots, 0) = 0$. Therefore $p = (x_1, x_2, \cdots, x_n)$ is $\mathbb{G}_m$-unstable if $x_1 = 0$ or $(x_2, x_3, \cdots, x_n) = 0$.

Let V be a finite dimensional vector space over K, we consider the action of the group of product of special linear groups $\mathrm{SL}(V)^d = \mathrm{SL}(V) \times \mathrm{SL}(V) \times \cdots \times \mathrm{SL}(V)$ on the tensor product space $V^{\otimes d} = V \otimes V \otimes \cdots \otimes V$. In this paper, we are interested in the stability of tensors in $V^{\otimes d}$ under the action of $\mathrm{SL}(V)^d$. By Definition 1, a tensor $v \in V^{\otimes d}$ is $\mathrm{SL}(V)^d$*-unstable* if there is a 1-parameter subgroup $\lambda : \mathbb{G}_m \to \mathrm{SL}(V)^d$, such that $\lim_{t\to 0} \lambda(t) \cdot v = 0$. If no such 1-parameter subgroup exists, then v is $\mathrm{SL}(V)^d$*-semistable*.

Let G be a reductive algebraic group over K. By a G-scheme X we mean a separated, finite type scheme X over K as well as a morphism $G \times X \to X$ mapping (g, x) to $g \cdot x$, such that $g \cdot (h \cdot x) = (gh) \cdot x$, for all $g, h \in G$ and for all $x \in X$. A morphism $f : X \to Y$ between two G-schemes X and Y is G-equivariant if for all $g \in G$ and $x \in X$, we have $f(g \cdot x) = g \cdot f(x)$. A subscheme S of a G-scheme X is called a G-subscheme if S is a G-scheme and the immersion $S \hookrightarrow X$ is G-equivariant.

Throughout this paper, we will work over a perfect field K. In [7], Kempf proved a K-rational version of the *Hilbert-Mumford* criterion:

Theorem 3 ([7], Corollary 4.3)**.** *Let G be a reductive algebraic group. Suppose that X is a G-scheme and $x \in X$ is a K-point. Assume S is a closed G-subscheme of X which does not contain x and S meets the closure of the orbit $G \cdot x$. Then there exits a K-rational 1-parameter subgroup $\lambda : \mathbb{G}_m \to G$, such that*

$$\lim_{t\to 0} \lambda(t) \cdot x \in S.$$

If $G = \mathrm{SL}(V)^d$, $X = V^{\otimes d}$, then by Theorem 3, v is unstable if and only if 0 is in the closure of the orbit of v, i.e. $0 \in \overline{\mathrm{SL}(V)^d \cdot v}$.

A tensor $T \in V^{\otimes d}$ is called symmetric if it is invariant under the action of symmetric group S_d. Let $D^d V \subseteq V^{\otimes d}$ be the space of symmetric tensors. As a representation of $\mathrm{GL}(V)$, this is the space of divided powers, which is isomorphic to the d-th symmetric power $S^d V$ if the characteristic of K is 0 or $> d$.

It is interesting to look at the diagonal action of $\mathrm{SL}(V)$ on $D^d V$ via the diagonal embedding:

$$\Delta : \mathrm{SL}(V) \hookrightarrow \mathrm{SL}(V)^d. \tag{1}$$

Definition 4. A symmetric tensor $v \in D^d V$ is $\mathrm{SL}(V)$-*unstable* if there is a 1-parameter subgroup $\lambda : \mathbb{G}_m \to \mathrm{SL}(V)$, such that

$$\lim_{t \to 0} \lambda(t) \cdot v = 0.$$

Otherwise we say v is $\mathrm{SL}(V)$-*semistable*.

1.2. *G-stable rank for tensors.* In [2], Derksen introduced G-stable rank for tensors. Suppose the base field K is perfect. If $\lambda : \mathbb{G}_m \to \mathrm{GL}_n$ is a 1-parameter subgroup, then we can view $\lambda(t)$ as an invertible $n \times n$ matrix whose entries lie in the ring $K[t, t^{-1}]$ of Laurent polynomials. We say that $\lambda(t)$ is a polynomial 1-parameter subgroup of GL_n if all these entries lie in the polynomial ring $K[t]$. Consider the action of the group $G = \mathrm{GL}(V_1) \times \mathrm{GL}(V_2) \times \cdots \times \mathrm{GL}(V_d)$ on the tensor product space $W = V_1 \otimes V_2 \otimes \cdots \otimes V_d$. A 1-parameter subgroup $\lambda : \mathbb{G}_m \to G$ can be written as

$$\lambda(t) = (\lambda_1(t), \cdots, \lambda_d(t)),$$

where $\lambda_i(t)$ is a 1-parameter subgroup of $\mathrm{GL}(V_i)$ for all i. We say that $\lambda(t)$ is polynomial if and only if $\lambda_i(t)$ is a polynomial 1-parameter subgroup for all i.

The t-valuation $\mathrm{val}(a(t))$ of a polynomial $a(t) \in K[t]$ is the biggest integer n such that $a(t) = t^n b(n)$ for some $b(t) \in K[t]$. For $a(t), b(t) \in K[t]$, the t-valuation $\mathrm{val}\left(\frac{a(t)}{b(t)}\right)$ of the rational function $\frac{a(t)}{b(t)} \in K(t)$ is $\mathrm{val}\left(\frac{a(t)}{b(t)}\right) = \mathrm{val}(a(t)) - \mathrm{val}(b(t))$. For a tuple $u(t) = (u_1(t), u_2(t), \cdots, u_d(t)) \in K(t)^d$, we define the t-valuation of $u(t)$ as

$$\mathrm{val}(u(t)) = \min_i \{\mathrm{val}(u_i(t)) | 1 \le i \le d\}. \tag{2}$$

If λ is a 1-parameter subgroup of G and $v \in W$ is a tensor, then we have $\lambda(t) \cdot v \in K(t) \otimes W$. We view $K(t) \otimes W$ as a vector space over $K(t)$ and define the t-valuation $\mathrm{val}(\lambda(t) \cdot v)$ as in (2). Assume $\mathrm{val}(\lambda(t) \cdot v) > 0$, then for for any $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_d) \in \mathbb{R}^d_{>0}$, we define the slope

$$\mu_\alpha(\lambda(t), v) = \frac{\sum_{i=1}^d \alpha_i \,\mathrm{val}(\det(\lambda_i(t)))}{\mathrm{val}(\lambda(t) \cdot v)}. \tag{3}$$

The G-stable rank for $v \in W$ is the infimum of the slope with respect to all such 1-parameter subgroups. More precisely:

Definition 5 ([2], Theorem 2.4)**.** If $\alpha \in \mathbb{R}^d_{>0}$, then the G-stable rank $\mathrm{rk}^G_\alpha(v)$ is the infimum of $\mu_\alpha(\lambda(t), v)$ where $\lambda(t)$ is a polynomial 1-parameter subgroup of $G = \mathrm{GL}(V_1) \times \cdots \times \mathrm{GL}(V_n)$ and $\mathrm{val}(\lambda(t) \cdot v) > 0$. If $\alpha = (1, 1, \cdots, 1)$, we simply write $\mathrm{rk}^G(v)$.

Remark 6. In Definition 1.3 of the original paper [2], the G-stable rank is defined as the infimum of the slope $\mu_\alpha(g(t), v)$ over a more general family of group elements $g(t) \in G(K[[t]]) = \mathrm{GL}(V_1, K[[t]]) \times \mathrm{GL}(V_2, K[[t]]) \times \cdots \times \mathrm{GL}(V_n, K[[t]])$ with $\mathrm{val}(g(t) \cdot v) > 0$, where $\mathrm{GL}(V_i, K[[t]])$ is the group of $K[[t]]$-endomorphisms of the space $K[[t]] \otimes_K V_i$. It is then proved in [2] Theorem 2.4 that to compute the G-stable rank, it suffices to consider all polynomial 1-parameter subgroups. For the purpose of this paper, we use the latter as our definition of G-stable rank.

Let V be a finitely dimensional vector space over K. Let $D^dV \subset V^{\otimes d}$ be the space of all symmetric tensors. Assume the group $\mathrm{GL}(V)$ acts on D^dV via the diagonal embedding: $\mathrm{GL}(V) \hookrightarrow \mathrm{GL}(V)^d$.

Definition 7. Let $v \in D^dV$ be a symmetric tensor, the symmetric G-stable rank $\mathrm{symmrk}^G(v)$ of v is the infimum of $\mu(\lambda(t),v) = d\frac{\mathrm{val}(\det(\lambda(t)))}{\mathrm{val}(\lambda(t)\cdot v)}$, where $\lambda(t)$ is a polynomial 1-parameter subgroup of $\mathrm{GL}(V)$ and $\mathrm{val}(\lambda(t)\cdot v) > 0$.

Since any 1-parameter subgroup of $\mathrm{GL}(V)$ is also a 1-parameter subgroup of $\mathrm{GL}(V)^d$ via the diagonal embedding, we have $\mathrm{symmrk}^G(v) \geq \mathrm{rk}^G(v)$ for any $v \in D^dV$. It turns out that the other inequality is also true,

Theorem 8. *Let $v \in D^dV$ be a symmetric tensor, then we have*

$$\mathrm{symmrk}^G(v) = \mathrm{rk}^G(v).$$

Example 9. Suppose that $V = K^2$, and $v = e_2 \otimes e_1 \otimes e_1 + e_1 \otimes e_2 \otimes e_1 + e_1 \otimes e_1 \otimes e_2 \in V^{\otimes 3}$, where $\{e_1, e_2\}$ is the standard basis of $V = K^2$. Let $\lambda(t) = \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}$ be a polynomial 1-parameter subgroup of $\mathrm{GL}(K^2)$. Then $\lambda(t)\cdot v = t^2 v$, $\det(\lambda(t)) = t$, the slope is

$$\mu(\lambda(t),v) = 3\frac{\mathrm{val}(\det(\lambda(t)))}{\mathrm{val}(\lambda(t)\cdot v)} = \frac{3}{2}.$$

Therefore we have $\mathrm{symmrk}^G(v) \leq \frac{3}{2}$. It was proved in [2] that $\mathrm{rk}^G(v) = \frac{3}{2}$. Hence by the fact $\mathrm{symmrk}^G(v) \geq \mathrm{rk}^G(v)$ we have $\mathrm{symmrk}^G(v) = \frac{3}{2}$.

A 1-parameter subgroup of $\mathrm{SL}(V)$ is also a 1-parameter subgroup of $\mathrm{SL}(V)^d$ via the diagonal embedding. It follows that if a symmetric tensor $v \in D^dV$ is $\mathrm{SL}(V)$-unstable, then v is also $\mathrm{SL}(V)^d$-unstable. Equivalently, if v is $\mathrm{SL}(V)^d$-semistable, then v is also $\mathrm{SL}(V)$-semistable. It follows from Theorem 8 that the converse direction is also true:

Corollary 10. *Let $v \in D^dV$ be a symmetric tensor, then v is $\mathrm{SL}(V)^d$-semistable if and only if it is $\mathrm{SL}(V)$-semistable.*

1.3. *G-stable rank for ideals and log canonical threshold.* Let V be an n-dimensional vector space over a perfect field K. By choosing a basis of V and a dual basis $\{x_1, x_2, \ldots, x_n\}$ of $V^\star$, we have an isomorphism of algebras $SV^\star \cong K[x_1, \cdots, x_n]$, where $SV^\star$ is the symmetric algebra on the vector space $V^\star$. We have defined the symmetric G-stable rank for symmetric tensors, it is natural to extend this idea to polynomials and more generally to ideals in the polynomial ring $K[x_1, \cdots, x_n]$. Furthermore, let X be a smooth irreducible affine variety with coordinate ring $K[X]$, and let $\mathfrak{a} \subset K[X]$ be an ideal. We can define the G-stable rank $\mathrm{rk}^G(P, \mathfrak{a})$ for the ideal $\mathfrak{a}$ at a point $P \in V(\mathfrak{a})$. We postpone the precise definition of G-stable rank for ideals to Section 5. It turns out that the G-stable rank $\mathrm{rk}^G(P, \mathfrak{a})$ of an ideal $\mathfrak{a}$ at P is closely related to the *log canonical threshold* $\mathrm{lct}_P(\mathfrak{a})$ of the ideal $\mathfrak{a}$ at the point $P \in V(\mathfrak{a})$.

Log canonical threshold is an invariant of singularities in algebraic geometry, [8] gives a comprehensive introduction to this subject. Let $K = \mathbb{C}$ be the complex field. Let $H \subset \mathbb{C}^n$ be a hypersurface defined by a polynomial $f \in \mathbb{C}[x_1, \cdots, x_n]$, and let $P \in H$ be a closed point. The log canonical threshold $\mathrm{lct}_P(f)$ of f at the point P tells us how singular f is at the point P. More precisely, $\mathrm{lct}_P(f)$ is a rational number bounded above by 1, and equal to 1 if P is a smooth point of H.

There are several equivalent ways to define the log canonical threshold, here we give an analytic definition, which we will use later.

Definition 11. Let X be a smooth irreducible affine variety. Let $\mathfrak{a} = (f_1, \cdots, f_r) \subset \mathbb{C}[X]$ be an ideal, and $P \in V(\mathfrak{a})$ is a closed point. The *log canonical threshold* $\mathrm{lct}_P(\mathfrak{a})$ of the ideal $\mathfrak{a}$ at P is

$$\mathrm{lct}_P(\mathfrak{a}) = \sup\left\{ s > 0 \;\middle|\; \frac{1}{(\sum_{i=0}^{r} |f_i|^2)^s} \text{ is integrable around } P \right\}. \tag{4}$$

The log canonical threshold $\mathrm{lct}_P(f)$ of a polynomial $f \in \mathbb{C}[X]$ is the log canonical threshold of the principle ideal $\mathfrak{a} = (f)$. We have the following relation between the log canonical threshold and the G-stable rank:

Theorem 12. *In the situation of Definition 11, the log canonical threshold of $\mathfrak{a}$ is less than or equal to the G-stable rank of $\mathfrak{a}$ at P:*

$$\mathrm{lct}_P(\mathfrak{a}) \leq \mathrm{rk}^G(P, \mathfrak{a}). \tag{5}$$

When $\mathfrak{a}$ is a monomial ideal, i.e. $\mathfrak{a}$ is generated by monomials, the equality holds.

Theorem 13. *Suppose $\mathfrak{a} \subset \mathbb{C}[x_1, \cdots, x_n]$ is a proper nonzero ideal generated by monomials and $P = (0, \cdots, 0)$ is the origin. Then we have*

$$\mathrm{lct}_P(\mathfrak{a}) = \mathrm{rk}^G(P, \mathfrak{a}). \tag{6}$$

Our results indicate that the G-stable rank for tensors is a useful tool for attacking the stability of tensors. Corollary 10 shows the equivalence of $\mathrm{SL}(V)^d$-semistability and $\mathrm{SL}(V)$-semistability for symmetric tensors, one can ask the same question for stability, i.e. is the subset of $\mathrm{SL}(V)^d$-stable symmetric tensors the same as the subset of $\mathrm{SL}(V)$-stable symmetric tensors? On the other hand, the G-stable rank for ideals gives a numerical upper bound for the log canonical threshold, this provides a different perspective for the study of complex singularities. In the mean time, we noticed that [6] defined the same notion for homogeneous polynomials and it was used to give a sharp bound on the change of the slice rank of polynomials under field extensions.

2. Kempf's theory of optimal subgroups

Let G be a reductive algebraic group over a perfect field K. In our case, G is one of $\mathrm{SL}(V)^d, \mathrm{SL}(V), \mathrm{GL}(V)^d$ or $\mathrm{GL}(V)$, depend on the situation. Let $\Gamma(G)$ denote the set of all 1-parameter subgroups of G. In [7], Kempf provided a way to approach the boundary of an orbit. Following [7], we have the definition:

Definition 14. Let X be a G-scheme over a perfect field K and let $x \in X$ be a K-point. We define $|X, x|$ to be the set of all 1-parameter subgroups of G such that $\lim_{t\to 0} \lambda(t) \cdot x$ exists in X. Assume S is a G-invariant closed sub-scheme of X not containing x, we define a subset $|X, x|_S \subset |X, x|$ by

$$|X, x|_S = \{\lambda \in |X, x| \mid \lim_{t\to 0} \lambda(t) \cdot x \in S\}. \tag{7}$$

Remark 15. If $S \cap \overline{G \cdot x} \neq \varnothing$, then by Theorem 3, there exists a 1-parameter subgroup $\lambda(t) \in \Gamma(G)$, such that $\lim_{t\to 0} \lambda(t) \cdot x \in S$. Hence $|X, x|_S \neq \varnothing$. If $X = V^{\otimes d}$, $S = 0$ and $v \in V^{\otimes d}$ is $\mathrm{SL}(V)^d$-unstable, then $|V^{\otimes d}, v|_{\{0\}} \neq \varnothing$.

Let $\lambda \in |X, x|$ be a 1-parameter subgroup of G, we get a morphism $\phi_\lambda : \mathbb{A}^1 \to X$ by $\phi_\lambda(t) = \lambda(t) \cdot x$ if $t \neq 0$ and $\phi_\lambda(0) = \lim_{t\to 0} \lambda(t) \cdot x$. Assume S is a G-invariant closed sub-scheme of X not containing x, the inverse image $\phi_\lambda^{-1}(S)$ is an effective divisor supported inside $t = 0$. Let $a_{S,x}(\lambda)$ denote the degree of the divisor $\phi_\lambda^{-1}(S)$ for $\lambda \in |X, x|$. Note that we have a natural conjugate action of G on the set of 1-parameter subgroups $\Gamma(G)$ by $(g \cdot \lambda)(t) = g\lambda(t)g^{-1}$, where $g \in G$, $\lambda \in \Gamma(G)$.

Definition 16. A *length* function $\|\cdot\|$ is a non-negative real-valued function on $\Gamma(G)$ such that

(1) $\|g \cdot \lambda\| = \|\lambda\|$ for any $\lambda \in \Gamma(G)$ and $g \in G$.
(2) For any maximal torus $T \subseteq G$, we have $\Gamma(T) \subseteq \Gamma(G)$, the restriction of $\|\cdot\|$ on $\Gamma(T)$ is integral valued and extends to a *norm* on the vector space $\Gamma(T) \otimes_{\mathbb{Z}} \mathbb{R}$.

Remark 17. Such a length function exists. Let T be a maximal torus of G. Let N be the normalizer of T. Then the Weyl group with respect to T is defined by $W = N/T$. By the fact that $\Gamma(G)/G \cong \Gamma(T)/W$, it suffices to define a W-invariant norm on $\Gamma(T) \otimes_{\mathbb{Z}} \mathbb{R}$. Since W is a finite group, any norm on $\Gamma(T) \otimes_{\mathbb{Z}} \mathbb{R}$ and then average over W will work.

Remark 18. In the original paper [7], Kempf defined a length function $\|\cdot\|$ that satisfies a different condition (2): for any maximal torus T of G, there is a positive definite integral-valued bilinear form $(\ ,\ )$ on $\Gamma(T)$, such that $(\lambda, \lambda) = \|\lambda\|^2$ for any λ in $\Gamma(T)$. But the proof in [7] of the theorem below is also valid for our slightly weaker definition of length function.

Theorem 19 (Kempf [7])**.** *Let X be an affine G-scheme over a perfect field K. Let $x \in X$ be a K-point. Assume S is an G-invariant closed sub-scheme not containing x such that $S \cap \overline{G \cdot x} \neq \varnothing$. Fix a length function $\|\cdot\|$ on $\Gamma(G)$, then we have*

(1) The function $\frac{a_{S,x}(\lambda)}{\|\lambda\|}$ has a maximum positive value $B_{S,x}$ on the set of non-trivial 1-parameter subgroups in $|X,x|$.
(2) Let $\Lambda_{S,x}$ be the set of indivisible 1-parameter subgroups $\lambda \in |X,x|$ such that $a_{S,x}(\lambda) = B_{S,x} \cdot \|\lambda\|$, then we have
(a) $\Lambda_{S,x} \neq \varnothing$.
(b) For $\lambda \in \Lambda_{S,x}$, Let $P(\lambda) = \{g \in G | \lim_{t \to 0} \lambda(t) \cdot g \cdot \lambda(t)^{-1}$ exists$\}$, then $P(\lambda)$ is a parabolic subgroup and independent of λ. We denote it by $P_{S,x}$.
(c) Any maximal torus of $P_{S,x}$ contains a unique member of $\Lambda_{S,x}$.

3. G-stable rank and symmetric G-stable rank

Let V be an n dimensional vector space over K, fix a maximal torus T of $\mathrm{GL}(V)$, we have an isomorphism $\Gamma(T) \cong \mathbb{Z}^n$. Any 1-parameter subgroup λ of the maximal torus T is given by a tuple of n integers $(v_1, \cdots, v_n)$, we define a function on $\Gamma(T) \cong \mathbb{Z}^n$ by

$$\|\lambda\| = \sum_{i=1}^{n} |v_i|. \tag{8}$$

This function extends linearly to a norm on the vector space $\Gamma(T) \otimes_{\mathbb{Z}} \mathbb{R}$. The Weyl group of $\mathrm{GL}(V)$ with respect to T is the symmetric group S_n. It is clear that the function is invariant under the action of S_n by permutation, therefore by Remark 17, it defines a length function on $\Gamma(\mathrm{GL}(V))$. Let $G = \mathrm{GL}(V)^d$, fix a maximal torus $T_i \subset \mathrm{GL}(V)$ for each component of $\mathrm{GL}(V)^d$, then $T = T_1 \times \cdots \times T_d$ is a maximal torus of G. We have $\Gamma(T) \cong (\mathbb{Z}^n)^d$, Let $\lambda = (\lambda_1, \cdots, \lambda_d)$ be a 1-parameter subgroup of T, where

$$\lambda_i = (\lambda_{i,1}, \cdots, \lambda_{i,n}),\ \lambda_{i,j} \in \mathbb{Z} \text{ for all } j$$

is a tuple of n integers. We define a function on $\Gamma(T)$ by

$$\|\lambda\| = \sum_{i=1}^{d} \|\lambda_i\|, \tag{9}$$

where $\|\lambda_i\| = \sum_{j=1}^{n} |\lambda_{i,j}|$. This extends to a length function on $\Gamma(G) = \Gamma(\mathrm{GL}(V)^d)$.

Let $G = \mathrm{GL}(V)^d$, $X = V^{\otimes d}$ and $S = \{0\}$. Recall the definition of t-valuation in equation (2).

Lemma 20. *Let $v \in D^dV \subset V^{\otimes d}$ be a symmetric tensor, and $G = \mathrm{GL}(V)^d$ acts on $V^{\otimes d}$ in the usual way. If $\lambda(t)$ is a 1-parameter subgroup of G, then*

(1) $|X, v| = \{\lambda \in \Gamma(G) \mid \mathrm{val}(\lambda(t)\cdot v) \geq 0\}$.
(2) $|X, v|_{\{0\}} = \{\lambda \in \Gamma(G) \mid \mathrm{val}(\lambda(t)\cdot v) > 0\}$.
(3) $a_{\{0\},v}(\lambda) = \mathrm{val}(\lambda(t)\cdot v)$ *for* $\lambda \in |X, v|$.

Proof. This follows immediately from the definition. □

Lemma 21. *Let $v \in D^dV \subset V^{\otimes d}$ be a symmetric tensor, then the function $\frac{\mathrm{val}(\lambda(t)\cdot v)}{\|\lambda\|} : \Gamma(G) \to \mathbb{R}$ attains its maximal value at some 1-parameter subgroup $\lambda \in \Gamma(\mathrm{GL}(V)^d)$. There exists a maximal torus $T \subset \mathrm{GL}(V)^d$, such that $\lambda \in \Gamma(T)$ and under the isomorphism $\Gamma(T^d) \cong (\mathbb{Z}^n)^d$, we can write $\lambda = (\lambda_1, \cdots, \lambda_d)$, where $\lambda_i = (\lambda_{i,1}, \cdots, \lambda_{i,n})$ such that $\lambda_{i,j} \in \mathbb{Z}$ and $\lambda_{i,j} \geq 0$ for all $1 \leq i \leq d, 1 \leq j \leq n$, in other words, λ is a polynomial 1-parameter subgroup.*

Proof. By Theorem 19, the maximal value of $\frac{\mathrm{val}(\lambda(t)\cdot v)}{\|\lambda\|}$ exists. Let $T \subset G$ be a maximal torus and $\lambda \in \Gamma(T)$ such that the function attains its maximum at λ. Assume λ is of the form in the lemma and $\lambda_{i,j} < 0$ for some i and j. If we replace $\lambda_{i,j}$ by $-\lambda_{i,j}$, $\mathrm{val}(\lambda(t)\cdot v)$ never decreases and $\|\lambda\|$ does not change. Therefore by the maximality of $\frac{\mathrm{val}(\lambda(t)\cdot v)}{\|\lambda\|}$, the value $\frac{\mathrm{val}(\lambda(t)\cdot v)}{\|\lambda\|}$ does not change after the replacement. Hence without loss of generality we can assume all $\lambda_{i,j} \geq 0$. □

Recall that for a tensor $v \in V^{\otimes d}$ and a polynomial 1-parameter subgroup λ of $G = \mathrm{GL}(V)^d$ such that $\mathrm{val}(\lambda(t)\cdot v) > 0$, we have the slope function

$$\mu(\lambda(t), v) = \frac{\sum_{i=1}^d \mathrm{val}(\det(\lambda_i(t)))}{\mathrm{val}(\lambda(t)\cdot v)}. \tag{10}$$

Let $\lambda = (\lambda_1, \cdots, \lambda_d) \in \Gamma(G)$ be a polynomial 1-parameter subgroup of $G = \mathrm{GL}(V)^d$, then by Lemma 21, $\sum_{i=1}^d \mathrm{val}(\det(\lambda_i(t)))$ is the restriction of the length function defined by equation (9). Let S_d be the symmetric group acting on $G = \mathrm{GL}(V)^d$ by permuting the d components. Then the length function defined by equation (9) is invariant under the action of S_d. From now on, fix this length function on $\Gamma(G)$. We have a corollary following from Theorem 19:

Corollary 22. *Let $v \in D^dV \subset V^{\otimes d}$ be a symmetric tensor. Let $\Lambda_{\{0\},v}$ be the set of indivisible 1-parameter subgroups $\lambda \in |V^{\otimes d}, v|$ such that $\frac{\mathrm{val}(\lambda(t)\cdot v)}{\|\lambda\|}$ attains the maximum value. Then we have*

(1) $\Lambda_{\{0\},v}$ *is invariant under* S_d.
(2) $P_{\{0\},v}$ *is* S_d *invariant. In other words,* $P_{\{0\},v} = P^d \subset \mathrm{GL}(V)^d$ *for some parabolic subgroup* $P \subset \mathrm{GL}(V)$.

Proof.

(1) It is clear that $\frac{\mathrm{val}(\lambda(t)\cdot v)}{\|\lambda\|}$ is S_d invariant. Indeed, Let $\sigma \in S_d$, since $v \in D^dV$ is a symmetric tensor and $\|\cdot\|$ is S_d invariant, we have

$$\frac{\mathrm{val}((\sigma\lambda)(t)\cdot v)}{\|\sigma\lambda\|} = \frac{\mathrm{val}((\sigma\lambda)(t)\cdot(\sigma v))}{\|\sigma\lambda\|} = \frac{\mathrm{val}(\sigma(\lambda(t)\cdot v))}{\|\sigma\lambda\|} = \frac{\mathrm{val}(\lambda(t)\cdot v)}{\|\lambda\|}.$$

Therefore if $\lambda \in \Lambda_{\{0\},v}$, so is $\sigma(\lambda)$.

(2) Since $G = \mathrm{GL}(V)^d$, the parabolic subgroup $P_{\{0\},v}$ is a product of parabolic subgroups of $\mathrm{GL}(V)$, the symmetric group S_d acts on $P_{\{0\},v}$ by permuting the components. Let $\lambda \in \Lambda_{\{0\},v}$, for any $\sigma \in S_d$, we have

$$\sigma(P_{\{0\},v}) = P(\sigma(\lambda)) = P_{\{0\},v}.$$

We used the fact that $P_{\{0\},v} = P(\lambda)$ is independent of $\lambda \in \Lambda_{\{0\},v}$ and $\sigma(\lambda) \in \Lambda_{\{0\},v}$. So $P_{\{0\},v}$ is S_d invariant and we can find a parabolic subgroup $P \subset \mathrm{GL}(V)$ such that $P_{\{0\},v} = P^d$.

□

Let $T \subset P \subset \mathrm{GL}(V)$ be a maximal torus, then T^d is a maximal torus of $P^d = P_{\{0\},v}$. By (2.c) in Theorem 19 and Lemma 21, there is a polynomial 1-parameter subgroup $\lambda = (\lambda_1, \cdots, \lambda_n)$ of $T^d \subset \mathrm{GL}(V)^d$, such that the slope function

$$\mu(\lambda(t), v) = \frac{\sum_{i=1}^d \mathrm{val}(\det(\lambda_i(t)))}{\mathrm{val}(\lambda(t) \cdot v)} = \frac{\|\lambda\|}{\mathrm{val}(\lambda(t) \cdot v)}$$

has a minimum value at λ. The minimal value of $\mu(\lambda(t), v)$ is by definition the G-stable rank $\mathrm{rk}^G(v)$ of v. In rest of the section, we fix such a maximal torus $T \subset \mathrm{GL}(V)$. Let

$$\lambda = (\lambda_1, \cdots, \lambda_d) \tag{11}$$

be a polynomial 1-parameter subgroup of $T^d \subset \mathrm{GL}(V)^d$, then

$$\gamma = \Pi_{i=1}^d \lambda_i \tag{12}$$

is a polynomial 1-parameter subgroup of $\mathrm{GL}(V)$. Furthermore, γ acts on $v \in D^d V$ via the diagonal embedding $\mathrm{GL}(V) \hookrightarrow \mathrm{GL}(V)^d$. We have the following lemma:

Lemma 23. *For any symmetric tensor $v \in D^d V$ and polynomial 1-parameter subgroup λ of $T^d \subset \mathrm{GL}(V)^d$ as in (11), let γ be the polynomial 1-parameter subgroup defined in (12), we have* $\mathrm{val}(\gamma(t) \cdot v) \geq d \cdot \mathrm{val}(\lambda(t) \cdot v)$.

Proof. Let $C = \mathrm{val}(\lambda(t) \cdot v)$, we define a subspace W of $V^{\otimes}$ as following

$$W = \{w \in V^{\otimes d} | \mathrm{val}(\sigma(\lambda(t)) \cdot w) \geq C, \forall \sigma \in S_d\}.$$

Since $\mathrm{val}(\sigma(\lambda(t)) \cdot v) = \mathrm{val}(\sigma(\lambda(t) \cdot v)) = \mathrm{val}(\lambda(t) \cdot v) = C$, we have $v \in W$. For any $\sigma \in S_d$, it is clear that $\sigma(\lambda(t)) \cdot W \subset t^C K[t] \cdot W$. We can write

$$\begin{aligned}
\gamma(t) \cdot v &= (\Pi_{i=1}^d \lambda_i, \cdots, \Pi_{i=1}^d \lambda_i) \cdot v \\
&= (\lambda_1, \lambda_2, \cdots, \lambda_d)(\lambda_2, \lambda_3, \cdots, \lambda_d, \lambda_1) \cdots (\lambda_d, \lambda_1, \cdots, \lambda_{d-1}) \cdot v \\
&= (\lambda_1, \lambda_2, \cdots, \lambda_d) \sigma(\lambda_1, \lambda_2, \cdots, \cdots, \lambda_d) \cdots \sigma^{d-1}(\lambda_1, \lambda_2, \cdots, \lambda_d) \cdot v \\
&= \lambda \sigma(\lambda) \cdots \sigma^{d-1}(\lambda) \cdot v,
\end{aligned}$$

where $\sigma \in S_d$ satisfies $\sigma(1) = 2, \sigma(2) = 3, \cdots, \sigma(d) = 1$. Therefore $\gamma(t) \cdot v \in t^{dC} K[t] \cdot W$, hence we get $\mathrm{val}(\gamma(t) \cdot v) \geq dC = d \cdot \mathrm{val}(\lambda(t) \cdot v)$. □

Next we prove that the symmetric G-stable rank is the same as the G-stable rank for symmetric tensors.

Proof of Theorem 8. Let T be the chosen maximal torus of $\mathrm{GL}(V)$ as above. Let $\lambda = (\lambda_1, \cdots, \lambda_d)$ be a polynomial 1-parameter subgroup of $T^d \subset \mathrm{GL}(V)^d$ such that the slope function $\mu(\lambda(t), v)$ attains its minimum value. In other words, λ computes the G-stable rank $\mathrm{rk}^G(v)$ of v:

$$\mathrm{rk}^G(v) = \frac{\sum_{i=1}^d \mathrm{val}(\det(\lambda_i(t)))}{\mathrm{val}(\lambda(t) \cdot v)}.$$

Let $\gamma = \Pi_{i=1}^d \lambda_i$ as above, then we have

$$\mathrm{symmrk}^G(v) \leq \frac{\sum_{i=1}^d \mathrm{val}(\det(\gamma(t)))}{\mathrm{val}(\gamma(t) \cdot v)} \leq \frac{d \sum_{i=1}^d \mathrm{val}(\det(\lambda_i(t)))}{d \cdot \mathrm{val}(\lambda(t) \cdot v)} = \mathrm{rk}^G(v).$$

On the other hand, it is clear that $\mathrm{symmrk}^G(v) \geq \mathrm{rk}^G(v)$. Therefore $\mathrm{symmrk}^G(v) = \mathrm{rk}^G(v)$, this completes the proof. □

4. Stability of symmetric tensors

As a result of Theorem 8, we prove Corollary 10, which says that for a symmetric tensor $v \in D^dV$, v is $\mathrm{SL}(V)^d$-semistable if and only if v is $\mathrm{SL}(V)$-semistable. It is clear that $\mathrm{SL}(V)^d$-semistability implies $\mathrm{SL}(V)$-semistability. To prove the other direction, we will use a result which relates semistability with G-stable rank.

Proposition 24 ([2], Proposition 2.6)**.** *Suppose that* $\alpha = (\frac{1}{n_1}, \cdots \frac{1}{n_d})$ *where* $n_i = \dim V_i$*. For* $v \in V_1 \otimes V_2 \otimes \cdots \otimes V_d$ *we have* $\mathrm{rk}^G_\alpha(v) \leq 1$*. Moreover,* $\mathrm{rk}^G_\alpha(v) = 1$ *if and only if* v *is semistable with respect to the group* $H = \mathrm{SL}(V_1) \times \mathrm{SL}(V_2) \times \cdots \times \mathrm{SL}(V_d)$.

If $v \in D^dV$ is a symmetric tensor, $\alpha = (1, 1, \cdots, 1)$ and $n = \dim V$, then by the above proposition, $\mathrm{rk}^G(v) = n$ if and only if v is $\mathrm{SL}(V)^d$-semistable. We have a similar result for symmetric G-stable rank.

Proposition 25. *For a symmetric tensor* $v \in D^dV$*, we have* $\mathrm{symmrk}^G(v) \leq n$*, where* $n = \dim V$*. Moreover,* $\mathrm{symmrk}^G(v) = n$ *if and only if* v *is* $\mathrm{SL}(V)$*-semistable.*

Proof. The first statement is clear from Theorem 8. If $\mathrm{symmrk}^G(v) = n$, by Proposition 24, we have $\mathrm{rk}^G(v) = n$ and v is $\mathrm{SL}(V)^d$-semistable, hence v is $\mathrm{SL}(V)$-semistable. On the other hand, assume v is $\mathrm{SL}(V)$-semistable. Let λ be a polynomial 1-parameter subgroup of $\mathrm{GL}(V)$ such that $\lim_{t\to 0} \lambda(t) \cdot v = 0$. Then we can define another 1-parameter subgroup $\lambda'(t) = \lambda(t)^n t^{-e}$, where $\det(\lambda(t)) = t^e$, such that $\det(\lambda') = 1$ and $\lambda' \in \mathrm{SL}(V)$. Since v is $\mathrm{SL}(V)$-semistable, we have $\mathrm{val}(\lambda'(t) \cdot v) \leq 0$. It follows that

$$\mathrm{val}(\lambda'(t) \cdot v) = \mathrm{val}(\lambda(t)^d t^{-e} \cdot v) = n\,\mathrm{val}(\lambda(t) \cdot v) - ed \leq 0.$$

The slope function

$$\mu(\lambda(t), v) = \frac{d\,\mathrm{val}(\det(\lambda(t)))}{\mathrm{val}(\lambda(t) \cdot v)} = \frac{de}{\mathrm{val}(\lambda(t) \cdot v)} \geq n.$$

We get $\mathrm{symmrk}^G(v) = n$. □

Proof of Corollary 10. It suffices to prove that if v is $\mathrm{SL}(V)$-semistable, then v is $\mathrm{SL}(V)^d$-semistable. Let us assume v is $\mathrm{SL}(V)$-semistable, then by Proposition 25 and Theorem 8,

$$\mathrm{rk}^G(v) = \mathrm{symmrk}^G(v) = n.$$

It follows from Proposition 24 that v is $\mathrm{SL}(V)^d$-semistable. □

5. G-stable rank for ideals and log canonical threshold

5.1. ***G-stable rank for ideals.*** Let $X = \mathrm{Spec}(R)$ be a nonsingular irreducible complex affine algebraic variety of dimension n, and $\mathfrak{a} \subset R$ be a nonzero ideal, and let $P \in V(\mathfrak{a})$ be a closed point, $\mathscr{O}_P$ be the local ring at P and $\mathfrak{m}_P$ be the maximal ideal corresponding to P.

Definition 26 ([11])**.** Functions $x_1, \cdots, x_n \in \mathscr{O}_P$ are *a system of local parameters* at P if each $x_i \in \mathfrak{m}_P$, and the images of $x_1, \cdots, x_n$ form a basis of the vector space $\mathfrak{m}_P/\mathfrak{m}_P^2$.

Let $T = \{x_1, x_2, \cdots, x_n\}$ be a system of local parameters at P. Let $\mathbb{C}\{x_1, x_2, \ldots, x_n\}$ be the ring of convergent power series in $x_1, x_2, \ldots, x_n$. The ring $\mathscr{O}_P$ is contained in $\mathbb{C}\{x_1, x_2, \ldots, x_n\}$. If $y_1, y_2, \ldots, y_n$ is any system of local parameters, then $\mathbb{C}\{x_1, x_2, \ldots, x_n\} = \mathbb{C}\{y_1, y_2, \ldots, y_n\}$. For any $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) \in \mathbb{Z}^n_{\geq 0}$, we have a natural action of $\mathbb{C}^*$ on $\mathbb{C}\{x_1, x_2, \cdots, x_n\}$ by $t \cdot x_i = t^{\lambda_i} x_i$ for any $t \in \mathbb{C}^*$.

Definition 27. Let $T = \{x_1, x_2, \cdots, x_n\}$ be a system of local parameters at P and $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) \in \mathbb{Z}^n_{\geq 0}$ a tuple of non-negative integers. Let $f \in \mathbb{C}\{x_1, x_2, \cdots, x_n\}$ be a convergent power series. We define the *valuation* of f with respect to T by

$$\mathrm{val}^T_\lambda(f) = \max\{k | f(t^{\lambda_1}x_1, t^{\lambda_2}x_2, \cdots, t^{\lambda_n}x_n) = t^k g(x_1, \cdots, x_n, t), \text{for some } g \in \mathbb{C}\{x_1, \cdots, x_n, t\}\}. \tag{13}$$

Let $\mathfrak{a} \subset R$ be a nonzero ideal as before, the *order* of $\mathfrak{a}$ with respect to this system of local parameters T and $\lambda \in \mathbb{Z}^n_{\geq 0}$ is defined as

$$\mathrm{ord}^T_\lambda(\mathfrak{a}) = \min\{\mathrm{val}^T_\lambda(f) | f \in \mathfrak{a}\}. \tag{14}$$

Remark 28. If $\mathfrak{a}$ is generated by $f_1, \cdots, f_r$, then

$$\mathrm{ord}^T_\lambda(\mathfrak{a}) = \min\{\mathrm{val}^T_\lambda(f_i) | i = 1, \cdots, r\}.$$

Indeed, it is clear that $\min\{\mathrm{val}^T_\lambda(f) | f \in \mathfrak{a}\} \leq \min\{\mathrm{val}^T_\lambda(f_i) | i = 1, \cdots, r\}$. On the other hand, if $f \in \mathfrak{a}$ computes $\mathrm{ord}^T_\lambda(\mathfrak{a})$, then we can write $f = \sum_i a_i f_i$, for some $a_i \in R$, we have $\mathrm{val}^T_\lambda(f) = \mathrm{val}^T_\lambda(\sum_i a_i f_i) \geq \min\{\mathrm{val}^T_\lambda(a_i f_i) | i = 1, \cdots, r\} \geq \min\{\mathrm{val}^T_\lambda(f_i) | i = 1, \cdots, r\}$.

Definition 29. Assume $T = \{x_1, \cdots, x_n\}$ is a system of local parameters at P and $\lambda = \{\lambda_1, \cdots, \lambda_n\} \in \mathbb{Z}^n_{\geq 0}$, we define the *slope* function $\mu_P(\lambda, \mathfrak{a})$ at P as

$$\mu_P(\lambda, \mathfrak{a}) = \frac{\sum_{i=1}^n \lambda_i}{\mathrm{ord}^T_\lambda(\mathfrak{a})}. \tag{15}$$

The *T-stable rank* of $\mathfrak{a}$ at P is the infimum of the slope function $\mu_P(\lambda, \mathfrak{a})$ with respect to the tuple $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) \in \mathbb{Z}^n_{\geq 0}$,

$$\mathrm{rk}^T(P, \mathfrak{a}) = \inf_\lambda \mu_P(\lambda, \mathfrak{a}) = \inf_\lambda \frac{\sum_{i=1}^n \lambda_i}{\mathrm{ord}^T_\lambda(\mathfrak{a})}. \tag{16}$$

The *G-stable rank* of $\mathfrak{a}$ is defined by taking the infimum of T-stable rank with respect to all system of local parameters T at P,

$$\mathrm{rk}^G(P, \mathfrak{a}) = \inf_T(\mathrm{rk}^T(P, \mathfrak{a})). \tag{17}$$

If $P \notin V(\mathfrak{a})$, we define $\mathrm{rk}^G(P, \mathfrak{a}) = \infty$, we write $\mathrm{rk}^G(\mathfrak{a})$ and $\mathrm{rk}^T(\mathfrak{a})$ if P is known in the context. In the following example, we see that an ideal $\mathfrak{a}$ can have different T-stable rank with respect to different system of local parameters T at a point P.

Example 30. Let $R = \mathbb{C}[x, y]$, $T = \{x, y\}$ and assume $\mathfrak{a} = (x^2 + 2xy + y^2)$ is a principle ideal generated by a polynomial $f(x, y) = x^2 + 2xy + y^2$, $P = (0, 0)$ is the origin. Let $\lambda = (\lambda_1, \lambda_2) \in \mathbb{Z}^2_{\geq 0}$, then we have

$$\mathrm{rk}^T(f) = \inf_\lambda \frac{\lambda_1 + \lambda_2}{\min(2\lambda_1, \lambda_1 + \lambda_2, 2\lambda_2)} = 1$$

Let us choose a different system of local parameters $T' = \{u = x + y, v = x - y\}$, then $\mathfrak{a} = (u^2)$, and $f(u, v) = u^2$, then

$$\mathrm{rk}^{T'}(f) = \inf_\lambda \frac{\lambda_1 + \lambda_2}{2\lambda_1} = \frac{1}{2}$$

In fact, $\mathrm{lct}_P(u^2) = \frac{1}{2}$, and by Theorem 12, we have $\mathrm{lct}_P(\mathfrak{a}) \leq \mathrm{rk}^G(\mathfrak{a})$, therefore we get $\mathrm{rk}^G(f) = \frac{1}{2}$.

Example 31. Let $\mathfrak{a} = (x^2y, y^2z, z^2x) \subset \mathbb{C}[x, y, z]$, $T = \{x, y, x\}$, $P = (0, 0, 0)$, then we get

$$\mathrm{rk}^T(\mathfrak{a}) = \inf_\lambda \frac{\lambda_1 + \lambda_2 + \lambda_3}{\min(2\lambda_1 + \lambda_2, 2\lambda_2 + \lambda_3, 2\lambda_3 + \lambda_1)} = 1$$

The ideal $\mathfrak{a} = (x^2y, y^2z, z^2x)$ is a monomial ideal and we will see later that for a monomial ideal $\mathfrak{a}$, we have $\mathrm{rk}^G(\mathfrak{a}) = \mathrm{lct}_P(\mathfrak{a})$. Using the fact that $\mathrm{lct}_P(\mathfrak{a}) = 1$, we obtain $\mathrm{rk}^G(\mathfrak{a}) = 1$.

Remark 32. We have a short exact sequence

$$1 \to K \to \mathrm{Aut}(\mathbb{C}\{x_1, \cdots, x_n\}) \to \mathrm{GL}(n) \to 1, \tag{18}$$

where K is a normal subgroup of the group $\mathrm{Aut}(\mathbb{C}\{x_1, \cdots, x_n\})$ of local holomorphic automorphisms. The morphism $\mathrm{Aut}(\mathbb{C}\{x_1, \cdots, x_n\}) \to \mathrm{GL}(n)$ is given by computing the Jacobian matrix at $(0, \cdots, 0)$. Furthermore, this sequence splits, we have $\mathrm{Aut}(\mathbb{C}\{x_1 \cdots, x_n\}) = K \rtimes \mathrm{GL}(n)$.

By Remark 32, there is an action of $\mathrm{GL}(n)$ on the set of system of local parameters. Assume $T = \{x_1, \cdots, x_n\}$ is a system of local parameters at P. For $g \in \mathrm{GL}(n)$, $g \cdot T$ is another system of local parameters. We say a system of local parameters $T = \{x_1, \cdots, x_n\}$ is *good* for $\mathfrak{a}$ if $\mathrm{rk}^G(P, \mathfrak{a}) = \mathrm{rk}^{g \cdot T}(P, \mathfrak{a})$ for some $g \in \mathrm{GL}(n)$. In other words, to compute the G-stable rank of $\mathfrak{a}$, it is enough to consider all systems of local parameters obtained from T by actions of $\mathrm{GL}(n)$.

Example 33. Let $f(x,y) = x + y^2 \in \mathbb{C}[x,y]$, $P = (0,0)$, we take $T = \{x, y\}$. It can be shown that $\mathrm{rk}^T(f) = \frac{3}{2}$. However, if we choose another system of local parameters $T' = \{u = x + y^2, v = y\}$, then $f(u,v) = u$, and $\mathrm{rk}^{T'}(f) = 1$. Indeed, this system of local parameters is optimal, in other words, we can compute the G-stable rank in this system of local parameters, and we have $\mathrm{rk}^G(f) = 1$.

Proposition 34. *If $\mathfrak{a}$ is homogeneous in local parameters $T = \{x_1, \cdots, x_n\}$, then T is good for $\mathfrak{a}$.*

Proof. Since $\mathfrak{a}$ is a homogeneous ideal, we can find a set of generators which are homogeneous polynomials. By Remark 28, it is enough to assume that $\mathfrak{a}$ is generated by a single homogeneous polynomial f. Let $g \in K \subseteq \mathrm{Aut}(\mathbb{C}\{x_1, \cdots, x_n\})$, we can write the action of g on T as following

$$g(x_i) = x_i + p_i(x_1, \cdots, x_n),$$

where $p_i \in \mathbb{C}\{x_1 \cdots, x_n\}$ with no constant and degree 1 terms. Since f is a homogeneous polynomial, we have

$$\mathrm{val}^T_\lambda(f(g(x_1), \cdots, g(x_n)) \leq \mathrm{val}^T_\lambda(f(x_1, \cdots, x_n)).$$

Let T' be the system of local parameters obtained from T by the action of g, then we have

$$\frac{\sum_{i=1}^n \lambda_i}{\mathrm{ord}^{T'}_\lambda(f)} \geq \frac{\sum_{i=1}^n \lambda_i}{\mathrm{ord}^T_\lambda(f)}. \tag{19}$$

By Lemma 32, given any $h \in \mathrm{Aut}(\mathbb{C}\{x_1, \cdots, x_n\})$, we can decompose the action of h into an action of K following by an action of $\mathrm{GL}(n)$. By inequality (19), in the system of local parameters obtained by the action of K from T, we have larger slope than the slope computed in T, hence to compute the G-stable rank of f, it suffices to consider the action of $\mathrm{GL}(n)$. This shows that T is good for $\mathfrak{a}$. □

Corollary 35. *If $f \in \mathbb{C}[x_1, \cdots, x_n]$ is a homogeneous polynomial of degree $d \geq 2$ and f has isolated singularity at $P = (0, \cdots, 0)$. Then $\mathrm{rk}^G(f) = \frac{n}{d}$.*

Proof. By Corollary 34, the system of local parameters $T = \{x_1, \cdots, x_n\}$ is good for f, therefore we only need to consider the group action of $\mathrm{GL}(n)$.

We claim that f is $\mathrm{SL}(n)$-semistable in the sense of Definition 4. Indeed, since f has an isolated singularity at origin, $\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n}$ only have a common zero at origin. By [9], chapter 13, their resultant $\mathrm{Res}(\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n})$ is nonzero and invariant under the action of $\mathrm{SL}(n)$. Now assume there is a one parameter subgroup $\lambda : \mathbb{G}_m \to \mathrm{SL}(n)$, such that

$$\lim_{t \to 0} \lambda(t) \cdot v = 0.$$

Then the resultant $\mathrm{Res}(\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n})$ is 0, which is impossible. This proves the claim.

By the claim that f is $\mathrm{SL}(n)$-semistable, the corollary follows immediately from Proposition 2.6 in [2] □

5.2. ***Relation to log canonical threshold.*** Let $X = \mathrm{Spec}(R)$ be a nonsingular irreducible complex affine variety, $\mathfrak{a} \subset R$ is an ideal, $P \in V(\mathfrak{a})$. If $\mathfrak{a} = (f_1, \cdots, f_r) \subseteq R$ is a nonzero ideal, recall the Definition 11, the log canonical threshold $\mathrm{lct}_P(\mathfrak{a})$ of the ideal $\mathfrak{a}$ at point P is

$$\mathrm{lct}_P(\mathfrak{a}) = \sup\{s > 0 | \frac{1}{(\sum_{i=0}^r |f_i|^2)^s} \text{ is integrable around } P\}. \tag{20}$$

Theorem 12 says that the log canonical threshold of an ideal is less than or equal to the G-stable rank of that ideal

$$\mathrm{lct}_P(\mathfrak{a}) \leq \mathrm{rk}^G(P, \mathfrak{a}). \tag{21}$$

Proof of Theorem 12. Let $s > 0$ be such that $\frac{1}{(\sum_{i=0}^r |f_i|^2)^s}$ is integrable around P, then there is a neighborhood U_P of P, such that

$$\int_{U_P} \frac{dV}{(\sum_{i=0}^r |f_i|^2)^s} < C < \infty,$$

for some constant C. Choose a system of local parameters $T = \{x_1, \cdots, x_n\}$ at P, let $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{Z}^n_{\geq 0}$, then let $t \in \mathbb{C}^*$ act on the coordinates by $x_i \to t^{\lambda_i} x_i$. We denote $t \cdot U_P$ for the image of U_P under this action. If $|t| < 1$, we have $t \cdot U_P \subset U_P$, therefore

$$\int_{t \cdot U_P} \frac{dV}{(\sum_{i=0}^r |f_i|^2)^s} < \int_{U_P} \frac{dV}{(\sum_{i=0}^r |f_i|^2)^s} < C \tag{22}$$

Let $y_i = t^{-\lambda_i} x_i$ for $i = 1, \cdots, n$, then we have

$$\int_{t \cdot U_P} \frac{dx_1 d\bar{x}_1 \cdots dx_n d\bar{x}_n}{(\sum_{i=0}^r |f_i(x_1, \cdots, x_n)|^2)^s} = \int_{U_P} \frac{|t|^{2\sum_{i=1}^n \lambda_i} dy_1 d\bar{y}_1 \cdots dy_n d\bar{y}_n}{(\sum_{i=1}^r |f_i(t^{\lambda_1} y_1, \cdots, t^{\lambda_n} y_n)|^2)^s} < C \tag{23}$$

Recall the Definition 27 for the valuation (13) and order of an ideal (14), we can write

$$\sum_{i=0}^r |f_i(t^{\lambda_1} y_1, \cdots, t^{\lambda_n} y_n)|^2 = |t|^{2\min_i(\mathrm{val}_\lambda^T(f_i))} (\sum_{i=1}^r |\tilde{f}_i(y_1, \cdots, y_n, t)|^2)$$

$$= |t|^{2\,\mathrm{ord}_\lambda^T(\mathfrak{a})} (\sum_{i=1}^r |\tilde{f}_i(y_1, \cdots, y_n, t)|^2),$$

for some $\tilde{f}_i(y_1, \cdots, y_n, t) \in \mathbb{C}\{y_1, y_2, \cdots, y_n, t\}$. In particular, we know that $\sum_{i=1}^r |\tilde{f}_i(y_1, \cdots, y_n, 0)|^2$ is not constantly zero in U_P. So we can find a point $Q \in U_P$ such that $0 < \sum_{i=1}^r |\tilde{f}_i(Q, 0)|^2 < B$ for some constant $B > 0$. By the continuity, there is a neighborhood U_Q such that $Q \in U_Q \subset U_P$ and some $\varepsilon > 0$, such that $0 < \sum_{i=1}^r |\tilde{f}_i(y_1, \cdots, y_n, t)|^2 < B$ for any $(y_1, \cdots, y_n) \in U_Q$ and $0 \leq t < \varepsilon$. We can write integral (23) as

$$\int_{U_P} \frac{|t|^{2\sum_{i=1}^n \lambda_i} dy_1 d\bar{y}_1 \cdots dy_n d\bar{y}_n}{|t|^{2\,\mathrm{ord}_\lambda^T(\mathfrak{a})} (\sum_{i=1}^r |\tilde{f}_i(y_1, \cdots, y_n, t)|^2)^s} = \int_{U_P} \frac{|t|^{2(\sum_{i=1}^n \lambda_i - s\,\mathrm{ord}_\lambda^T(\mathfrak{a}))} dV}{(\sum_{i=1}^r |\tilde{f}_i(y_1, \cdots, y_n, t)|^2)^s} < C. \tag{24}$$

Therefore we get

$$|t|^{2(\sum_{i=1}^n \lambda_i - s\,\mathrm{ord}_\lambda^T(\mathfrak{a}))} \int_{U_Q} \frac{dV}{B^s} = \int_{U_Q} \frac{|t|^{2(\sum_{i=1}^n \lambda_i - s\,\mathrm{ord}_\lambda^T(\mathfrak{a}))} dV}{B^s} < \int_{U_P} \frac{|t|^{2(\sum_{i=1}^n \lambda_i - s\,\mathrm{ord}_\lambda^T(\mathfrak{a}))} dV}{(\sum_{i=1}^r |\tilde{f}_i(y_1, \cdots, y_n, t)|^2)^s} < C$$

for all $t \in [0, \varepsilon)$, where $\varepsilon > 0$. Since $\int_{U_Q} \frac{dV}{B^s} > 0$, we must have

$$\sum_{i=1}^{n} \lambda_i - s\,\mathrm{ord}_\lambda^T(\mathfrak{a}) \geq 0.$$

Hence we get $s \leq \frac{\sum_{i=1}^n \lambda_i}{\mathrm{ord}_\lambda^T(\mathfrak{a})}$. This holds for any system of local coordinates and $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{Z}^n_{\geq 0}$, therefore

$$\mathrm{lct}_P(\mathfrak{a}) \leq \mathrm{rk}^G(P, \mathfrak{a}).$$

This completes the proof of Theorem 12. □

Example 36. Let $f = x_1^{u_1} + x_2^{u_2} + \cdots + x_n^{u_n} \in \mathbb{C}[x_1, \cdots, x_n]$, $P = (0, \cdots, 0)$ be the origin. It was shown in [8] that $\mathrm{lct}_P(f) = \min(1, \sum_{i=1}^n \frac{1}{u_i})$. However, we will show that $\mathrm{rk}^G(f) = \sum_{i=1}^n \frac{1}{u_i}$. So we get $\mathrm{lct}_P(f) \leq \mathrm{rk}^G(f)$. If $n = 3$, $u_1 = u_2 = u_3 = 2$, then we have $f = x_1^2 + x_2^2 + x_3^2$ and $\mathrm{lct}_P(f) = 1 < \mathrm{rk}^G(f) = \frac{3}{2}$.

Suppose $\mathfrak{a} = (m_1, \cdots, m_r) \subset \mathbb{C}[x_1, \cdots, x_n]$ is a proper nonzero ideal generated by monomials $\{m_1, \cdots, m_r\}$ and let $P = (0, \cdots, 0)$ be the origin. Given $u = (u_1, \cdots, u_n) \in \mathbb{Z}^n_{\geq 0}$, we write $x^u = x_1^{u_1} \cdots x_n^{u_n}$. The *Newton Polyhedron* of $\mathfrak{a}$ is

$$P(\mathfrak{a}) = \text{convex hull } (\{u \in \mathbb{Z}^n_{\geq 0} | x^u \in \mathfrak{a}\}). \tag{25}$$

It was shown in [8] that

$$\mathrm{lct}_P(\mathfrak{a}) = \max\{\nu \in \mathbb{R}_{\geq 0} | (1, 1, \cdots, 1) \in \nu \cdot P(\mathfrak{a})\}. \tag{26}$$

In other words, $\mathrm{lct}_P(\mathfrak{a})$ is equal to the largest ν such that $\sum_{i=1}^n \lambda_i \geq \nu \cdot \min_{u \in P(\mathfrak{a})} \langle u, \lambda \rangle$ for any $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{Z}^n_{\geq 0}$, where we use the standard inner product $\langle u, \lambda \rangle = \sum_{i=1}^n u_i \lambda_i$.

Theorem 13 says that the log canonical threshold is equal to the G-stable rank for monomial ideals. More precisely, suppose $\mathfrak{a} \subset \mathbb{C}[x_1, \cdots, x_n]$ is a proper nonzero ideal generated by monomials and $P = (0, \cdots, 0)$ is the origin. Then

$$\mathrm{lct}_P(\mathfrak{a}) = \mathrm{rk}^G(\mathfrak{a}). \tag{27}$$

Proof of Theorem 13. Let $\mathfrak{a} = (m_1, \cdots, m_r)$ and $\{m_i = x_1^{l_{i1}} x_2^{l_{i2}} \cdots x_n^{l_{in}}\}_{i=1,\cdots,r}$ be a set of generators, where $l_i = (l_{i1}, \cdots, l_{in}) \in \mathbb{Z}^n_{\geq 0}$, we have

$$\begin{aligned}
\mathrm{lct}_P(\mathfrak{a}) &= \max\{\nu \in \mathbb{R}_{\geq 0} | (1, \cdots, 1) \in \nu \cdot P(\mathfrak{a})\} \\
&= \max\{\nu \in \mathbb{R}_{\geq 0} | \sum_j \lambda_j \geq \nu \cdot \min_{u \in P(\mathfrak{a})} \langle u, \lambda \rangle, \forall \lambda \in \mathbb{Z}^n_{\geq 0}\} \\
&= \max\{\nu \in \mathbb{R}_{\geq 0} | \sum_j \lambda_j \geq \nu \cdot \min_i \langle l_i, \lambda \rangle, \forall \lambda \in \mathbb{Z}^n_{\geq 0}\} \\
&= \max\{\nu \in \mathbb{R}_{\geq 0} | \nu \leq \frac{\sum_j \lambda_j}{\min_i(\sum_j l_{ij} \lambda_j)}, \forall \lambda \in \mathbb{Z}^n_{\geq 0}\}.
\end{aligned}$$

In this system of local parameters $T = \{x_1, \cdots, x_n\}$, we have $\mathrm{ord}_\lambda^T(\mathfrak{a}) = \min_i(\sum_j l_{ij} \lambda_j)$, therefore we get

$$\begin{aligned}
\mathrm{lct}_P(\mathfrak{a}) &= \max\{\nu \in \mathbb{R}_{\geq 0} | \nu \leq \frac{\sum_i \lambda_i}{\mathrm{ord}_\lambda^T(\mathfrak{a})}, \forall \lambda \in \mathbb{Z}^n_{\geq 0}\} \\
&= \max\{\nu \in \mathbb{R}_{\geq 0} | \nu \leq \mathrm{rk}^T(\mathfrak{a})\} = \mathrm{rk}^T(\mathfrak{a}).
\end{aligned}$$

Since log canonical threshold does not depend on the local coordinates, hence we have

$$\mathrm{lct}_P(\mathfrak{a}) = \mathrm{rk}^G(\mathfrak{a}).$$

This completes the proof of Theorem 13. □

Example 37. Suppose $\mathfrak{a} = (x_1^{u_1}, \cdots, x_n^{u_n})$ and $P = (0, \cdots, 0)$, then $\mathrm{lct}_P(\mathfrak{a}) = \sum_{i=1}^n \frac{1}{u_i}$. Therefore we also have $\mathrm{rk}^G(\mathfrak{a}) = \sum_{i=1}^n \frac{1}{u_i}$.

5.3. *Some properties of G-stable rank for ideals.* Some results for log canonical threshold can be found in [8]. Here, we also prove similar results for the G-stable rank of ideals.

Proposition 38. *If $\mathfrak{a} \subseteq \mathfrak{b}$ are nonzero ideals on X, then we have $\mathrm{lct}_P(\mathfrak{a}) \leq \mathrm{lct}_P(\mathfrak{b})$ and $\mathrm{rk}^G(P, \mathfrak{a}) \leq \mathrm{rk}^G(P, \mathfrak{b})$.*

Proof. The first inequality was shown in [8]. If $P \notin V(\mathfrak{b})$, it is trivial. Assume $P \in V(\mathfrak{b})$, let T be a system of local parameters at P and $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{Z}^n_{\geq 0}$. Since $\mathfrak{a} \subseteq \mathfrak{b}$, we have $\mathrm{ord}_\lambda^T(\mathfrak{b}) \leq \mathrm{ord}_\lambda^T(\mathfrak{a})$, therefore $\mu_P(\lambda, \mathfrak{a}) \leq \mu_P(\lambda, \mathfrak{b})$, it follows immediately that $\mathrm{rk}^G(P, \mathfrak{a}) \leq \mathrm{rk}^G(P, \mathfrak{b})$. □

Proposition 39. *We have $\mathrm{lct}_P(\mathfrak{a}^r) = \frac{\mathrm{lct}_P(\mathfrak{a})}{r}$ and $\mathrm{rk}^G(P, \mathfrak{a}^r) = \frac{\mathrm{rk}^G(P,\mathfrak{a})}{r}$ for every $r \geq 1$.*

Proof. The first claim was shown in [8] and the second claim follows from the fact that $\mathrm{ord}_\lambda^T(\mathfrak{a}^r) = r \cdot \mathrm{ord}_\lambda^T(\mathfrak{a})$ and Definition 29. □

Proposition 40. *If $\mathfrak{a}$ and $\mathfrak{b}$ are ideals of X, then*

$$\frac{1}{\mathrm{lct}_P(\mathfrak{a} \cdot \mathfrak{b})} \leq \frac{1}{\mathrm{lct}_P(\mathfrak{a})} + \frac{1}{\mathrm{lct}_P(\mathfrak{b})}, \quad \frac{1}{\mathrm{rk}^G(P, \mathfrak{a} \cdot \mathfrak{b})} \leq \frac{1}{\mathrm{rk}^G(P, \mathfrak{a})} + \frac{1}{\mathrm{rk}^G(P, \mathfrak{b})}.$$

Proof. The first inequality was shown in [8], we show the second inequality. Let T be a system of local parameters at P and $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{Z}^n_{\geq 0}$, then

$$\mathrm{ord}_\lambda^T(\mathfrak{a} \cdot \mathfrak{b}) = \mathrm{ord}_\lambda^T(\mathfrak{a}) + \mathrm{ord}_\lambda^T(\mathfrak{b}).$$

Therefore, we have

$$\begin{aligned} \sup_{T,\lambda} \frac{\mathrm{ord}_\lambda^T(\mathfrak{a} \cdot \mathfrak{b})}{\sum \lambda_i} &= \sup_{T,\lambda} \left(\frac{\mathrm{ord}_\lambda^T(\mathfrak{a})}{\sum \lambda_i} + \frac{\mathrm{ord}_\lambda^T(\mathfrak{b})}{\sum \lambda_i} \right) \\ &\leq \sup_{T,\lambda} \frac{\mathrm{ord}_\lambda^T(\mathfrak{a})}{\sum \lambda_i} + \sup_{T,\lambda} \frac{\mathrm{ord}_\lambda^T(\mathfrak{b})}{\sum \lambda_i}. \end{aligned}$$

□

The following two propositions are from [8]. A lot of evidence suggests the same results for G-stable rank of ideals, we give them as conjectures.

Proposition 41. *If $H \subset X$ is a nonsingular hypersurface such that $\mathfrak{a} \cdot \mathcal{O}_H$ is nonzero, then $\mathrm{lct}_P(\mathfrak{a} \cdot \mathcal{O}_H) \leq \mathrm{lct}_P(\mathfrak{a})$.*

Proposition 42. *If $\mathfrak{a}$ and $\mathfrak{b}$ are ideals on X, then*

$$\mathrm{lct}_P(\mathfrak{a} + \mathfrak{b}) \leq \mathrm{lct}_P(\mathfrak{a}) + \mathrm{lct}_P(\mathfrak{b})$$

for every $P \in X$.

Conjecture 43. *If $H \subset X$ is a nonsingular hypersurface such that $\mathfrak{a} \cdot \mathcal{O}_H$ is nonzero, then $\mathrm{rk}^G(P, \mathfrak{a} \cdot \mathcal{O}_H) \leq \mathrm{rk}^G(P, \mathfrak{a})$ for any $P \in H$.*

Conjecture 44. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be two nonzero proper ideals of X, then for any point $P \in X$, we have*

$$\mathrm{rk}^G(P, \mathfrak{a} + \mathfrak{b}) \leq \mathrm{rk}^G(P, \mathfrak{a}) + \mathrm{rk}^G(P, \mathfrak{b}).$$

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