BASS AND BETTI NUMBERS OF A MODULE AND ITS DEFICIENCY MODULES

THIAGO FIEL AND RAFAEL HOLANDA

Abstract. This paper aims to provide several relations between Bass and Betti numbers of a given module and its deficiency modules. Such relations and the tools used throughout allow us to generalize some results of Foxby, characterize Cohen-Macaulay modules in equidimensionality terms, study the Cohen-Macaulay and complete intersection properties of a ring, and furnish a case for the Auslander-Reiten conjecture.

1. Introduction

In the celebrated paper [10], Foxby proved that over a Gorenstein local ring R of dimension d, a Cohen-Macaulay R-module M of dimension t is such that

$$\beta_j(M) = \mu^{j+t}(\operatorname{Ext}_R^{d-t}(M,R)) \quad \text{and} \quad \mu^j(M) = \beta_{j-t}(\operatorname{Ext}_R^{d-t}(M,R))$$

for all $j \geq 0$, where $\beta_j(N)$ and $\mu^j(N)$ denote, respectively, the j-th Betti and Bass numbers of the R-module N. In particular, $\operatorname{pd}_R M < \infty$ if and only if $\operatorname{id}_R \operatorname{Ext}_R^{d-t}(M,R) < \infty$, and $\operatorname{id}_R M < \infty$ if and only if $\operatorname{pd}_R \operatorname{Ext}_R^{d-t}(M,R) < \infty$. Recently, Freitas and Jorge-Pérez [11] generalized the first equivalence for local rings which are factors of Gorenstein local rings. In this paper, we shall look at these results in a wider situation as follows.

Schenzel [20] generalized the notion of canonical module in the following sense. Given a Noetherian local ring R which is a factor ring of a s-dimensional Gorenstein local ring S and a finite R-module M, the *j-th deficiency module of M* is defined as

$$K^j(M) = \operatorname{Ext}_S^{s-j}(M,S)$$

for all $j = 0, ..., \dim_R M$. Local duality assures that these modules are well-defined. Particularly, $K(M) := K^{\dim_R M}(M)$ is called the *canonical module of M*. The deficiency modules of M measure the extent of the failure of M to be Cohen-Macaulay in the sense that M is Cohen-Macaulay if and only if $K^j(M) = 0$ for all $j \neq \dim_R M$.

In this paper, we shall look for relations between Bass and Betti numbers of a given module and its deficiency modules. As Foxby provided the relations above for Cohen-Macaulay modules over a Gorenstein local ring, we furnish the same relations for generalized Cohen-Macaulay and canonically Cohen-Macaulay modules with zeroth and first deficiency modules of positive depth over a local ring

Date: September 7, 2023.
2010 *Mathematics Subject Classification.* Primary 13C14, 13D45; Secondary 13H10, 14B15.
Key words and phrases. Generalized Cohen-Macaulay module, homological dimensions, deficiency modules, Auslander-Reiten conjecture.
The second-named author was supported by a CAPES Doctoral Scholarship.

which is a factor of a Gorenstein local ring, see Theorem 3.3. Furthermore, theorems 3.6 and 4.11 show the same relations hold for arbitrary finite R-modules when certain homological conditions over its deficiency modules are imposed.

Besides such generalizations, we exhibit bounds for the Bass numbers (Betti numbers) of a module in terms of the Betti numbers (Bass numbers) of its deficiency modules, see theorems 3.1 and 4.1. They provide several applications that are worked out through this paper. Three examples of such applications are Corollary 3.4, providing the Cohen-Macaulay property of a local ring in terms of homological conditions over deficiency modules, Corollary 4.6 furnishing a characterization of the complete intersection property in terms of the first and second Bass numbers of the residue field, and Corollary 4.10 that states that the Auslander-Reiten conjecture holds for modules whose deficiency modules have finite injective dimensions, generalizing then a similar application given quite recently in [11].

Our methods are especially concerned with studying the behavior of some spectral sequences that we chose to consider in the first quadrant. The first of them is called the Foxby spectral sequence 2.1, as it was used by Foxby in [10]. The first applications of such spectral sequences regard general information on the canonical module of a generalized Cohen-Macaulay module or an equidimensional module, see Theorem 2.3 and Proposition 2.8. These results provide sufficient conditions for when the module is also canonically Cohen-Macaulay and its canonical module is generalized Cohen-Macaulay, see the Corollaries 2.4 and 2.5, also a characterization of Cohen-Macaulay modules in Corollary 2.10 and a version for generalized Cohen-Macaulay modules of a Schenzel's result, see Corollary 2.11.

2. Generalized Cohen-Macaulay modules

Setup. Throughout this paper, R will always denote a commutative Noetherian local ring with non-zero unity, maximal ideal $\mathfrak{m}$, and residue class field k. Also, R is supposed to be a factor of a Gorenstein local ring S of dimension s, i.e., there exists a surjective ring homomorphism $S \to R$. We say that an R-module M is *finite* if it is a finitely generated R-module and denote by $M^\vee := \operatorname{Hom}_R(M, E_R(k))$ its Matlis dual, where $E_R(k)$ is the injective hull of k.

For an R-module M, $\operatorname{pd}_R M$ and $\operatorname{id}_R M$ denote, respectively, the projective dimension and injective dimension of M. Further, $\beta_i(M) = \dim_k \operatorname{Tor}_i^R(k, M)$ is the i-th Betti number of M, $\mu^i(M) = \dim_k \operatorname{Ext}_R^i(k, M)$ is the i-th Bass number of M and $\mathrm{r}(M) = \dim_k \operatorname{Ext}_R^{\operatorname{depth}_R M}(k, M)$ is its type.

As far as we know, the following spectral sequences have first been used the way we do here in the [10]. Also, we slightly extend them by considering a change of rings.

Lemma 2.1 (Foxby spectral sequences). *Given a finite R-module X, an R-module Y and a S-module Z, if either $\operatorname{pd}_R X < \infty$ or $\operatorname{id}_S Z < \infty$, then there exist a graded R-module H and first quadrant spectral sequences*

$$E_2^{p,q} = \operatorname{Ext}_S^p(\operatorname{Ext}_R^q(X, Y), Z) \Rightarrow_p H^{q-p}$$

and

$${}'E_2^{p,q} = \operatorname{Tor}_p^R(X, \operatorname{Ext}_S^q(Y, Z)) \Rightarrow_p H^{p-q}.$$

7 Sep 2023 12:09:00 PDT
230123-Holanda Version 2 - Submitted to J. Comm. Alg.

Proof. Let $F_\bullet$ be a free R-resolution of X and let $E^\bullet$ be an injective S-resolution of Z. Following the terminology in [19, Chapter 10], the desired spectral sequences are the ones obtained from the first and second filtrations of the first quadrant double complexes

$$\operatorname{Hom}_S(\operatorname{Hom}_R(F_\bullet, Y), E^\bullet) \simeq F_\bullet \otimes_R \operatorname{Hom}_S(Y, E^\bullet).$$

□

The first application of the Foxby spectral sequences 2.1 is a generalization of a well-known result about Cohen-Macaulay modules and their canonical modules, see [20, Theorem 1.14]. First, we need an auxiliary lemma.

We say that a finite R-module M satisfies *Serre's condition* S_k, for k being a non-negative integer, provided

$$\operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} \geq \min\{k, \dim_{R_\mathfrak{p}} M_\mathfrak{p}\}$$

for all $\mathfrak{p} \in \operatorname{Supp} M$.

Lemma 2.2. [20, Lemma 1.9] *Let M be a finite R-module of dimension t. The modules $K^j(M)$ satisfy the following properties.*

(i) $\dim_R K^j(M) \leq j$ *for all integers* j *and* $\dim_R K(M) = t$*;*

(ii) Suppose that M is equidimensional. Then, M satisfies Serre's condition S_k if and only if $\dim_R K^j(M) \leq j - k$*, for all* $0 \leq j < t$*.*

A finite R-module M is said to be *generalized Cohen-Macaulay* if $K^j(M)$ is of finite length for all $j < \dim_R M$.

Theorem 2.3. *Let M be a generalized Cohen-Macaulay R-module of dimension t. The following statements hold.*

(i) There exists an isomorphism

$$K^0(K(M)) \simeq \operatorname{Tor}^S_{-t}(M, S);$$

(ii) There exists a five-term exact sequence

$$\begin{tikzcd} \operatorname{Tor}^S_{-t+2}(M,S) \arrow[r] & K^2(K(M)) \arrow[r] & K^0(K^{t-1}(M)) \arrow[dl] & \\ & \operatorname{Tor}^S_{-t+1}(M,S) \arrow[r] & K^1(K(M)) \arrow[r] & 0 \end{tikzcd}$$

(iii) There exists an exact sequence

$$0 \to K^0(K^0(M)) \to M \to K(K(M)) \to K^0(K^1(M)) \to 0;$$

(iv) If $t \geq 3$*, then there exist isomorphisms*

$$K^{t-j}(K(M)) \simeq K^0(K^{j+1}(M))$$

for all $1 \leq j \leq t-2$*.*

Proof. Consider the Foxby spectral sequences 2.1 by taking $X = M$ as an S-module and $Y = Z = S$

7 Sep 2023 12:09:00 PDT
230123-Holanda Version 2 - Submitted to J. Comm. Alg.

$$E_2^{p,q} = \mathrm{Ext}_S^p(\mathrm{Ext}_S^q(M,S),S) \Rightarrow_p H^{q-p}$$

and

$$'E_2^{p,q} = \mathrm{Tor}_p^S(M, \mathrm{Ext}_S^q(S,S)) \Rightarrow_p H^{p-q}.$$

Since $'E_2^{p,q} = 0$ for all $q \neq 0$, we have

$$H^j \simeq {'E_2^{j,0}} = \mathrm{Tor}_j^S(M,S)$$

for all $j \geq 0$, and

$$E_2^{p,q} = \mathrm{Ext}_S^p(\mathrm{Ext}_S^q(M,S),S) \Rightarrow_p \mathrm{Tor}_{q-p}^S(M,S).$$

Since $H_{\mathfrak{m}}^j(M)$ has finite length for all $j < t$, so has $K^j(M)$, and by local duality

$$\mathrm{Ext}_S^p(\mathrm{Ext}_S^q(M,S),S) = \mathrm{Ext}_S^p(K^{s-q}(M),S) = 0$$

for all $q > s-t$ and for all $p \neq s$. Also, Lemma 2.2 (i) assures that $\dim_R K(M) = t$. Thus, E_2 has the following shape

$$\begin{array}{cccccc}
0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & \mathrm{Ext}_S^s(K^0(M),S) & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \mathrm{Ext}_S^s(K^{t-1}(M),S) & 0 \\
0 & K(K(M)) & \mathrm{Ext}_S^{s-(t-1)}(K(M),S) & \cdots & \mathrm{Ext}_S^s(K(M),S) & 0 \\
0 & 0 & 0 & 0 & 0 & 0.
\end{array}$$

By convergence, there are isomorphisms

$$K^0(K(M)) = \mathrm{Ext}_S^s(K(M),S) \simeq E_\infty^{s,s-t} \simeq \mathrm{Tor}_{-t}^S(M,S),\ K^1(K(M)) = \mathrm{Ext}_S^{s-1}(K(M),S) \simeq E_\infty^{s-1,s-t}$$

and

$$K^0(K^0(M)) = \mathrm{Ext}_S^s(K^0(M),S) \simeq E_\infty^{s,s}.$$

Thus we get item (i) and by applying Matlis dual one has isomorphisms

$$H_{\mathfrak{m}}^1(K(M)) \simeq (E_\infty^{s-1,s-t})^\vee \text{ and } H_{\mathfrak{m}}^0(K^0(M)) \simeq (E_\infty^{s,s})^\vee.$$

The convergence again gives us short exact sequences

$$0 \to E_\infty^{s,s-j} \to \mathrm{Tor}_{-j}^S(M,S) \to E_\infty^{s-(t-j),s-t} \to 0 \tag{2.1}$$

for all $j \geq 0$. Further, as we move through the pages of E, the differentials between the vertical and horizontal lines in the diagram above come out. In other words, there is an exact sequence

$$0 \to E_\infty^{s-(t-j),s-t} \to \operatorname{Ext}_S^{s-(t-j)}(K(M), S) \to \operatorname{Ext}_S^s(K^{j+1}(M), S) \to E_\infty^{s,s-(j+1)} \to 0 \tag{2.2}$$

for all $0 \leq j \leq t-2$.

Item (ii) is exactly the five-term exact sequence of E. For item (iii), by taking $j = 0$ in both above exact sequences, we have the following exact sequences

$$0 \to \operatorname{Ext}_S^s(K^0(M), S) \to M \to E_\infty^{s-t,s-t} \to 0$$

and

$$0 \to E_\infty^{s-t,s-t} \to K(K(M)) \to \operatorname{Ext}_S^s(K^1(M), S) \to E_\infty^{s,s-1} \to 0.$$

The result follows by splicing these sequences and noticing that $E_\infty^{s,s-1} \subseteq \operatorname{Tor}_{-1}^S(M, S) = 0$.

The exact sequence 2.1 assures that $E_\infty^{s-(t-j),s-t} = E_\infty^{s,s-j} = 0$ for all $j > 0$, so that, by the exact sequence 2.2,

$$K^{t-j}(K(M)) = \operatorname{Ext}_S^{s-(t-j)}(K(M), S) \simeq \operatorname{Ext}_S^s(K^{j+1}(M), S) = K^0(K^{j+1}(M))$$

for all $1 \leq j \leq t-2$. □

The concept of *canonically Cohen-Macaulay module* was introduced by Schenzel [21]. We say that a finite R-module M is canonically Cohen-Macaulay if its canonical module $K(M)$ is Cohen-Macaulay.

Corollary 2.4. *Let M be a generalized Cohen-Macaulay R-module of dimension t. The following statements hold.*

(i) If $t > j$ with $j \in \{0, 1\}$, then $\operatorname{depth}_R K(M) > j$*;*
(ii) If $t = 1$, then M is canonically Cohen-Macaulay and there exists the short exact sequence

$$0 \to K^0(K^0(M)) \to M \to K(K(M)) \to 0;$$

(iii) If $t = 2$, then M is canonically Cohen-Macaulay;
(iv) If $t \geq 3$, then $K(M)$ is generalized Cohen-Macaulay.

Proof. Item (i) follows immediately from Theorem 2.3 (i) and (ii). For item (ii), item (i) assures that $K(M)$ is Cohen-Macaulay and Theorem 2.3 (iii) is the desired exact sequence. As to item (iii), item (i) again assures that $K(M)$ is Cohen-Macaulay. Item (iv) follows directly from item (i) and Theorem 2.3 (iv). □

Corollary 2.5. *If M is generalized Cohen-Macaulay, then so is $K(M)$.*

Corollary 2.5 inspires us to ask the following.

Question 2.6. *Given a finite R-module M, when is $K(M)$ generalized Cohen-Macaulay?*

As Corollary 2.4 assures that generalized Cohen-Macaulay of dimension at most two are canonically Cohen-Macaulay, Theorem 2.3 (iv) recovers a characterization [6] for the case where the dimension is at least three.

Recall that for an ideal I of R, the I-transform of an R-module M is defined to be $D_I(M) := \varinjlim_{n\in\mathbb{N}} \operatorname{Hom}_R(I^n, M)$, see [7] for more details.

Corollary 2.7. [6, Corollary 2.7] *Let M be a generalized Cohen-Macaulay R-module of dimension $t \geq 3$. Then, the following statements are equivalent:*

(i) M is canonically Cohen-Macaulay;
(ii) $H^j_{\mathfrak{m}}(M) = 0$ for all $j = 2, ..., t-1$;
(iii) The $\mathfrak{m}$-transform functor $D_{\mathfrak{m}}(M)$ is a Cohen-Macaulay R-module.

Proposition 2.8. *Let M be a finite R-module of dimension t. The following statements hold.*

(i) If M is generalized Cohen-Macaulay R-module with depth at least two, then $M \simeq K(K(M))$.
(ii) Suppose M is equidimensional. If M satisfies Serre's condition S_{k+1} for some positive integer k, then

$$K^j(K(M)) \simeq \operatorname{Tor}^S_{-t+j}(M, S)$$

for all $t-k+1 \leq j \leq t$.

Proof. Item (i) follows immediately from Theorem 2.3 (iii) and from the fact that $K^0(M) = K^1(M) = 0$ in case of $g \geq 2$.

For item (ii), consider the Foxby spectral sequences given in Theorem 2.3

$$E_2^{p,q} = \operatorname{Ext}^p_S(\operatorname{Ext}^q_S(M, S), S) \Rightarrow_p \operatorname{Tor}^S_{q-p}(M, S).$$

By Lemma 2.2 (ii) and local duality, we have

$$E_2^{s-i,s-j} = \operatorname{Ext}^{s-i}_S(K^j(M), S) = 0$$

for all $0 \leq j < t$ and $i > j-k-1$. In other words, all modules $E_2^{p,q}$ such that $q \neq s-t$ above the dotted line in the below diagram must be zero

$$\begin{array}{ccccc}
0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \iddots & \operatorname{Ext}^s_S(K^{k+1}(M), S) & 0 \\
0 & \operatorname{Ext}^{s-(t-k-2)}_S(K^{t-1}(M), S) & \cdots & \vdots & 0 \\
\operatorname{Ext}^{s-(t-k-1)}_S(K(M), S) & \operatorname{Ext}^{s-(t-k-2)}_S(K(M), S) & \cdots & \operatorname{Ext}^s_S(K(M), S) & 0 \\
0 & \cdots & 0 & 0 & 0.
\end{array}$$

The result follows from the convergence. ☐

Our results also retrieve the well-known fact that every Cohen-Macaulay module is canonically Cohen-Macaulay, see [20, Theorem 1.14].

7 Sep 2023 12:09:00 PDT
230123-Holanda Version 2 - Submitted to J. Comm. Alg.

Corollary 2.9. *If M is Cohen-Macaulay of dimension t, then so is $K(M)$ and $K(K(M)) \simeq M$.*

Proof. There are two immediate ways of proving the desired result. Indeed the result follows directly from Theorem 2.3 as well as from Proposition 2.8 (ii). ☐

Proposition 2.8 provides a characterization for the Cohen-Macaulay property.

Corollary 2.10. *If M is a finite R-module, then M is Cohen-Macaulay if and only if M is equidimensional canonically Cohen-Macaulay satisfying Serre's condition S_{k+1} for some positive integer k.*

Proof. It is well-known that a Cohen-Macaulay module is equidimensional and satisfies Serre's condition S_k for any k. Corollary 2.9 assures that such a module is also canonically Cohen-Macaulay. Conversely, by taking $j = t$ in Proposition 2.8 (ii), we have the isomorphism $K(K(M)) \simeq M$. Since $K(M)$ is Cohen-Macaulay, Corollary 2.9 again assures that $M \simeq K(K(M))$ is Cohen-Macaulay. ☐

The next corollary is a version of Corollary 2.9 for generalized Cohen-Macaulay modules.

Corollary 2.11. *If M is a generalized Cohen-Macaulay module with depth at least two, then so is $K(M)$ and $M \simeq K(K(M))$.*

Proof. It follows directly from Theorem 2.3, Corollary 2.5 and Proposition 2.8 (i). ☐

3. Bounding Bass numbers

The Foxby spectral sequences 2.1 are fundamental tools in our work. They provide the main result of this section.

Theorem 3.1. *If M is a finite R-module of depth g and dimension t, then the following inequality holds for all $j \geq 0$*

$$\mu^j(M) \leq \sum_{i=g}^{t} \beta_{j+i}(K^i(M)).$$

Moreover, $\mathrm{r}(M) = \beta_0(K^g(M))$ *and*

$$\mu^{g+2}(M) - \mu^{g+1}(M) \leq \beta_2(K^g(M)) - \beta_1(K^g(M)) - \beta_0(K^{g+1}(M)).$$

Proof. Consider the Foxby spectral sequences 2.1 by taking $X = k, Y = M$ and $Z = S$

$$E_2^{p,q} = \mathrm{Ext}_S^p(\mathrm{Ext}_R^q(k, M), S) \Rightarrow_p H^{q-p}$$

and

$$'E_2^{p,q} = \mathrm{Tor}_p^R(k, \mathrm{Ext}_S^q(M, S)) \Rightarrow_p H^{p-q}.$$

Since $\mathrm{Ext}_R^q(k, M)$ is of finite length, we must have $E_2^{p,q} = 0$ for all $p \neq s$, so that

$$H^j \simeq E_2^{s,j+s} = \mathrm{Ext}_S^s(\mathrm{Ext}_R^{j+s}(k, M), S)$$

for all integer j. Once $K^{s-q}(M) = \mathrm{Ext}_S^q(M, S)$ for all $q \geq 0$, we conclude that

$$'E_2^{p,q} = \mathrm{Tor}_p^R(k, K^{s-q}(M)) \Rightarrow_p \mathrm{Ext}_S^s(\mathrm{Ext}_R^{p-q+s}(k, M), S). \tag{3.1}$$

Now, since $\operatorname{Ext}_S^s(k,S)^\vee \simeq k$, where $\_^\vee$ denotes the Matlis dual of R, we have

$$\operatorname{Ext}_S^s(\operatorname{Ext}_R^j(k,M),S) \simeq \operatorname{Ext}_S^s(k,S)^{\mu^j(M)} \simeq k^{\mu^j(M)}$$

as k-vector spaces. Therefore, by the convergence of the spectral sequence (3.1),

$$\mu^j(M) \leq \sum_{j=p-q+s} \beta_p(K^{s-q}(M)) = \sum_{i=g}^{t} \beta_{j+i}(K^i(M))$$

for all $j \geq 0$.

Now, since $K^i(M) = \operatorname{Ext}_S^{s-i}(M,S) = 0$ for all $i < g$, then ${}'E_2$ has the following corner

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
\cdots & \operatorname{Tor}_2^R(k,K^{g+1}(M)) & \operatorname{Tor}_2^R(k,K^g(M)) \arrow[ddl] & 0 & \cdots \\
\cdots & \operatorname{Tor}_1^R(k,K^{g+1}(M)) & \operatorname{Tor}_1^R(k,K^g(M)) & 0 & \cdots \\
\cdots & k \otimes_R K^{g+1}(M) & k \otimes_R K^g(M) & 0 & \cdots
\end{tikzcd}$$

Therefore,

$$k \otimes_R K^g(M) = {}'E_2^{0,s-g} \simeq H^{g-s} \simeq \operatorname{Ext}_S^s(\operatorname{Ext}_R^g(k,M),S)$$

so that $\mathrm{r}(M) = \beta_0(K^g(M))$ and there exists a five-term-type exact sequence

$$\begin{tikzcd}
\operatorname{Ext}_S^s(\operatorname{Ext}_R^{g+2}(k,M),S) \arrow[r] & \operatorname{Tor}_2^R(k,K^g(M)) \arrow[r] & k \otimes_R K^{g+1}(M) \arrow[dl] & \\
 & \operatorname{Ext}_S^s(\operatorname{Ext}_R^{g+1}(k,M),S) \arrow[r] & \operatorname{Tor}_1^R(k,K^g(M)) \arrow[r] & 0
\end{tikzcd}$$

whence the desired formula. $\square$

Corollary 3.2. *Let M be a finite R-module of depth g and dimension t. If $\operatorname{pd}_R K^i(M) < \infty$ for all $i = g,...,t$, then $\operatorname{id}_R M < \infty$.*

Proof. The hypothesis means that $\beta_l(K^i(M)) = 0$ for all $l \gg 0$ and by Theorem 3.1 one has

$$\mu^j(M) \leq \sum_{i=g}^{t} \beta_{j+i}(K^i(M)) = 0$$

for $j \gg 0$, i.e., $\operatorname{id}_R M < \infty$. $\square$

Theorem 3.3. *If M is a generalized Cohen-Macaulay canonically Cohen-Macaulay R-module of dimension t and depth at least two, then*

$$\beta_j(M) = \mu^{j+t}(K(M)) \quad \text{and} \quad \mu^j(M) = \beta_{j-t}(K(M))$$

for all $j \geq 0$. In particular, $\operatorname{pd}_R M < \infty$ if and only if $\operatorname{id}_R K(M) < \infty$ and $\operatorname{id}_R M < \infty$ if and only if $\operatorname{pd}_R K(M) < \infty$.

7 Sep 2023 12:09:00 PDT
230123-Holanda Version 2 - Submitted to J. Comm. Alg.

Proof. By Lemma 2.2 (i), $K(M)$ is Cohen-Macaulay of dimension t and by Proposition 2.8 (i), $K(K(M)) \simeq M$, that is, $K^i(K(M)) = 0$ for all $i \neq t$ and $K^t(K(M)) \simeq M$. The spectral sequence 3.1

$$'E_2^{p,q} = \operatorname{Tor}_p^R(k, K^{s-q}(K(M))) \Rightarrow_p \operatorname{Ext}_S^s(\operatorname{Ext}_R^{p-q+s}(k, K(M)), S)$$

degenerates, so that

$$\operatorname{Tor}_j^R(k, M) \simeq \operatorname{Tor}_j^R(k, K(K(M))) = {}'E_2^{j,s-t} \simeq \operatorname{Ext}_S^s(\operatorname{Ext}_R^{j+t}(k, K(M)), S)$$

for all $j \geq 0$. Therefore,

$$\beta_j(M) = \dim_k \operatorname{Tor}_j^R(k, M) = \dim_k \operatorname{Ext}_S^s(\operatorname{Ext}_R^{j+t}(k, K(M)), S) = \mu^{j+t}(K(M))$$

for all $j \geq 0$. The other equality follows from the fact $K(K(M)) \simeq M$. □

It is well-known that a local ring must be Cohen-Macaulay provided one of the following holds: it admits either a Cohen-Macaulay finite module of finite projective dimension or a finite module of finite injective dimension. The second hypothesis is known as Bass' conjecture and both results follow from the Peskine-Szpiro intersection theorem [17], which was generalized by Roberts in [18].

The next corollary follows from Corollary 3.2, Theorem 3.3 and Bass' conjecture. It is worth comparing it the first hypothesis above.

Corollary 3.4. *Let M be a finite R-module of depth g and dimension t. Then R is Cohen-Macaulay if any of the following statements hold:*

(i) $\operatorname{pd}_R K^i(M) < \infty$ *for all $i = g, ..., t$.*

(ii) M is generalized Cohen-Macaulay canonically Cohen-Macaulay with finite projective dimension and $g \geq 2$.

Theorem 3.3 generalizes [10, Corollary 3.6] and improves [11, Corollary 3.3]. We record this in the next corollary.

Corollary 3.5. *If M is Cohen-Macaulay R-module of dimension t, then*

$$\beta_j(M) = \mu^{j+t}(K(M)) \quad \textit{and} \quad \mu^j(M) = \beta_{j-t}(K(M))$$

for all $j \geq 0$. In particular, $\operatorname{pd}_R M < \infty$ *if and only if* $\operatorname{id}_R K(M) < \infty$ *and* $\operatorname{id}_R M < \infty$ *if and only if* $\operatorname{pd}_R K(M) < \infty$.

Proof. If $t \geq 2$, then the result follows from Theorem 3.3. Otherwise, Corollary 2.9 and the spectral sequence argument given in the proof of Theorem 3.3 asserts the result. □

The next theorem is an attempt to extend part of Theorem 3.3 to arbitrary modules. In the next section, we work on the other part.

Theorem 3.6. *Let M be a finite R-module of depth g and dimension t. If* $\operatorname{pd}_R K^i(M) < \infty$ *for all $g \leq i < t$, then*

$$\mu^j(M) = \beta_{j-t}(K(M))$$

for all $j > \operatorname{depth} R + t$. *In particular,* $\operatorname{id}_R M < \infty$ *if and only if* $\operatorname{pd}_R K(M) < \infty$.

Proof. The spectral sequence 3.1 is such that ${}'E_2^{p,q} = 0$ for all $p > \operatorname{depth} R$ and $g \leq q < t$, so that

$$\operatorname{Tor}_j^R(k, K(M)) = {}'E_2^{j,s-t} \simeq \operatorname{Ext}_S^s(\operatorname{Ext}_R^{j+t}(k, M), S),$$

whence the result. □

We derive other consequences of Theorem 3.1. In particular, we say exactly when the type of a finite module is one in terms of its deficiency modules.

Corollary 3.7. *Let M be a finite R-module of depth g and dimension t. The following statements hold.*

(i) If M is Cohen-Macaulay of dimension t, then

$$\mu^{t+2}(K(M)) - \mu^{t+1}(K(M)) \geq \beta_2(M) - \beta_1(M).$$

In particular, if $\operatorname{pd}_R M < \infty$ then $\beta_1(M) \geq \beta_2(M)$.

(ii) If $\operatorname{id}_R M < \infty$, then

$$\beta_0(K^{g+1}(M)) \geq \beta_2(K^g(M)) - \beta_1(K^g(M)).$$

In particular, if M is also Cohen-Macaulay, then $\beta_1(K(M)) \geq \beta_2(K(M))$.

(iii) $\mathrm{r}(M) = 1$ if and only if $K^g(M)$ is cyclic.

Proof. Item (*iii*) follows directly from Theorem 3.1. Item (*i*) follows from Corollary 2.9, Theorem 3.1 and Corollary 3.5, and item (*ii*) follows from [8, Theorem 3.7], Corollaries 2.9 and 3.5 and item (*i*). □

The spectral sequence 3.1 provides more information when the module involved has only two (possibly) non-zero deficiency modules.

Proposition 3.8. *Let M be a finite R-module of depth g and dimension t. Suppose $K^i(M) = 0$ for all $i \neq g, t$. If $\operatorname{id}_R M < \infty$ then $\beta_j(K^g(M)) = \beta_{j+g-t-1}(K(M))$ for all $j > \operatorname{depth} R - g + 1$.*

Proof. Write $t = g + r$. The spectral sequence 3.1 has only two vertical lines as the following diagram shows

$$\begin{tikzcd}
 & & \vdots & \vdots & & \vdots & \vdots & & \\
\cdots & 0 & \operatorname{Tor}_{r+1}^R(k, K(M)) & 0 & \cdots & 0 & \operatorname{Tor}_{r+1}^R(k, K^g(M)) \arrow[ddddllll] & 0 & \cdots \\
 & & \vdots & & \iddots & & \vdots & & \\
\cdots & 0 & \operatorname{Tor}_2^R(k, K(M)) & 0 & \cdots & 0 & \operatorname{Tor}_2^R(k, K^g(M)) & 0 & \cdots \\
\cdots & 0 & \operatorname{Tor}_1^R(k, K(M)) & 0 & \cdots & 0 & \operatorname{Tor}_1^R(k, K^g(M)) & 0 & \cdots \\
\cdots & 0 & k \otimes_R K(M) & 0 & \cdots & 0 & k \otimes_R K^g(M) & 0 & \cdots
\end{tikzcd}$$

From convergence, we obtain an exact sequence

$$\operatorname{Ext}^s_S(\operatorname{Ext}^{j+g}_R(k,M),S) \to \operatorname{Tor}^R_j(k,K^g(M)) \to \operatorname{Tor}^R_{p-r-1}(k,K(M)) \to \operatorname{Ext}^s_S(\operatorname{Ext}^{j+g-1}_R(k,M),S)$$

for all $j \geq 0$. Thus, since $\operatorname{id}_R M = \operatorname{depth} R$ (see [8, Theorem 3.7.1]), we conclude that

$$\operatorname{Tor}^R_j(k,K^g(M)) \simeq \operatorname{Tor}^R_{j-r-1}(k,K(M))$$

for all $j > \operatorname{depth} R - g + 1$, whence the result. □

Based on Corollary 3.2 and Proposition 3.8, we finish this section by asking the following.

Question 3.9. *Let M be a finite R-module of depth g and dimension t. Is it true that*

$$\operatorname{id}_R M < \infty \Leftrightarrow \operatorname{pd}_R K^i(M) < \infty, \forall i = g, ..., t?$$

4. Bounding Betti numbers

In the last section, we bounded the Bass numbers of a module in terms of the Betti numbers of the deficiency modules. In this section, we get a dual version of Theorem 3.1 in the following sense.

Theorem 4.1. *For a finite R-module M of depth g and dimension t, the following inequality holds true for all $j \geq 0$*

$$\beta_j(M) \leq \sum_{i=g}^{t} \mu^{j+i}(K^i(M)).$$

Moreover, $\mu^0(K(M)) = \beta_{-t}(M)$ and

$$\beta_{-t+2}(M) - \beta_{-t+1}(M) \geq \mu^2(K(M)) - \mu^1(K(M)) - \mu^0(K^{t-1}(M)).$$

Proof. By taking a free R-resolution $F_\bullet$ of k and an injective S-resolution $E^\bullet$ of S, the tensor-hom adjunction induces a first quadrant double complex isomorphism

$$\operatorname{Hom}_S(F_\bullet, \operatorname{Hom}_S(M,E^\bullet)) \simeq \operatorname{Hom}_S(F_\bullet \otimes_R M, E^\bullet)$$

which yields two spectral sequences as follows

$$E_2^{p,q} = \operatorname{Ext}^p_R(k, \operatorname{Ext}^q_S(M,S)) \Rightarrow_p H^{p+q}$$

and

$$'E_2^{p,q} = \operatorname{Ext}^p_S(\operatorname{Tor}^R_q(k,M),S) \Rightarrow_p H^{p+q}.$$

Since $\operatorname{Tor}^R_q(k,M)$ is of finite length for all $q \geq 0$, due to local duality, we must have $'E_2^{p,q} = 0$ for all $p \neq s$, so that

$$H^j \simeq {}'E_2^{s,j-s} = \operatorname{Ext}^s_S(\operatorname{Tor}^R_{j-s}(k,M),S)$$

for all $j \geq 0$. Once $K^{s-q}(M) = \operatorname{Ext}^q_R(M,S)$ for all $q \geq 0$, one has spectral sequence

$$E_2^{p,q} = \operatorname{Ext}^p_R(k,K^{s-q}(M)) \Rightarrow_p \operatorname{Ext}^s_S(\operatorname{Tor}^R_{p+q-s}(k,M),S). \tag{4.1}$$

7 Sep 2023 12:09:00 PDT
230123-Holanda Version 2 - Submitted to J. Comm. Alg.

Once $\beta_j(M) = \dim_k \operatorname{Ext}^s_S(\operatorname{Tor}^R_{(j+s)-s}(k, M), S)$, by convergence we conclude that

$$\beta_j(M) \leq \sum_{p+q=j+s} \dim_k \operatorname{Ext}^p_R(k, K^{s-q}(M)) = \sum_{i=g}^{t} \mu^{i+j}(K^i(M)).$$

Now, since $K^i(M) = 0$ for all $i < g$ or $i > t$, then $E_2^{p,q} = 0$ for all $q < s - t$ or $q > s - g$. In particular, E_2 has a corner as follows

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \\
\operatorname{Hom}_R(k, K^{t-1}(M)) \arrow[drr] & \operatorname{Ext}^1_R(k, K^{t-1}(M)) & \operatorname{Ext}^2_R(k, K^{t-1}(M)) & \cdots \\
\operatorname{Hom}_R(k, K(M)) & \operatorname{Ext}^1_R(k, K(M)) & \operatorname{Ext}^2_R(k, K(M)) & \cdots \\
0 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots &
\end{tikzcd}$$

Therefore, there exists the isomorphism

$$\operatorname{Hom}_R(k, K(M)) = E_2^{0,s-t} \simeq \operatorname{Ext}^s_S(\operatorname{Tor}^R_{-t}(k, M), S)$$

and a five-term type exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Ext}^1_R(k, K(M)) \arrow[r] & \operatorname{Ext}^s_S(\operatorname{Tor}^R_{-t+1}(k, M), S) \arrow[r] & \operatorname{Hom}_R(k, K^{t-1}(M)) \arrow[dl] \\
 & & \operatorname{Ext}^2_R(k, K(M)) \arrow[r] & \operatorname{Ext}^s_S(\operatorname{Tor}^R_{-t+2}(k, M), S)
\end{tikzcd}$$

whence the result. □

Remark 4.2. *It should be noticed that the estimate $\beta_j(M) \leq \sum_{i=g}^{t} \mu^{j+i}(K^i(M))$ is already known, see* [20, Theorem 3.2].

Corollary 4.3. *The following statements hold.*

(i) *If $t = 0$, then $\beta_0(M) = \mu^0(K(M))$ and*

$$\beta_2(M) - \beta_1(M) \geq \mu^2(K(M)) - \mu^1(K(M)).$$

Otherwise, $\operatorname{depth}_R K(M) > 0$*;*

(ii) *If $t = 1$, then $\beta_1(M) - \beta_0(M) \geq \mu^2(K(M)) - \mu^1(K(M)) - \mu^0(K^0(M))$;*

(iii) *If $t = 2$, then $\beta_0(M) \geq \mu^2(K(M)) - \mu^1(K(M)) - \mu^0(K^1(M))$;*

(iv) *If $t > 2$, then $\mu^0(K^{t-1}(M)) \geq \mu^2(K(M)) - \mu^1(K(M))$.*

Proof. Properties (i)-(iv) follow directly from Theorem 4.1. □

Corollary 4.4. *If M is a finite Artinian R-module, then*

$$\beta_2(M) - \beta_1(M) = \mu^2(K(M)) - \mu^1(K(M)).$$

Proof. By the corollaries 3.7 (i) and 4.3 (i),

$$\mu^2(K(M)) - \mu^1(K(M)) \geq \beta_2(M) - \beta_1(M) \geq \mu^2(K(M)) - \mu^1(K(M)).$$

$\square$

Lemma 4.5. [12, Proposition 2.8.4] *Suppose R is d-dimensional with embedding dimension e. Then $\beta_1(R/\mathfrak{m}) = e$ and the following statements are equivalent.*

(i) $\beta_2(R/\mathfrak{m}) = \binom{e}{2} + e - d$;

(ii) R is a complete intersection.

Corollary 4.6. *If R is d-dimensional of embedding dimension e, then*

$$\mu^2(k) - \mu^1(k) = \binom{e}{2} - d$$

if and only if R is a complete intersection.

Proof. It follows directly from Corollary 4.4 and Lemma 4.5. $\square$

Corollary 4.7. *Let M be a finite R-module of depth g and dimension t. If $\operatorname{id}_R K^i(M) < \infty$ for all $i = g, ..., t$, then $\operatorname{pd}_R M < \infty$.*

Proof. By hypothesis, we have $\mu^l(K^i(M)) = 0$ for all $l \gg 0$ and by Theorem 4.1 one has

$$\beta_j(M) \leq \sum_{i=g}^{t} \mu^{j+i}(K^i(M)) = 0$$

for all $j \gg 0$, whence $\mu^j(M) = 0$ for all $j \gg 0$, that is, $\operatorname{pd}_R M < \infty$. $\square$

The *Auslander-Reiten conjecture* [3] states the following. Given a finite R-module M, if

$$\operatorname{Ext}^j_R(M, M \oplus R) = 0$$

for all $j > 0$, then M is free. This long-standing conjecture has been largely studied and several positive answers are already known, see for instance [1, 2, 4, 9, 11, 13, 14, 16]. Corollary 4.7 provides another positive answer for the Auslander-Reiten conjecture for a class of modules. But first, we need a lemma.

Lemma 4.8. [15, Lemma 1 (iii)] *Let R be a local ring and let M and N be finite R-modules. If $\operatorname{pd}_R M < \infty$ and $N \neq 0$, then*

$$\operatorname{pd}_R M = \sup\{j : \operatorname{Ext}^j_R(M, N) \neq 0\}.$$

Theorem 4.9. *Let M be a finite R-module of depth g and dimension t. If $n \leq d$ is a positive integer, then $\operatorname{pd}_R M < n$ provided the following statements hold.*

(i) $\operatorname{id}_R K^i(M) < \infty$ *for all* $i = g, ..., t$;

(ii) There exists an R-module N such that $\operatorname{Ext}^j_R(M, N) = 0$ for all $j = n, ..., d$.

Proof. It follows directly from Corollary 4.7 and Lemma 4.8. $\square$

The next corollary proves the Auslander-Reiten conjecture for a certain class of modules. It generalizes the case of the conjecture obtained in [11].

Corollary 4.10. *The Auslander-Reiten conjecture holds for finite modules having deficiency modules of finite injective dimension over local rings which are factors of Gorenstein local rings.*

Proof. It follows immediately from Theorem 4.9 by taking $n = 1$. $\square$

In the next theorem, such as Theorem 3.6, we furnish another attempt to remove the generalized Cohen-Macaulayness hypothesis from Theorem 3.3.

Theorem 4.11. *Let M be a finite R-module of depth g and dimension t. If $\operatorname{id}_R K^i(M) < \infty$ for all $g \le i < t$, then*

$$\beta_j(M) = \mu^{j+t}(K(M))$$

for all $j > s + \operatorname{depth} R - t - g$. In particular, $\operatorname{pd}_R M < \infty$ if and only if $\operatorname{id}_R K(M) < \infty$.

Proof. Consider the spectral sequence 4.1

$$E_2^{p,q} = \operatorname{Ext}_R^p(k, K^{s-q}(M)) \Rightarrow_p \operatorname{Ext}_S^s(\operatorname{Tor}_{p+q-s}^R(k, M), S).$$

The hypothesis and [8, Theorem 3.7.1] assures that $E_2^{p,q} = 0$ for all $p > \operatorname{depth} R$ and for all $s - t < q \ge s - g$. Therefore, the convergence of E implies that

$$\operatorname{Ext}_R^j(k, K(M)) \simeq \operatorname{Ext}_S^s(\operatorname{Tor}_{j-t}^R(k, M), S)$$

for all $j > s - \operatorname{depth} R - g$, whence the result. $\square$

The next proposition is an attempt to understand the converse of Corollary 4.7.

Proposition 4.12. *Assume $K^i(M) = 0$ for all $i \neq g, t$. If $\operatorname{pd}_R M < \infty$, then $\mu^j(K^g(M)) = \mu^{j-g+t+1}(K(M))$ for all $j > \operatorname{pd}_R M + 1$.*

Proof. The spectral sequence 4.1 has only two lines as follows

$$\begin{tikzcd}
0 & 0 & \cdots & 0 & \cdots \\
\vdots & \vdots & & \vdots & \\
\operatorname{Hom}_R(k, K^g(M)) \arrow[ddrrr] & \operatorname{Ext}_R^1(k, K^g(M)) & \cdots & \operatorname{Ext}_R^{p+r+1}(k, K^g(M)) & \cdots \\
\vdots & \vdots & \ddots & \vdots & \\
\operatorname{Hom}_R(k, K(M)) & \operatorname{Ext}_R^1(k, K(M)) & \cdots & \operatorname{Ext}_R^{p+r+1}(k, K(M)) & \cdots \\
0 & 0 & \cdots & 0 & \cdots \\
\vdots & \vdots & & \vdots &
\end{tikzcd}$$

Such a shape and convergence yields an exact sequence

$$\operatorname{Ext}_S^s(\operatorname{Tor}_{j-g}^R(k,M),S) \to \operatorname{Ext}_R^j(k,K^g(M)) \to \operatorname{Ext}_R^{j+r+1}(k,K(M)) \to \operatorname{Ext}_S^s(\operatorname{Tor}_{j-g+1}^R(k,M),S)$$

for all $j \geq 0$. Thus, if $j > \operatorname{pd}_R M + 1$, then

$$\operatorname{Ext}_R^j(k,K^g(M)) \simeq \operatorname{Ext}_R^{j+r+1}(k,K(M))$$

and, in particular, $\mu^j(K^g(M)) = \mu^{j+r+1}(K(M))$. □

Corollary 4.7 and Proposition 4.12 lead us to ask the following.

Question 4.13. *Let M be a finite R-module of depth g and dimension t. Is it true that*

$$\operatorname{pd}_R M < \infty \Leftrightarrow \operatorname{id}_R K^i(M) < \infty, \forall i = g, ...t?$$

Acknowledgements

Both authors thank Marc Chardin and Victor Hugo Jorge Pérez for valuable comments, suggestions, and encouragement. We thank Thiago Henrique de Freitas for suggesting Corollary 4.6 and for his valuable comments. Also, the authors are grateful to the referee for remarks that led to several improvements in the text.

References

[1] T. Araya, *The Auslander–Reiten conjecture for Gorenstein rings*, Proc. Amer. Math. Soc. **137** (2009), no. 6, 1941–1944. 13

[2] T. Araya, Y. Yoshino, *Remarks on a depth formula, a grade inequality and a conjecture of Auslander*, Comm. Algebra, **26**(11):3793–3806, 1998. 13

[3] M. Auslander, I., Reiten, *On a generalized version of the Nakayama conjecture*, Proc. Amer. Math. Soc. **32**, 69-74 (1975). 13

[4] L. L. Avramov, R.-O. Buchweitz, *Support varieties and cohomology over complete intersections*, Invent. Math. **142**(2) (2000):285–318. 13

[5] H. Bass, *On the ubiquity of Gorenstein rings*, Math. Zeitschr., **82** (1963), 9-28.

[6] M. Brodmann, L. T. Nhan, *On canonical Cohen–Macaulay modules*, Journal of Algebra, Volume 371, 2012, 480-491. 5, 6

[7] M.P. Brodmann, R.Y. Sharp, *Local Cohomology. An Algebraic Introduction with Geometric Applications*, 2nd ed. Cambridge: Cambridge University Press, (2013). 6

[8] W. Bruns, J. Herzog, *Cohen–Macaulay rings, revised edition*, Cambridge Studies in Advanced Mathematics, **39**, Cambridge University Press, Cambridge, 1998. 10, 11, 14

[9] H. Dao, M. Eghbali, J. Lyle, *Hom and Ext, revisited*, J. Algebra **571** (2021), 75–93. 13

[10] H.-B., Foxby, *On the μ^i in a minimal injective resolution*, Math. Scand. **29**, 175-186 (1971-1972). 1, 2, 9

[11] T. H. Freitas, V. H. Jorge-Pérez, *When does the canonical module of a module have finite injective dimension?*, Arch. Math. **117**, 485–494 (2021). 1, 2, 9, 13, 14

[12] M. Hashimoto, *Auslander-Buchweitz Approximations of Equivariant Modules*, London Mathematical Society Lecture Note Series, Cambridge: Cambridge University Press, 2000. 13

[13] C. Huneke, G. J. Leuschke, *On a conjecture of Auslander and Reiten*, J. Algebra **275** (2004), no. 2, 781–790. 13

[14] J. Lyle, J. Montaño, *Extremal growth of Betti numbers and trivial vanishing of (co)homology*, Trans. Amer. Math., Soc. **373** (2020), no. 11, 7937–7958. 13

[15] H. Matsumura, *Commutative ring theory*, Translated from the Japanese by M. Reid, Second edition, Cambridge Studies in Advanced Mathematics **8**, Cambridge University Press, Cambridge, 1989. 13

[16] S. Nasseh, S. Sather-Wagstaff, *Vanishing of Ext and Tor over fiber products.* Proc. Amer. Math. Soc. **145** (2017), 4661–4674. 13

[17] C. Peskine, L. Szpiro, *Dimension projective finie et cohomologie locale. Applications à la démonstration de conjectures de M. Auslander, H. Bass et A. Grothendieck*, Inst. Hautes, Études Sci. Publ. Math. **42**, 47-119 (1973). 9

[18] P. Roberts, *Le théorème d'intersection*, C. R. Acad. Sc. Paris Sér. I Math. **304**(7), 177-180 (1987). 9

[19] J. J. Rotman, *An introduction to homological algebra*, second ed. Universitext. Springer, New York, 2009. 3

[20] P. Schenzel, *On the use of local cohomology in algebra and geometry*, In: Six Lectures on Commutative Algebra (Bellaterra, 1996), pp. 241-292. Progr. Math., 166 Birkhäuser, Basel (1998). 1, 3, 6, 12

[21] P. Schenzel, *On birational Macaulayfications and Cohen–Macaulay canonical modules*, J. Algebra **275** (2004) 751–770. 5

Departamento de Matemática, Universidade Federal da Paraíba - 58051-900, João Pessoa, PB, Brazil
Email address: thiagofieldacostacabral@gmail.com

Email address: rferreir@sissa.it, rf.holanda@gmail.com