DIFFERENTIAL MODULES WITH COMPLETE INTERSECTION HOMOLOGY

MAYA BANKS AND KELLER VANDEBOGERT

Abstract. Differential modules are natural generalizations of complexes. In this paper, we study differential modules with complete intersection homology, comparing and contrasting the theory of these differential modules with that of the Koszul complex. We construct a Koszul differential module that directly generalizes the classical Koszul complex and investigate which properties of the Koszul complex can be generalized to this setting.

1. Introduction

A *differential module* is a module equipped with a square-zero endomorphism. While initially introduced at least as far back as the classical treatise of Cartan and Eilenberg [CE16, Chapter 4], differential modules have become a topic of recent interest in commutative algebra motivated in part by their connections to the Buchsbaum–Eisenbud–Horrocks and Carlsson conjectures [ABI07, Car86, BE77, Har79], the BGG correspondence and Tate resolutions [BE21], and the representation theory of algebras ([Rou06], [RZ17]). Differential modules are a natural generalization of chain complexes, and their study can thus provide a novel perspective on familiar objects such as free resolutions. More generally, there is an ever-expanding literature focusing on the use of differential modules to provide new insight on old conjectures (see for instance [BD10], [ŞÜ19], and [IW18]), and also on the development of a general theory of differential modules for their own sake (see [Sta17], [Wei15], and [XYY15], and the references therein).

Our work is motivated in particular by recent work of Brown and Erman [BE22], in which they extend the notion of a *minimal free resolution* to differential modules. Moreover, they prove a theorem indicating that the classical theory of minimal free resolutions still plays a significant role in understanding the structure of minimal free resolutions of differential modules. In particular, they show that for a differential module D with homology $H(D)$, there is a *free flag* F whose structure and differential are partially controlled by the minimal free resolution of $H(D)$ and where there is a quasi-isomorphism $F \to D$ (see Theorem 2.10 for the precise statement). This result begs the question of whether properties of the minimal free resolution of the homology $H(D)$ can be 'lifted' to the free flag F. Brown and Erman examined this in the case where the homology is a Cohen-Macaulay codimension 2 algebra. We explore this question in the case where $H(D)$ is the quotient by an ideal generated by a regular sequence.

In the classical theory of minimal free resolutions, the Koszul complex is one of the most fundamental objects of study for the simple reason of its sheer ubiquity; it is well-known

Date: October 12, 2023.

2020 *Mathematics Subject Classification.* 13D02, 13D07, 13C13.

Key words and phrases. Differential modules, complete intersections, Koszul complex, DG-algebras, minimal free resolutions.

to be a minimal free resolution of ideals generated by regular sequences, but even for non-complete intersections, properties of the Koszul homology of an ideal make their appearance in relation to DG-algebra techniques, the study of Rees algebras, and in the construction of more subtle types of complexes. In this paper, we begin developing a parallel theory for differential modules by first constructing a differential module analog of the Koszul complex, directly generalizing the classical case. We also investigate which properties of the Koszul complex lift to the minimal free resolution of differential modules whose homology is a complete intersection. We ask three main questions, the first of which is the following:

Question 1.1. For R a graded-local ring, what are the differential modules D whose homology is equal to the residue field $R/\mathfrak{m}$? More broadly, can we classify the differential modules with homology R/I a complete intersection?

Example 1.2. Let $S = k[x_1, \dots, x_n]$ for k a field. Let $D = S^4$ with differential given by the matrix

$$\begin{pmatrix} 0 & x_1 & x_2 & 0 \\ 0 & 0 & 0 & -x_2 \\ 0 & 0 & 0 & x_1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

This differential module has homology $S/(x_1, x_2)$. In fact, it is the differential module we obtain by taking the Koszul complex on the regular sequence (x_1, x_2) and viewing it as a differential module whose underlying module is the sum of the free modules in the Koszul complex and whose differential is the direct sum of the Koszul differentials. However, we can alter the differential by adding a nonzero entry to the top right corner without changing the homology of the differential module. That is, we get a family of differential modules D_f with the same underlying module and differential given by

$$\begin{pmatrix} 0 & x_1 & x_2 & f \\ 0 & 0 & 0 & -x_2 \\ 0 & 0 & 0 & x_1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Furthermore, by [BE22], every differential module with homology $S/(x_1, x_2)$ admits a quasi-isomorphism from some such D_f. However, not all choices of f yield nonisomorphic differential modules. For instance for $f \in (x_1, x_2)$ we can perform row and column operations to show that D_f is isomorphic to our original D, but this is not the case if (for instance) $f = 1$.

In the above example, we see the structure of the Koszul complex itself mirrored in the structure of the differential modules D_f. This motivates our next two questions, which have to do with how much of this structure is actually preserved when we pass from resolutions to differential modules.

Question 1.3. The differential of the Koszul complex corresponds to a multiplication by a single element in a particular exterior algebra. For a differential module D with complete intersection homology, does every free flag resolution of the type described in [BE22] arise via a similar construction?

Question 1.4. The classical Koszul complex is well-known to admit the structure of a DG-algebra. Does the generalization of the Koszul complex to differential modules admit any kind of analogous structure?

Our results give a partial answer to Question 1.1—while we do not give a classification of *all* differential modules with complete intersection homology, we do find constraints on such differential modules and prove results that simplify the classification question. We show that with some additional hypotheses, Question 1.3 can be answered affirmatively (see Theorem 4.10), but that absent these hypotheses we can construct examples of free flags with complete intersection homology that are not of the "expected" form. In contrast to the previous two questions, the answer to Question 1.4 seems to be a resounding "no", and indicates that generalizations of the classical notions of DG-algebra/module structures for differential modules will require much subtler formulations.

1.1. **Results.** Let R be a commutative graded local ring, D a module over R and $d : D \to D$ an R-module endomorphism that squares to 0. We define the homology of D to be $H(D) = \operatorname{Ker}(d)/\operatorname{Im} d$. If the underlying module D has the form $\bigoplus_{i\geqslant 0} F_i$ where F_i is free and the differential d satisfies that $d(F_j) \subseteq \bigoplus_{j<i} F_i$ then we call D a *free flag.* Note that any bounded below free complex can automatically be considered as a free flag. A core result of [BE22] states that any differential module D with finitely generated homology admits a quasi-isomorphism from a free flag (F, d) where

$$F_0 \xleftarrow{\delta} F_1 \xleftarrow{\delta} F_2 \leftarrow \cdots$$

is a minimal free resolution of $H(D)$ and the d restricts to δ when considered as a map $F_i \to F_{i-1}$. In this case, we can represent d via a block matrix

$$\begin{pmatrix}
0 & \delta & A_{2,0} & A_{3,0} & \cdots & A_{m,0} & \cdots \\
0 & 0 & \delta & A_{3,1} & \cdots & A_{m,1} \cdots & \\
\vdots & & & \ddots & & \vdots & \\
0 & 0 & 0 & 0 & \cdots & A_{m,m-2} & \cdots \\
0 & 0 & 0 & 0 & \cdots & \delta & \cdots \\
0 & 0 & 0 & 0 & \cdots & 0 & \ddots \\
\vdots & \vdots & \vdots & \vdots & \cdots & \vdots &
\end{pmatrix}$$

where $A_{i,j} : F_i \to F_j$. We refer to free flags of this type as *anchored free flags.* Since every differential module admits an anchored free flag resolution, it is both convenient and reasonable to focus on differential modules of this form. To classify anchored free flag resolutions amounts to classifying the possibilities for $A_{i,j}$ that yield nonisomorphic differential modules. Our first result simplifies this task in the case where the homology is a complete intersection.

Theorem 1.5. *Let D be a free flag differential module with homology $H(D)$ a complete intersection where the differential d is given by a block matrix as above. Let F be the minimal free resolution of $H(D)$ considered as a differential module. Then $D \cong F$ if and only if $\operatorname{Im} A_{i,0} \subseteq \operatorname{Im} \delta$ for all $i \geqslant 2$.*

This result says that, in the case of a homology induced free flag with complete intersection homology, the property of being isomorphic to a minimal free resolution of the homology can be detected by only the first row of blocks in the differential. This further tells us that we can at least partially characterize the differential modules with given complete intersection homology R/I by the choices of nonzero maps $F_i \to R/I$.

We prove Theorem 1.5 in Section 3, along with a discussion of minimal free resolutions of differential modules with homology k. In particular, we also show via explicit construction that the total Betti numbers of a differential module with homology k may be strictly smaller than the sum of the Betti numbers of k.

In Section 4, we show how graded commutative algebras admitting divided powers can be used to construct differential modules. This allows us to construct a Koszul differential module–a family of differential modules generalizing the Koszul complex (see Construction 4.4). In much the same way as the Koszul complex provides a valuable source of examples in the study of minimal free resolutions, Koszul differential modules generate a large set of examples of free flag differential modules. Moreover, we prove the following theorem which shows that in certain cases we can guarantee that anchored free flags with complete intersection homology are isomorphic to the Koszul differential module.

Theorem 1.6 (See Theorem 4.10 for the more general statement)**.** *Let R be a Noetherian graded local ring with maximal ideal $\mathfrak{m}$, D a differential R-module with $H(D)$ a complete intersection and let $F = \bigoplus_{i \geqslant 0} F_i \to D$ be an anchored free flag resolution with differential d^F. If $\operatorname{Im}(d^F) \cap F_0$ is generated by a regular sequence, then F is isomorphic to a Koszul differential module.*

Finally, in Section 5 we consider the existence of DG-module structures on free flag resolutions as posited in Question 1.4. This leads us to consider free flag resolutions that can be given the structure of a DG-module over the minimal free resolution of their homology. In the classical case of complexes, it is well-known that every free resolution admits the structure of a (possibly non-associative) DG-algebra structure, and hence the existence of such a structure is guaranteed. Our main result of this section is the following theorem which says that the existence of such a DG-module structure on an arbitrary free flag resolution is in fact a much rarer property.

Theorem 1.7. *Let F be an anchored free flag with complete intersection homology. If F admits the structure of a DG-module over the minimal free resolution of $H(F)$, then F is isomorphic to the Koszul complex considered as a differential module.*

This theorem gives an example of a property of free resolutions that does *not* generalize to the setting of differential modules. The inability to generalize this property tells us that the DG-algebra structure of free resolutions, at least for complete intersections, relies on structure that is unique to free resolutions rather than structure that can be extended to free flags. This is in contrast with properties that are successfully generalized to free flags in [ABI07] and [BE22].

Acknowledgements. We thank Daniel Erman and Michael Brown for many helpful conversations and comments throughout all stages of our work. We also thank the anonymous referee for many valuable comments and corrections.

2. Background

In this section, we introduce some background and notation on differential modules that will be used throughout the paper. This includes a straightforward reformulation of free flag differential modules without reference to matrices, which will be useful for avoiding complicated matrix computations; this formulation is implicit in the work of Avramov, Buchweitz, and Iyengar [ABI07], but we state it here since it will be used frequently throughout the paper.

12 Oct 2023 15:05:43 PDT
230718-VandeBogert Version 4 - Submitted to J. Comm. Alg.

Notation 2.1. Throughout this paper, R will denote a graded local ring. More precisely, $R = \bigoplus_{i \geqslant 0} R_i$ is an $\mathbb{N}$-graded Noetherian ring with R_0 local. Moreover, all differential modules throughout this paper will be assumed to have finitely generated homology.

Definition 2.2. A *differential module* (D, d) or (D, d^D) is an R-module D equipped with an R-endomorphism $d = d^D \colon D \to D$ that squares to 0. A differential module is $\mathbb{Z}$-graded of degree a if D is equipped with a $\mathbb{Z}$ grading over R such that $d \colon D \to D(a)$ is a graded map.

The *homology* of a differential module (D, d) is defined to be $\operatorname{Ker}(d)/\operatorname{Im}(d)$. If D is $\mathbb{Z}$-graded of degree a, then the homology is defined to be the quotient $\operatorname{Ker}(d)/\operatorname{Im}(d(-a))$.

A differential module is *free* if the underlying module D is a free R-module. The differential module D is *minimal* if $d \otimes k = 0$.

A morphism of differential modules $\phi \colon (D, d^D) \to (D', d^{D'})$ is a morphism of R-modules $D \to D'$ satisfying $d^{D'} \circ \phi = \phi \circ d^D$. Notice that morphisms of differential modules induce well-defined maps on homology in an identical fashion to the case of complexes. A morphism of differential modules is a *quasi-isomorphism* if the induced map on homology is a quasi-isomorphism.

The category of degree a differential R-modules will be denoted $\operatorname{DM}(a, R)$. The notation $\operatorname{DM}(R)$ will denote the full category of differential modules (without any concern for grading).

Remark 2.3. The collection of differential modules and their morphisms forms a category, denoted $\operatorname{DM}(R)$. Notice that a differential R-module (D, d) is equivalently a module over the ring $R[x]/(x^2)$, as mentioned in the introduction. In particular, the category $\operatorname{DM}(R)$ is equivalently the category of $R[x]/(x^2)$-modules, and as such $\operatorname{DM}(R)$ is an abelian category. The category $\operatorname{DM}(a, R)$ is also equivalently graded modules over $R[x]/(x^2)$ in the case that x is given degree a.

The following definition will play an essential role throughout the paper, and it allows us to view the category of complexes as a subcategory of the category of differential modules, though it is important to note that this is *not* necessarily a full subcategory:

Definition 2.4. Given any complex F, there is a functor

$$\begin{aligned} &\operatorname{Fold} \colon \operatorname{Com}(R) \to \operatorname{DM}(R), \\ &\qquad (F, d^F) \mapsto \Big(\bigoplus_{i \in \mathbb{Z}} F_i, \bigoplus_{i \in \mathbb{Z}} d_i^F \Big), \\ &\{\phi \colon F_\bullet \to G_\bullet\} \mapsto \Big\{ \bigoplus_{i \in \mathbb{Z}} \phi_i \colon \operatorname{Fold}(F_\bullet) \to \operatorname{Fold}(G_\bullet) \Big\}. \end{aligned}$$

The object $\operatorname{Fold}(F_\bullet)$ will often be referred to as the *fold* of the complex $F_\bullet$.

The following definition introduces *free flags*. These are a proper subclass of differential modules that still generalize complexes of free modules, and in general are better behaved than arbitrary differential modules. One way to think of free flags is as differential modules admitting a finite length filtration whose associated graded pieces are themselves free R-modules.

Definition 2.5. Let D be a differential module. Then D is a *free flag* if D admits a splitting $D = \bigoplus_{i \in \mathbb{Z}} F_i$, where each F_i is a free R-module, $F_i = 0$ for $i < 0$, and $d_D(F_i) \subseteq \bigoplus_{j < i} F_j$.

Given a free flag D with associated splitting $D = \bigoplus_{i \in \mathbb{Z}} F_i$, define $D^i := \bigoplus_{j \leqslant i} F_j$. This will be referred to as the *flag filtration* on F. By definition of a free flag, one has $d_D(D^i) \subset$

D^{i-1}, implying that the associated graded object associated to the flag filtration is a chain complex.

Associated to a free flag D, there are maps $A_{i,j}\colon F_i \to F_j$ induced by splitting the maps $d_D\colon F_i \to D^{i-1}$ with the isomorphism $\operatorname{Hom}(F_i, D^{i-1}) = \bigoplus_{j<i} \operatorname{Hom}(F_i, F_j)$.

A core theme of [ABI07] is that the flag structure on a differential module allows many proofs from classical homological algebra to be generalized to differential modules by substituting the homological grading for the grading induced by the flag filtration. As such, it is useful to pass from general differential modules to free flags, which we can do using the following definition.

Definition 2.6. For any differential module D, a *free flag resolution* F of D is a free flag F equipped with a quasi-isomorphism $F \to D$. A *minimal free resolution* of D is a quasi-isomorphism $M \to D$ that factors through a free flag resolution F such that $M \to F$ is a split injection and M is minimal.

Remark 2.7. Notice that if $F_\bullet \to M$ is a minimal free resolution of a module M, then $\operatorname{Fold}(F_\bullet) \to M$ (where M is viewed as having the 0 endomorphism) is a minimal free flag resolution of M (since $H(\operatorname{Fold}(F_\bullet)) = \bigoplus_{i\in\mathbb{Z}} H_i(F_\bullet)$). In general, there may be free flag resolutions of M that do not arise as the fold of a complex, and it is an interesting question as to when a free flag is isomorphic to the fold of some complex. We give a characterization of this property for certain classes of free flag resolutions in Section 3 which turns out to be quite effective in proving some general statements about free flag resolutions.

One way to think of free flags is as strictly upper triangular block matrices (as in the setup to Theorem 1.5). This can be useful for explicit computations and more matrix-theoretic methods, but we will also find it useful to think of free flags in a way that is not reliant on matrices. The following observation is a coordinate-free reformulation of the definition of a free flag that highlights the data of the maps $A_{i,j}\colon F_i \to F_j$ which determine the flag.

Observation 2.8. A free flag is equivalently the data of a collection of free modules $\{F_i \mid i \in \mathbb{Z}_{\geqslant 0}\}$ and maps $\{A_{i,j}\colon F_i \to F_j \mid j < i\}$, such that for all $j < i$, one has the relation

$$\sum_{j<k<i} A_{k,j} A_{i,k} = 0.$$

Remark 2.9. In order to distinguish the structure maps $A_{i,j}$ given in Observation 2.8, we will often use the more precise notation $A^D_{i,j}$ to specify that these maps determine the differential module D.

The theory of minimal free resolutions of arbitrary differential modules turns out to be quite subtle. However, the following result of Brown and Erman shows that the classical theory of minimal free resolutions of modules still plays an important role in understanding the homological properties of differential modules.

Theorem 2.10 ([BE22, Theorem 3.2])**.** *Let D be a differential module with finitely generated homology and $(F_\bullet, d) \to H(D)$ a minimal free resolution of $H(D)$. Then D admits a free flag resolution $\widetilde{F}$ where the underlying free module is $F_\bullet$ and where, in the notation of Observation 2.8, one has $A_{i,i-1} = d_i$ for all i.*

3. Anchored Free Flags and Folds of Complexes

In this section, we prove Theorem 3.6, which is our first structural result about free flag resolutions whose homology is a complete intersection. We make a note about terminology here:

($*$) Throughout the paper, a "complete intersection" will be a quotient of a commutative Noetherian ring by a regular sequence. This is a slight loosening of the more standard definition, where it is assumed that the ambient ring is a regular ring. This decision is made for sake of conciseness of presentation.

Specifically, we consider a free flag resolution F as in Theorem 2.10 whose homology is a complete intersection, and we show that the question of whether or not F is trivial—i.e. where F is isomorphic to the fold of a Koszul complex—is entirely determined by an analysis of the "top row" of the differential. As an application, we then completely classify all differential modules D where $H(D)$ is isomorphic to the residue field k and R is regular.

We conclude the section with some interesting examples illustrating the subtlety of minimal free resolutions of differential modules. In particular, we show that if $R = k[x_1, \dots, x_n]$ is a standard graded polynomial ring over a field, then k viewed as a differential R-module in degree 2 has total Betti numbers *strictly* less than the total Betti numbers of k when viewed as an R-module in the usual fashion.

Definition 3.1. Let $\widetilde{F}$ be a free flag and $(F_\bullet, d) \to H(\widetilde{F})$ a minimal free resolution of $H(\widetilde{F})$. If $\widetilde{F}$ arises as in the statement of Theorem 2.10, then $\widetilde{F}$ will be called an *anchored free flag resolution*. The complex $(F_\bullet, d)$ is called the *anchor* of $\widetilde{F}$.

Remark 3.2. An anchored free flag resolution has differential with off-diagonal blocks coming from the minimal free resolution of the homology. These off-diagonal blocks can be thought of as "anchors" for the maps $A_{i,j}$ for $i - j \geqslant 2$; more precisely, we have complete freedom to choose the "higher-up" maps $A_{i,j}$ up to the constraint that these maps must still make the corresponding differential square to 0.

Conceptually, the following lemma shows that if one can perform column operations on the matrix representation of the differential of an anchored free flag to cancel a term $A_{i,0}$, then one can in fact perform column operations to cancel all other terms appearing along the associated diagonal.

Lemma 3.3. *Let D be an anchored free flag and assume $\operatorname{Im} A_{i,0}^D \subset \operatorname{Im} d_1$ for all $2 \leqslant i \leqslant m$ for some given m. Then D is isomorphic to a free flag D' satisfying $A_{i,\ell}^{D'} = 0$ for all $2 \leqslant i - \ell \leqslant m$.*

Proof. The assumption $\operatorname{Im} A_{i,0}^D \subset \operatorname{Im} d_1$ for all $2 \leqslant i \leqslant m$ implies that D is isomorphic to a differential module D' with the same underlying module determined by maps $\{A_{i,j}^{D'} : F_i \to F_j \mid i < j\}$, but satisfying $A_{i,0}^{D'} = 0$ for all $i \leqslant m$ and $A_{i,i-1}^{D'} = d_i$ for all i.

Let $2 \leqslant i - 1 \leqslant m$. Then there is the relation

$$\sum_{j<i} A_{j,0}^{D'} \cdot A_{i,j}^{D'} = 0.$$

Since $i \leqslant m + 1$ and $j < i$, one has that $A_{j,0}^{D'} = 0$ for each $j > 1$ appearing in the above equality. Thus $d_1 \circ A_{i,1}^{D'} = 0$, and exactness implies that $\operatorname{Im} A_{i,1}^{D'} \subset \operatorname{Im} d_2$ for each $i \leqslant m+1$. Replacing D with D' and iterating this argument, the result follows. □

In particular, the above gives a criterion for D to be isomorphic to the fold of the minimal free resolution of its homology. One might hope that this is in fact an equivalence—that is, that any differential module isomorphic to the fold of the minimal free resolution of its homology can be identified in this way. This is in general not the case, as we see in the following example.

12 Oct 2023 15:05:43 PDT
230718-VandeBogert Version 4 - Submitted to J. Comm. Alg.

Example 3.4. Let $R = k[x_1, x_2]$ and E be a rank 2 free module on the basis e_1, e_2. Let $D = \bigwedge^\bullet E$ be the free flag with differential

$$\begin{pmatrix} 0 & x_1^2 & x_1x_2 & x_1 \\ 0 & 0 & 0 & -x_2 \\ 0 & 0 & 0 & x_1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let $K_\bullet$ denote the minimal free resolution of $H(D)$

$$K_\bullet = \quad \bigwedge^0 E \xleftarrow{\begin{pmatrix} x_1^2 & x_1x_2 \end{pmatrix}} \bigwedge^1 E \xleftarrow{\begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}} \bigwedge^2 E$$

Then the morphism of differential modules $D \to \operatorname{Fold}(K_\bullet)$ induced by

$$1 \mapsto 1, \qquad e_1 \mapsto e_1, \quad e_2 \mapsto e_2, \qquad e_1 \wedge e_2 \mapsto e_1 \wedge e_2 + e_1,$$

is an isomorphism, even though $x_1 \notin (x_1^2, x_1x_2) = \operatorname{Im} d_1^K$.

Note that the isomorphism described in the above example corresponds to performing *row* operations on the differential to cancel out the x_1 in the corner. In this case, we are able to cancel via row operations but not by column operations. This is explained by the lack of symmetry in the differentials of the minimal free resolution of $H(D)$. In fact, when the minimal free resolution of $H(D)$ is given by the Koszul complex—i.e. when $H(D)$ is a complete intersection—this scenario cannot arise, as we will show next.

First, notice that any morphism $\phi\colon (D, d) \to (D', d')$ of free flags with underlying free modules $\bigoplus_{i\in\mathbb{Z}} F_i$ and $\bigoplus_{i\in\mathbb{Z}} G_i$, respectively, decomposes as a direct sum of maps $\phi_{i,j}\colon F_i \to G_j$ for every $i, j \in \mathbb{Z}$.

Lemma 3.5. *Let D be a free flag anchored on the minimal free resolution of a finitely generated R-module M. Assume that there exists an isomorphism $\phi\colon D \to \operatorname{Fold}(F_\bullet)$, and assume that $\phi_{i-1,0}(d_i^F(F_i)) \subset \operatorname{Im} d_1^F$ for all $i \geqslant 2$. Then $\operatorname{Im} A_{i,0} \subset \operatorname{Im} d_1^F$ for all $i \geqslant 2$.*

Proof. The map ϕ decomposes as a sum of maps of the form

$$\phi = \sum_{i,j=1}^{n} \phi_{i,j},$$

where $\phi_{i,j}\colon F_i \to F_j$ and n is the length of the flag (note $n = \infty$ is allowed here). We first claim that $\phi_{0,0} : F_0 \to F_0$ may be chosen to be the identity. To see this, notice that the fact that $\phi \circ d^D = d^{\operatorname{Fold}(F_\bullet)} \circ \phi$ implies that there are equalities

$$d_i^F \circ \phi_{0,i} = 0 \quad \text{for all } i \geqslant 1.$$

Since $F_\bullet$ is a resolution, it follows that $\phi_{0,i} = d_{i+1} \circ \phi'_{0,i+1}$ for some $\phi'_{0,i+1} : F_0 \to F_{i+1}$. On the other hand, let ψ denote the inverse of ϕ. By definition there is an equality

$$(*) \quad \phi_{0,0} \circ \psi_{0,0} + \phi_{1,0} \circ \psi_{0,1} + \cdots + \phi_{n,0} \circ \psi_{0,n} = \mathrm{id}_{F_0}.$$

Applying the functor $- \otimes_R k$ to $(*)$, the fact that $\operatorname{Im} \phi_{0,i} \subset \mathfrak{m}F_i$ for each $i \geqslant 1$ (by the minimality assumption on $F_\bullet$) implies that $\phi_{0,0} \circ \psi_{0,0} \otimes_R k = \mathrm{id}_{F_0} \otimes_R k$. By Nakayama's lemma, the map $\phi_{0,0}$ is a surjective endomorphism of F_0 and hence an isomorphism. Changing bases as necessary, it is thus of no loss of generality to assume $\phi_{0,0}$ is the identity.

12 Oct 2023 15:05:43 PDT
230718-VandeBogert Version 4 - Submitted to J. Comm. Alg.

Now, let $f_i \in F_i$ be any element; the assumption that ϕ is a morphism of differential modules then yields:

$$\begin{aligned}\phi(d^D(f_i)) &= \phi\Big(\sum_{0<j<i} A_{i,j}(f_i)\Big) + \phi \circ A_{i,0}(f_i) \\ &= \sum_{i,j=1}^{n} d_j^F(\phi_{i,j}(f_i)) \\ &= d^F(\phi(f_i)).\end{aligned}$$

Comparing the above equality restricted to the direct summand F_0, one obtains the equality

$$d_1^F(\phi_{i,1}(f_i)) = A_{i,0}(f_i) + \sum_{0<j<i} \phi_{j,0}(A_{i,j}(f_i)). \tag{3.1}$$

Now, we proceed by induction on i to prove the desired statement. When $i = 2$, the above equality becomes

$$d_1^F(\phi_{2,1}(f_2)) = A_{2,0}(f_2) + \phi_{1,0}(d_2^F(f_2)).$$

The assumption that $\phi_{1,0}(d_2^F(f_2)) \in \operatorname{Im} d_1^F$ implies that $A_{2,0}(F_2) \subset \operatorname{Im} d_1^F$, and Lemma 3.3 implies that D may be replaced with a differential module satisfying $A_{i,i-2} = 0$ for all $i \geqslant 2$. Proceeding inductively, assume $i > 2$; by induction, we may assume that $A_{j,k} = 0$ for all $1 < j - k < i$. The equality (3.1) then reduces to

$$d_1^F(\phi_{i,1}(f_i)) = A_{i,0}(f_i) + \phi_{i-1,0}(d_i^F(f_i)),$$

and again the assumption $\phi_{i-1,0}(d_i^F(f_i)) \in \operatorname{Im} d_1^F$ implies that $A_{i,0}(f_i) \in \operatorname{Im} d_1^F$, and Lemma 3.3 allows us to replace D with a differential module satisfying $A_{j,k} = 0$ for all $j - k \leqslant i$. Iterating this argument, the result follows. $\square$

The above holds, in particular, when $H(D)$ is a complete intersection.

Theorem 3.6. *Let D be an anchored free flag and assume that $H(D)$ is a complete intersection; that is, $H(D) \cong R/I$ where I is generated by a regular sequence. Then*

$$D \cong \operatorname{Fold}(F_\bullet) \quad \iff \quad \operatorname{Im} A_{i,0}^D \subset \operatorname{Im} d_1 \text{ for all } i \geqslant 2.$$

Proof. $\implies$: Let $H(D) = R/\mathfrak{a}$ where $\mathfrak{a}$ is generated by a regular sequence. By definition of the Koszul complex, the minimal free resolution of $R/\mathfrak{a}$ satisfies $d_i^F \otimes R/\mathfrak{a} = 0$ for all i, whence the assumption $\phi_{i-1,0}(d_i^F(F_i)) \subset \operatorname{Im} d_1^F$ of Lemma 3.5 is trivially satisfied.

$\impliedby$: This implication holds without any further assumptions by Lemma 3.3. $\square$

Under the perspective of free flags as upper triangular block matrices, this says that the property of D being isomorphic to the resolution of its homology is completely detectable via only the top row of the matrix. On the other hand, in the case where D is graded, the degrees of the entries of the top row may be deduced by using this grading. Adding this additional structure gives strong restrictions on the possibilities for differential modules D with complete intersection homology. In the most restrictive case, when $H(D) \cong k$, we have the following.

Corollary 3.7. *Assume R is a regular graded local ring. Let $D \in \operatorname{DM}(R, a)$ be an anchored free flag with $H(D) \cong k$. If $a \neq 2$, then $D \cong \operatorname{Fold}(K_\bullet)$, where $K_\bullet$ denotes the Koszul complex resolving k.*

Proof. Since D has degree a, the minimal free resolution of $H(D)$ is given by the Koszul complex $K_\bullet$ with the i^{th} free module with a degree shift by ia (so that the maps in the complex are all homogeneous of degree a). The maps $A_{i,j} \colon R(ia - a) \to R(ja - j + a)$ in the differential on D therefore have degree $(ja - j + a) - (ia - i)$. When $\deg A_{i,0} \neq 0$, we have $\operatorname{Im} A_{i,0} \otimes k = 0$. On the other hand, $(ja - j + a) - (ia - i) = 0$ only when $a = 2$ and $i - j = 2$. Thus for $a \neq 2$ the result follows from Corollary 3.6. $\square$

The above corollary implies that in degree $a \neq 2$, all differential R-modules with homology k have isomorphic anchored free flag resolutions, and that furthermore this resolution is minimal and isomorphic to the Koszul complex. However, this is not true for differential modules of degree 2. In general, any R-module may be viewed as a degree a differential module, for any integer a, and the following result shows that the homological invariants of $M \in \mathrm{DM}(a, R)$ can vary as the degree a varies.

Proposition 3.8. *Let $S := k[x_1, \ldots, x_n]$ be a standard graded polynomial ring over a field k, where $n \geqslant 2$. Then there exists a degree 2 free differential module D of rank 2^{n-1} with $H(D) \cong k$.*

Proof. We construct the differential module inductively. For $n = 2$, let $D = S^2$ with differential given by the matrix $\begin{pmatrix} x_1x_2 & -x_2^2 \\ x_1^2 & -x_1x_2 \end{pmatrix}$. One can check that D is a degree 2 differential module, and moreover that $H(D) \cong k$.

For $n \geqslant 3$, let

$$D_n := \operatorname{Cone}(D_{n-1} \otimes_k k[x_n] \xrightarrow{x_n} D_{n-1} \otimes_k k[x_n]).$$

By the inductive hypothesis, there is an equality $H(D_{n-1} \otimes_k k[x_n]) \cong k \otimes_k k[x_n] = k[x_n]$, and by [ABI07, 1.2] the mapping cone induces an exact triangle

$$\begin{tikzcd} k[x_n] \cong H(D_{n-1} \otimes_k k[x_n]) \arrow[rr, "x_n"] & & H(D_{n-1} \otimes_k k[x_n]) \cong k[x_n] \arrow[dl] \\ & H(D_n) \arrow[ul] & \end{tikzcd}$$

Since multiplication by x_n is injective, the above exact triangle degenerates to a short exact sequence:

$$0 \to k[x_n] \xrightarrow{x_n} k[x_n] \to H(D_n) \to 0,$$

in which case $H(D_n) \cong k$, as desired. Moreover, the rank of D_n is precisely $2 \cdot \operatorname{rank}(D_{n-1}) = 2 \cdot 2^{n-2} = 2^{n-1}$. Since the differential on D_n is given by the block matrix $\begin{pmatrix} -d \otimes_R S & x_n \mathrm{id} \\ 0 & d \otimes_R S \end{pmatrix}$, we can see furthermore that the degree of D_n is 2. $\square$

If we define the *Betti numbers* of a differential module as in [BE22], then the sum of the Betti numbers of D is equal to the rank of the minimal free resolution of D. We next show that the differential module constructed in 3.8 is actually a minimal free resolution. This will imply that the sum of the betti numbers of a degree 2 differential module with homology k may be at least as small as 2^{n-1}, strictly smaller than the total rank of a minimal free resolution with the same homology. Although this differential module is certainly free and minimal, it is not immediately obvious that it is a minimal free *resolution*, since this requires it to be a summand of a free flag resolution. To prove that this is indeed the case, we leverage the mapping cone structure of the differential module constructed in 3.8.

We will first need a lemma; in the following, recall that a differential module F is *contractible* if the identity map is homotopic to 0. A homotopy h for which $\mathrm{id}_F = d^F h + h d^F$ is called a *contracting homotopy.*

Lemma 3.9. *Let F and G be contractible differential modules with contracting homotopies h^F and h^G, respectively. If $\phi : F \to G$ is a morphism of differential modules satisfying $\phi \circ h^F = h^G \circ \phi$, then the mapping cone* $\mathrm{Cone}(\phi)$ *is contractible with contracting homotopy*

$$h^{\mathrm{Cone}(\phi)} := \begin{pmatrix} -h^F & 0 \\ 0 & h^G \end{pmatrix}.$$

Proof. Recall that $\mathrm{Cone}(\phi)$ has underlying free module isomorphic to $F \oplus G$ equipped with the differential whose block form is given by

$$d^{\mathrm{Cone}(\phi)} = \begin{pmatrix} -d^F & 0 \\ -\phi & d^G \end{pmatrix}.$$

Using this, we compute:

$$\begin{aligned} d^{\mathrm{Cone}(\phi)} h^{\mathrm{Cone}(\phi)} + h^{\mathrm{Cone}(\phi)} d^{\mathrm{Cone}(\phi)} &= \begin{pmatrix} -d^F & 0 \\ -\phi & d^G \end{pmatrix} \begin{pmatrix} -h^F & 0 \\ 0 & h^G \end{pmatrix} + \begin{pmatrix} -h^F & 0 \\ 0 & h^G \end{pmatrix} \begin{pmatrix} -d^F & 0 \\ -\phi & d^G \end{pmatrix} \\ &= \begin{pmatrix} d^F h^F & 0 \\ \phi h^F & d^G h^G \end{pmatrix} + \begin{pmatrix} h^F d^F & 0 \\ -h^G \phi & h^G d^G \end{pmatrix} \\ &= \begin{pmatrix} \mathrm{id}_F & 0 \\ 0 & \mathrm{id}_G \end{pmatrix}. \end{aligned}$$

By definition, $h^{\mathrm{Cone}(\phi)}$ is a contracting homotopy. □

Corollary 3.10. *Let D_n be the rank 2^{n-1} free differential module over $k[x_1, \dots, x_n]$ defined in Proposition 3.8. Then D_n is its own minimal free resolution.*

Proof. Proceed by induction on n. For $n = 2$, the module $S \oplus S$ with differential $\begin{pmatrix} xy & x^2 \\ -y^2 & xy \end{pmatrix}$ is known to be its own minimal free resolution (see [BE22], Example 5.8). In particular, it is a minimal free summand of the free S-module $S \oplus S(1)^2 \oplus S(2)$ with differential

$$\begin{pmatrix} 0 & x & y & 1 \\ 0 & 0 & 0 & -y \\ 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let F^2 denote this free flag differential module, and notice that F^2 is anchored on the length 2 Koszul complex on variables x_1, x_2. For $n > 2$, define

$$F^n := \mathrm{Cone}\left(x_n : F^{n-1} \otimes_k k[x_n] \to F^{n-1} \otimes_k k[x_n] \right).$$

By the inductive hypothesis, the free flag F^{n-1} is anchored on the Koszul complex for $x_1, \dots, x_{n-1}$. Since F^n is obtained by taking the mapping cone of multiplication by x_n, the differential module F^n is anchored on the Koszul complex for $x_1, \dots, x_n$.

Moreover, by induction the free flag F^{n-1} splits as a direct sum

$$F^{n-1} \cong D_{n-1} \oplus T^{n-1},$$

where T^{n-1} is some contractible differential $k[x_1, \dots, x_{n-1}]$-module and D_{n-1} denotes the differential module of Proposition 3.8. Combining this information, we find that there is an

12 Oct 2023 15:05:43 PDT
230718-VandeBogert Version 4 - Submitted to J. Comm. Alg.

isomorphism:

$$\begin{aligned} F^n &= \operatorname{Cone}\left(x_n : F^{n-1} \otimes_k k[x_n] \to F^{n-1} \otimes_k k[x_n]\right) \\ &= \underbrace{\operatorname{Cone}(D_{n-1} \otimes_k k[x_n] \xrightarrow{x_n} D_{n-1} \otimes_k k[x_n])}_{=D_n} \oplus \underbrace{\operatorname{Cone}(T^{n-1} \otimes_k k[x_n] \xrightarrow{x_n} T^{n-1} \otimes_k k[x_n])}_{:=T^n}. \end{aligned}$$

The image of any contractible object under an additive functor remains contractible, so the differential module $T^{n-1} \otimes_k k[x_n]$ is also contractible. Since scalar multiplication commutes with any choice of homotopy, the hypotheses of Lemma 3.9 are satisfied for the morphism $x_n : T^{n-1} \otimes_k k[x_n] \to T^{n-1} \otimes_k k[x_n]$. It thus follows from Lemma 3.9 that T^n is contractible and F^n splits as the direct sum of D_n and a contractible differential module. By definition, the differential module D_n is its own minimal free resolution. □

Note that because the differential module D constructed in Proposition 3.8 is not a free flag, it does not itself contradict the (disproven) conjecture of Avramov–Buchweitz–Iyengar that the rank of a free flag over R with finite length homology is at least $2^{\dim(R)}$ [ABI07, Conj. 5.3]. In fact, the example when $n = 2$ appears in [ABI07], and this construction directly generalizes their example.

4. A Generalization of the Koszul Complex for Differential Modules

In this section, we introduce a differential module analog of the Koszul complex and show that under certain hypotheses, all anchored free flags are isomorphic to this Koszul differential module. We also provide an example showing that in general not all such free flags with complete intersection homology are obtained by this Koszul differential module; to begin this section, we recall the definition of the Koszul complex that will be most convenient for our purposes.

Let R be a ring and E be a free R-module on basis $e_1, \dots, e_n$ and $\psi \colon E \to R$ any R-module homomorphism. The notation e_I will be shorthand for the basis element $e_{i_1} \wedge \cdots \wedge e_{i_k}$, where $I = \{i_1 < \cdots < i_k\}$ is an indexing set of the appropriate size. Recall that the Koszul complex can be constructed as the complex with the ith exterior power $\bigwedge^i E$ sitting in homological degree i and differential

$$e_{j_1,\dots,j_i} \mapsto \sum_{k=1}^{i} (-1)^{k+1} \psi(e_{j_k}) e_{j_1,\dots,\widetilde{j_k},\dots,j_i}.$$

Another way to view this map is as the composition

$$\begin{aligned} \bigwedge^i E &\xrightarrow{\text{comultiplication}} E \otimes \bigwedge^{i-1} E \\ &\xrightarrow{\psi \otimes 1} R \otimes \bigwedge^{i-1} E \cong \bigwedge^{i-1} E. \end{aligned}$$

This is equivalently described as multiplication by an element $f \in E^*$ (recall that $\bigwedge^\bullet E$ is a graded $\bigwedge^\bullet E^*$-module and vice versa). Choose a map $A_{i,0} \colon \bigwedge^i E \to R$ (the notation here is intentionally reminiscent of Observation 2.8). Notice that such a map is equivalently induced by multiplication by an element $f_i \in \bigwedge^i E^*$. This is because the pairing $\bigwedge^i E \otimes \bigwedge^{n-i} E \to \bigwedge^n E$ is perfect.

Our goal will now be to generalize the exterior algebra structure of the Koszul complex to define a "Koszul differential module". In order to make the general construction more clear, we illustrate with an example:

12 Oct 2023 15:05:43 PDT
230718-VandeBogert Version 4 - Submitted to J. Comm. Alg.

Example 4.1. Given an integer n, use the notation $[n] := \{1, \ldots, n\}$. Let $R = k[x_I \mid I \subset [4],\ I \neq \varnothing]$ and let f_i for $i = 1, \ldots, 4$ be the elements of $\bigwedge^i E^*$ induced by the maps

$$\begin{array}{cccc} E \to R & \bigwedge^2 E \to R & \bigwedge^3 E \to R & \bigwedge^4 E \to R \\ e_i \mapsto x_i & e_{ij} \mapsto \begin{cases} 0 & \text{if } i = 1, \\ x_{ij} & \text{otherwise.} \end{cases} & e_{ijk} \mapsto x_{ijk} & e_{1234} \mapsto x_{1234}. \end{array}$$

Define a free flag F whose underlying module is $\bigwedge^\bullet E$ and whose differential is given by the maps $A_{i,j} : \bigwedge^i E \to \bigwedge^j E$ defined by $A_{i,j}(g) = (-1)^{ij} f_{i-j} g$. To check that this indeed defines a differential module amounts to checking that for each $i > j$

$$\sum_{j<k<i} A_{k,j} A_{i,k} = \sum_{j<k<i} (-1)^{kj} (-1)^{ik} f_{k-j} f_{i-k} = 0$$

In this case, one just needs to verify the relations

$$f_1^2 = 0, \quad f_1 f_2 - f_2 f_1 = 0, \quad f_1 f_3 + f_3 f_1 + f_2^2 = 0.$$

We can generalize the construction in Example 4.1 to obtain a differential module in a similar way given a suitably chosen bialgebra. Recall that an algebra *admits divided powers* if the subalgebra generated by elements of even degree satisfies the axioms of a divided power algebra (for the definition of a divided power algebra see, for instance, [ABW82]). A canonical example of such an algebra (and the only example we will use in this paper) to keep in mind is the exterior algebra on a free module, where the elements of even degree are the divided power elements.

Proposition 4.2. *Let T denote any graded-cocommutative R-bialgebra such that T^* admits divided powers (where T^* denotes the graded dual). Given any $f_i \in T_i^*$, the notation $f_i \colon T_\ell \to T_{\ell-i}$ will denote the left-multiplication map. Assume either:*

(1) $\operatorname{char} R = 2$*, or*

(2) $\operatorname{char} R \neq 2$ *and $f_i \cdot f_j = 0$ if both i and j are even.*

Define $A_{i,j} := (-1)^{ij} f_{i-j} \colon T_i \to T_j$. Then the data

$$\{T_i,\ A_{i,j} \colon T_i \to T_j \mid j < i,\ i \in \mathbb{Z}\}$$

determines a differential module.

Proof. One only needs to verify that $\sum_{j<k<i} A_{k,j} A_{i,k} = 0$ for all choices of i and j, and this is a straightforward computation. Assume that $i + j$ is odd; one computes:

$$\begin{aligned} \sum_{k=j+1}^{i-1} A_{k,j} \cdot A_{i,k} &= \sum_{k=j+1}^{(i+j-1)/2} (A_{k,j} A_{i,k} + A_{i+j-k,j} A_{i,i+j-k}) \\ &= \sum_{k=j+1}^{(i+j-1)/2} \Big((-1)^{jk+ik} + (-1)^{j(i+j-k)+i(i+j-k)+(i-k)(k-j)} \Big) f_{k-j} f_{i-k}. \end{aligned}$$

Thus it suffices to show that the coefficient

$$(-1)^{jk+ik} + (-1)^{j(i+j-k)+i(i+j-k)+(i-k)(k-j)}$$

is 0 if $i - k$ or $k - j$ is odd. Since the above expression is symmetric in i and j modulo 2, it is of no loss of generality to assume that $i \equiv_2 k + 1$. One computes:

$$jk + ik \equiv_2 jk, \quad \text{and}$$

$$j(i+j-k)+i(i+j-k)+(i-k)(k-j) \equiv_2 j(j+1)+(k+1)(j+1)+j+k \equiv_2 jk+1.$$

Thus the coefficient is 0 if $i-k$ and $k-j$ are not both even, and if they are both even, then $f_{i-k}f_{k-j}=0$ or appears with coefficient 2, implying that these terms vanish as well.

If $i+j$ is even, then the computation is identical, with the only difference being that the term $f^2_{(i+j)/2}$ appears. If $\operatorname{char} k \neq 2$, then this term vanishes by assumption. If $\operatorname{char} k = 2$, then the assumption that T admits divided powers implies that $f^2_{(i+j)/2} = 2f^{(2)}_{(i+j)/2} = 0$, so all terms again vanish. □

Notation 4.3. Let $f_i \colon T_i \to R$ for $i = 1, \dots, n$ be a collection of maps induced by multiplication by $f_i \in (T_i)^*$, where $T_\bullet$ is any graded-cocommutative bialgebra as in the statement of Proposition 4.2. The notation $K(f_1, \dots, f_n)$ will denote the (not necessarily differential) module induced by the data $\{T_i,\ (-1)^{ij} f_{i-j}\}$.

We can also construct a "generic" Koszul differential module; we will see that the theorems below give criteria for which free flags of the appropriate form are obtained as specializations on the generic Koszul differential module.

Construction 4.4 (Generic Koszul Differential Module)**.** Let $n \in \mathbb{N}$ and $A = \mathbb{Z}[x_I \mid I \subset [n],\ I \neq \varnothing]$. Let $E = \bigoplus_{i=1}^n Ae_i$ and let $f_i \in \bigwedge^i E^*$ be the generic maps

$$f_i = \sum_{|I|=i} x_I e_I^*.$$

Next, let I be the ideal generated the relations $f_i f_j = 0$ for $i,\ j$ both even. Then define $S := A/I$. Notice by construction $A/I \otimes K(f_1, \dots, f_n)$ is a differential module with homology isomorphic to $\mathbb{Z}$.

Use the notation $K^{\text{gen}} := A/I \otimes K(f_1, \dots, f_n)$.

Remark 4.5. The relations induced by imposing the condition $f_i \cdot f_j = 0$ for i and j both even are in general quite complicated. The first case for which we obtain nontrivial equations is when $n = 4$ in Construction 4.4, in which case we are imposing the relation $f_2^{(2)} = 0$. Choosing bases, notice that f_2 is represented as a generic 4×4 skew symmetric matrix and the condition $f_2^{(2)} = 0$ means we are taking the quotient by the 4×4 pfaffian of this matrix representation.

Notice that in general, the ideal I appearing in Construction 4.4 is always generated by quadratic equations in the f_i.

Example 4.6. Assume $n = 6$ in the notation of Construction 4.4. Then the relations imposed come from setting

$$f_2^{(2)} = 0, \quad \text{and} \quad f_2 \cdot f_4 = 0.$$

The relations $f_2^{(2)} = 0$ are precisely the equations of the 4×4 pfaffians of the 6×6 matrix representation of the map f_2. The additional relation $f_2 \cdot f_4 = 0$ contributes, after choosing bases, the single quadratic equation

$$\sum_{\substack{I \subset [6], \\ |I|=2}} \operatorname{sgn}(I \subset [6]) x_I x_{[6]\setminus I} = 0.$$

It may be tempting to believe that all anchored free flags with complete intersection homology arise as specializations of Construction 4.4. We will see that this is true if the ring has characteristic 2, but the following example shows that this is not the case in general.

Example 4.7. Let $E = \bigoplus_{i=1}^4 Re_i$, where $R = k[x_1, \dots, x_4]$ and k is a field of characteristic $\neq 2$. Let F be the free flag defined by the following data:

$$A_{1,0} = A_{2,1} = A_{3,2} = A_{4,3} = x_1^3 e_1^* + x_2^3 e_2^* + x_3^3 e_3^* + x_4^3 e_4^*,$$

$$A_{2,0} = -A_{3,1} = A_{4,2} = x_1 x_2 e_{12}^* + x_2^2 e_{34}^*,$$

$$A_{3,0} = -2x_1 e_{134}^*, \quad A_{4,1} = A_{4,0} = 0.$$

Under a choice of basis, we can write $F = R^1 \oplus F^4 \oplus R^6 \oplus R^4 \oplus R^1$ and express the differential d^F as a block matrix.

$$\begin{array}{c} \\ R^1 \\ R^4 \\ R^6 \\ R^4 \\ R^1 \end{array} \begin{array}{c} \begin{array}{ccccc} R^1 & R^4 & R^6 & R^4 & R^1 \end{array} \\ \begin{pmatrix} 0 & A_{1,0} & A_{2,0} & A_{3,0} & A_{4,0} \\ 0 & 0 & A_{2,1} & A_{3,1} & A_{4,1} \\ 0 & 0 & 0 & A_{3,2} & A_{4,2} \\ 0 & 0 & 0 & 0 & A_{4,3} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

The blocks $A_{i,i-1}$ on the first off-diagonal are the matrices appearing in the Koszul complex on $(x_1^3, x_2^3, x_3^3, x_4^3)$. The $A_{i,i-2}$ maps are given by the following:

$$A_{2,0} = \begin{pmatrix} x_1x_2 & 0 & 0 & 0 & 0 & x_2^2 \end{pmatrix} \quad A_{3,1} = \begin{pmatrix} 0 & 0 & -x_2^2 & 0 \\ 0 & 0 & 0 & -x_2^2 \\ -x_1x_2 & 0 & 0 & 0 \\ 0 & -x_1x_2 & 0 & 0 \end{pmatrix} \quad A_{4,2} = \begin{pmatrix} x_2^2 \\ 0 \\ 0 \\ 0 \\ 0 \\ x_1x_2 \end{pmatrix}$$

and we have $A_{3,0} = \begin{pmatrix} 0 & 0 & 2x_1 & 0 \end{pmatrix}$.

One can check that d^F squares to 0 so this data determines a well-defined free flag. However, $A_{3,0} \neq A_{4,1}$ since $A_{3,0}$ is given by multiplication by a nonzero element and $A_{4,1}$ is not. This means that the free flag induced by the above data is not of the form $K(f_1, f_2, f_3, f_4)$ for any choice of f_i.

The above example hinges on the observation that if we let $A_{1,0} = \begin{pmatrix} x_1^3 & x_2^3 & x_3^3 & x_4^3 \end{pmatrix}$ and $A_{4,3} = \begin{pmatrix} -x_4^3 & x_3^3 & -x_2^3 & x_1^3 \end{pmatrix}^T$ be the first and last matrices in the Koszul complex on $(x_1^3, x_2^3, x_3^3, x_4^3)$, then $A_{2,0}A_{4,2} + A_{3,0}A_{4,3} = 0$. The condition for F to be a differential module requires that $A_{1,0}A_{4,1} + A_{2,0}A_{4,2} + A_{3,0}A_{4,3} = 0$, which in this case forces $A_{4,1}$ to be 0 modulo $(x_1^3, x_2^3, x_3^3, x_4^3)$. This "forcing" occurred due to cancellation in $A_{2,0}A_{4,2} + A_{3,0}A_{4,3}$. Our next result shows that absent this cancellation, it is indeed the case that anchored free flags with complete intersection homology as in Construction 4.4 are Koszul differential modules.

This intuition is clarified in the following definition which presents a technical condition that will be needed as a hypothesis for the proof of Theorem 4.10.

Definition 4.8. Let R be a ring and $I \subset R$ an ideal. The ideal I is *completely Tor-independent* with respect to a family of ideals $J_1, \dots, J_\ell$ if for all $1 \leqslant i_1 < \cdots < i_k \leqslant \ell$, one has

$$\operatorname{Tor}_{>0}^R \left(\frac{R}{I}, \frac{R}{J_{i_1} + \cdots + J_{i_k}} \right) = 0.$$

Conceptually, Tor-independent objects do not have any nontrivial homological interaction; at the level of resolutions, this means that tensoring a resolution of a module M with a Tor-independent quotient ring preserves exactness.

The next theorem gives a partial characterization of anchored free flags, and is the main result of this section. The abridged version of this result states that if the homology $H(D)$ is sufficiently Tor-independent with respect to a subfamily of the ideals $\operatorname{Im}(f_i \colon \bigwedge^i E \to R)$, then differential modules with complete intersection homology must arise as in Proposition 4.2.

Remark 4.9. Some remarks about the statement of Theorem 4.10 are necessary before the full statement: we can assume that the differential module D has free flag resolution anchored on a Koszul complex $K_\bullet$ resolving the complete intersection $H(D)$. Then the components of the flag differentials mapping to $K_0 = R$ are R-module homomorphisms $f_i : \bigwedge^i E \to R$. These are the elements $f_i : \bigwedge^i E \to R$ as written in the statement of the theorem, and the conclusion of the theorem is that D is isomorphic to the generalized Koszul flag $K(f_1, \dots, f_n)$ where the $f_i \in \bigwedge^i E^*$ arise as just mentioned.

Theorem 4.10. *Let D be a differential R-module with $H(D)$ a complete intersection, viewed as the cokernel of some map $f_1 : E \to R$ (where E is a free R-module). Let $F \to D$ be a free flag resolution anchored on the Koszul complex associated to $f_1 \in E^*$ and assume that* $\operatorname{Im}(f_1 \colon E \to R)$ *is completely Tor-independent with respect to the set*

$$\Big\{ \operatorname{Im}(f_i \colon \bigwedge^i E \to R) \mid i \text{ is even and } i \leqslant \operatorname{rank}(E)/2 \Big\}.$$

Then F is isomorphic to the differential module of Proposition 4.2, where $T = \bigwedge^\bullet E$.

Proof. Proceed by induction on $i - j - 1$, where i, j are the indices of the component $A_{i,j}$ of the differential of D. When $i - j - 1 = 0$, this is the statement of Theorem 2.10 since it is of no loss of generality to assume that D is anchored on the Koszul complex resolving $H(D)$, associated to some element $f_1 \in E^*$. Let $e_1, \dots, e_n$ denote a basis for E. Assume now that $i - j - 1 \geqslant 1$. Inducting also on j, one may assume that for all $k > 0$, there is the equality $A_{i-k,j+1-k} = (-1)^{(i-k)(j-k)} f_{i-j-1}$ (the base case is for $k = j + 1$, which holds by assumption). Using this, one computes:

$$\begin{aligned} 0 &= \sum_{k=j+1}^{i-1} A_{k,j} A_{i,k} \\ (*) \qquad &= 2 \cdot \sum_{\substack{i-k,\ j-k \text{ even} \\ j<k\leqslant\lfloor (i+j)/2 \rfloor}} f_{k-j} \cdot f_{i-k} + f_1\big((-1)^{(i+1)j+i+j} f_{i-j-1} - A_{i,j+1}\big). \end{aligned}$$

If $i - j > \operatorname{rank} E$, then the equality $(*)$ reduces to $f_1\big((-1)^{(i+1)j+i+j} f_{i-j-1} - A_{i,j+1}\big) = 0$, and exactness of multiplication by f_1 implies that $(-1)^{(i+1)j+i+j} f_{i-j-1} - A_{i,j+1} \in \operatorname{Im} f_1$. If $i - j \leqslant \operatorname{rank} E$, then notice that for each $j < k \leqslant \lfloor (i + j)/2 \rfloor$, one has $i - k$ or $k - j \leqslant \operatorname{rank}(E)/2$. Define the ideal

$$\mathfrak{a} := \big(\operatorname{Im}(f_\ell \colon \bigwedge^\ell E \to R) \mid \ell \text{ is even and } \ell < i - j - 1 \big) \subset R.$$

Let I be any indexing set with $|I| = i - j - 2$ and let $e_I^* \in \bigwedge^{i-j-2} E^*$ denote the basis element dual to $e_I \in \bigwedge^{i-j-2} E$. Multiplying the equality $(*)$ on the right by e_I^*, one obtains

$$f_1 \cdot (b \cdot e_I^*) \in \mathfrak{a}, \tag{4.1}$$

where $b := (-1)^{(i+1)j+i+j} f_{i-j-1} - A_{i,j+1}$. By the Tor-independence assumption, notice that if $K_\bullet$ denotes the Koszul complex induced by f_1, then the complex $K_\bullet \otimes_R R/\mathfrak{a}$ must remain exact. Combining this with (4.1) implies that $b \cdot e_I^*$ is a cycle in $K_\bullet \otimes_R R/\mathfrak{a}$, and hence by exactness there exists some $a \in \bigwedge^2 E$ such that

$$b \cdot e_I^* - f_1 \cdot a \in \mathfrak{a}E.$$

Multiplying the above on the right by e_i^* for $i = 1, \dots, n$, it follows that for all indexing sets J of size $i-j-1$, there exist elements $a_\ell^J \in \bigwedge^\ell E$ such that

$$b \cdot e_J^* = f_1 \cdot a_1^J + \underbrace{\sum_{\substack{\ell \text{ even}, \\ \ell < i-j-1}} f_\ell \cdot a_\ell^J}_{\in \mathfrak{a}}. \tag{4.2}$$

Recall that the identity map $\bigwedge^{i-j-1} E \to \bigwedge^{i-j-1} E$ is equivalently represented as right multiplication by the trace element $\sum_{|J|=i-j-1} e_J^* \otimes e_J$, whence:

$$\begin{aligned}
b &= b \cdot \Big(\sum_{|J|=i-j-1} e_J^* \otimes e_J \Big) \\
&= \sum_{|J|=i-j-1} b e_J^* \otimes e_J \\
&= \sum_{|J|=i-j-1} (f_1 \cdot a_1^J) e_J + \sum_{|J|=i-j-1} \sum_{\substack{\ell \text{ even}, \\ \ell < i-j-1}} (f_\ell \cdot a_\ell^J) e_J \quad \text{(by 4.2)} \\
&= f_1 \cdot \Big(\sum_{|J|=i-j-1} a_1^J \cdot e_J \Big) + \sum_{\substack{\ell \text{ even}, \\ \ell < i-j-1}} f_\ell \cdot \Big(\sum_{|J|=i-j-1} a_\ell^J \cdot e_J \Big) \\
&\in \operatorname{Im}\Big(f_1 \colon \bigwedge^{i-j} E \to \bigwedge^{i-j-1} E \Big) + \sum_{\substack{\ell \text{ even}, \\ \ell < i-j-1}} \operatorname{Im}\Big(f_\ell \colon \bigwedge^{i-j-1+\ell} E \to \bigwedge^{i-j-1} E \Big).
\end{aligned}$$

Recalling that $b := (-1)^{(i+1)j+i+j} f_{i-j-1} - A_{i,j+1}$, this means that the differential component $A_{i,j+1}$ differs from $(-1)^{(i+1)j+i+j} f_{i-j-1}$ by multiplication by the elements f_ℓ, where $\ell < i-j-1$. It follows that one may perform row/column operations on the matrix representation of the square-zero endomorphism of D to ensure that $A_{i,j+1} = (-1)^{(i+1)j+i+j} f_{i-j-1}$. This completes the proof. $\square$

The hypotheses of Theorem 4.10 are stated in a decent level of generality, but this is because we needed a condition that was general enough to capture a family of the special cases for which there existed an isomorphism to a Koszul flag. The following corollary makes explicit a list of common cases for which the hypotheses of Theorem 4.10 are satisfied:

Corollary 4.11. *The assumptions of Theorem 4.10 are satisfied in the following cases:*

1. *The free module E satisfies* $\operatorname{rank} E \leqslant 3$.
2. *$(R, \mathfrak{m})$ is a Noetherian local ring and*

$$\operatorname{Im}\Big(f_1 \colon E \to R \Big) + \sum_{\substack{\ell \leqslant \operatorname{rank}(E)/2 \\ \ell \text{ even}}} \operatorname{Im}\Big(f_\ell \colon \bigwedge^\ell E \to R \Big)$$

is generated by a regular sequence contained in $\mathfrak{m}$.

(3) R is a graded ring and

$$\operatorname{Im}\left(f_1\colon E \to R\right) + \sum_{\substack{\ell \leqslant \operatorname{rank}(E)/2 \\ \ell \text{ even}}} \operatorname{Im}\left(f_\ell\colon \bigwedge^\ell E \to R\right)$$

is generated by a homogeneous regular sequence of positive degree.

(4) $R = k[x_1, \dots, x_n]$ and the image of each $f_i\colon \bigwedge^i E \to R$ for $i = 1$ and $i \leqslant \operatorname{rank}(E)/2$ even lie in polynomial rings in disjoint variables.

In particular, if either:

(1) $(R, \mathfrak{m})$ is a Noetherian local ring and the first row of the matrix representation of the square-zero endomorphism of D generates a complete intersection contained in $\mathfrak{m}$, or

(2) R is a graded ring and the first row of the matrix representation of the square-zero endomorphism of D generates a homogeneous complete intersection of positive degree,

then the assumptions of Theorem 4.10 are satisfied.

Interestingly, if we assume that R has characteristic 2, then the statement of Theorem 4.10 can be generalized significantly.

Theorem 4.12. *Assume R is a ring of characteristic 2. Let F be an anchored free flag with $H(F)$ a complete intersection. Then F isomorphic to the differential module of Proposition 4.2 for some choice of $f_i \in \bigwedge^i E^*$, where $T = \bigwedge^\bullet E$.*

Proof. Do the computation of the previous proof, but notice that all other extraneous terms cancel by the characteristic assumption (since one of the terms of the equation $(*)$ in the proof of Theorem 4.10 has coefficient 2). $\square$

5. DG-module Structures on Free Flags

In this section, we study differential modules that can be given the structure of a DG-module over some DG-algebra. Our main motivation for considering this question is based on the philosophy that the homological properties of a differential module are tightly linked to those of the homology, as suggested by Theorem 2.10. One natural direction related to this question is the extent to which additional structure on the minimal free resolution of the homology can be "lifted" to the differential module. Our results here indicate that there are some very restrictive obstructions to lifting algebra structures to the level of differential modules.

It is evident that if a free resolution $F_\bullet$ of the homology $H(D)$ of an anchored free flag D admits the structure of an associative DG-algebra structure and $D \cong \operatorname{Fold}(F_\bullet)$, then the algebra structure on $F_\bullet$ can be transferred to a DG-module structure on D. We prove even further that if the homology $H(D)$ is a complete intersection, then this becomes an equivalence; more precisely: an anchored free flag D with $H(D)$ a complete intersection admits the structure of a DG-module over $K_\bullet$ *if and only if* $D \cong \operatorname{Fold}(K_\bullet)$, where $K_\bullet$ denotes the Koszul complex resolving $H(D)$.

We conclude the section with questions about DG-module structures on more general free flag resolutions. In particular, we know of no example of a free flag admitting a DG-module structure over the minimal free resolution of its homology that is *not* isomorphic to the fold of some complex, and are very interested in any such example.

Definition 5.1. A (graded commutative) *differential graded algebra* (F, d) (or *DG-algebra*) over a commutative Noetherian ring R is a complex of finitely generated free R-modules with differential d and with a unitary, associative multiplication $F \otimes_R F \to F$ satisfying

(a) $F_i F_j \subseteq F_{i+j}$,
(b) $d_{i+j}(f_i f_j) = d_i(f_i)f_j + (-1)^i f_i d_j(f_j)$,
(c) $f_i f_j = (-1)^{ij} f_j f_i$, and
(d) $f_i^2 = 0$ if i is odd,

where $f_k \in F_k$.

Remark 5.2. It is worth mentioning that this is a stricter definition of DG-algebra for the purposes of this paper, but there are more general definitions in the literature.

There does not exist a tensor product between arbitrary differential modules (or even free flags) that directly generalizes the tensor product of complexes, but, it is possible to construct such a product between a *complex* and a differential module.

Definition 5.3. Let $F_\bullet$ be a complex and D a differential module. Then the *box product* $F_\bullet \boxtimes_R D$ is defined to be the differential module with underlying module $\bigoplus_{i \in \mathbb{Z}} F_i \otimes_R D$ and differential

$$d^{F \boxtimes D}(f_i \otimes d) := d^F(f_i) \otimes d + (-1)^i f_i \otimes d^D(d).$$

Remark 5.4. This notion of a box product was introduced in the [ABI07, Subsection 1.9].

Definition 5.5. Let D be a differential module whose homology is a cyclic R-module and let $F_\bullet$ be a minimal free resolution of $H(D)$ admitting the structure of a DG-algebra. Then D is a DG-module over $F_\bullet$ if there exists a morphism of differential modules

$$p \colon F_\bullet \boxtimes_R D \to D$$

extending the R-module action on D. In such a case, the notation $f_i \cdot_D d := p(f_i \otimes d)$ will be used.

Remark 5.6. Sometimes the simpler notation $\cdot$ will be used over $\cdot_D$ when it is clear which product is being considered. It is important to note that there is almost no hope for an appropriate generalization of a DG-algebra even for general free flag resolutions. This is for at least two reasons: firstly, as already mentioned, there is no natural candidate for the tensor product of two differential modules, so one cannot employ a definition similar to Definition 5.5. Secondly, the "degree" of an element is not well-defined if it is induced by the flag filtration of arbitrary free flag $D = \bigoplus_{i \in \mathbb{Z}} F_i$, since any given $f_i \in F_i$ is contained in D^j for all $j \geqslant i$.

Observation 5.7. If $F_\bullet$ is a complex admitting the structure of a DG-algebra, then $\mathrm{Fold}(F_\bullet)$ is a DG-module over $F_\bullet$.

Proof. Just define the action on $\mathrm{Fold}(F_\bullet)$ via the product on $F_\bullet$. □

Observation 5.8. Let $\phi \colon D \to D'$ be an isomorphism of differential modules and $F_\bullet$ a DG-algebra minimal free resolution of $H(D)$. If D' is a DG-module over $F_\bullet$, then D is a DG-module over $F_\bullet$ with the induced product:

$$f_i \cdot_D d := \phi^{-1}\big(f_i \cdot_{D'} \phi(d)\big).$$

Moreover, ϕ becomes a morphism of DG-modules with this product.

Proof. The induced product is defined by making the following diagram commute:

$$\begin{tikzcd} F_\bullet \boxtimes D \arrow[r, "1\boxtimes\phi"] \arrow[d, "\cdot_D"] & F_\bullet \boxtimes D' \arrow[d, "\cdot_{D'}"] \\ D & D' \arrow[l, "\phi^{-1}"] \end{tikzcd}$$

□

Example 5.9. Let $R = k[x_1, x_2]$ and E be a rank 2 free module on the basis e_1, e_2. Let $D = \bigwedge^\bullet E$ be the free flag with differential

$$\begin{pmatrix} 0 & x_1 & x_2 & x_1^2 + x_2^2 \\ 0 & 0 & 0 & -x_2 \\ 0 & 0 & 0 & x_1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then D admits the structure of a DG-module over the Koszul complex $K_\bullet$ with the following product (any product not listed is understood to be 0):

$$1 \cdot_D d = d \text{ for all } d \in D, \quad e_1 \cdot_D 1 = e_1, \; e_2 \cdot_D 1 = e_2,$$
$$e_{12} \cdot_D 1 = e_{12} - x_1e_1 - x_2e_2,$$
$$e_1 \cdot_D e_2 = e_{12} - x_1e_1 - x_2e_2,$$
$$e_1 \cdot_D e_{12} = x_2e_{12} - x_1x_2e_1 - x_2^2e_2, \quad \text{and}$$
$$e_2 \cdot_D e_{12} = -x_1e_{12} + x_1^2e_1 + x_1x_2e_2.$$

Recall that by Theorem 3.6, the differential module D as in Example 5.9 is isomorphic to the fold of the Koszul complex on (x_1, x_2). In fact, the product given above is induced by this isomorphism. This construction works in general, that is if we have an isomorphism to a *DG*-algebra, we can obtain a *DG*-module structure in the same way. This leads to a string of immediate corollaries to Observation 5.8.

Corollary 5.10. *Let D be an anchored free flag and assume that $D \cong \mathrm{Fold}(F_\bullet)$ where $F_\bullet \to H(D)$ is a minimal free resolution. If $F_\bullet$ admits the structure of a DG-algebra, then D admits the structure of a DG-module over $F_\bullet$.*

Proof. This follows from Observation 5.8 and Observation 5.7. □

Corollary 5.11. *Let D be an anchored free flag and assume that the minimal free resolution $F_\bullet$ of $H(D)$ admits the structure of a DG-algebra. If the matrices $A_{i,0}$ satisfy $\operatorname{Im} A_{i,0} \subset \operatorname{Im} d_1$ for each $i \geqslant 2$, then D is a DG-module over $F_\bullet$.*

Proof. If $\operatorname{Im} A_{i,0} \subset \operatorname{Im} d_1$, then $D \cong \mathrm{Fold}(F_\bullet)$ by Lemma 3.3, so employ Corollary 5.10. □

Corollary 5.12. *Assume R is a regular graded local ring. Let D be a degree 0 anchored free flag with $H(D) \cong k$. Then D admits the structure of a DG-module over the minimal free resolution of k.*

Proof. The assumption that D has degree 0 implies that each matrix $A_{i,0}$ has entries in $\mathfrak{m}$, so employ Corollary 5.11. □

The above string of corollaries are all proved by reducing to the case that the differential module being considered may be realized as the folding of a DG-algebra resolution. The following theorem shows that this assumption is not only sufficient, but also *necessary* for free flags with complete intersection homology.

Theorem 5.13. *Let D be an anchored free flag with $H(D)$ a complete intersection. Let $K_\bullet$ denote the Kosul complex resolving $H(D)$. Then D is a DG-module over $K_\bullet$ if and only if $D \cong \operatorname{Fold}(K_\bullet)$.*

Proof. $\Longleftarrow$: This is clear by Observation 5.8.

$\Longrightarrow$: Choose $i := \min\{i > 1 \mid f_i \neq 0\}$, where $A_{i,0} =: f_i \in \bigwedge^i E^*$ are the defining data of components of the differential; if no such i exists, then $D \cong \operatorname{Fold}(K_\bullet)$ by Lemma 3.3 and there is nothing to prove, so assume i exists. Otherwise, Lemma 3.3 implies that we may assume $A_{k,j} = 0$ for all $k - j < i$. In particular, for all indexing sets I of size i, one has

$$d(e_I) = f_1(e_I) + f_i(e_I).$$

Assume that D has a DG-module structure over $K_\bullet$. By assumption $H(D) \cong R/\mathfrak{a}$ for some ideal $\mathfrak{a}$ that is generated by a regular sequence, and $f_1 \otimes R/\mathfrak{a} = 0$. One can choose the algebra structure such that $e_I \cdot e_J = e_I \wedge e_J + t_{i-1}$ for all $|I| + |J| \leqslant i$, where t_{i-1} is some element of $\bigoplus_{j \leqslant i-1} \bigwedge^j E$. This is because exactness of multiplication by f_1 forces $e_I \wedge e_J - p_{|I|+|J|}(e_I \cdot_D e_J)$ to be a boundary in the Koszul complex induced by f_1, where $p_{|I|+|J|} \colon D \to \bigwedge^{|I|+|J|} E$ denotes the projection onto the corresponding direct summand (and 0 is the only boundary with k-coefficients). Let e_I be any basis vector such that $f_i(e_I) \otimes R/\mathfrak{a} \neq 0$ (such an element must exist by selection of i), where $|I| = i$. Let ℓ be the first element of I and notice that:

$$\begin{aligned} f_1(e_I) + f_i(e_I) &= d(e_I) \\ &= d(e_\ell \cdot_D e_{I \setminus \ell} + t_{i-1}) \\ &= d(e_\ell) \cdot_D e_{I \setminus \ell} - e_\ell \cdot_D d(e_{I \setminus \ell}) + d(t_{i-1}). \end{aligned}$$

Tensoring the above relation with $R/\mathfrak{a}$, it follows that $f_i(e_I) \in \mathfrak{a}$, which is a contradiction to the assumptions. It follows that no DG-module structure can exist. $\square$

In view of Theorem 5.13, it follows that DG-module structures over the minimal free resolution of the homology are actually quite rare. Indeed, after running many examples it seems that the only time such a DG-module structure exists is if the free flag arises as the fold of the minimal free resolution, indicating that DG-module structures can be used to distinguish free flags that are in the isomorphism class of a complex. It is an interesting question as to whether there exists a family of structures, similar to a DG-module structure, that can be used to detect the isomorphism class of any anchored free flag. We conclude with the following (likely easier) question:

Question 5.14. Does there exist an anchored free flag D admitting the structure of a module over the minimal free resolution of its homology $F_\bullet$ that is *not* isomorphic to $\operatorname{Fold}(F_\bullet)$?

References

[ABI07] Luchezar L. Avramov, Ragnar-Olaf Buchweitz, and Srikanth Iyengar, *Class and rank of differential modules*, Invent. Math. **169** (2007), no. 1, 1–35. MR 2308849

[ABW82] Kaan Akin, David A. Buchsbaum, and Jerzy Weyman, *Schur functors and Schur complexes*, Adv. in Math. **44** (1982), no. 3, 207–278. MR 658729

[BD10] Adam Boocher and Justin W DeVries, *On the rank of multigraded differential modules*, arXiv preprint arXiv:1011.2167 (2010).

[BE77] David A Buchsbaum and David Eisenbud, *Algebra structures for finite free resolutions, and some structure theorems for ideals of codimension 3*, American Journal of Mathematics **99** (1977), no. 3, 447–485.

[BE21] Michael K. Brown and Daniel Erman, *Tate resolutions on toric varieties*, To Appear, Journal of the European Mathematical Society, arXiv preprint arXiv:2108.03345 (2021).

12 Oct 2023 15:05:43 PDT
230718-VandeBogert Version 4 - Submitted to J. Comm. Alg.

[BE22] Michael Brown and Daniel Erman, *Minimal free resolutions of differential modules*, Transactions of the American Mathematical Society **375** (2022), no. 10, 7509–7528.

[Car86] Gunnar Carlsson, *Free $\mathbb{Z}/2k$-actions and a problem in commutative algebra*, Transformation Groups Poznań 1985 (Berlin, Heidelberg) (Stefan Jackowski and Krzysztof Pawałowski, eds.), Springer Berlin Heidelberg, 1986, pp. 79–83.

[CE16] Henry Cartan and Samuel Eilenberg, *Homological algebra (pms-19), volume 19*, Princeton university press, 2016.

[Har79] Robin Hartshorne, *Algebraic vector bundles on projective spaces: A problem list*, Topology **18** (1979), no. 2, 117–128.

[IW18] Srikanth B Iyengar and Mark E Walker, *Examples of finite free complexes of small rank and small homology*, Acta Mathematica **221** (2018), no. 1, 143–158.

[Rou06] Raphaël Rouquier, *Representation dimension of exterior algebras*, Inventiones mathematicae **165** (2006), no. 2, 357–367.

[RZ17] Claus Michael Ringel and Pu Zhang, *Representations of quivers over the algebra of dual numbers*, Journal of Algebra **475** (2017), 327–360.

[Sta17] Torkil Stai, *Differential modules over quadratic monomial algebras*, Algebras and Representation Theory **20** (2017), no. 5, 1239–1247.

[ŞÜ19] Berrin Şentürk and Özgün Ünlü, *Carlsson's rank conjecture and a conjecture on square-zero upper triangular matrices*, Journal of Pure and Applied Algebra **223** (2019), no. 6, 2562–2584.

[Wei15] Jiaqun Wei, *Gorenstein homological theory for differential modules*, Proceedings of the Royal Society of Edinburgh Section A: Mathematics **145** (2015), no. 3, 639–655.

[XYY15] Huabo Xu, Shilin Yang, and Hailou Yao, *Gorenstein theory for n-th differential modules*, Periodica Mathematica Hungarica **71** (2015), no. 1, 112–124.