ON D-ALGEBRAS BETWEEN $D[X]$ AND $\mathrm{Int}(D)$

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Abstract. The aim of this paper is to study conditions on an integral domain D such that any D-algebra between the polynomial ring $D[X]$ and the ring of integer-valued polynomials $\mathrm{Int}(D)$ is (locally) free. These results are then extended to several indeterminates.

Introduction

Throughout this paper, we let D be an integral domain with quotient field K. The *ring of integer-valued polynomials* on D is defined as follows:

$$\mathrm{Int}(D) := \{f \in K[X] \mid f(D) \subseteq D\}.$$

The ring $\mathrm{Int}(\mathbb{Z})$ and, more generally, the rings $\mathrm{Int}(\mathcal{O}_K)$ where $\mathcal{O}_K$ denotes the ring of integers of a number field K, were first studied by A. Ostrowski [16] and G. Pólya [17] in 1919 and, after a century of research, has become a classical object in commutative ring theory, number theory, and further areas of active research in mathematics. Particularly, Cahen *et al.* in [6] asked whether $\mathrm{Int}(D)$ is always (locally) free, or at least flat, as a D-module.

In a chronological overview of contributions concerning the module structure of integer-valued polynomial rings, Pólya [17] established in 1919 that $\mathrm{Int}(D)$ is a free D-module for all principal ideal domains D. Later, in 1971, Cahen & Chabert showed in [4, consequence of Corollaires (3), page 303] that $\mathrm{Int}(D)$ is a faithfully flat D-module for all Dedekind domains D. A year after, Cahen [3] proved that $\mathrm{Int}(D)$ is projective for all Dedekind domains D, while the first author [7] established that $\mathrm{Int}(D)$ is a free D-module with a regular basis, that is, a basis with exactly one polynomial for each degree, for all unique factorization domains D. In 1996, Cahen & Chabert noted in [5, Remark II.3.7] that the D-module $\mathrm{Int}(D)$ is free for all Dedekind domains D. In 2009, Elliott [11] showed that $\mathrm{Int}(D)$ is locally free if D is a PvMD such that $\mathrm{Int}(D)_\mathfrak{p} = \mathrm{Int}(D_\mathfrak{p})$ for every prime ideal $\mathfrak{p}$ of D. That includes the case where D is a Krull domain or, more generally, a Krull-type domain. Finally, the

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second author showed in [15] that, for any locally essential domain D, the D-module $\mathrm{Int}(D)$ is always flat, and it is locally free under a hypothesis of good behavior under localization. Notably, the hypothesis of good behavior under localization plays a crucial role in investigating the local freeness of $\mathrm{Int}(D)$ as a D-module. The reader may consult the survey [19] for more information about the module structure of the integer-valued polynomial rings.

The main goal of this paper is to investigate the (local) freeness of D-algebras between $D[X]$ and $\mathrm{Int}(D)$. In particular, we show that every algebra $\mathbb{B}$ over a locally essential domain D such that $D[X] \subseteq \mathbb{B} \subseteq \mathrm{Int}(D)$ is locally free without any hypothesis of good behavior under localization.

Before starting the next section, it seems convenient to introduce some remarkable D-algebras that lie between $D[X]$ and $\mathrm{Int}(D)$.

∘ For every overring R of D, the ring $\mathrm{Int}_R(D)$ of *D-valued R-polynomials*

$$\mathrm{Int}_R(D) := \{f \in R[X] \mid f(D) \subseteq D\}. \quad [1]$$

∘ The rings $\mathrm{Int}^{\{k\}}(D)$ of polynomials that are integer-valued together with their *divided differences* up to the order k [2].

∘ The rings $\mathrm{Int}^{(k)}(D)$ (resp., $\mathrm{Int}^{[k]}(D)$) of polynomials that are integer-valued on D together with their *derivatives* (resp., *finite differences*) of order up to k [5]. Also, the rings $\mathrm{Int}^{(\infty)}(D)$ (resp., $\mathrm{Int}^{[\infty]}(D)$) of polynomials which are integer-valued together with their derivatives (resp., finite differences) of all orders.

∘ The *Bhargava ring* over D at x where x is any element of D

$$\mathbb{B}_x(D) := \{f \in K[X] \mid \forall a \in D,\ f(xX + a) \in D[X]\} \quad [20].$$

∘ The ring of integer-valued polynomials on a torsion-free D-algebra A such that $A \cap K = D$ with coefficients in K defined by

$$\mathrm{Int}_K(A) := \{f \in K[X] \mid f(A) \subseteq A\} \quad [14].$$

∘ In particular, the rings $\mathrm{Int}_K(\mathcal{M}_n(D))$ and $\mathrm{Int}_K(T_n(D))$ defined by

$$\mathrm{Int}_K(\mathcal{M}_n(D)) := \{f \in K[X] \mid f(\mathcal{M}_n(D)) \subseteq \mathcal{M}_n(D)\}$$

$$\text{and} \quad \mathrm{Int}_K(T_n(D)) := \{f \in K[X] \mid f(T_n(D)) \subseteq T_n(D)\}$$

where $\mathcal{M}_n(D)$ denotes the ring of $n \times n$ matrices with coefficients in D and $T_n(D)$ the subring of $\mathcal{M}_n(D)$ formed by triangular matrices [13].

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1. Local study

We start by recalling some concepts and facts. So, let D be an integral domain with quotient field K and let $\mathbb{B}$ be a D-algebra such that $D[X] \subseteq \mathbb{B} \subseteq K[X]$. To avoid the trivial case, we will assume that $D \neq K$.

Following [17], a basis of the D-module $\mathbb{B}$ is said to be a *regular basis* if it is formed by exactly one polynomial of each degree.

Recall that the *characteristic ideal* of index n of the D-algebra $\mathbb{B}$, denoted by $\mathfrak{I}_n(\mathbb{B})$, is defined to be the set formed by 0 and the leading coefficients of the polynomials in $\mathbb{B}$ of degree n. Note that $D \subseteq \mathfrak{I}_n(\mathbb{B}) \subseteq K$, and $\mathfrak{I}_n(\mathbb{B}_1) \subseteq \mathfrak{I}_n(\mathbb{B}_2)$ for any two D-algebras $\mathbb{B}_1$ and $\mathbb{B}_2$ such that $\mathbb{B}_1 \subseteq \mathbb{B}_2$. Moreover, it is known that $\mathbb{B}$ admits a regular basis if and only if the D-modules $\mathfrak{I}_n(\mathbb{B})$ are principal fractional ideals of D (cf. [5, Proposition II.1.4]). In particular, for any principal ideal domain D, the D-algebra $\mathbb{B}$ has a regular basis. More details on these concepts can be found in [9].

Based on the observation made in [5, Remark II.2.14], we can state the following lemma.

Lemma 1.1. *Let D be a local domain whose maximal ideal is principal generated by π and whose residue field is finite with cardinality q. Let $a_0, a_1, \ldots, a_{q-1}$ be a set of representatives of D modulo πD, and consider the sequence $\underline{a} = \{a_n\}_{n\geqslant 0}$ defined by*

$$a_n = a_{n_0} + a_{n_1}\pi + \cdots + a_{n_r}\pi^r$$

when

$$n = n_0 + n_1 q + \cdots + n_r q^r \ \text{ where } \ 0 \leqslant n_i < q.$$

Then,

1. *for every $x \in D$, $\pi^{w_q(n)}$ divides $\prod_{k=0}^{n-1}(x-a_k)$ where $w_q(n) = \sum_{k\geqslant 1}\left[\frac{n}{q^k}\right]$,*
2. *the polynomials $\binom{X}{0}_{\underline{a}} = 1$ and, for $n > 0$, $\binom{X}{n}_{\underline{a}} = \prod_{k=0}^{n-1}\frac{X-a_k}{a_n-a_k}$ form a regular basis of $\mathrm{Int}(D)$,*
3. *for every $n \geqslant 0$, $\mathfrak{I}_n(\mathrm{Int}(D)) = \frac{1}{\pi^{w_q(n)}}D$,*
4. *for every $f \in K[X]$ of degree n, we have:*

$$f \in \mathrm{Int}(D) \Leftrightarrow f(a_0), f(a_1), \ldots, f(a_n) \in D.$$

We state now our first main result.

Theorem 1.2. *If D is a local domain whose maximal ideal is principal generated by π, then every D-algebra $\mathbb{B}$ such that $D[X] \subseteq \mathbb{B} \subseteq \mathrm{Int}(D)$ admits a regular basis. More precisely, for every n, the ideal $\mathfrak{I}_n(\mathbb{B})$ is of the form $\pi^{-s_n}D$ where $0 \leqslant s_n \leqslant w_q(n)$ when the residue field of D is finite with cardinality q.*

Proof. Let $\mathbb{B}$ be a D-algebra such that $D[X] \subseteq \mathbb{B} \subseteq \mathrm{Int}(D)$. If the residue field of D is infinite, then $\mathbb{B} = D[X] = \mathrm{Int}(D)$, and the conclusion is trivial. Thus, we may assume that the residue field of D is finite with cardinality q.

From the obvious containments $D \subseteq \mathfrak{J}_n(\mathbb{B}) \subseteq \frac{1}{\pi^{w_q(n)}}D$, we deduce that

$$\pi^{w_q(n)}D \subseteq \pi^{w_q(n)}\mathfrak{J}_n(\mathbb{B}) \subseteq D.$$

Let us prove that the entire ideal $\mathfrak{a} = \pi^{w_q(n)}\mathfrak{J}_n(\mathbb{B})$ which contains $\pi^{w_q(n)}$ is of the form $\pi^r D$ where $0 \leqslant r \leqslant w_q(n)$. Clearly, if an integer s is such that $\pi^{w_q(n)} \in \pi^s D$, then $s \leqslant w_q(n)$, and hence, there is a greatest non-negative integer r such that $\mathfrak{a} \subseteq \pi^r D$. Assume that $\mathfrak{a} \neq \pi^r D$ and let $x \in \mathfrak{a} = \mathfrak{a} \cap \pi^r D$, then $x = a\pi^r$ for some $a \in D$. If a is invertible, then $\pi^r \in \mathfrak{a}$, and this implies $\mathfrak{a} = \pi^r D$, contradicting our assumption. Thus, for all $x \in \mathfrak{a}$, $x = a\pi^r$ with $a \in \pi D$, which means that $x \in \pi^{r+1}D$, this is a contradiction since $\mathfrak{a} \not\subseteq \pi^{r+1}D$. Therefore, $\mathfrak{a} = \pi^r D$.

Consequently, we have $\mathfrak{J}_n(\mathbb{B}) = \frac{1}{\pi^{w_q(n)-r}}D$ and so $\mathfrak{J}_n(\mathbb{B})$ is a principal fractional ideal of D for all n. By [5, Proposition II.1.4], we deduce that $\mathbb{B}$ admits a regular basis. $\square$

In [5, Remark II.2.14], it is pointed out that: for any valuation domain V, the V-module $\mathrm{Int}(V)$ has a regular basis. This result has been recently generalized by the second author in [18] to the case of $\mathrm{Int}_R(V)$, where R is an overring of V. As an application of Theorem 1.2, we can now extend this result to any V-algebra $\mathbb{B}$ satisfying $V[X] \subseteq \mathbb{B} \subseteq \mathrm{Int}(V)$.

Corollary 1.3. *Let V be a valuation domain. Then every V-algebra $\mathbb{B}$ such that $V[X] \subseteq \mathbb{B} \subseteq \mathrm{Int}(V)$ admits a regular basis.*

Proof. Let $\mathfrak{m}$ be the maximal ideal of V and let $\mathbb{B}$ be a V-algebra between $V[X]$ and $\mathrm{Int}(V)$. If $\mathfrak{m}$ is not principal or its residue field is infinite, then $\mathrm{Int}(V)$ is just $V[X]$ by [5, Proposition I.3.16]. In this case, $\mathbb{B} = V[X]$ has a regular basis. On the other hand, if $\mathfrak{m}$ is principal and its residue field is finite, then Theorem 1.2 implies that $\mathbb{B}$ has a regular basis. $\square$

Remark 1.4. We could wonder whether the previous study in the local case, where $\mathfrak{m} = \pi D$, may be extended, or not, to D-algebras contained in rings larger than $\mathrm{Int}(D)$, such as rings of the form $\mathrm{Int}(E, D) = \{f \in K[X] \mid f(E) \subseteq D\}$ where E is a subset of D. Obviously, a necessary condition is that $\mathrm{Int}(E, D)$ itself admits a regular basis. In particular, E has to be infinite since else, if $E = \{a_1, \ldots, a_r\}$ then, for every $g \in K[X]$, $g(X)\prod_{j=1}^{r}(X - a_j)$ belongs to $\mathrm{Int}(E, D)$, which implies that $\mathfrak{J}_n(\mathrm{Int}(E, D)) = K$ for all $n \geq r$. On the other hand, the following assertion gives a sufficient condition that allows to extend Theorem 1.2.

Proposition 1.5. *If D is a local domain whose maximal $\mathfrak{m}$ is principal and if E is a subset of D which meets infinitely many distinct residue classes of D modulo*

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$\mathfrak{p} = \cap_{n\geq 0}\mathfrak{m}^n$, *then every D-algebra $\mathbb{B}$ such that $D[X] \subseteq \mathbb{B} \subseteq \mathrm{Int}(E, D)$ admits a regular basis.*

Proof. Let $\pi \in D$ be such that $\mathfrak{m} = \pi D$. The fact that E meets infinitely many distinct residue classes of D modulo the prime ideal $\mathfrak{p} = \cap_{n\geq 0}\mathfrak{m}^n$ implies that $\mathrm{Int}(E, D) \subseteq D_{\mathfrak{p}}[X]$, but also that the characteristic subsets $\mathfrak{I}_n(\mathrm{Int}(E, D))$ are all of the form $\frac{1}{\pi^s}D$ for some non-negative integer s. Indeed, let $\{a_n\}_{n\geq 0}$ be an infinite sequence of elements of E that are in distinct classes modulo $\mathfrak{p}$ and, for each n, consider the Vandermonde determinant $V(a_0, a_1, \ldots, a_n) = \prod_{0\leq i<j\leq n}(a_j - a_i)$. As $V(a_0, a_1, \ldots, a_n) \notin \mathfrak{p}$, there exists an integer r such that $V(a_0, a_1, \ldots, a_n) \in \mathfrak{m}^r \backslash \mathfrak{m}^{r+1}$, and hence, $\pi^r\, \mathfrak{I}_n(\mathrm{Int}(E, D)) \subseteq D$ by [5, Proposition I.3.1]. Thus, $\mathfrak{I}_n(\mathrm{Int}(E, D)) = \frac{1}{\pi^s}D$ and we may conclude as in Theorem 1.2. □

In the case where $\cap_{n\geq 0}\mathfrak{m}^n = (0)$, the previous sufficient condition on E just means that E is infinite which is a necessary condition by Remark 1.4. The next assertion shows such an example.

Proposition 1.6 ([5, Corollary II.1.6])**.** *Let D be a principal ideal domain and let E be a subset of D. Every D-algebra $\mathbb{B}$ such that $D[X] \subseteq \mathbb{B} \subseteq \mathrm{Int}(E, D)$ has a regular basis if and only if E is infinite.*

This is a global result that naturally leads us to our next section.

2. Globalization

Let us first recall some concepts.

A prime ideal $\mathfrak{p}$ of D is called *int prime* if $\mathrm{Int}(D) \nsubseteq D_{\mathfrak{p}}[X]$ and it is called *polynomial prime* if $\mathrm{Int}(D) \subseteq D_{\mathfrak{p}}[X]$. If $\mathfrak{p}$ is a polynomial prime we also have that $\mathrm{Int}(D)_{\mathfrak{p}} = D_{\mathfrak{p}}[X]$.

For a D-module $\mathbb{B}$, we say that $\mathbb{B}$ is *locally free* if, for each maximal ideal $\mathfrak{m}$ of D, the $D_{\mathfrak{m}}$-module $\mathbb{B}_{\mathfrak{m}}$ is free. Moreover, if $D[X] \subseteq \mathbb{B} \subseteq K[X]$, we say that $\mathbb{B}$ has *locally a regular basis* if, for each maximal ideal $\mathfrak{m}$ of D, the $D_{\mathfrak{m}}$-algebra $\mathbb{B}_{\mathfrak{m}}$ has a regular basis. From these definitions, we deduce immediately the following implications:

$$\mathbb{B} \text{ has locally a regular basis} \Rightarrow \mathbb{B} \text{ is locally free} \Rightarrow \mathbb{B} \text{ is (faithfully) flat.}$$

Proposition 2.1. *Let D be an integral domain such that $\mathfrak{m}D_{\mathfrak{m}}$ is principal for every int prime ideal $\mathfrak{m}$ of D. Then every D-algebra $\mathbb{B}$ such that $D[X] \subseteq \mathbb{B} \subseteq \mathrm{Int}(D)$ has locally a regular basis and so it is locally free.*

Proof. Let $\mathfrak{m}$ be a maximal ideal of D and let $\mathbb{B}$ be a D-algebra between $D[X]$ and $\mathrm{Int}(D)$. We have then two possible cases:

Case 1: $\mathfrak{m}$ is a polynomial prime of D. Since $\mathrm{Int}(D)_{\mathfrak{m}} = D_{\mathfrak{m}}[X]$ and $D_{\mathfrak{m}}[X] \subseteq \mathbb{B}_{\mathfrak{m}} \subseteq \mathrm{Int}(D)_{\mathfrak{m}}$, we deduce that $\mathbb{B}_{\mathfrak{m}} = D_{\mathfrak{m}}[X]$, and then $\mathbb{B}_{\mathfrak{m}}$ has a regular basis.

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Case 2: $\mathfrak{m}$ is an int prime of D. By assumption, the maximal ideal of $D_{\mathfrak{m}}$ is principal and then it follows from Theorem 1.2 that $\mathbb{B}_{\mathfrak{m}}$ has a regular basis because $D_{\mathfrak{m}}[X] \subseteq \mathbb{B}_{\mathfrak{m}} \subseteq \mathrm{Int}(D)_{\mathfrak{m}} \subseteq \mathrm{Int}(D_{\mathfrak{m}})$.

Therefore, in both cases, $\mathbb{B}_{\mathfrak{m}}$ has a regular basis, and thus the D-algebra $\mathbb{B}$ has locally a regular basis, as wanted. □

Remark 2.2. Let $D = \mathbb{Z}[\sqrt{-5}]$ be the ring of integers of the number field $\mathbb{Q}(\sqrt{-5})$. In [5, Exercise II.30], it is shown that $\mathrm{Int}(D)$ is free as a D-module but has no regular basis. However, Proposition 2.1 implies that $\mathrm{Int}(D)$ has locally a regular basis.

We say that D is an *essential domain* if there exists a set $\mathcal{P}$ consisting of prime ideals of D such that the following two properties hold:

(1) $D = \cap_{\mathfrak{p}\in\mathcal{P}} D_{\mathfrak{p}}$ and
(2) $D_{\mathfrak{p}}$ is a valuation domain for all $\mathfrak{p} \in \mathcal{P}$,

in which case we say that D is *essential with defining family* $\mathcal{P}$. In addition, D is called *locally essential* if the localization of D at any prime ideal is essential. Relevant examples of locally essential domains include PvMDs and almost Krull domains (Recall that an *almost Krull domain* is an integral domain whose localizations at maximal ideals are Krull domains, and a *PvMD* is an integral domain whose localizations at maximal t-ideals are valuation domains).

Theorem 2.3. *Let D be a locally essential domain. Then every D-algebra $\mathbb{B}$ such that $D[X] \subseteq \mathbb{B} \subseteq \mathrm{Int}(D)$ has locally a regular basis and hence it is locally free.*

The assertion is an easy consequence of the following lemma together with Corollary 1.3.

Lemma 2.4. *If $\mathfrak{m}$ is a maximal ideal of a locally essential domain D then, either $D_{\mathfrak{m}}$ is a valuation domain, or $\mathrm{Int}(D_{\mathfrak{m}}) = D_{\mathfrak{m}}[X]$.*

Proof. Assume that $D_{\mathfrak{m}}$ is not a valuation domain, then $D_{\mathfrak{m}} = \cap_{\mathfrak{p}\in\mathrm{Spec}(D),\, \mathfrak{p}\subsetneq\mathfrak{m}} D_{\mathfrak{p}}$ since $D_{\mathfrak{m}}$ is an essential domain. Consequently,

$$\mathrm{Int}(D_{\mathfrak{m}}) = \mathrm{Int}(D, D_{\mathfrak{m}}) = \cap_{\mathfrak{p}\subsetneq\mathfrak{m}} \mathrm{Int}(D, D_{\mathfrak{p}}) = \cap_{\mathfrak{p}\subsetneq\mathfrak{m}} \mathrm{Int}(D_{\mathfrak{p}}) = \cap_{\mathfrak{p}\subsetneq\mathfrak{m}} D_{\mathfrak{p}}[X] = D_{\mathfrak{m}}[X].$$

The two first equalities follows from [5, Corollary I.2.6] and the penultimate equality follows from the fact that $\mathrm{Int}(D)_{\mathfrak{p}} = D_{\mathfrak{p}}[X]$ for all primes $\mathfrak{p} \neq \mathfrak{m}$ since being non-maximal these ideals have infinite residue rings. □

Proof of Theorem 2.3. Once more, for every maximal ideal $\mathfrak{m}$ of D, we have the containments $D_{\mathfrak{m}}[X] \subseteq \mathbb{B}_{\mathfrak{m}} \subseteq \mathrm{Int}(D)_{\mathfrak{m}} \subseteq \mathrm{Int}(D_{\mathfrak{m}})$. By Lemma 2.4, either $D_{\mathfrak{m}}$ is a valuation domain and we conclude with Corollary 1.3, or $\mathrm{Int}(D_{\mathfrak{m}}) = D_{\mathfrak{m}}[X]$ and $\mathbb{B}_{\mathfrak{m}} = D_{\mathfrak{m}}[X]$. □

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Remark 2.5. In [11, Proposition 3.5], [12, Theorem 1.2(a)], [15, Theorem 1], and [18, Theorem 18(3)], the authors assumed that $\mathrm{Int}(D)$ behaves well under localization, that is, $\mathrm{Int}(D)_{\mathfrak{p}} = \mathrm{Int}(D_{\mathfrak{p}})$ for $\mathfrak{p}$ in a specified subset of $\mathrm{Spec}(D)$, when dealing with the local freeness of $\mathrm{Int}(D)$, or more generally $\mathrm{Int}_R(D)$. However, Examples 6.2 and 6.5 of [8] provide almost Dedekind domains D with finite residue fields such that $\mathrm{Int}(D)$ does not behave well under localization, i.e., $\mathrm{Int}(D_{\mathfrak{m}}) \neq \mathrm{Int}(D)_{\mathfrak{m}}$ for some maximal ideals $\mathfrak{m}$ of D. Nonetheless, we see easily that the assumption of good behavior under localization is not necessary for the cited results at the beginning of this remark.

We next provide some illustrative examples.

Example 2.6. Let $D = \mathbb{Z} + T\mathbb{Q}[T]$, where T is an indeterminate over $\mathbb{Q}$, and let $\mathbb{B}$ be a D-algebra between $D[X]$ and $\mathrm{Int}(D)$. It is known that the integral domain D is Prüfer, thus it follows from Theorem 2.3 that $\mathbb{B}$ is a locally free D-module.

Example 2.7. Let $\mathcal{E}$ be the ring of entire functions and set $D := \mathcal{E} + T\mathcal{E}_S[T]$, where T is an indeterminate over $\mathcal{E}$ and S is the multiplicative subset generated by the prime elements of $\mathcal{E}$. Let $\mathbb{B}$ be a D-algebra between $D[X]$ and $\mathrm{Int}(D)$.

According to [21, Example 2.6], D is a locally essential domain which is neither PvMD nor almost Krull. By Theorem 2.3, $\mathbb{B}$ is locally free as a D-module .

3. Several indeterminates

The previous results may be extended to several indeterminates. Let n be a fix positive integer and consider the ring of integer-valued polynomials on D in n variables:

$$\mathrm{Int}(D^n, D) = \{f \in K[X_1, \dots, X_n] \mid f(D^n) \subseteq D\}.$$

More generally, for every subset $\underline{E}$ of D^n, we consider the ring

$$\mathrm{Int}(\underline{E}, D) = \{f \in K[X_1, \dots, X_n] \mid f(\underline{E}) \subseteq D\}.$$

Lemma 3.1. *Let $\underline{E}$ be a subset of D^n of the form $\prod_{j=1}^n E_j$ where $E_j \subseteq D$. Assume that, for $0 \leq j \leq n$, $\mathrm{Int}(E_j, D)$ admits a regular basis $\{f_{j,k}\}_{k\geq 0}$. Then, the D-module $\mathrm{Int}(\underline{E}, D)$ admits the regular basis $\{\prod_{j=1}^n f_{j,k_j}\}_{\underline{k}=(k_1,\dots,k_n)\in\mathbb{N}^n}$.*

Proof. The proof is very similar to that of [5, Proposition XI.1.13]. Since there is one and only one polynomial of each multi-degree $\underline{k}$ in $\{\prod_{j=1}^n f_{j,k_j}\}_{(k_1,\dots,k_n)\in\mathbb{N}^n}$, this is a basis of the K-vector space $K[X_1, \dots, X_n]$. Let $h \in \mathrm{Int}(\underline{E}, D)$ and write $h(X_1, \dots, X_n) = \sum_{\underline{k}=(k_1,\dots,k_n)\in\mathbb{N}^n} c_{\underline{k}}\, f_{1,k_1}(X_1) \cdots f_{n,k_n}(X_n)$ with $c_{\underline{k}}$'s belonging to K. For simplicity we prove that the $c_{\underline{k}}$'s, that are uniquely determined, belong to D in the case $n = 2$.

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Thus, with obvious notation, let $h(X,Y) = \sum_{k,l} c_{k,l} f_k(X) g_l(Y) \in \mathrm{Int}(E \times F, D)$. For every element $a \in E$, $h(a,Y) = \sum_{k,l} c_{k,l} f_k(a) g_l(Y) \in \mathrm{Int}(F,Y)$, and hence, for each l, $\sum_k c_{k,l} f_k(a) \in D$, that is, $\sum_k c_{k,l} f_k(X) \in \mathrm{Int}(E,D)$. Consequently, $c_{k,l} \in D$ for all k and l. □

To study the case of subsets $\underline{E}$ of D^n that are not of the previous form, we consider a total order on the monomials of $K[X_1,\dots,X_n]$, for instance, the lexicographic order on $\mathbb{N}^n$, that is, $\underline{k} < \underline{h}$ if and only if there exists a smallest j such that $k_j \neq h_j$ and, for this j, $k_j < h_j$.

Notation. For every D-algebra $\mathbb{B}$ such that $D[X_1,\dots,X_n] \subseteq \mathbb{B} \subseteq \mathrm{Int}(D^n,D)$ and every $\underline{k} = (k_1,\dots,k_n) \in \mathbb{N}^n$, we denote by $\mathfrak{J}_{\underline{k}}(\mathbb{B})$ the fractional ideal formed by the leading coefficients, with respect to the lexicographic order, of the polynomials of $\mathbb{B}$ of multi-degree $\underline{k}$.

For $\mathbb{B} = \mathrm{Int}(D^n,D)$, we write $\mathfrak{J}_{\underline{k}}$ instead of $\mathfrak{J}_{\underline{k}}(\mathrm{Int}(D^n,D))$.

Lemma 3.2. *Let $\mathbb{B}$ be a D-algebra such that $D[X_1,\dots,X_n] \subseteq \mathbb{B} \subseteq K[X_1,\dots,X_n]$. The D-module $\mathbb{B}$ admits a regular basis if and only if $\mathfrak{J}_{\underline{k}}(\mathbb{B})$ is a principal fractional ideal of D for all $\underline{k} = \{k_1,\dots,k_n\} \in \mathbb{N}^n$.*

Proof. The necessary condition is obvious. Let us prove that it is enough. For each $\underline{k} = (k_1,\dots,k_n) \in \mathbb{N}^n$, let $f_{\underline{k}} \in \mathbb{B}$ whose leading coefficient generates the principal fractional ideal $\mathfrak{J}_{\underline{k}}(\mathbb{B})$. The $f_{\underline{k}}$'s form a basis of $K[X_1,\dots,X_n]$. Thus, if $h \in \mathbb{B}$, we may write $h = \sum_{\underline{k}} c_{\underline{k}} f_{\underline{k}}$ where the $c_{\underline{k}}$'s $\in K$ are uniquely determined. Let $\underline{k}_0$ be the largest multi-degree such that $c_{\underline{k}_0} \neq 0$. By definition of $\mathfrak{J}_{\underline{k}_0}(\mathbb{B})$, $c_{\underline{k}_0} \in D$. Hence, $c_{\underline{k}_0} f_{\underline{k}_0} \in \mathbb{B}$ and $f_1 = f - c_{\underline{k}_0} f_{\underline{k}_0} \in \mathbb{B}$ also. Then, $\underline{\deg} f_1 < \underline{\deg} f$. If $f_1 \neq 0$, we consider the largest multi-degree of f_1 such that $c_{\underline{k}_1} \neq 0$. And so on, until we obtain 0. Then, we have proved that all the $c_{\underline{k}}$'s are in D. □

The analog of Theorem 1.2 becomes:

Theorem 3.3. *If D is a local domain whose maximal ideal is principal generated by π with finite residue field of cardinality q, then*

1. *$\mathrm{Int}(D^n,D)$ admits a regular basis. More precisely, for every $\underline{k} = \{k_1,\dots,k_n\}$, the ideal $\mathfrak{J}_{\underline{k}}$ is equal to $\prod_{j=1}^n \mathfrak{J}_{k_j} = \pi^{-\sum_{j=1}^n w_q(k_j)} D$, and hence, is principal.*
2. *Every D-algebra $\mathbb{B}$ such that $D[X_1,\dots,X_n] \subseteq \mathbb{B} \subseteq \mathrm{Int}(D^n,D)$ admits a regular basis. More precisely, for every $\underline{k} = \{k_1,\dots,k_n\}$, the ideal $\mathfrak{J}_{\underline{k}}(\mathbb{B})$ is of the form $\pi^{-\sum_{j=1}^n s_j} D$ where $0 \leq s_j \leq w_q(k_j)$, and hence, is principal.*

Proof. The first assertion follow from Lemmas 1.1 and 3.1. For the second assertion note first that $D \subseteq \mathfrak{J}_{\underline{k}}(\mathbb{B}) \subseteq \mathfrak{J}_{\underline{k}}$, and hence, the last sentence of Theorem 3.3 follows from the proof of Theorem 1.2. We may end with Lemma 3.2. □

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In the case where the residue field of the maximal ideal $\mathfrak{m}$ of a local domain D is infinite, we use the following result.

Proposition 3.4 ([5, Proposition XI.1.10])**.** *If* $\mathrm{Int}(D)$ *is trivial, then* $\mathrm{Int}(D^n, D)$ *is trivial for every* n*.*

Corollary 3.5. *If D is a local domain whose maximal ideal is principal, then every D-algebra $\mathbb{B}$ such that $D[X_1, \ldots, X_n] \subseteq \mathbb{B} \subseteq \mathrm{Int}(D^n, D)$ admits a regular basis.*

Proof. This is a consequence of Theorem 3.3 and Proposition 3.4. □

Corollary 3.6. *Let V be a valuation domain. Then every V-algebra $\mathbb{B}$ such that $V[X_1, \ldots, X_n] \subseteq \mathbb{B} \subseteq \mathrm{Int}(V^n, V))$ admits a regular basis.*

Proof. This is a consequence of Corollary 3.5 if the maximal ideal of V is principal. If not, we know that $\mathrm{Int}(D) = D[X]$, and hence, by Proposition 3.4, $\mathrm{Int}(D^n, D) = D[X_1, \ldots, X_n]$. Thus, for each maximal ideal $\mathfrak{m}$ of D, we have:

$$D_\mathfrak{m}[X_1, \ldots, X_n] \subseteq \mathbb{B} \subseteq \mathrm{Int}(D^n, D)_\mathfrak{m} \subseteq \mathrm{Int}(D^n, D_\mathfrak{m}) = D_\mathfrak{m}[X_1, \ldots, X_n]. \qquad \square$$

Thanks to Lemma 2.4, Corollary 3.6 leads by globalization to the following.

Theorem 3.7. *Let D be a locally essential domain. Then every D-algebra $\mathbb{B}$ such that $D[X_1, \ldots, X_n] \subseteq \mathbb{B} \subseteq \mathrm{Int}(D^n, D)$ has locally a regular basis, and hence, is locally free.*

Remark 3.8. For any set of variables $\underline{X}$, let:

$$\mathrm{Int}(D^{\underline{X}}, D) := \{f \in K[\underline{X}] \mid f(D^{\underline{X}}) \subseteq D\}.$$

As any fixed polynomial contains only finitely many variables, we can write:

$$\mathrm{Int}(D^{\underline{X}}, D) = \bigcup_{\substack{\underline{Y} \subseteq \underline{X} \\ \underline{Y} \text{ finite}}} \mathrm{Int}(D^{\underline{Y}}, D) \quad \text{(See [10, Lemma 2.3]).}$$

Thus, assertions obtained for finitely many variables may be easily extended to infinitely many. For instance, for any set $\underline{X}$ of variables, if D is a local domain whose maximal ideal is principal, $\mathrm{Int}(D^{\underline{X}}, D)$ admits a regular basis.

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