EXISTENCE AND CONTINUITY RESULTS FOR A NONLINEAR FRACTIONAL LANGEVIN EQUATION WITH A WEAKLY SINGULAR SOURCE

NGUYEN MINH DIEN

Abstract. We study a nonlinear Langevin equation involving a Caputo fractional derivatives of a function with respect to another function in a Banach space. Unlike previous papers, we assume the source function having a singularity. Under a regularity assumption of solution of the problem, we show that the problem can be transformed to a Volterra integral equation with two parameters Mittag-Leffler function in the kernel. Based on the obtained Volterra integral equation, we investigate the existence and uniqueness of the mild solution of the problem. Moreover, we show that the mild solution of the problem is dependent continuously on the inputs: initial data, fractional orders, appropriate function, and friction constant. Meanwhile, a new Henry-Gronwall type inequality is established to prove the main results of the paper. Examples illustrating our results are also presented.

1. Introduction

1.1. History and motivations. In recent decades, the subject of fractional calculus has emerged as a powerful and efficient mathematical instrument and its applications have found in fields science and engineering such as physics, chemistry, electrical engineering, quantum mechanics, the fluid-dynamics traffic model, electro-dynamics, pollution control, turbulence, etc, we refer to [9, 20, 21, 28] and the references therein. There are various definitions for fractional derivatives such as Riemann-Liouville, Caputo, Caputo-Katugampola, Caputo-Hadamard, Riesz. The derivatives of a function with respect to another function were introduced in the pioneering work of Kilbas et al [18]. From this idea, Almeida [3] presented a Caputo fractional derivative of a function with respect to another function that concept is generalized from the well-known mentioned fractional derivatives.

The Langevin equation was proposed to study problems related to Brownian motion, the random movement of a particle in a fluid due to collisions with the molecules of the fluid [19, 23, 34]. Besides, the Langevin equation has been used extensively in many different application areas, such as modelling the evacuation processes, the stock market, the fluid-dynamics traffic, protein dynamics, physical chemistry and electrical engineering [7, 9, 16, 25, 36]. But, the classical Langevin equation does not appropriate for some complex systems. There are many methods that were proposed to overcome this issue, one of them is replacing the ordinary derivative in the Langevin equation by a fractional derivative. The obtained equation is called the fractional Langevin equation.

2010 Mathematics Subject Classification. 34A08; 26A33; 34A12.
Key words and phrases. Langevin equation; fractional derivatives; weakly singular source; existence and uniqueness; continuity.
In the literature, there are many authors studied the nonlinear fractional Langevin equations with variety of boundary conditions, e.g. [1, 4, 5, 6, 11, 12, 13, 14, 22, 26, 29, 33, 37, 38]. In the mentioned papers, to obtain the existence of solution of problems, they used various methods such as contraction mapping, and Schauder fixed point theorem, or variational methods, etc. Usually, they assume that the source function satisfies some given conditions. In order to compare our work with the previous ones, let us define a class of weakly singular functions. Specifically, let $\rho > 0$, and $L_1, L_2$ are positive functions on $[a, b]$, and let $G : [a, b] \times \mathcal{B} \to \mathcal{B}$. The function $f(t, \chi, u)$ is called weakly singular non-Lipschitz if

$$
(1.1) \quad \| f(t, \chi, u) \| \leq L_1(t)(\chi(t) - \chi(a))^{-\rho}\|G(t, u)\| \quad \text{for any } u, v \in \mathcal{B}, \ t \in (a, b],
$$

and weakly singular Lipschitz if

$$
(1.2) \quad \| f(t, \chi, u) - f(t, \chi, v) \| \leq L_2(t)(\chi(t) - \chi(a))^{-\rho}\|u - v\| \quad \text{for any } u, v \in \mathcal{B}, \ t \in (a, b].
$$

When the function $f$ satisfies (1.1) or (1.2) with $\rho = 0$, we say that $f$ is a non-Lipschitz or Lipschitz function, respectively.

In the above mentioned papers, authors always assumed that $f$ is a non-Lipschitz or Lipschitz function. Furthermore, they also assumed that the source function $f$ satisfies some restrictions. For instance, Yu al et [37] and Baghani [4] considered the problem (1.4) in the case $\chi(t) = t, a = 0, b = 1$, and assumed that $L_1 = \sup_{t \in [0, 1]} \left( \int_0^t (t-s)^{\alpha+\beta-1} L_1(s) \, ds + \frac{|\lambda|}{\Gamma(\alpha+1)} \right) < 1$, or $L_2 = \sup_{t \in [0, 1]} \left( \int_0^t (t-s)^{\alpha+\beta-1} L_2(s) \, ds + \frac{|\lambda|}{\Gamma(\alpha+1)} \right) < 1$. Hence, if $|\lambda| \geq \Gamma(\alpha + 1)$ or $L_1, L_2$ large enough then we can not discuss the existence and uniqueness of solution of the problem. Fazli et al [14] have tried to relax these restrictions by transforming the problem to Volterra equation with Mittag-Leffler function in the kernel.

Recently, some authors have studied problems related to weakly singular sources. Indeed, Sin et al [27] considered the existence solution of the problem

$$
(1.3) \quad ^C D_t^\alpha u = f(t, u), \ u^{(i)}(0) = \mu_i, \ (i = 1, 2, \ldots, |\alpha| - 1),
$$

where $^C D_t^\alpha$ is the Caputo fractional derivatives. The authors assumed that the source function of the problem (1.3) satisfies some weakly conditions as follows

$$
|f(t, u)| \leq \begin{cases} A + Bt^{-\kappa}|u|^q \quad \text{for any } t \in (0, L_1], \ u \in \mathbb{R}, \\ C + Du^p \quad \text{for any } t \in (L_1, L_2], \ u \in \mathbb{R}, \end{cases}
$$

where $p, q \in (0, 1)$ and $\kappa \in (0, \min\{1, \alpha\})$, or

$$
|f(t, u) - f(t, v)| \leq \begin{cases} At^{-\kappa}|u - v| \quad \text{for any } t \in (0, L_1], \ u \in \mathbb{R}, \\ B|u - v| \quad \text{for any } t \in (0, L_1], \ u \in \mathbb{R}, \end{cases}
$$

where $\kappa \in (0, \min\{1, \alpha\})$. In 2019, Webb [35] consider existence results for the following initial fractional differential equation

$$
^C D_t^\alpha u = t^{-\kappa}f(t, u), \ u(0) = \mu,
$$

where $\alpha \in (0, 1]$ and $\kappa < \alpha$. Very recently, in [10], we consider the nonlinear fractional diffusion equations with Nagumo-type source. However, we can not find any papers dealt with fractional nonlinear Langevin equations having weakly singular sources. Besides, the Langevin equation involving Caputo fractional derivatives of
EXISTENCE AND CONTINUITY RESULTS

a function with respect to another function is still under consideration. This is the first motivation of the paper.

In the real world of applications, the inputs: initial data, fractional orders, appropriate function, and friction constant can be determined experimentally, or from the mathematical models (see in [2, 8]). Therefore, the inputs are obtained approximate values. Due to this issue, the continuity of solution of the problem with respect to inputs is worth considering. To the best of our knowledge, the related papers on this topic are quite rare, readers can see in [10, 30, 32]. Moreover, we can’t find any paper dealt with the continuity of solution of nonlinear Langevin fractional equation with respect to inputs. In the current paper, we would like to fill a part of this gap. Here is the second motivation of our paper.

1.2. Mathematical statements. Let \((\mathfrak{B}, \|\|)\) be a Banach space, and let \(\alpha, \beta \in (0, 1]\). Let \(\lambda, a, b \in \mathbb{R}\) with \(a < b\). We consider the following Langevin equation

\[
C^{\chi, \alpha}_{a^+} (C^{\chi, \beta}_{a^+} + \lambda) u(t) = f(t, \chi, u(t)), \quad a < t \leq b
\]

subject to the initial conditions

\[
u(a) = \mu, \quad C^{\chi, \alpha}_{a^+} u(a) = \eta,
\]

where \(C^{\chi, \alpha}_{a^+}, C^{\chi, \beta}_{a^+}\) are fractional derivatives of a function with respect to a function \(\chi\) (see definitions in section 2).

Note that if \(\alpha = 1, 0 < \beta \leq 1,\) and \(\chi(t) = t,\) we obtain the Langevin equation with friction memory kernel \(\lambda t^{\beta-1}/\Gamma(1 - \beta).\) According to [17, section B], in this case, the resulting motion is in fact subdiffusive. Moreover, it is worth noting that there is no physical meaning for our problem if \(\alpha + \beta > 2.\) For this reason, in this work, we concentrate on the case \(\alpha, \beta \in (0, 1].\)

Here we study the problem related to a weakly singular source function and raise the following questions. Under what conditions does the problem have at least one solution/a unique solution \(u \in C([a, b], \mathfrak{B})?\) Does the solution of the problem depend continuously on the inputs: initial data, fractional orders, appropriate function, and friction constant?

1.3. Outline of the paper. Summarizing the above discussions, in this work, under a regularity assumption of solution of the problem (1.4) and (1.5), we transform the problem to a Volterra integral equation with two parameters Mittag-Leffler function in the kernel. Besides, we also establish a new Henry-Gronwall type inequality. Using the results just mentioned, we will prove the following results.

- If the source function satisfies the weakly singular non-Lipschitz condition then the problem has at least one mild solution.
- If the source function satisfies the weakly singular Lipschitz condition then the problem has a unique mild solution.
- The mild solution of the problem is dependent continuously on the inputs: initial data, fractional orders, appropriate function, and friction constant.

The current paper is structured as follows. In section 2, we present some notations, definitions, and some preliminary results. Section 3 presents the main results of the paper. Section 4 is devoted to examples illustrating the theoretical findings. Finally, conclusions are given in section 5.
2. Mathematical preliminaries and notations

In this section we set up some notations, definitions, and introduce Lemmas which use throughout the current paper.

Let us denote \( D = [a, b], \mathbb{R}^+ = \{ x : x \geq 0 \} \), and define a class of function

\[
H^+(D) = \{ \chi : \chi \in C^1(D) \text{ and } \chi'(t) > 0 \text{ for all } t \in D \}.
\]

For \( \varphi \in C(D), w \in C(D; \mathbb{B}) \), we denote \( \| \varphi \|_4 = \sup_{a \leq t \leq b} |\varphi(t)| \) and \( \| w \| = \sup_{t \in D} \| w(t) \| \). For \( a < b \), we define

\[
\Delta = \{ (\tau, t) : a \leq \tau < t \leq b \}.
\]

Let us recall the Mittag-Leffler function defined by

\[
E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + \beta)}, \quad \alpha, \beta > 0, \quad z \in \mathbb{C},
\]

where \( \Gamma(\cdot) \) is the Gamma function. For \( \beta = 1 \), we also denote \( E_{\alpha,1}(z) = E_{\alpha}(z) \).

We can list here some properties of the Mittag-Leffler function

**Lemma 2.1.** Let \( 0 < \alpha_* < \alpha^* < 2, \alpha^* < 2\alpha_*, \) and \( 0 < \beta_* < \beta^* \), and let \( \alpha, \alpha \in [\alpha_*, \alpha^*], \) and \( \beta, \beta \in [\beta_*, \beta^*] \). For any \( z_0 \in \mathbb{R} \) there exists a positive constant \( C = C(\alpha_*, \alpha^*, \beta_*, \beta^*) \) such that

(i). \( |E_{\alpha,\beta}(z)| + \left| \frac{\partial E_{\alpha,\beta}}{\partial z}(z) \right| + \left| \frac{\partial^2 E_{\alpha,\beta}}{\partial z^2}(z) \right| \leq \frac{C}{1+|z|} \) for all \( z \leq z_0 \).

(ii). \( |E_{\alpha,\beta}(z_1) - E_{\alpha,\beta}(z_2)| \leq C|z_1 - z_2| \) for every \( z_1, z_2 \in (-\infty, z_0] \).

(iii). \( |E_{\alpha,\beta}(z) - E_{\alpha,\beta}(z)| \leq C \left( |\alpha - \alpha| + |\beta - \beta| \right) \) for all \( z \leq z_0 \).

(iv). \( \left( \frac{d}{dz} \right)^m \left[ z^{\beta-1} E_{\alpha,\beta}(z^\lambda) \right] = z^{\beta-1-m} E_{\alpha,\beta-m}(z^\lambda) \) for \( m \geq 1 \). Consequently,

\[
\int_0^z \tau^{\beta-1} E_{\alpha,\beta}(\lambda \tau^\lambda) \, d\tau = z^{\beta} E_{\alpha,\beta+1}(\lambda z^\lambda)
\]

for any \( \lambda \in \mathbb{R} \).

**Proof.** Readers can find the proofs of part (i) and part (ii) in [31], and part (iv) in [18, page 58]. Using part (i) together with the mean value theorem, we obtain the desired result of part (iii). \( \square \)

Throughout the current paper, for \( \alpha > 0, \kappa \in \mathbb{R}, \) and \( (\tau, t) \in \Delta \), we denote

\[
K_{\alpha,\kappa}(\tau, t, \chi) = \chi'(\tau)(\chi(t) - \chi(\tau))^\kappa - 1(\chi(\tau) - \chi(\alpha))^{-\kappa}.
\]

For \( \kappa = 0 \), we also denote \( K_{\alpha,0}(\tau, t, \chi) = K_{\alpha}(\tau, t, \chi) \).

Now, we introduce the definitions of fractional derivatives and fractional integrals of a function \( f \) with respect to function \( \chi \).

**Definition 2.2** (see [3, 18]). Let \( \alpha > 0, a < b, \chi \in H^+(D) \), and \( f \in L^1(D) \). The fractional integral of a function \( f \) with respect to function \( \chi \) is defined by

\[
I_{a}^{\alpha,\chi} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \chi'(\tau)(\chi(t) - \chi(\tau))^{\alpha-1} f(\tau) \, d\tau = \frac{1}{\Gamma(\alpha)} \int_a^t K_{\alpha}(\tau, t, \chi) f(\tau) \, d\tau,
\]

where \( \Gamma(\cdot) \) is the Gamma function.
Definition 2.3 (\(\chi\)-Caputo fractional derivative). Let \(\alpha > 0, \chi \in H^+(\mathcal{D})\), and \(f \in C^n(\mathcal{D})\). The left Caputo fractional derivative of a function \(f\) with respect to function \(\chi\) is defined as in [3]

\[
C^D_{a+}^{\chi,\alpha}f(t) = I_{a+}^{n-\alpha,\chi} \left( \frac{1}{\chi'(t)} \frac{d}{dt} \right)^n f(t),
\]

where \(n = \lfloor \alpha \rfloor + 1\) for \(n \neq \mathbb{N}\) and \(n = \alpha\) for \(\alpha \in \mathbb{N}\).

Below, we list some properties of fractional integrals and \(\chi\)-Caputo fractional derivatives.

Lemma 2.4 (see [3, 18]). Let \(n \in \mathbb{N}^+, \alpha \in (n-1,n],\) and \(\chi \in H^+(\mathcal{D})\). We have
(i). If \(g,h \in C^n(\mathcal{D})\), then \(C^D_{a+}^{\chi,\alpha}g(t) = C^D_{a+}^{\chi,\alpha}h(t)\) if and only if \(g(t) = h(t) + \sum_{k=0}^{n-1} c_k (\chi(t) - \chi(a))^k\).
(ii). If \(g \in C^1(\mathcal{D})\), then \(C^D_{a+}^{\chi,\alpha}I_{a+}^{\chi}g(t) = g(t)\).
(iii). \(I_{a+}^{\chi,\alpha}(\chi(t) - \chi(a)) = \frac{\Gamma(\sigma+1)}{\Gamma(\alpha+\sigma+1)} (\chi(t) - \chi(a))^\alpha\) for any \(\sigma \geq 0\).
(iv). \(I_{a+}^{\chi,\alpha}I_{a+}^{\beta,\chi}g(t) = I_{a+}^{\alpha+\beta,\chi}g(t)\) for any \(\beta > 0\).

For the convenience, we denote
\[
\Phi_{a,\kappa}(\zeta,\xi,t,\chi) = \int_{\xi}^{\zeta} \chi'(\tau)(\chi(t) - \chi(\tau))^{\alpha-1}(\chi(\tau) - \chi(a))^{-\kappa} \, d\tau = \int_{\zeta}^{\xi} K_{a,\kappa}(\tau,t,\chi) \, d\tau
\]
for any \(a \leq \zeta < \xi \leq t \leq b\) and \(\chi \in H^+(\mathcal{D})\).

Lemma 2.5. Let \(a,b \in \mathbb{R}, b > a, \alpha > 0, \kappa < \min\{1, \alpha\}\), and let \(\chi \in H^+(\mathcal{D})\). For \(a \leq \zeta \leq \xi \leq t \leq b\), we have
\[
\Phi_{a,\kappa}(\zeta,\xi,t,\chi) = (\chi(t) - \chi(a))^{\alpha-\kappa} \int_{\ell(\zeta)}^{\ell(\xi)} (1 - \tau)^{\alpha-1-\kappa} \, d\tau,
\]
where \(\ell(s) = (\chi(s) - \chi(a))/(\chi(t) - \chi(a))\). Consequently,
\[
\Phi_{a,\kappa}(a,t,t,\chi) = (\chi(t) - \chi(a))^{\alpha-\kappa} B(\alpha,1-\kappa) \text{ for any } t \in \mathcal{D},
\]
where \(B(\cdot, \cdot)\) is the Beta function.

Proof. The proof of Lemma is very easy by change the variable of integration from \(\tau\) to \(t\) where \(\ell(\tau) = (\chi(\tau) - \chi(a))/(\chi(t) - \chi(a))\). So we omit it here. \(\square\)

Remark 2.6. From Lemma 2.5, we can verify that
(i). \(\Phi_{a,\kappa}(a,t,t,\chi) \leq \Phi_{a,\kappa}(a,b,b,\chi)\) for any \(t \in \mathcal{D}\).
(ii). \(\Phi_{a,\kappa}(\zeta,\xi,t,\chi) \to 0\) as \(|\zeta - \xi| \to 0\).
(iii). \(\Phi_{a,\kappa}(a,\zeta,\xi,\chi) \to 0\) as \(\zeta \to \zeta\).

Let \(\chi \in H^+(\mathcal{D}), \lambda \in \mathbb{R}, \) and \((\tau,t) \in \Delta\). We define
\[
C_{a,\beta}(t,\tau,\lambda,\chi) = \chi'(\tau)(\chi(t) - \chi(\tau))^{\alpha+\beta-1} E_{a,\alpha+\beta} (-\lambda(\chi(t) - \chi(\tau))^\alpha).
\]
For \(\beta = 0\), we denote \(C_{a,0}(t,\tau,\lambda,\chi) = C_{a}(t,\tau,\lambda,\chi)\). Let \(0 < \alpha_s < \alpha_s < 2\alpha_s \leq 2, 0 < \beta_s < \beta_s \leq 1, \) and \(\lambda_s < \lambda_s\). We define
\[
W = \{(\alpha, \beta, \lambda) \in \mathbb{R}^3 : \alpha_s \leq \alpha < \alpha_s, \beta_s \leq \beta \leq \beta_s, \lambda_s \leq \lambda \leq \lambda_s\}.
\]
Using the above notations, we state and prove the following lemmas.
Lemma 2.7. Let \((\alpha, \beta, \lambda), (\alpha_i, \beta_i, \lambda_i) \in W, \) and \(\chi, \chi_k \in H^+(D)\) for any \(k \in \mathbb{N}\). Suppose that \((\alpha_i, \beta_i, \lambda_i) \to (\alpha, \beta, \lambda), \|\chi_k - \chi\| \to 0\) and \(\|\chi_k' - \chi'\|_{L^1(D)} \to 0\), then we have

\[
\int_a^t |\mathcal{C}_{\alpha, \beta}(t, \tau, \lambda, \chi) - \mathcal{C}_{\alpha, \beta}(t, \tau, \lambda_k, \chi_k)| \, d\tau \to 0 \quad \text{as} \quad k \to \infty.
\]

Proof. For \((\tau, t) \in \Delta, \) using Lemma 2.1 and by directly computes, we have

\[
\mathcal{C}_{\alpha, \beta}(t, \tau, \lambda, \chi) - \mathcal{C}_{\alpha, \beta_i}(t, \tau, \lambda_i, \chi_i) \leq \lambda_i'(\tau)(\chi_i(t) - \chi_i(\tau))^{\alpha_i + \beta_i - 1} H_i
\]

\[
+ C|\lambda_i'(\tau)(\chi_i(t) - \chi_i(\tau))^{\alpha_i + \beta_i - 1} - \lambda'(\tau)(\chi(t) - \chi(\tau))^{\alpha + \beta - 1}|,
\]

where

\[
H_i = \sup_{a \leq t \leq b} |E_{\alpha, \alpha + \beta} (-\lambda(\chi(t) - \chi(a))^\alpha) - E_{\alpha_i, \alpha_i + \beta_i} (-\lambda_i(\chi_i(t) - \chi_i(a))^{\alpha_i})|,
\]

and \(C\) defined in Lemma 2.1. The latter inequality gives

\[
\int_a^t |\mathcal{C}_{\alpha, \beta}(t, \tau, \lambda, \chi) - \mathcal{C}_{\alpha, \beta_i}(t, \tau, \lambda_i, \chi_i)| \, d\tau \leq H_{i,1}(t) + CH_{i,2}(t),
\]

where

\[
H_{i,1}(t) = \Phi_{\alpha_i + \beta_i, 0}(a, t, t, \chi_i) H_i,
\]

\[
H_{i,2}(t) = \int_a^t |\chi_i'(\tau)(\chi_i(t) - \chi_i(\tau))^{\alpha_i + \beta_i - 1} - \chi'(\tau)(\chi(t) - \chi(\tau))^{\alpha + \beta - 1}| \, d\tau.
\]

Continuously, we find some estimations for \(H_{i,1}, H_{i,2}.\)

Firstly, we find an estimation for \(H_{i,1}.\) Using part (ii) and part (iii) of Lemma 2.1, we can easily verify that \(\lim_{i \to \infty} H_i = 0.\) This leads to

\[
H_{i,1}(t) = \Phi_{\alpha_i + \beta_i, 0}(a, t, t, \chi_i) H_i \leq \Phi_{\alpha_i + \beta_i, 0}(a, b, b, \chi_i) H_i \to 0 \quad \text{as} \quad i \to \infty.
\]

Secondly, we find an estimation for \(H_{i,2}.\) For \(\epsilon \in (0, b - a),\) we consider two cases

For \(a \leq t \leq a + \epsilon, \) using the fact that \(|x - y| \leq x + y\) for any \(x, y \geq 0,\) we get

\[
H_{i,2}(t) = \int_a^t |\chi_i'(\tau)(\chi_i(t) - \chi_i(\tau))^{\alpha_i + \beta_i - 1} - \chi'(\tau)(\chi(t) - \chi(\tau))^{\alpha + \beta - 1}| \, d\tau
\]

\[
\leq \Phi_{\alpha + \beta, \alpha_i + \beta_i, 0}(a, t, t, \chi_i)
\]

\[
\leq \Phi_{\alpha_i + \beta_i, \alpha + \beta, \alpha_i + \beta_i}(a, b, b, \chi_i) := L_i(\epsilon) \to 0
\]

as \(i \to \infty\) for any \(a \leq t \leq a + \epsilon.\)

For \(a + \epsilon < t \leq b: \) we find that

\[
\int_a^t |\chi'(\tau)(\chi(t) - \chi(\tau))^{\alpha + \beta - 1} - \chi_i'(\tau)(\chi_i(t) - \chi_i(\tau))^{\alpha_i + \beta_i - 1}| \, d\tau
\]

\[
\leq J_i(t, \epsilon) + J_{i,1}(t, \epsilon) + J_{i,2}(t, \epsilon) + J_{i,3}(t, \epsilon),
\]
where

\[ J_i(t, \epsilon) = \int_{t-\epsilon}^{t} |\chi'(\tau) (\chi(t) - \chi(\tau))^{\alpha+\beta-1} - \chi'_i(\tau)(\chi_i(t) - \chi_i(\tau))^{\alpha_i+\beta_i-1}| \, d\tau \]

\[ J_{1,i}(t, \epsilon) = \int_{a}^{t-\epsilon} \chi'(\tau) |(\chi(t) - \chi(\tau))^{\alpha+\beta-1} - (\chi_i(t) - \chi_i(\tau))^{\alpha_i+\beta_i-1}| \, d\tau, \]

\[ J_{2,i}(t, \epsilon) = \int_{a}^{t-\epsilon} \chi'(\tau) - \chi'_i(\tau)((\chi(t) - \chi(\tau))^{\alpha+\beta-1} \, d\tau, \]

\[ J_{3,i}(t, \epsilon) = \int_{a}^{t-\epsilon} \chi'_i(\tau)|(\chi(t) - \chi_i(\tau))^{\alpha+\beta-1} - (\chi_i(t) - \chi_i(\tau))^{\alpha_i+\beta_i-1}| \, d\tau. \]

From Lemma 2.5, one has

\[ J_i(t, \epsilon) \leq \Phi_{\alpha+\beta,0}(t - \epsilon, t, \chi) + \Phi_{\alpha_i+\beta_i,0}(t - \epsilon, t, \chi_i) := A_i(t, \epsilon). \]

Here we note that \( \lim_{t \to \infty} A_i(t, \epsilon) = 2\Phi_{\alpha+\beta,0}(t - \epsilon, t, \chi) \) and \( \lim_{\epsilon \to 0} \Phi_{\alpha+\beta,0}(t - \epsilon, t, \chi) = 0 \) uniform for \( t \in D \) for any \( i \in \mathbb{N} \). On the other hand, by elementary computations, we have

\[ J_{1,i}(t, \epsilon) \leq \|\chi'\|_{L^1(D)} \sup_{a \leq \tau \leq t - \epsilon} \left| (\chi(t) - \chi(\tau))^{\alpha+\beta-1} - (\chi_i(t) - \chi_i(\tau))^{\alpha_i+\beta_i-1} \right| \to 0, \quad \text{as} \quad i \to \infty, \]

\[ J_{2,i}(t, \epsilon) \leq \|\chi'\|_{L^1(D)} \sup_{a \leq \tau \leq t - \epsilon} (\chi_i(t) - \chi_i(\tau))^{\alpha_i+\beta_i-1} \to 0, \quad \text{as} \quad i \to \infty, \]

\[ J_{3,i}(t, \epsilon) \leq \|\chi'\|_{L^1(D)} \sup_{a \leq \tau \leq t - \epsilon} \left| (\chi(t) - \chi_i(\tau))^{\alpha+\beta-1} - (\chi_i(t) - \chi_i(\tau))^{\alpha_i+\beta_i-1} \right| \to 0, \quad \text{as} \quad i \to \infty. \]

Using (2.6), (2.7), (2.8), and (2.9) into (2.5), we conclude that

\[ H_{1,2}(t) = \int_{a}^{t} |\chi'(\tau) (\chi(t) - \chi(\tau))^{\alpha+\beta-1} - \chi'_i(\tau)(\chi_i(t) - \chi_i(\tau))^{\alpha_i+\beta_i-1}| \, d\tau \]

\[ \to 0 \quad \text{as} \quad i \to \infty \]

for any \( t \in [a + \epsilon, b] \). From (2.4) and (2.10), we obtain

\[ H_{1,2}(t) \to 0 \quad \text{as} \quad i \to \infty \quad \text{for any} \quad t \in [a, b]. \]

Substituting (2.3) and (2.11) into (2.2), we obtain the desired result. \( \square \)

**Lemma 2.8.** Let \( a < b \), \( (\alpha, \beta, \lambda) \in W, \) \( (\tau, t) \in \Delta, \) and let \( \chi \in H^+(D), \) \( w \in C(D; \mathcal{B}) \). We denote

\[ Qw(t) = \int_{a}^{t} \mathcal{E}_{\alpha,\beta}(t, \tau, \lambda, \chi) f(\tau, \chi, w(\tau)) \, d\tau. \]

Suppose that there exist \( L_R > 0 \) and \( \rho < \alpha + \beta \) such that

\[ \|f(\tau, \chi, w(\tau))\| \leq L_R(\chi(t) - \chi(\tau))^{-\rho} \quad \text{for any} \quad t \in (a, b) \quad \text{and} \quad \|w\| \leq R. \]

Then the function \( Qw(t) \) is equicontinuous in the closed ball \( B_R = \{w \in C(D; \mathcal{B}) : \|w\| \leq R\} \).
PROOF. For $a \leq \zeta < \xi \leq b$, we have

$$\|Q w(\zeta) - Q w(\xi)\| \leq I_1(\zeta, \xi) + I_2(\zeta, \xi),$$

where

$$I_1(\zeta, \xi) = \int_{\alpha}^{\xi} |C_{\alpha, \beta}(\xi, \tau, \lambda, \chi) - C_{\alpha, \beta}(\zeta, \tau, \lambda, \chi)||f(\tau, \chi, w(\tau))|| \, d\tau,$$

$$I_2(\zeta, \xi) = \int_{\xi}^{\lambda} C_{\alpha, \beta}(\zeta, \tau, \lambda, \chi)||f(\tau, \chi, w(\tau))|| \, d\tau.$$

We estimate for $I_1(\zeta, \xi)$ and $I_2(\zeta, \xi)$ one by one.

**Estimate for $I_1(\zeta, \xi)$**. We consider two cases:

**The first case**: $0 < \alpha + \beta \leq 1$. We use the fact that $|x^{\alpha + \beta} - y^{\alpha + \beta}| \leq |x - y|^{\alpha + \beta}$ for $x, y \geq 0$. Therefore, using Lemma 2.1 and direct computations, we have

$$|C_{\alpha, \beta}(\xi, \tau, \lambda, \chi) - C_{\alpha, \beta}(\zeta, \tau, \lambda, \chi)|$$

$$\leq C x'(\tau)((\chi(\zeta) - \chi(\xi))^{\alpha + \beta - 1} - (\chi(\xi) - \chi(\tau))^{\alpha + \beta - 1})$$

$$+ C \lambda x'(\tau)\chi(\zeta) - \chi(\tau))^{\alpha + \beta - 1}((\chi(\xi) - \chi(\tau))^{\alpha} - (\chi(\xi) - \chi(\tau))^{\alpha})$$

$$\leq C x'(\tau)((\chi(\xi) - \chi(\tau))^{\alpha + \beta - 1} - (\chi(\xi) - \chi(\tau))^{\alpha + \beta - 1})$$

$$+ C M x'(\tau)\chi(\xi) - \chi(\tau))^{\alpha + \beta - 1}((\chi(\xi) - \chi(\tau))^{\alpha} - (\chi(\xi) - \chi(\tau))^{\alpha})$$

where $C = C(\alpha, \alpha^*, \beta, \beta^*, \lambda, \lambda^*, \chi)$, and $M_{\lambda} = \max\{|\lambda|, |\lambda^*|\}$. The latter inequality yields

$$I_1(\zeta, \xi) \leq CLR(\Phi_{\alpha + \beta, \rho}(a, \zeta, \xi, \chi) - \Phi_{\alpha + \beta, \rho}(a, \xi, \xi, \chi))$$

$$+ CLR M_{\lambda}(\chi(\xi) - \chi(\zeta))^{\alpha + \beta - 1}\Phi(a, \xi, \xi, \chi)$$

Observe that $\Phi_{\alpha + \beta, \rho}(a, \zeta, \xi, \chi) = \Phi_{\alpha + \beta, \rho}(a, \xi, \xi, \chi) - \Phi_{\alpha + \beta, \rho}(a, \xi, \xi, \chi)$, hence, using Remark 2.6, we obtain

$$I_1(\zeta, \xi) \leq CLR(\Phi_{\alpha + \beta, \rho}(a, \zeta, \xi, \chi) - \Phi_{\alpha + \beta, \rho}(a, \xi, \xi, \chi) + CLR(\Phi_{\alpha + \beta, \rho}(a, \zeta, \xi, \chi)$$

$$+ CLR M_{\lambda}(\chi(\xi) - \chi(\zeta))^{\alpha + \beta - 1}\Phi(a, \xi, \xi, \chi) \to 0 \text{ uniform as } \zeta \to \xi.$$

**The second case**: $\alpha + \beta > 1$. We can use the fact that $|u^{\alpha + \beta} - v^{\alpha + \beta}| \leq (\alpha + \beta) \max\{|u^{\alpha + \beta - 1} - v^{\alpha + \beta - 1}| \}|u - v|$ for any $u, v > 0$. So similar to the first case, using Lemma 2.1, we have

$$|C_{\alpha, \beta}(\xi, \tau, \lambda, \chi) - C_{\alpha, \beta}(\zeta, \tau, \lambda, \chi)|$$

$$\leq C x'(\tau)((\chi(\zeta) - \chi(\xi))^{\alpha + \beta - 1} - (\chi(\xi) - \chi(\tau))^{\alpha + \beta - 1}) + L_1 \lambda x'(\tau)(\chi(\xi) - \chi(\zeta))^{\alpha},$$

where $C = C(\alpha, \alpha^*, \beta, \beta^*, \lambda, \lambda^*, \chi)$, and $L_1 = C M_{\lambda}(\chi(\xi) - \chi(\zeta))^{2(\alpha + \beta - 1)}$ with $M_{\lambda} = \max\{|\lambda|, |\lambda^*|\}$. This leads to

$$I_1(\zeta, \xi) \leq CLR(\Phi_{\alpha + \beta, \rho}(a, \zeta, \xi, \chi) - \Phi_{\alpha + \beta, \rho}(a, \xi, \xi, \chi) + CLR(\Phi_{\alpha + \beta, \rho}(a, \zeta, \xi, \chi)$$

$$= CLR(\Phi_{\alpha + \beta, \rho}(a, \xi, \xi, \chi) - \Phi_{\alpha + \beta, \rho}(a, \xi, \xi, \chi)) - CLR(\Phi_{\alpha + \beta, \rho}(a, \xi, \xi, \chi)$$

$$+ CLR M_{\lambda}(\chi(\xi) - \chi(\zeta))^{\alpha + \beta - 1}\Phi(a, \xi, \xi, \chi) \to 0 \text{ uniform as } \zeta \to \xi.$$

**Estimate for $I_2(\zeta, \xi)$**. By Lemma 2.1 and Remark 2.6, by direct computation, we have

$$I_2(\zeta, \xi) \leq CLR(\Phi_{\alpha + \beta, \rho}(\zeta, \xi, \xi, \chi) \to 0 \text{ uniform as } \zeta \to \xi.$$
Combining (2.12) with (2.13), (2.14), and (2.15), we obtain the desired result of Lemma.

Finally, we present a new Henry-Gronwall type inequality which plays a dominant role in the proof of the main results of the present paper.

**Lemma 2.9** (Henry-Gronwall type inequality). Let $0 < \alpha \leq 1$, $\kappa \in \{1, \alpha\}$, and $\chi \in H^+(\mathcal{D})$, and let $q$ be a continuous, positive and non-decreasing function. Suppose that $w$ is a continuous, non-negative function in $\mathcal{D}$ and satisfies the following integral inequality

$$w(t) \leq q(t) + \int_a^t K_{\alpha,\kappa}(\tau, t, \chi) L(t) w(\tau) \: d\tau \quad \text{for any } t \in \mathcal{D}.$$ 

Then, for any $0 < r < \min\{1, 1 - \kappa\}$ and $r \leq \alpha - \kappa$, we have

$$u(t) \leq 2^{1-r} q(t) \exp \left( M \int_a^t \chi(\tau) L^{1 \over r}(t) \: d\tau \right),$$

where $M = \frac{2^{1-r} B^{1-r} (\alpha - \kappa - r - 1)}{1 - r} (\chi(b) - \chi(a))^{a - \kappa - r}$. As a consequence, if $q(t) = 0$ for all $t \in \mathcal{D}$, then $u \equiv 0$ in $\mathcal{D}$.

**Proof.** Let us denote $\bar{K}_{x,y,z}(\tau, t, \chi) = (\chi'(\tau))^x (\chi(t) - \chi(\tau))^{y-1} (\chi(\tau) - \chi(a))^{-z}$. It is obvious that $K_{\alpha,\kappa}(\tau, t, \chi) = (\chi'(\tau))^\kappa \bar{K}_{1-r,\alpha,\kappa}(\tau, t, \chi)$. Therefore, applying the Holder inequality, we have

$$u(t) \leq q(t) + \int_a^t (\chi'(\tau))^\kappa \bar{K}_{1-r,\alpha,\kappa}(\tau, t, \chi) u(\tau) \: d\tau$$

$$\leq q(t) + \left( \int_a^t \left( \bar{K}_{1-r,\alpha,\kappa}(\tau, t, \chi) \right)^{1 \over 1-r} \: d\tau \right)^{1-r} \left( \int_a^t \chi'(\tau) L^{1 \over r}(t) u^{1 \over r}(\tau) \: d\tau \right)^r$$

$$= q(t) + \left( \int_a^t K_{\alpha,\kappa}(\tau, t, \chi) \: d\tau \right)^{1-r} \left( \int_a^t \chi'(\tau) L^{1 \over r}(t) u^{1 \over r}(\tau) \: d\tau \right)^r$$

due to $\bar{K}_{1-r,\alpha,\kappa}(\tau, t, \chi) = K_{\alpha,\kappa}(\tau, t, \chi)$. Using Lemma 2.5, we get

$$u(t) \leq q(t) + M(t) \left( \int_a^t \chi'(\tau) L^{1 \over r}(t) u^{1 \over r}(\tau) \: d\tau \right)^r,$$

where $M(t) = B^{1-r} (\alpha - \kappa - r - 1) (\chi(t) - \chi(a))^{a - \kappa - r}$. Using the fact that $(x + y)^p \leq 2^{p-1} (x^p + y^p)$ for any $x, y > 0$ and $p \geq 1$, we obtain

$$u^{1 \over r}(t) \leq 2^{1 \over r-1} \left( q^{1 \over r}(t) + M_1 \int_a^t \chi'(\tau) L^{1 \over r}(t) u^{1 \over r}(\tau) \: d\tau \right),$$

where $M_1 = \frac{2^{1-r} B^{1-r} (\alpha - \kappa - r - 1)}{1 - r} (\chi(b) - \chi(a))^{a - \kappa - r}$. Using the Gronwall inequality [24, p. 12], we obtain

$$u^{1 \over r}(t) \leq 2^{1-r} q^{1 \over r}(t) \exp \left( M_1 \int_a^t \chi'(\tau) L^{1 \over r}(t) \: d\tau \right).$$

The latter inequality leads to the desired result of Lemma.
3. Main results

This section presents the main results of the current paper. We investigate the existence of mild solution of the problem with weakly singular non-Lipschitz/Lipschitz source. Importantly, we prove that mild solution of the problem dependent continuously on the inputs: initial data, fractional orders, appropriate function, and friction constant. Let us begin with the following lemma.

Lemma 3.1. Let $0 < \alpha \leq 1$, $g \in C^1(D)$, and let $w \in C^1(D; \mathcal{B})$ be a solution of the following problem

\[
(CD_{a+}^{\chi,\alpha} + \lambda) w(t) = g(t), \quad t > a
\]

\[
w(a) = \eta
\]

then

\[
w(t) = \eta E_\alpha (\chi(t) - \chi(a)) + \int_a^t C_\alpha (t, \tau, \chi) g(\tau) \, d\tau.
\]

Proof. If $w \in C^1(D, \mathcal{B})$ then the problem (3.1) can be transformed to the integral equation

\[
w(t) = \mu - \lambda I_{a+}^{\chi,\alpha} w(t) + I_{a+}^{\chi,\alpha} g(t)
\]

We use the method of successive approximations to solve the above integral equation

\[
w_0(t) = \mu, \quad w_n(t) = w_0 - \lambda I_{a+}^{\chi,\alpha} w_{n-1}(t) + I_{a+}^{\chi,\alpha} g(t), \quad n \in \mathbb{N}.
\]

Using Lemma 2.5, we have

\[
w_1(t) = w_0 - \lambda I_{a+}^{\chi,\alpha} w_0(t) + I_{a+}^{\chi,\alpha} g(t) = \mu \sum_{n=0}^{1} \frac{(-\lambda)^n (\chi(t) - \chi(a))^{n\alpha}}{\Gamma(n\alpha + 1)} + I_{a+}^{\chi,\alpha} g(t)
\]

due to $I_{a+}^{\chi,\alpha} \lambda = \frac{\lambda (\chi(t) - \chi(a))^{\alpha}}{\Gamma(\alpha + 1)}$. Using Lemmas 2.4 and 2.5, by direct computation, we have

\[
w_2(t) = w_0 - \lambda I_{a+}^{\chi,\alpha} w_1(t) + I_{a+}^{\chi,\alpha} g(t)
\]

\[
= \mu \sum_{n=0}^{2} \frac{(-\lambda)^n (\chi(t) - \chi(a))^{n\alpha}}{\Gamma(n\alpha + 1)} + \sum_{n=0}^{1} \frac{(-\lambda)^n I_{a+}^{\chi,\alpha+n\alpha} g(t).}
\]

Continuously this process, we get

\[
w_k(t) = \mu \sum_{n=0}^{k} \frac{(-\lambda)^n (\chi(t) - \chi(a))^{n\alpha}}{\Gamma(n\alpha + 1)} + \sum_{n=0}^{k-1} \frac{(-\lambda)^n I_{a+}^{\chi,\alpha+n\alpha} g(t).}
\]

Letting $k \to \infty$, we obtain

\[
w(t) = \mu \sum_{n=0}^{\infty} \frac{(-\lambda)^n (\chi(t) - \chi(a))^{n\alpha}}{\Gamma(n\alpha + 1) + \int_a^t \chi(t) C_\alpha (t, \tau, \chi) \left( \sum_{n=0}^{\infty} \frac{(-\lambda)^n (\chi(t) - \chi(\tau))^{n\alpha}}{\Gamma(n\alpha + \alpha)} \right) g(\tau) \, d\tau
\]

\[
= \mu E_\alpha (\chi(t) - \chi(a)) + \int_a^t C_\alpha (t, \tau, \chi) g(\tau) \, d\tau
\]

due to $I_{a+}^{\chi,\alpha+n\alpha} g(t) = \frac{1}{\Gamma(n\alpha + \alpha)} \int_a^t K_\alpha (\tau, t, \chi) g(\tau) \, d\tau$ for any $n \in \mathbb{N}$. This completes the proof of Lemma. \qed
Now, we present a formulation for the problem (1.4) and (1.5).

**Lemma 3.2.** Let $\alpha, \beta \in (0, 1]$, and let $u$ be a solution of the problem (1.4) and (1.5). Suppose that $w = (C D^{\chi, \alpha}_{a+} + \lambda) u \in C^1(\mathcal{D})$ and $f(t, \chi, u) \in C^1(\mathcal{D})$. Then $u$ is a solution of the following Volterra integral equation

$$u(t) = \mu E_{\alpha} (-\lambda(\chi(t) - \chi(a))^\alpha) + (\lambda \mu + \eta)(\chi(t) - \chi(a))^\alpha E_{\alpha, \alpha+1} (-\lambda(\chi(t) - \chi(a))^\alpha)$$

$$+ \int_a^t \mathcal{C}_{\alpha, \beta}(t, \tau, \lambda, \chi) f(\tau, \chi, u(\tau)) \, d\tau,$$

(3.2)

where $\mathcal{C}_{\alpha, \beta}(t, \tau, \lambda, \chi)$ defined as in (2.1).

**Proof.** If $w = (C D^{\chi, \alpha}_{a+} + \lambda) u \in C^1(\mathcal{D})$, using Lemma 2.4, the equation can be transformed to the equation

$$(C D^{\chi, \alpha}_{a+} + \lambda) u(t) = c + I^{\chi, \beta}_a f(t, \chi, u(t)).$$

From the initial conditions (1.5), we get

$$c = \lambda \mu + \eta.$$

According to Lemma 3.1, we obtain

$$u(t) = \mu E_{\alpha} (-\lambda(\chi(t) - \chi(a))^\alpha)$$

$$+ \int_a^t \mathcal{C}_{\alpha}(t, \tau, \lambda, \chi) \left( (\lambda \mu + \eta) + I^{\chi, \beta}_a f(\tau, \chi, u(\tau)) \right) \, d\tau.$$

(3.3)

Using Lemma 2.1 (part (iv)), we can find that

$$\int_a^t \mathcal{C}_{\alpha}(t, \tau, \lambda, \chi) \, d\tau = (\chi(t) - \chi(a))^\alpha E_{\alpha, \alpha+1} (-\lambda(\chi(t) - \chi(a))^\alpha).$$

(3.4)

Using the definition of Mittag-Leffler function and direct computation, we have

$$\int_\tau^t (\chi(s) - \chi(\tau))^\beta-1 \mathcal{C}_{\alpha}(t, s, \lambda, \chi) \, ds$$

$$= \int_\tau^t \chi'(s)(\chi(t) - \chi(s))^\alpha-1(\chi(s) - \chi(\tau))^\beta-1 E_{\alpha, \alpha} (-\lambda(\chi(t) - \chi(s))^\alpha) \, ds$$

$$= \int_\tau^t \chi'(s)(\chi(t) - \chi(s))^\alpha-1(\chi(s) - \chi(\tau))^\beta-1 \sum_{i=0}^\infty \frac{(-\lambda(\chi(t) - \chi(s))^\alpha}{\Gamma(i\alpha + \alpha)} ds$$

$$= \sum_{i=0}^\infty \frac{(-\lambda)^i}{\Gamma(i\alpha + \alpha)} \int_\tau^t \chi'(s)(\chi(t) - \chi(s))^{(i+1)\alpha-1}(\chi(s) - \chi(\tau))^\beta-1 \, ds.$$
Using Lemma 2.5 and the identity $B((i + 1)\alpha, \beta) = \Gamma(i\alpha + \alpha)\Gamma(\beta) / \Gamma(i\alpha + (\alpha + \beta))$, we get

\[
\int_t^\tau (\chi(s) - \chi(\tau))^{\beta - 1} \mathcal{C}_\alpha(t, s, \lambda, \chi) \, ds
\]

\[
= \sum_{i=0}^{\infty} (-\lambda)^i \frac{B((i + 1)\alpha, \beta)}{\Gamma(i\alpha + \alpha)} (\chi(t) - \chi(s))^{(i+1)\alpha - \beta - 1}
\]

\[
= \frac{\Gamma(\beta)}{\Gamma(i\alpha + (\alpha + \beta))} (\chi(t) - \chi(\tau))^{\alpha + \beta - 1} \sum_{i=0}^{\infty} \frac{(-\lambda(\chi(t) - \chi(\tau)))^i}{\Gamma(i\alpha + (\alpha + \beta))}
\]

(3.5)

\[
= \frac{\Gamma(\beta)}{\Gamma(i\alpha + (\alpha + \beta))} (\chi(t) - \chi(\tau))^{\alpha + \beta - 1} \mathcal{E}_{\alpha, \alpha + \beta} (-\lambda(\chi(t) - \chi(\tau)))^\alpha).
\]

Pushing (3.4) and (3.5) into (3.3), we obtain the desired result of Lemma.

\[\square\]

**Definition 3.3.** The function $u \in C(\mathcal{D}; \mathcal{B})$ satisfying the Eq. (3.2) is called mild solution of the problem (1.4) and (1.5).

In order to study the continuity result of the mild solution of the problem with respect to inputs, we consider the approximate problem of problem (1.4) and (1.5) in the following form

\[
C^D_a^{\chi_i; \beta_i} \left( C^D_a^{\chi_i; \alpha_i} + \lambda_i \right) u_i(t) = f(t, \chi_i, u_i(t)),
\]

subject to the conditions

\[
u_i(a) = \mu_i, \quad C^D_a^{\chi_i; \alpha_i} u_i(a) = \eta_i.
\]

We can transform the problem (3.6) and (3.7) to the following integral equation

\[
u_i(t) = \mu_i \mathcal{E}_{\alpha_i} (-\lambda_i(\chi(t) - \chi(a))^{\alpha_i})
\]

\[
+ (\lambda_i \mu_i + \eta_i)(\chi(t) - \chi(a))^{\alpha_i} \mathcal{E}_{\alpha_i, \alpha_i + 1} (-\lambda_i(\chi(t) - \chi(a))^{\alpha_i})
\]

\[
+ \int^t_\tau \mathcal{C}_{\alpha_i, \beta_i}(t, \tau, \lambda_i, \chi)f(\tau, \chi_i, u_i(\tau)) \, d\tau.
\]

We list here some assumptions which we will use in this paper.

- **Assumption (A):** There are two numbers $\kappa, \rho \in \mathbb{R}$ such that

\[
\|f(t, \chi, u) - f(t, \chi, v)\| \leq (\chi(t) - \chi(a))^{-\kappa}\|H(u, v)\|, \quad u, v \in \mathcal{B}, \ t > a,
\]

\[
\|f(t, \chi, v)\| \leq (\chi(t) - \chi(a))^{-\rho}\|G(t, v)\|, \quad v \in \mathcal{B}, \ t > a,
\]

where $H : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ with $\|H(u, v)\| \to 0$ as $u \to v$, and $G \in C(\mathcal{D} \times \mathcal{B}, \mathcal{B})$.

- **Assumption (B):** There are two numbers $\kappa \in \mathbb{R}$, and $L_\chi : \mathcal{D} \to \mathbb{R}^+$ such that

\[
\|f(t, \chi, u) - f(t, \chi, v)\| \leq L_\chi(t)(\chi(t) - \chi(a))^{-\kappa}\|u - v\|
\]

\[
\|f(t, \chi, 0)\| \leq L_\chi(t)(\chi(t) - \chi(a))^{-\kappa}
\]

for any $u, v \in \mathcal{B}$ and $t > a$.

- **Assumption (C):** There are two numbers $\kappa_i, \rho_i \in \mathbb{R}$, and functions $L_{\chi_i} : \mathcal{D} \to \mathbb{R}^+$ such that

\[
\|f(t, \chi_i, u) - f(t, \chi_i, v)\| \leq L_{\chi_i}(t)(\chi_i(t) - \chi_i(a))^{-\kappa_i}\|u - v\|,
\]

\[
\|f(t, \chi_i, 0)\| \leq L_{\chi_i}(t)(\chi_i(t) - \chi_i(a))^{-\kappa_i}
\]

for any $u, v \in \mathcal{B}$ and $t > a$. 
Firstly, we consider the weakly singular non-Lipschitz source. In this case, we have the following result.

**Theorem 3.4.** Let $0 < \alpha, \beta \leq 1$, and $\kappa, \rho \in \mathbb{R}$ with $\kappa, \rho < \min\{1, \alpha + \beta\}$. Let $f \in C(D \times H^+(D) \times \mathfrak{B})$, and $G \in C(D \times \mathfrak{B}, \mathfrak{B})$, $\chi \in H^+(D)$. Suppose that the assumption $(A)$ holds. If there exist $M_1, M_2 \in C(D, \mathbb{R}^+)$ such that

$$
\|G(t, u)\| \leq M_1(t)\|u\| + M_2(t) \quad \text{for any } u \in \mathfrak{B},
$$

then the problem (1.4) and (1.5) has at least one mild solution in $C(D; \mathfrak{B})$.

**Proof.** We define an operator

$$
Q_w(t) = \mu E_\alpha(-\lambda(\chi(t) - \chi(a))^\alpha) + (\lambda \mu + \eta)(\chi(t) - \chi(a))^\alpha E_{\alpha, \alpha + 1}(-\lambda(\chi(t) - \chi(a))^\alpha)
$$

$$
+ \int_a^t C_{\alpha, \beta}(t, \tau, \lambda, \chi)f(\tau, \chi, u(\tau)) \, d\tau.
$$

We denote $W = \{w \in \mathfrak{B} : \|w\| \leq R\}$, and put $M_R = \sup_{t \in D, w \in W} \|G(t, w)\|$. From the assumption $(A)$, we find that

$$
\|f(\tau, \chi, u(\tau))\| \leq M_R(\chi(\tau) - \chi(a))^{-\rho} \quad \text{for any } \|u\| \leq R.
$$

Using Lemma 2.1 and the latter inequality, we have

$$
\|Qw(t)\| \leq \|h\|_b + CM_R \int_a^t K_{\alpha + \beta, \rho}(\tau, t, \chi) \, d\tau
$$

$$
= \|h\|_b + CM_R \Phi_{\alpha + \beta, \rho}(a, t, t, \chi) \leq \|h\|_b + CM_R \Phi_{\alpha + \beta, \rho}(a, b, b, \chi),
$$

where $h(t) = \|\mu\| E_\alpha(-\lambda(\chi(t) - \chi(a))^\alpha) + \|\lambda \mu + \eta\|((\chi(t) - \chi(a))^\alpha E_{\alpha, \alpha + 1}(-\lambda(\chi(t) - \chi(a))^\alpha)$. This shows that $Q$ is bounded. Next, we verify that $Q$ is a continuous operator on $C([a, b]; \mathfrak{B})$. In fact, from the assumption $(A)$, for $\epsilon > 0$ arbitrary, there exists $\delta > 0$ such that $\|H(w_1(\tau), w_2(\tau))\| \leq \epsilon/M$ for any $\|w_1(\tau) - w_2(\tau)\| \leq \delta$ with $M = C \Phi_{\alpha + \beta, \kappa}(a, b, b, \chi)$ and $C$ defined in Lemma 2.1. Using Lemma 2.5, we find that

$$
\|Qw_1(t) - Qw_2(t)\| \leq \int_a^t C_{\alpha, \beta}(t, \tau, \lambda, \chi)((\chi(t) - \chi(a))^{-\kappa}\|H(w_1(\tau), w_2(\tau))\| \, d\tau
$$

$$
\leq \frac{\epsilon}{M} C \Phi_{\alpha + \beta, \kappa}(a, t, t, \chi) \leq \epsilon
$$

due to $|C_{\alpha, \beta}(t, \tau, \lambda, \chi)| \leq C \chi'(\tau)(\chi(t) - \chi(\tau))^{\alpha + \beta - 1} = C \Phi_{\alpha + \beta, \kappa}(t, \chi)$. This shows that $Q$ is a continuous operator. Finally, from the assumption $(A)$ and Lemma 2.8, we conclude that $Q$ is equicontinuous on $B_R = \{w \in \mathfrak{B} : \|w\| \leq R\}$ for any $R > 0$. Thus, we conclude that $Q$ is a completely continuous operator.

Fix $r \in \mathbb{R}$ such that $0 < r < \min\{1, 1 - \rho\}$ and $r \leq \alpha + \beta - \rho$, let us denote

$$
M_0 = 2^{1-r} C^{1/r} B^{1-\frac{1}{r}} \left( \frac{\alpha + \beta - r}{1 - r}, \frac{1 - \rho - r}{1 - r} \right) (\chi(b) - \chi(a))^{\frac{\alpha + \beta - r}{1 - r}},
$$

where $C$ defined as in Lemma 2.1. We put

$$
R = 2^{1-r} (\|h\|_b + \|M_2\|_{\beta} \Phi_{\alpha + \beta, \rho}(a, b, b, \chi)) \exp \left( M_0 \int_a^b \chi'(\tau) \eta_{\tau} \, d\tau \right) + 1,
$$

where $h(t) = \|\mu\| E_\alpha(-\lambda(\chi(t) - \chi(a))^\alpha) + \|\lambda \mu + \eta\|((\chi(t) - \chi(a))^\alpha E_{\alpha, \alpha + 1}(-\lambda(\chi(t) - \chi(a))^\alpha)$, and $C$ defined in Lemma 2.1. Now, we suppose by contradiction that there
does not exist mild solution of the problem (1.4) in $B_R$. Using the nonlinear Leray-Schauder alternatives fixed point theorem \[15\], we deduce that there exist $t \in (0, 1)$ and $u \in \partial B_R$, i.e. $\|u\| = R$, such that $u = tQu$. This leads to

$$
\|u(t)\| \leq h(t) + C \int_a^t K_{\alpha+\beta}(\tau, t)\|f(\tau, \chi, u(\tau))\| d\tau
\leq h(t) + C \int_a^t K_{\alpha+\beta,\rho}(\tau, t, \chi)\|G(\tau, u(\tau))\| d\tau
\leq \|h\|_t + C\|M_2\|\Phi_{\alpha+\beta,\rho}(a, t, t, \chi) + C \int_a^t K_{\alpha,\rho}(\tau, t, \chi)M_1(\tau)\|u(\tau)\| d\tau,
$$

where $C$ defined in Lemma 2.1. Note that $g(t) = \|h\|_t + C\|M_2\|\Phi_{\alpha+\beta,\rho}(a, t, t, \chi)$ positive and non-decreasing function, hence, applying Lemma 2.9, we have

$$
\|u(t)\| \leq 2^{1-r}(\|h\|_t + C\|M_2\|\Phi_{\alpha+\beta,\rho}(a, t, t, \chi)) \exp\left(M_0 \int_a^t \chi'(\tau)M_1^2(\tau) d\tau\right) < R,
$$

where $M_0 = 2^{1-r}C^{1/r}B^{1-r} \left(\frac{\alpha + \beta - r}{1-r}, \frac{1 - \kappa - r}{1-r}\right) (\chi(b) - \chi(a))^{\frac{\alpha + \beta - r}{r}}$. The latter inequality contradicts to $u \in \partial B_R$. Therefore, we conclude that $Q$ has a fixed point in $B_R$. \[QED\]

If the source function of the problem is a weakly singular Lipschitz, we can prove that the problem has a unique mild solution in $C(\mathcal{D}; \mathfrak{B})$. More precisely, we have the following result.

**Theorem 3.5.** Let $0 < \alpha, \beta \leq 1$, $\chi \in H^+(\mathcal{D})$, and let $\kappa \in \mathbb{R}$ with $\kappa < \min\{1,\alpha + \beta\}$. Suppose that the assumption (B) holds. If $L_\chi \in C(\mathcal{D}, \mathbb{R}^+)$, then the problem (1.4) and (1.5) has a unique mild solution in $C(\mathcal{D}; \mathfrak{B})$. Moreover, for some $r \in \mathbb{R}$ such that $0 < r < \min\{1,1 - \rho\}$ and $r \leq \alpha + \beta - \rho$, we have an upper bounded estimate

$$
\|u(t)\| \leq 2^{1-r}\|g\|_t \exp\left(M \int_a^t \chi'(\tau)L_\chi^{1/r}(\tau) d\tau\right),
$$

where $C$ defined as in Lemma 2.1, and

$$
M = 2^{1-r}C^{1/r}B^{1-r} \left(\frac{\alpha + \beta - r}{1-r}, \frac{1 - \kappa - r}{1-r}\right) (\chi(b) - \chi(a))^{\frac{\alpha + \beta - r}{r}},
$$

$$
g(t) = \|h\|_t + \int_a^t K_{\alpha+\beta,\kappa}(\tau, t, \chi)L_\chi(\tau) d\tau
$$

with $h(t) = \|\mu\|E_\alpha(-\lambda(\chi(t) - \chi(a))^\alpha) + \|\lambda\mu + \eta\| (\chi(t) - \chi(a))^\alpha|E_{\alpha,\alpha+1}(-\lambda(\chi(t) - \chi(a))^\alpha)|$.

**Proof.** From the assumption (B), we find that

$$
\|f(t, \chi, w)\| \leq \|f(t, \chi, w) - f(t, \chi, 0)\| + \|f(t, \chi, 0)\|
\leq L_\chi(t)(\chi(t) - \chi(a))^{-\kappa}(\|w\| + 1)
$$

for all $w \in \mathfrak{B}$.

Therefore, according to Theorem 3.4, we conclude that the problem (1.4) and (1.5) has at least one mild solution in $C(\mathcal{D}, \mathfrak{B})$. Next, we consider the uniqueness of mild solution of the problem (1.4) and (1.5). Suppose that $u_1, u_2$ are two mild solutions
of the problem (1.4) and (1.5), by directly computes, we have
\[
\|u_1(t) - u_2(t)\| \leq C \int_a^t K_{\alpha + \beta}(t, \tau) \|f(\tau, \chi, u_1(\tau)) - f(\tau, \chi, u_2(\tau))\| \, d\tau \\
\leq C \int_a^t K_{\alpha + \beta, \kappa}(t, \tau) L_\chi(\tau) \|u_1(\tau) - u_2(\tau)\| \, d\tau,
\]
where \(C\) defined as in Lemma 2.1. Applying Lemma 2.9 we conclude that \(u_1 - u_2 \equiv 0\) or \(u_1 = u_2\). Finally, we prove the upper-bounded estimation. Using (3.11), Lemma 2.5, and Lemma 2.5, we have
\[
\|u(t)\| \leq \|h\|_t + C \int_a^t K_{\alpha + \beta}(t, \tau) \|f(\tau, \chi, u(\tau))\| \, d\tau \\
\leq \|g\|_t + C \int_a^t K_{\alpha + \beta, \kappa}(t, \tau) L_\chi(\tau) \|u(\tau)\| \, d\tau,
\]
where \(h(t) = \|\|E_a(-\lambda(\chi(t) - \chi(a))^\alpha) + \|\lambda \mu + \eta\|((\chi(t) - \chi(a))^{\alpha})|E_a,\alpha+1(-\lambda(\chi(t) - \chi(a))^\alpha)|\) and \(g(t) = \|h\|_t + C \int_a^t K_{\alpha + \beta, \kappa}(t, \tau) L_\chi(\tau) \|u(\tau)\| \, d\tau\). Applying Lemma 2.9, we obtain the desired result. This completes the proof of Theorem. \(\square\)

**Remark 3.6.** We can use Banach fixed point theorem to show that the problem (1.4) and (1.5) has a unique mild solution in \(C(\mathcal{D}, \mathcal{B})\). In fact, by induction and using Lemma 2.5, we can verify that
\[
\|Q^k v(t) - Q^k w(t)\| \leq \frac{D_k \Gamma(1 - \kappa)}{\Gamma(k(\alpha + \beta - \kappa) + 1)} \times (C\|L_\chi\|_b \Gamma(\alpha + \beta))^{k} (\chi(t) - \chi(a))^{k(\alpha + \beta - \kappa)} \|v - w\|,
\]
(3.12)
where \(Q\) defined in (3.10), \(C\) defined as in Lemma 2.1, and \(D_k = \frac{\Gamma(k(\alpha - \kappa) - \kappa + 1)}{\Gamma(k(\alpha + \beta) - \kappa + 1)} D_{k-1}\), \(D_0 = 1\). From (3.12) we can find an integer number \(k_0\) such that \(Q^k\) the contraction mapping for any \(k \geq k_0\). This deduces the equation (3.2) has a unique solution in \(C(\mathcal{D}, \mathcal{B})\) or the problem (1.4) and (1.5) has a unique mild solution in \(C(\mathcal{D}, \mathcal{B})\).

Finally, we state and prove the continuity result of the mild solution of the problem with respect to inputs.

**Theorem 3.7.** Let \((\alpha, \beta, \chi), (\alpha_i, \beta_i, \chi_i) \in \mathcal{W}_i, \chi, \chi_i \in H^+(\mathcal{D}), L_\chi, L_{\chi_i} \in C(\mathcal{D}).\) Let \(\kappa, \kappa_i \in \mathbb{R}, \quad \kappa < \min\{1, \alpha + \beta\}\). Suppose that the assumptions (B) and (C) hold. Suppose further that \((\alpha_i, \beta_i, \lambda_i, \kappa_i, \mu_i, \eta_i) \to (\alpha, \beta, \lambda, \kappa, \mu, \eta), \quad \|L_\chi - L_{\chi_i}\|_b \to 0, \quad \|\chi_i - \chi\|_b \to 0, \quad \|\chi'_i - \chi'\|_{L(\mathcal{D})} \to 0, \) and
\[
\sup_{t [a + \epsilon, b], \|w\| \leq \Lambda} \|f(t, \chi, w) - f(t, \chi_i, w)\| \to 0 \quad \text{as} \quad i \to \infty
\]
for every \(\epsilon \in (0, b - a)\) for any positive number \(\Lambda\). Then, the problem (1.4) and (1.5) has a unique mild solution \(u \in C(\mathcal{D}, \mathcal{B})\). Besides, we can find an integer number \(i_*\) such that the problem (3.6) and (3.7) has a unique mild solution \(u_i \in C(\mathcal{D}, \mathcal{B})\) for any \(i \geq i_*\). Furthermore, one has
\[
u_i \to u \quad \text{as} \quad i \to \infty \quad \text{in} \quad C(\mathcal{D}, \mathcal{B}).
\]
**Proof.** According to Theorem 3.5, we find that the problems (1.4) and (1.5) has a unique mild solution \(u \in C(\mathcal{D}, \mathcal{B})\). Since \((\alpha_i, \beta_i) \to (\alpha, \beta), \kappa_i \to \kappa, \) and \(\kappa < \min\{1, \alpha + \beta\}\), hence, we can find \(i_* \in \mathbb{N}\) such that \(\kappa_i < \min\{1, \alpha_i + \beta_i\}\) for any \(i \geq i_*\). From Theorem 3.5, we conclude that the problem (3.6) has a unique mild solution \(u_i \in \)
\[ \|u\|, \|u_i\| \leq \Lambda \]

for any \( i \geq i_* \). Base on this upper estimate in Theorem 3.5, there exists a positive constant \( \Lambda \) independent of \( i \)

\[ (3.14) \]

Estimate for \( I_{1,i}(t) \). Using properties of Mittag-Leffler function in Lemma 2.1, we can easily verify that

\[ (3.15) \quad M_i := \sup_{t \in \mathbb{D}} \|F(\mu, \eta, \lambda, \alpha, t, \chi) - F(\mu_i, \eta, \lambda_i, \alpha_i, t, \chi_i)\| \to 0 \quad \text{as} \quad i \to \infty. \]

Estimate for \( I_{2,i}(t) \). The estimation is separate in two cases.

The first case: \( a \leq t \leq a + \epsilon \). Since \( \|u\|, \|u_i\| \leq \Lambda \), this leads to

\[ (3.16) \]

By the virtue of Lemma 2.1 together with (3.16), there exists a positive constatnt \( C \) such that

\[ (3.17) \]

Pushing (3.15), (3.18) into (3.14) and according to Remark 2.6, we obtain

\[ (3.19) \quad \sup_{a \leq t \leq a + \epsilon} \|u_i(t) - u(t)\| \leq \sup_{a \leq t \leq a + \epsilon} I_{1,i}(t) + \sup_{a \leq t \leq a + \epsilon} I_{2,i}(t) \leq M_{1,i}(\epsilon) \to 0 \]

as \( \epsilon \to 0, i \to \infty. \)

The second case: \( a + \epsilon < t \leq b \). We find that

\[ (3.20) \quad I_{2,i}(t) \leq \Theta_{1,i}(t) + \Theta_{2,i}(t, \epsilon) + \Theta_{3,i}(t, \epsilon), \]
where
\[ \Theta_{1,i}(t) = \int_a^t |\mathcal{C}_{\alpha,\beta}(t, \tau, \lambda, \chi)| \|f(\tau, \chi, u_i(\tau)) - f(\tau, \chi_i, u_i(\tau))\| \, d\tau, \]
\[ \Theta_{2,i}(t) = \int_a^t |\mathcal{C}_{\alpha,\beta}(t, \tau, \lambda, \chi)| \|f(\tau, \chi, u(\tau)) - f(\tau, \chi, u_i(\tau))\| \, d\tau, \]
\[ \Theta_{3,i}(t) = \int_a^t |\mathcal{C}_{\alpha,\beta}(t, \tau, \lambda, \chi) - \mathcal{C}_{\alpha_i,\beta_i}(t, \tau, \lambda_i, \chi_i)| \|f(\tau, \chi, u_i(\tau))\| \, d\tau. \]

Firstly, we estimate for the first term \( \Theta_{1,i}(t) \). Using Lemma 2.1, Lemma 2.5, and Remark 2.6, we have
\[ \Theta_{1,i}(t) \leq \int_a^t |\mathcal{C}_{\alpha,\beta}(t, \tau, \lambda, \chi)| \|f(\tau, \chi, u(\tau)) - f(\tau, \chi_i, u_i(\tau))\| \, d\tau \]
\[ \leq C \sup_{a+\epsilon \leq t \leq b, \|v\| \leq \Lambda} \|f(t, \chi, v) - f(t, \chi_i, v)\| \int_a^t \mathcal{K}_{\alpha+\beta}(\tau, t, \chi) \, d\tau \]
\[ = C \Phi_{\alpha+\beta,0}(a, b, b) \sup_{a+\epsilon \leq t \leq b, \|v\| \leq \Lambda} \|f(t, \chi, v) - f(t, \chi_i, v)\| := M_{2,i}(\epsilon) \to 0 \]
as \( i \to \infty \).

Secondly, we find an estimation for the term \( \Theta_{2,i}(t) \). In the view of Lemma 2.1, we have
\[ \Theta_{2,i}(t) \leq C \int_a^t \mathcal{K}_{\alpha+\beta,\kappa}(\tau, t, \chi) L_\lambda(\tau) \|u_i(\tau) - t(\tau)\| \, d\tau. \]

Lastly, we find an estimation for \( \Theta_{3,i}(t) \). Using the inequality (3.17) and Lemma 2.7, we obtain
\[ \Theta_{3,i}(t) \leq \mathfrak{M}_i(\epsilon) \int_a^t |\mathcal{C}_{\alpha,\beta}(t, \tau, \lambda, \chi) - \mathcal{C}_{\alpha_i,\beta_i}(t, \tau, \lambda_i, \chi_i)| \, d\tau := M_{3,i}(\epsilon) \to 0 \]
as \( i \to \infty \), where \( \mathfrak{M}_i(\epsilon) = M_0(A+1) \sup_{a+\epsilon \leq t \leq b} (\chi_i(t) - \chi_i(a))^{-\gamma} \). Pushing (3.21), (3.22), and (3.23) into (3.20), we get
\[ I_{2,i}(t) \leq M_{2,i}(\epsilon) + M_{3,i}(\epsilon) + C \int_a^t \mathcal{K}_{\alpha+\beta,\kappa}(\tau, t, \chi) L_\lambda(\tau) \|u(\tau) - u_i(\tau)\| \, d\tau \]
Combining (3.15), (3.24), and (3.14), we obtain
\[ \|u(t) - u_i(t)\| \leq M_{4,i}(\epsilon) + C \int_a^t \mathcal{K}_{\alpha+\beta,\kappa}(\tau, t, \chi) L_\lambda(\tau) \|u(\tau) - u_i(\tau)\| \, d\tau, \]
where \( M_{4,i}(\epsilon) = M_1 + M_{2,i}(\epsilon) + M_{3,i}(\epsilon) \). Fixed \( r \in \mathbb{R} \) such that \( 0 < r < \min\{1, 1-\kappa\} \) and \( r \leq \alpha + \beta - \kappa \), applying Lemma 2.9, we obtain
\[ \|u(t) - u_i(t)\| \leq 2^{1-r} M_{4,i}(\epsilon) \exp \left( M \int_a^t \left( \chi'(\tau) \frac{1}{\lambda(\tau)} \right) \, d\tau \right) \]
where \( M = 2^{1-r} C_\epsilon^\rho B^\rho \left( \frac{\alpha+\beta-r}{1-r}, \frac{1-\kappa-r}{1-r}\right) (\chi(b) - \chi(a))^{\alpha+\beta-r}. \) This leads to
\[ \sup_{a+\epsilon \leq t \leq b} \|u(t) - u_i(t)\| \leq 2^{1-r} M_{4,i}(\epsilon) \exp \left( M \int_a^b \left( \chi'(\tau) \frac{1}{\lambda(\tau)} \right) \, d\tau \right) \to 0. \]
From (3.19) and (3.25), we obtain the desired result of Theorem. \( \square \)
4. Examples

In this section, we introduce two examples illustrating theoretical results of the present paper.

Example 4.1. For $\chi \in H^+(\mathcal{D})$, we consider the problem of finding a function $u : \mathcal{D} \to \mathbb{R}$ satisfying

\begin{equation}
C D_{a_+}^{\chi,4/5} \left( C D_{a_+}^{\chi,9/10} + 1 \right) u(t) = \frac{(2t + 1)u^{4/5}(t)}{(\chi(t) - \chi(a))^{3/8}} + \frac{t^2 + 1}{(\chi(t) - \chi(a))^{1/2}}, \quad a < t \leq b
\end{equation}

subject to the conditions

\begin{equation}
u(a) = \mu, \quad C D_{a_+}^{\chi,9/10} u(a) = \eta.
\end{equation}

Herein we have $\alpha = 9/10$, $\beta = 4/5$, $\lambda = 1$, and $f(t, \chi, u) = \frac{(2t + 1)u^{4/5}(t)}{(\chi(t) - \chi(a))^{3/8}} + \frac{t^2 + 1}{(\chi(t) - \chi(a))^{1/2}}$. We find that $f : \mathcal{D} \times H^+(\mathcal{D}) \times \mathbb{R} \to \mathbb{R}$ and

\begin{align*}
|f(t, \chi, u) - f(t, \chi, v)| &\leq (\chi(t) - \chi(a))^{-3/5}|G(t, u)|, \\
|f(t, \chi, u) - f(t, \chi, v)| &\leq (\chi(t) - \chi(a))^{-3/5}|H(u(t), v(t))|,
\end{align*}

where $G(t, u) = (2t + 1)u^{4/5}(t) + (t^2 + 1)(\chi(t) - \chi(a))^{1/10}$ and $H(u(t), v(t)) = (1 + 2 \max\{|a|, |b|\}) (u^{4/5}(t) - v^{4/5}(t))$. It is obvious that $\kappa = \rho = 3/5 < 1 < \alpha + \beta$ and $G \in C(\mathcal{D} \times \mathbb{R}, \mathbb{R})$, $H(u, v) \to 0$ as $u \to v$. Hence, the assumption (A) holds. Furthermore, we can see that

$$|G(t, u)| \leq M_1(t)|u(t)| + M_2(t),$$

where $M_1(t) = |2t + 1|$ and $M_2(t) = |2t + 1| + (t^2 + 1)(\chi(t) - \chi(a))^{1/10}$. Thus, the condition (3.9) in Theorem 3.4 is satisfied. Therefore, we conclude that the problem (4.1) and (4.2) has at least one mild solution in $C(\mathcal{D}, \mathbb{R})$.

Remark 4.2. In the example 4.1, we have $\lambda = 1 > \Gamma(1 + \alpha) = \Gamma(19/10) = 0.961766$. Hence, the results in Yu et al. [37] or Baghani [4] cannot apply to discuss the existence solution of the problem (4.1) and (4.2).

Example 4.3. For $\chi \in H^+(\mathcal{D})$, we consider the problem of finding a function $u : \mathcal{D} \to \mathbb{R}$ satisfying the following equation

\begin{equation}
C D_{a_+}^{\chi,9/10} \left( C D_{a_+}^{\chi,8/9} + 11/10 \right) u(t) = \frac{1 + t^2}{(\chi(t) - \chi(a))^{7/8}} \frac{u}{u^2 + 1} + \frac{t}{(\chi(t) - \chi(a))^{2/3}}
\end{equation}

subject to the conditions

\begin{equation}
u(a) = \mu, \quad C D_{a_+}^{\chi,8/9} u(a) = \eta.
\end{equation}

We have $\alpha = 8/9$, $\beta = 9/10$, $\lambda = 11/10$, and $f(t, \chi, u) = \frac{1 + t^2}{(\chi(t) - \chi(a))^{7/8}} \frac{u}{u^2 + 1} + \frac{t}{(\chi(t) - \chi(a))^{2/3}}$. Using the fact that

$$\frac{1 - uv}{(u^2 + 1)(v^2 + 1)} \leq 1 \text{ for any } u, v \in \mathbb{R},$$

one has

$$|f(t, \chi, u) - f(t, \chi, v)| = \frac{1 + t^2}{(\chi(t) - \chi(a))^{7/8}} \left| \frac{(u - v)(1 - uv)}{(u^2 + 1)(v^2 + 1)} \right| \leq L_{\chi,1}(t)(\chi(t) - \chi(a))^{-7/8}|u - v|,$$

where

$$L_{\chi,1}(t)(\chi(t) - \chi(a))^{-7/8}.$$
where $L_{\chi,1}(t) = \frac{t^2+1}{2}$. We also have

\begin{equation}
|f(t, \chi, 0)| \leq L_{\chi,2}(t)(\chi(t) - \chi(a))^{-7/8},
\end{equation}

where $L_{\chi,2}(t) = t(\chi(t) - \chi(a))^{-5/24}$. Combining (4.5) with (4.6) we deduce that Assumption (B) is satisfied with $L_{\chi}(t) = \max\{L_{\chi,1}(t), L_{\chi,2}(t)\}$. So, we conclude that the problem (4.3) and (4.4) has a unique mild solution in $C(D, \mathbb{R})$.

5. Conclusions

In this paper, we have presented some results dealing with the existence and uniqueness of mild solutions for a nonlinear fractional Langevin equation involving fractional derivatives of a function with respect to another function in a Banach space. Unlike previous papers, in this paper, the source function has been assumed to have a singularity. Our obtained results have improved some of the previous ones. Furthermore, we have proved that mild solution of the problem is dependent continuously on the inputs (initial data, fractional orders, appropriate function, and friction constant). To prove the results just mentioned, we have transformed the problem to a Volterra integral equation with two parameters Mittag-Leffler function in the kernel and established a new Henry-Gronwall type inequality. For the illustration of the theoretical results, we have considered two examples.

Acknowledgements

We would like to thank the handling editor and the anonymous reviewers for their careful reading of our manuscript and their many constructive comments and suggestions, which led to improving our manuscript.

References


**Faculty of Education, Thu Dau Mot University, Binh Duong Province, Vietnam**

*Email address: diennm@tdmu.edu.vn*