FRACTIONAL ORDER MATHEMATICAL MODELING AND ANALYSIS OF MULTI INFECTIOUS DISEASES

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Abstract

The purpose of this paper to explore a multi-infection model involving Caputo Fab-Raizo fractional order derivatives. Existence, uniqueness, positivity, and boundedness of the solutions for the multi-infection type model are established. Further, an Adams-Bashforth method is applied to calculate solution of the proposed fractional order model. Finally, to show the influence of fractional order and model parameters, we present a detailed numerical simulation for different values used in the proposed fractional order model. The result shows the importance and convincing behavior of the fractional order and ensures that by including the memory effects in the model seems very appropriate for such an investigation. This study will help to understand the complexity of the co-infection model that is valid and reliable for both integer and non-integer orders.
Keywords: Multi-infection of Malaria, Ebola and Typhoid Epidemic model, Caputo-Fabrizio Fractional derivative, Non-singularity, Adams-Bashforth scheme Numerical simulation.

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1 Introduction

Due to the geographic overlap of the three most prevalent diseases of poverty Malaria, Ebola, and Typhoid multiple infections are very common. Their mimicking symptomatology frequently results in egregious misdiagnosis and mistreatment. This study was conducted to ascertain the prevalence of multiple infections with Typhoid, Ebola, and Malaria among the adult population in Unwana Community Afikpo-North Local Government Area, Ebonyi State. Five species of the protozoan parasite Plasmodium, including P. Falciparum, P.

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Vivax, P. Malariae, P. Ovale, and P. Knowlesi, cause the potentially fatal illness known as malaria. Humans contract malaria when an infected female anopheles mosquito bites them. [1, 2, 3]. P. falciparum, which is mostly found in sub-Saharan Africa, is thought to be the sole source of malaria-related mortality in humans. It is widely acknowledged as a public health issue with significant medical, social, and economic repercussions and is regarded as a disease of poverty [4]. The inhabitants of the Unwana village in Afikpo are not an exception to the fact that malaria imposes a severe cost on the most vulnerable and impoverished groups. Nevertheless, typhoid fever, sometimes referred to as "typhoid," is a bacterial infection brought on by Salmonella typhi, also known as Salmonella enterica serotype Typhi [5, 6]. Humans are the only animals that can contract it when they consume food or water that has been tainted with the bacteria Salmonella enteric serovar typhi from an infected person [1]. Poor sanitation and hygiene caused by poverty are risk factors [6]. Co-infections are quite frequent since the two infections occur in the same geographic area.

The Ebola virus (Ebola) outbreak that devastated many people in Western Africa in 2014 is now known as Ebola. In comparison to the 20 Ebola threats that have occurred since 1976, this epidemic was the worst with more than 16,000 clinically confirmed patients and roughly 70% death cases [7]. People live near to rainforests in Africa, especially in the areas where Ebola outbreaks occurred. They hunt bats and monkeys for food and gather forest fruits [8, 9]. In reference [10], a novel SIR model is proposed that accounts for both direct and indirect transmissions, ensuring the stability of Ebola virus transmission. The authors conduct a thorough numerical analysis of their model.

Numerous mathematical models have been developed to investigate the dynamics of Ebola and other infectious disease outbreaks from different perspectives [11, 12]. The SEIR model, which divides the population into susceptible, exposed, infectious, and recovered compartments, is a widely used model for characterizing disease epidemics, including Ebola [13]. Some extensions of this model incorporate explicit consideration of transmission from deceased Ebola hosts [1, 14], while others account for mismatches between symptoms and infectiousness. [14, 15].

Fractional calculus is an emerging field that delves into the realm of non-integer order derivatives and integrals. Initially, it was introduced by Abel to solve the Tautochrone problem, and it has since found applications in various fields, including physics, economics, biology, medicine, viscoelasticity, and control theory. Unlike the conventional derivative, which is a local operator, fractional order derivatives have a wider scope in determining the equilibrium field of dynamical systems. Fractional calculus is a parallel branch of calculus that cannot be considered a generalized version of integer order calculus [16, 17]. Fractional order systems are more appropriate than integer order systems in many fields and can express phenomena that are linked to memory and affected by hereditary properties [18, 19]. Fractional order models were gaining increasing attention in various fields of science and engineering due to their unique advantages and superiority over traditional integer order models. Here we are going to mentioned some of the keys advantages of fractional order over integer order models i,e Increased Flexibility, Better Approximation, Improved Memory Handling, Enhanced System Identification, Better Representation of Anomalous Behavior and Improved Stability Analysis etc. This makes it a crucial tool for developing a mathematical model to assess the dynamics and transmissibility of Malaria, Ebola, and Typhoid in a multi-infection setting.

2 Description and Formulation of the Model

In this section, we formulate a mathematical model of Malaria, Typhoid and Ebola Multi-infection is presented consisting of thirteen classes with whole human population represented by N. The class having healthy but likely to be infected individuals is represented by S, the class showing Ebola, Malaria, and Typhoid infected individuals is represented by $I_e, I_m and I_t$ respectively, and co-infectious individuals classes are represented by I_{em}, I_{et}, and I_{mt}, respectively. Recovered classes of Malaria, Ebola, and Typhoid individual are denoted by R_m, R_e, R_t and recovered classes of their co-infection denoted by R_{em}, R_{et}, R_{mt}, respectively. The total human population is $N = S + I_m + I_e + I_t + I_{em} + I_{et} + I_{mt} + R_e + R_m + R_t + R_{em} + R_{et} + R_{mt}$. Susceptible class increase by enrollment rate of Π and from Ebola, Typhoid and Malaria recovered classes rates of α, β, and γ and from their co-infection recovered class are δ, ζ, and η respectively. λ_1, λ_2 and λ_3 are force of infection of (Ebola & Malaria), (Malaria & Typhoid) respectively, where **a**, **b** and **c** is the interaction rate of

Malaria, Typhoid and Ebola respectively. Malaria only and co-infectious recovered class is increased due to the recovery rate of Malaria represented by σ_1, and τ_2, Typhoid only and co-infectious recovered compartments increase their number with a rate of recovery of σ_2, and τ_1, and Ebola only and co-infectious recovered class increase their number with a rate of recovery of σ_3 and τ_3, respectively.

In the co-infectious recovered class, individuals either recovered only from Malaria, Typhoid, Ebola or from all the three diseases with a probability of $\tau_1(1-e)$, $\tau_1 g(1-e)$ or $\tau_1(1-g)(1-e)$, $\tau_2(1-e)$, $\tau_2 g(1-e)$ or $\tau_2(1-g)(1-e)$, and $\tau_3(1-e)$, $\tau_3 g(1-e)$ or $\tau_3(1-g)(1-e)$, respectively, where τ_1, τ_2 τ_3 $\mathbf{e} \in (0,1)$ and $\mathbf{g} \in (0,1)$. The natural expiry rate is denoted by μ and Malaria, Typhoid and Ebola causing expiry rates are represented by ψ_1, ψ_2 and ψ_3, respectively. The parameters which we used in this model are positive. Thus, our mathematical model is consisting of the following system of differential equations is given by

$$
\begin{aligned}
\frac{dS}{dt} &= \Pi + \alpha R_e + \beta R_t + \gamma R_m + \delta R_{et} + \zeta R_{em} + \eta R_{mt} - (\lambda_1 + \lambda_2 + \lambda_3 + \mu)S, \\
\frac{dI_m}{dt} &= \lambda_2 S - \omega\lambda_1 I_m - \kappa\lambda_3 I_m - (\sigma_1 + \mu + \psi_2)I_m, \\
\frac{dI_e}{dt} &= \lambda_1 S - \chi\lambda_2 I_e - \rho\lambda_3 I_e - (\sigma_3 + \mu + \psi_1)I_e, \\
\frac{dI_t}{dt} &= \lambda_3 S - \phi\lambda_2 I_t - \vartheta\lambda_1 I_t - (\sigma_2 + \mu + \psi_3)I_t, \\
\frac{dI_{em}}{dt} &= \omega\lambda_1 I_m + \chi\lambda_2 I_e - (\tau_2 + \mu + \psi_1 + \psi_2)I_{em}, \\
\frac{dI_{et}}{dt} &= \vartheta\lambda_1 I_t + \rho\lambda_3 I_e - (\psi_1 + \psi_3 + \mu + \tau_3)I_{et}, \\
\frac{dI_{mt}}{dt} &= \phi\lambda_2 I_t + \kappa\lambda_3 I_m - (\mu + \psi_2 + \psi_3 + \tau_1)I_{mt}, \qquad (1) \\
\frac{dR_e}{dt} &= \sigma_3 I_e + e\tau_2 I_{em} + \tau_3 g(1-e)I_{et} - (\mu + \alpha)R_e, \\
\frac{dR_m}{dt} &= \sigma_1 I_m + \tau_1 e I_{mt} + \tau_2 g(1-e)I_{em} - (\gamma + \mu)R_m, \\
\frac{dR_t}{dt} &= \sigma_2 I_t + \tau_3 e I_{et} + \tau_1 g(1-e)I_{mt} - (\mu + \beta)R_t, \\
\frac{dR_{et}}{dt} &= \tau_3(1-e)(1-g)I_{et} - (\mu + \delta)R_{et}, \\
\frac{dR_{em}}{dt} &= \tau_2(1-g)(1-e)I_{em} - (\zeta + \mu)R_{em}, \\
\frac{dR_{mt}}{dt} &= \tau_1(1-g)(1-e)I_{mt} - (\eta + \mu)R_{mt}.
\end{aligned}
$$

Here,

$$
\lambda_1 = \frac{c(I_e(t) + I_{em}(t))}{N}, \quad
\lambda_2 = \frac{a(I_m(t) + I_{mt}(t))}{N}, \quad
\lambda_3 = \frac{b(I_t(t) + I_{et}(t))}{N},
$$

with initial conditions
$S(0) = S_0 \geq 0$, $I_e(0) = I_{e(0)} \geq 0$, $I_m(0) = I_{m(0)} \geq 0$, $I_t(0) = I_{t(0)} \geq 0$, $I_{em}(0) = I_{em(0)} \geq 0$, $R_e(0) = R_{e(0)} \geq 0$, $R_m(0) = R_{m(0)} \geq 0$, $R_t(0) = R_{t(0)} \geq 0$, $R_{em}(0) = R_{em(0)} \geq 0$, $R_{et}(0) = R_{et(0)} \geq 0$, $R_{mt}(0) = R_{mt(0)} \geq 0$.
Most natural phenomena including epidemiological dynamics involve time memory effect and are valuable to demonstrate the fact about nature related processes having non-local dynamics. Models with fractional derivatives handle these issues in batter way because non- integral order derivatives contain time dependent kernels.

3 Preliminaries

Definition 3.1. *Let $h(\tau) \in C^{\kappa}$, then for the fractional derivative with order α, in the Caputo sense is defined in [20] as,*

$$ {}^{C}D_{\tau}^{\alpha}h(\tau) = \frac{1}{\Gamma(\kappa-\alpha)} \int_0^{\tau} \frac{(\tau-\eta)^{\kappa-\alpha}h^{(\kappa)}(\eta)}{(\tau-\eta)} d\eta, $$

where $\kappa = \lceil \alpha \rceil + 1$ with $\lceil \alpha \rceil$ is the integer order part of real number α. Evidently, ${}^{C}D_{\tau}^{\alpha}h(\tau) \to \acute{h}(\tau)$ as $\alpha \to 1$

Definition 3.2. *Let $h \in H^1(a,b), a < b, a \in (-\infty, t)$, and $\alpha \in (0,1)$, [21] then the α $th-order$*

$$ {}^{CF}D^{\alpha}h(\tau) = \frac{M(\alpha)}{\Gamma(1-\alpha)} \int_a^{\tau} \acute{h} exp[-\frac{\alpha}{1-\alpha}](\tau-\eta)d\eta, $$

where $M(\alpha)$ is a normalizing function depending on α such that $M(0) = M(1) = 1$.

Definition 3.3. *For $0 < \imath$, consider the equation*

$$ {}^{CF}_{\alpha_1}D_t^{\imath}\hbar(t) = \hbar(t), $$

then the corresponding integral of order $\imath$ is defined as [16].

$$ {}^{CF}_{\alpha_1}I_t^{\imath}\hbar(t) = \frac{2(1-\imath)}{B(\imath)(2-\imath)}\hbar(t) + \frac{2\imath}{B(\imath)(2-\imath)} \int_0^t \hbar(\zeta)d\zeta, t \geq 0. $$

such that

$$ \frac{2}{2B(\imath) - \imath B(\imath)} = 1. $$

Solving for $B(\imath)$, we have $B(\imath) = \frac{2}{(2-\imath)}, 0 \leq \imath \leq 1$.

3.1 Extension of the Proposed Model to Fractional Order

Natural phenomena exhibit time memory effect, which can be observed in various processes, including epidemiological dynamics. These processes are characterized by non-local dynamics, indicating the involvement of long-range interactions. To accurately model these phenomena, fractional derivatives are often utilized as they can handle time-dependent kernels more effectively than integer-order derivatives. While numerous fractional derivatives exist in the literature, the Caputo fractional derivative is one of the most widely used techniques in fractional Calculus. By employing such techniques, we can gain deeper insights into the complex behaviors of natural systems, which can help us better understand and predict their behavior.

Utilizing the Caputo fractional derivative offers a significant benefit in that it maintains the same initial conditions as traditional derivatives. This means that fractional initial values are not necessary, simplifying the modeling process. Building upon this advantage, we have developed a fractional order mathematical modeling and show multiple infectious disease model analysis, presented in equation (1) in fractional form. We have adopted the Caputo fractional time derivative for this approach. To introduce a power-law correlation, we incorporated a time-dependent kernel into our model. These techniques enable us to better understand the dynamics of infectious diseases and the impact of fractional-order modeling on the co-infections dynamics in the community.

Now by introducing the time-dependent kernel, we define the power-law correlation in the following

$$ k(t-\tau) = \frac{1}{\Gamma(\imath-1)}(t-\tau)^{\imath-2}, \tag{2} $$

To write the system (1) of differential equations in terms of an integral, we use the concept of convolution. The convolution integral is defined as follows

$$ (f * g)(t) = \int_0^t (t-\tau)g(\tau)d\tau, $$

where $(f * g)(t)$ is the convolution of functions f(t) and g(t). Now, our task is to find the function k(t) that satisfies the given equation. By setting $f(t) = \frac{dS}{dt}$ and $g(t) = k(t)$ we can find k(t) is given by,

$$\frac{dS}{dt} = (f * g)(t) = \int_{t0}^{t} f(t-\tau)g(\tau)d\tau.$$

Plugging in the given expression for $\frac{dS}{dt}$,

$$\frac{dS}{dt} = \int_{t_0}^{t} k(t-\tau)[\Pi + \alpha R_e + \beta R_t + \gamma R_m + \delta R_{et} + \zeta R_{em} + \eta R_{mt} - (\lambda_1 + \lambda_2 + \lambda_3 + \mu)S]d\tau.$$

Now, let's differentiate both sides of the equation with respect to t,

$$\frac{d^2S}{dt^2} = \frac{d}{dt}\int_{t_0}^{t} k(t-\tau)[\Pi + \alpha R_e + \beta R_t + \gamma R_m + \delta R_{et} + \zeta R_{em} + \eta R_{mt} - (\lambda_1 + \lambda_2 + \lambda_3 + \mu)S]d\tau.$$

Next, we can swap the order of differentiation and integration,

$$\frac{d^2S}{dt^2} = \int_{t_0}^{t} [\frac{d}{dt}k(t-\tau)][\Pi + \alpha R_e + \beta R_t + \gamma R_m + \delta R_{et} + \zeta R_{em} + \eta R_{mt} - (\lambda_1 + \lambda_2 + \lambda_3 + \mu)S]d\tau.$$

To evaluate the derivative inside the integral, we have

$$\frac{d^2S}{dt^2} = \int_{t_0}^{t} [\frac{d}{dt}k(t-\tau)][\Pi + \alpha R_e + \beta R_t + \gamma R_m + \delta R_{et} + \zeta R_{em} + \eta R_{mt} - (\lambda_1 + \lambda_2 + \lambda_3 + \mu)S]d\tau.$$

Finally, we need to set this equal to $\frac{dS^2}{dt^2}$

$$\frac{d^2S}{dt^2} = \int_{t_0}^{t} [\frac{d}{dt}k(t-\tau)][\Pi + \alpha R_e + \beta R_t + \gamma R_m + \delta R_{et} + \zeta R_{em} + \eta R_{mt} - (\lambda_1 + \lambda_2 + \lambda_3 + \mu)S]d\tau.$$

We can also equate the integrands on both sides of the equation

$$\frac{d^2S}{dt^2} = \int_{t_0}^{t} [\frac{d}{dt}k(t-\tau)][\Pi + \alpha R_e + \beta R_t + \gamma R_m + \delta R_{et} + \zeta R_{em} + \eta R_{mt} - (\lambda_1 + \lambda_2 + \lambda_3 + \mu)S]d\tau = (f * g)(t).$$

Since, $\frac{dS^2}{dt^2} = (f * g)(t)$, we have $f(t) = \frac{dS}{dt}$, $g(t) = \frac{d}{dt}k(t)$,

$$(f * g)(t) = \Pi + \alpha R_e + \beta R_t + \gamma R_m + \delta R_{et} + \zeta R_{em} + \eta R_{mt} - (\lambda_1 + \lambda_2 + \lambda_3 + \mu)S.$$

Now, you can see that the original differential equation can be represented in the form of the convolution integral

$$\frac{dS}{dt} = \int_{t_0}^{t} k(t-\tau)][\Pi + \alpha R_e + \beta R_t + \gamma R_m + \delta R_{et} + \zeta R_{em} + \eta R_{mt} - (\lambda_1 + \lambda_2 + \lambda_3 + \mu)S]d\tau$$

and we express system (1) in term of integral as

$$
\begin{aligned}
\frac{dS}{dt} &= \int_{t_0}^{t} k(t-\tau)[\Pi + \alpha R_e + \beta R_t + \gamma R_m + \delta R_{et} + \zeta R_{em} + \eta R_{mt} - (\lambda_1 + \lambda_2 + \lambda_3 + \mu)S]d\tau, \\
\frac{dI_m}{dt} &= \int_{t_0}^{t} k(t-\tau)[\lambda_2 S - \omega\lambda_1 I_m - \kappa\lambda_3 I_m - (\sigma_1 + \mu + \psi_2)I_m]d\tau, \\
\frac{dI_e}{dt} &= \int_{t_0}^{t} k(t-\tau)[\lambda_1 S - \chi\lambda_2 I_e - \rho\lambda_3 I_e - (\sigma_3 + \mu + \psi_1)I_e]d\tau, \\
\frac{dI_t}{dt} &= \int_{t_0}^{t} k(t-\tau)[\lambda_3 S - \phi\lambda_2 I_t - \vartheta\lambda_1 I_t - (\sigma_2 + \mu + \psi_3)I_t]d\tau, \\
\frac{dI_{em}}{dt} &= \int_{t_0}^{t} k(t-\tau)[\omega\lambda_1 I_m + \chi\lambda_2 I_e - (\tau_2 + \mu + \psi_1 + \psi_2)I_{em}]d\tau, \\
\frac{dI_{et}}{dt} &= \int_{t_0}^{t} k(t-\tau)[\vartheta\lambda_1 I_t + \rho\lambda_3 I_e - (\psi_1 + \psi_3 + \mu + \tau_3)I_{et}]d\tau, \\
\frac{dI_{mt}}{dt} &= \int_{t_0}^{t} k(t-\tau)[\phi\lambda_2 I_t + \kappa\lambda_3 I_m - (\mu + \psi_2 + \psi_3 + \tau_1)I_{mt}]d\tau, \\
\frac{dR_e}{dt} &= \int_{t_0}^{t} k(t-\tau)[\sigma_3 I_e + e\tau_2 I_{em} + \tau_3 g(1-e)I_{et} - (\mu + \alpha)R_e]d\tau, \\
\frac{dR_m}{dt} &= \int_{t_0}^{t} k(t-\tau)[\sigma_1 I_m + \tau_1 e I_{mt} + \tau_2 g(1-e)I_{em} - (\gamma + \mu)R_m]d\tau, \\
\frac{dR_t}{dt} &= \int_{t_0}^{t} k(t-\tau)[\sigma_2 I_t + \tau_3 e I_{et} + \tau_1 g(1-e)I_{mt} - (\mu + \beta)R_t]d\tau, \\
\frac{dR_{et}}{dt} &= \int_{t_0}^{t} k(t-\tau)[\tau_3(1-e)(1-g)I_{et} - (\mu + \delta)R_{et}]d\tau, \\
\frac{dR_{em}}{dt} &= \int_{t_0}^{t} k(t-\tau)[\tau_2(1-g)(1-e)I_{em} - (\zeta + \mu)R_{em}]d\tau, \\
\frac{dR_{mt}}{dt} &= \int_{t_0}^{t} k(t-\tau)[\tau_1(1-g)(1-e)I_{mt} - (\eta + \mu)R_{mt}]d\tau.
\end{aligned}
\tag{3}
$$

Next, apply the Caputo type derivative having order $(\imath-1)$ and substituting (2) in (3) yields to,

$$\begin{aligned}
{}^{CF}D_t^{\imath-1}[\tfrac{dS}{dt}] &= {}^{CF}D_t^{\imath-1}I^{-(\imath-1)}[\Pi+\alpha R_e+\beta R_t+\gamma R_m+\delta R_{et}+\zeta R_{em}+\eta R_{mt}-(\lambda_1+\lambda_2+\lambda_3+\mu)S],\\
{}^{CF}D_t^{\imath-1}[\tfrac{dI_m}{dt}] &= {}^{CF}D_t^{\imath-1}I^{-(\imath-1)}[\lambda_2 S-\omega\lambda_1 I_m-\kappa\lambda_3 I_m-(\sigma_1+\mu+\psi_2)I_m],\\
{}^{CF}D_t^{\imath-1}[\tfrac{dI_e}{dt}] &= {}^{CF}D_t^{\alpha-1}I^{-(\imath-1)}[\lambda_1 S-\chi\lambda_2 I_e-\rho\lambda_3 I_e-(\sigma_3+\mu+\psi_1)I_e],\\
{}^{CF}D_t^{\imath-1}[\tfrac{dI_t}{dt}] &= {}^{CF}D_t^{\imath-1}I^{-(\imath-1)}[\lambda_3 S-\phi\lambda_2 I_t-\vartheta\lambda_1 I_t-(\sigma_2+\mu+\psi_3)I_t],\\
{}^{CF}D_t^{\alpha-1}[\tfrac{dI_{em}}{dt}] &= {}^{CF}D_t^{\imath-1}I^{-(\imath-1)}[\omega\lambda_1 I_m+\chi\lambda_2 I_e-(\tau_2+\mu+\psi_1+\psi_2)I_{em}],\\
{}^{CF}D_t^{\imath-1}[\tfrac{dI_{et}}{dt}] &= {}^{CF}D_t^{\imath-1}I^{-(\imath-1)}[\vartheta\lambda_1 I_t+\rho\lambda_3 I_e-(\psi_1+\psi_3+\mu+\tau_3)I_{et}],\\
{}^{CF}D_t^{\imath-1}[\tfrac{dI_{mt}}{dt}] &= {}^{CF}D_t^{\imath-1}I^{-(\imath-1)}[\phi\lambda_2 I_t+\kappa\lambda_3 I_m-(\mu+\psi_2+\psi_3+\tau_1)I_{mt}],\\
{}^{CF}D_t^{\imath-1}[\tfrac{dR_e}{dt}] &= {}^{CF}D_t^{\imath-1}I^{-(\imath-1)}[\sigma_3 I_e+e\tau_2 I_{em}+\tau_3 g(1-e)I_{et}-(\mu+\alpha)R_e],\\
{}^{CF}D_t^{\imath-1}[\tfrac{dR_m}{dt}] &= {}^{CF}D_t^{\imath-1}I^{-(\imath-1)}[\sigma_1 I_m+\tau_1 eI_{mt}+\tau_2 g(1-e)I_{em}-(\gamma+\mu)R_m],\\
{}^{CF}D_t^{\imath-1}[\tfrac{dR_t}{dt}] &= {}^{CF}D_t^{\imath-1}I^{-(\imath-1)}[\sigma_2 I_t+\tau_3 eI_{et}+\tau_1 g(1-e)I_{mt}-(\mu+\beta)R_t],\\
{}^{CF}D_t^{\imath-1}[\tfrac{dR_{et}}{dt}] &= {}^{CF}D_t^{\imath-1}I^{-(\imath-1)}[\tau_3(1-e)(1-g)I_{et}-(\mu+\delta)R_{et}],\\
{}^{CF}D_t^{\imath-1}[\tfrac{dR_{em}}{dt}] &= {}^{CF}D_t^{\imath-1}I^{-(\imath-1)}[\tau_2(1-g)(1-e)I_{em}-(\zeta+\mu)R_{em}],\\
{}^{CF}D_t^{\imath-1}[\tfrac{dR_{mt}}{dt}] &= {}^{CF}D_t^{\imath-1}I^{-(\imath-1)}[\tau_1(1-g)(1-e)I_{mt}-(\eta+\mu)R_{mt}].
\end{aligned} \tag{4}$$

In general the dimensions of the fractional derivatives may not coincide with the dimensions of the rates. The proposed model represents the population dynamics with dimension of time. So, the dimensions of the time dependent parameters in the integer-order model should be adjusted to have a balance of the dimensions. In order to do this, $\frac{d}{dt}$ has the dimension of day^{-1}, $\frac{d^{\imath}}{dt^{\imath}}$ has the unit of $(day)^{-\imath}$, taking $0<\imath\leq 1$ and $\imath$ a parameter that possesses the dimension of day, then the dimension of $[\frac{1}{\imath^{(1-\imath)}}\frac{d^{\imath}}{dt^{\imath}}]$ is $(day)^{-1}$ [36]. As a result, the fractional version of (4) can be introduced in the following way [?]

$$\begin{aligned}
\frac{1}{\tau^{1-\imath}}{}^{CF}D_t^{\imath}S(t) &= \bar{\Pi}^{\imath}+\bar{\alpha}^{\imath}R_e+\bar{\beta}^{\imath}R_t+\bar{\gamma}^{\imath}R_m+\bar{\delta}^{\imath}R_{et}+\bar{\zeta}^{\imath}R_{em}+\bar{\eta}^{\imath}R_{mt}-(\lambda_1+\lambda_2+\lambda_3+\bar{\mu}^{\imath})S,\\
\frac{1}{\tau^{1-\imath}}{}^{CF}D_t^{\imath}I_m(t) &= \lambda_2 S-\bar{\omega}^{\imath}\lambda_1 I_m-\bar{\kappa}^{\imath}\lambda_3 I_m-(\bar{\sigma}_1^{\imath}+\bar{\mu}^{\imath}+\bar{\psi}_2^{\imath})I_m,\\
\frac{1}{\tau^{1-\imath}}{}^{CF}D_t^{\imath}I_e(t) &= \lambda_1 S-\bar{\chi}^{\imath}\lambda_2 I_e-\bar{\rho}^{\imath}\lambda_3 I_e-(\bar{\sigma}_3^{\imath}+\bar{\mu}^{\imath}+\bar{\psi}_1^{\imath})I_e,\\
\frac{1}{\tau^{1-\imath}}{}^{CF}D_t^{\imath}I_t(t) &= \lambda_3 S-\bar{\phi}^{\imath}\lambda_2 I_t-\bar{\vartheta}^{\imath}\lambda_1 I_t-(\bar{\sigma}_2^{\imath}+\bar{\mu}^{\imath}+\bar{\psi}_3^{\imath})I_t,\\
\frac{1}{\tau^{1-\imath}}{}^{CF}D_t^{\imath}I_{em}(t) &= \bar{\omega}^{\imath}\lambda_1 I_m+\bar{\chi}^{\imath}\lambda_2 I_e-(\bar{\tau}_2^{\imath}+\bar{\mu}^{\imath}+\bar{\psi}_1^{\imath}+\bar{\psi}_2^{\imath})I_{em},\\
\frac{1}{\tau^{1-\imath}}{}^{CF}D_t^{\imath}I_{et}(t) &= \bar{\vartheta}^{\imath}\lambda_1 I_t+\bar{\rho}^{\imath}\lambda_3 I_e-(\bar{\psi}_1^{\imath}+\bar{\psi}_3^{\imath}+\bar{\mu}^{\imath}+\bar{\tau}_3^{\imath})I_{et},\\
\frac{1}{\tau^{1-\imath}}{}^{CF}D_t^{\imath}I_{mt}(t) &= \bar{\phi}^{\imath}\lambda_2 I_t+\bar{\kappa}^{\imath}\lambda_3 I_m-(\bar{\mu}^{\imath}+\bar{\psi}_2^{\imath}+\bar{\psi}_3^{\imath}+\bar{\tau}_1^{\imath})I_{mt}, \qquad (5)\\
\frac{1}{\tau^{1-\imath}}{}^{CF}D_t^{\imath}R_m(t) &= \bar{\sigma}_3^{\imath}I_e+\bar{e}\bar{\tau}_2^{\imath}I_{em}+\bar{\tau}_3^{\imath}\bar{g}^{\imath}(1-\bar{e}^{\imath})I_{et}-(\bar{\mu}^{\imath}+\bar{\alpha}^{\imath})R_e,\\
\frac{1}{\tau^{1-\imath}}{}^{CF}D_t^{\imath}R_e(t) &= \bar{\sigma}_1^{\imath}I_m+\bar{\tau}_1^{\imath}\bar{e}^{\imath}I_{mt}+\bar{\tau}_2^{\imath}\bar{g}^{\imath}(1-\bar{e}^{\imath})I_{em}-(\bar{\gamma}^{\imath}+\bar{\mu}^{\imath})R_m,\\
\frac{1}{\tau^{1-\imath}}{}^{CF}D_t^{\imath}R_t(t) &= \bar{\sigma}_2^{\imath}I_t+\bar{\tau}_3^{\imath}\bar{e}^{\imath}I_{et}+\bar{\tau}_1^{\imath}\bar{g}^{\imath}(1-\bar{e}^{\imath})I_{mt}-(\bar{\mu}^{\imath}+\bar{\beta}^{\imath})R_t,\\
\frac{1}{\tau^{1-\imath}}{}^{CF}D_t^{\imath}R_{em}(t) &= \bar{\tau}_3^{\imath}(1-\bar{e}^{\imath})(1-\bar{g}^{\imath})I_{et}-(\bar{\mu}^{\imath}+\bar{\delta}^{\imath})R_{et},\\
\frac{1}{\tau^{1-\imath}}{}^{CF}D_t^{\imath}R_{et}(t) &= \bar{\tau}_2^{\imath}(1-\bar{g}^{\imath})(1-\bar{e}^{\imath})I_{em}-(\bar{\zeta}^{\imath}+\bar{\mu}^{\imath})R_{em},\\
\frac{1}{\tau^{1-\imath}}{}^{CF}D_t^{\imath}R_{mt}(t) &= \bar{\tau}_1^{\imath}(1-\bar{g}^{\imath})(1-\bar{e}^{\imath})I_{mt}-(\bar{\eta}^{\imath}+\bar{\mu}^{\imath})R_{mt}.
\end{aligned}$$

The operators ${}^{c}D_t^{\imath-1}$ and $I^{-(\imath-1)}$ are inverse of each other and Naturally, if $\chi=\tau^{(1-\imath)}\bar{\chi}$, for every constant χ, we may rewrite the system as

$$
\begin{aligned}
{}^{CF}D_t^{\imath}S(t) &= \Pi+\alpha R_e+\beta R_t+\gamma R_m+\delta R_{et}+\zeta R_{em}+\eta R_{mt}-(\lambda_1+\lambda_2+\lambda_3+\mu)S,\\
{}^{CF}D_t^{\imath}I_m(t) &= \lambda_2 S-\omega\lambda_1 I_m-\kappa\lambda_3 I_m-(\sigma_1+\mu+\psi_2)I_m,\\
{}^{CF}D_t^{\imath}I_e(t) &= \lambda_1 S-\chi\lambda_2 I_e-\rho\lambda_3 I_e-(\sigma_3+\mu+\psi_1)I_e,\\
{}^{CF}D_t^{\imath}I_t(t) &= \lambda_3 S-\phi\lambda_2 I_t-\vartheta\lambda_1 I_t-(\sigma_2+\mu+\psi_3)I_t,\\
{}^{CF}D_t^{\imath}I_{em}(t) &= \omega\lambda_1 I_m+\chi\lambda_2 I_e-(\tau_2+\mu+\psi_1+\psi_2)I_{em},\\
{}^{CF}D_t^{\imath}I_{et}(t) &= \vartheta\lambda_1 I_t+\rho\lambda_3 I_e-(\psi_1+\psi_3+\mu+\tau_3)I_{et},\\
{}^{CF}D_t^{\imath}I_{mt}(t) &= \phi\lambda_2 I_t+\kappa\lambda_3 I_m-(\mu+\psi_2+\psi_3+\tau_1)I_{mt},\\
{}^{CF}D_t^{\imath}R_m(t) &= \sigma_3 I_e+e\tau_2 I_{em}+\tau_3 g(1-e)I_{et}-(\mu+\alpha)R_e,\\
{}^{CF}D_t^{\imath}R_e(t) &= \sigma_1 I_m+\tau_1 e I_{mt}+\tau_2 g(1-e)I_{em}-(\gamma+\mu)R_m,\\
{}^{CF}D_t^{\imath}R_t(t) &= \sigma_2 I_t+\tau_3 e I_{et}+\tau_1 g(1-e)I_{mt}-(\mu+\beta)R_t,\\
{}^{CF}D_t^{\imath}R_{em}(t) &= \tau_3(1-e)(1-g)I_{et}-(\mu+\delta)R_{et},\\
{}^{CF}D_t^{\imath}R_{et}(t) &= \tau_2(1-g)(1-e)I_{em}-(\zeta+\mu)R_{em},\\
{}^{CF}D_t^{\imath}R_{mt}(t) &= \tau_1(1-g)(1-e)I_{mt}-(\eta+\mu)R_{mt}.
\end{aligned}
\tag{6}
$$

Here ${}^{CF}D_t^{\imath}$ is the Caputo Fabrizio derivative of order $\imath$ for $\imath \in (0,1]$.

Theorem 3.1. *In the region R_{10}^{+}, the proposed fractional order model* (6) *exhibits a unique, bounded, and non-negative solution.*

Proof. According to Wei [20], the time interval $(0,\infty)$ ensures the existence and uniqueness of the solution to the model (6), while the non-negative region R_{13}^{+} must be considered as a positive invariant region. We observe from the model (6)

$$
\begin{aligned}
{}^{CF}D_t^{\imath}|_{S=0} &> 0 \quad , \\
{}^{CF}D_t^{\imath}|_{I_m=0} &= \frac{aI_{mt}}{S+I_e+I_t+I_{em}+I_{et}+I_{mt}+R_m+R_e+R_t+R_{em}+R_{et}+R_{mt}} > 0, \\
{}^{CF}D_t^{\imath}|_{I_e=0} &= \frac{cI_{em}}{S+I_m+I_t+I_{em}+I_{et}+I_{mt}+R_m+R_e+R_t+R_{em}+R_{et}+R_{mt}} > 0, \\
{}^{CF}D_t^{\imath}|_{I_t=0} &= \frac{bI_{et}}{S+I_e+I_m+I_{em}+I_{et}+I_{mt}+R_m+R_e+R_t+R_{em}+R_{et}+R_{mt}} > 0, \\
{}^{CF}D_t^{\imath}|_{I_{em}=0} &= \frac{\omega cI_eI_m}{S+I_e+I_t+I_m+I_{et}+I_{mt}+R_m+R_e+R_t+R_{em}+R_{et}+R_{mt}} + \lambda_2\chi I_e > 0, \\
{}^{CF}D_t^{\imath}|_{I_{et}=0} &= \frac{\rho bI_eI_t}{S+I_e+I_t+I_{em}+I_m+I_{mt}+R_m+R_e+R_t+R_{em}+R_{et}+R_{mt}} + \lambda_1\vartheta I_t > 0, \\
{}^{CF}D_t^{\imath}|_{I_{mt}=0} &= \frac{\phi aI_tI_m}{S+I_m+I_e+I_t+I_{em}+I_{et}+R_m+R_e+R_t+R_{em}+R_{et}+R_{mt}} + \lambda_3\kappa I_m > 0, \qquad (7) \\
{}^{CF}D_t^{\imath}|_{R_m=0} &= \sigma_3 I_e + e\tau_2 I_{em} + \tau_3 g(1-e)I_{et} > 0, \\
{}^{CF}D_t^{\imath}|_{R_e=0} &= \sigma_1 I_m + \tau_1 eI_{mt} + \tau_2 g(1-e)I_{em} > 0, \\
{}^{CF}D_t^{\imath}|_{R_t=0} &= \sigma_2 I_t + \tau_3 eI_{et} + \tau_1 g(1-e)I_{mt} > 0, \\
{}^{CF}D_t^{\imath}|_{R_{em}=0} &= \tau_3(1-e)(1-g)I_{et} > 0, \\
{}^{CF}D_t^{\imath}|_{R_{et}=0} &= \tau_2(1-g)(1-e)I_{em} > 0, \\
{}^{CF}D_t^{\imath}|_{R_{mt}=0} &= \tau_1(1-g)(1-e)I_{mt} > 0.
\end{aligned}
$$

From Remark 1 and system (7) the solution will remain
$\{S(0), I_m(0), I_e(0), I_t(0), I_{em}(0), I_{et}(0), I_{mt}(0), R_m(0), R_e(0), R_t(0), R_{em}(0), R_{et}(0), R_{mt}(0) \in R_{13}^+\}$.
Also in each line bounding the non-negative octant, the vector field points will remain in R_{13}^+. Therefore,the fractional model (6) of a solution
$\{S(t), I_m(t), I_e(t), I_t(t), I_{em}(t), I_{et}(t), I_{mt}(t), R_m(t), R_e(t), R_t(t), R_{em}(t), R_{et}(t), R_{mt}(t)\}$
is not negative if the initial condition is set positively invariant. □

3.2 Bounded and Existence of the System

Theorem 3.2. *The closed set*

$$
\Delta = \{S(t) + I_m(t) + I_e(t) + I_t(t) + I_{em}(t) + I_{et}(t) + I_{mt}(t) + R_m(t) + R_e(t) + R_t(t) + R_{em}(t) + R_{et}(t) + R_{mt}(t) \leq \tfrac{\Pi}{\mu}\}. \qquad (8)
$$

is positively invariant with respect to the model (6).

Proof. By summing up all the human equations of the model (6), the fractional derivative of the total human population can be expressed as

$$
\begin{aligned}
{}^{c}D_t^{\imath}N(t) &= \Pi - \mu N - \epsilon_1 I_e - \epsilon_2 I_m - \epsilon_3 I_t - (\epsilon_1+\epsilon_2)I_{em} - (\epsilon_1+\epsilon_3)I_{et} - (\epsilon_2-\epsilon_3)I_{mt} \\
&\leq \Pi - \mu N(t). \qquad (9)
\end{aligned}
$$

Taking Laplace transform of (9), we get

$$
S^{\alpha}N(s) - S^{(\alpha-1)}N(0) = \frac{\Pi}{S} - \mu N(s)
$$

by re-arrange that, we get

$$N(s) = \frac{\Pi}{S(S^\alpha + \mu)} + \frac{S^{(\alpha} - 1)}{S^\alpha + \mu} N(0). \tag{10}$$

Taking the inverse Laplace transform of (10), we have

$$N(t) = N(0)E_{\alpha,1}(-\mu t^\alpha) + \Pi t^\alpha E_{\alpha,\alpha+1}(-\mu t^\alpha), \tag{11}$$

where $E_{\alpha,\beta}$ is the Mittag-Leffler function. But the fact that the Mittag-Leffler function has an asymptotic behavior [18, 19], it follows that

$$E_{\alpha,1}N(t) = \Sigma_{k=0}^{\infty} \frac{N^k(t)}{\Gamma(\alpha k + 1)}, \alpha > 0, \tag{12}$$

$$E_{\alpha,\alpha+1}N(t) = \Sigma_{k=0}^{\infty} \frac{N^k(t)}{\Gamma(\alpha k + \alpha + 1)}, \alpha > 0. \tag{13}$$

Expanding (13), we have

$$E_{\alpha,1}N(t) = \frac{1}{\Gamma 1} + \frac{N(t)}{\Gamma(\alpha + 1)} + \frac{N^2(t)}{\Gamma(2\alpha + 1)} + ... \tag{14}$$

Expanding (14), we have

$$E_{\alpha,\alpha+1}N(t) = \frac{1}{\Gamma(\alpha + 1)} + \frac{N(t)}{\Gamma(2\alpha + 1)} + \frac{N^2(t)}{\Gamma(3\alpha + 1)} + ... \tag{15}$$

Since Mittag-Leffler function has an asymptotic property, we have

$$N(t) = 1 + O(N).$$

Taking limit as $k \to \infty$, we have $N(t) \approx 1$ Then, it is clear that Δ is a positive invariant set. Therefore, all solutions of the model with initial conditions in Δ remain in Δ for all $t > 0$. Hence, $\Delta = N(t) > 0$ implies that it is feasible with respect to the model (6). □

4 Existence and Uniqueness of the Presented Model

By using the fixed point theorem, we want to check the existence of the solution. From definition (1), we obtained that

$$\begin{aligned}
S(t)-S(0) &= {}_0^{CF}I_t^{\varepsilon}\{\Pi+\alpha R_e+\beta R_t+\gamma R_m+\delta R_{et}+\zeta R_{em}+\eta R_{mt}-(\lambda_1+\lambda_2+\lambda_3+\mu)S\},\\
I_m(t)-I_m(0) &= {}_0^{CF}I_t^{\varepsilon}\{\lambda_2 S-\omega\lambda_1 I_m-\kappa\lambda_3 I_m-(\sigma_1+\mu+\psi_2)I_m\},\\
I_e(t)-I_e(0) &= {}_0^{CF}I_t^{\varepsilon}\{\lambda_1 S-\chi\lambda_2 I_e-\rho\lambda_3 I_e-(\sigma_3+\mu+\psi_1)I_e\},\\
I_t(t)-I_t(0) &= {}_0^{CF}I_t^{\varepsilon}\{\lambda_3 S-\phi\lambda_2 I_t-\vartheta\lambda_1 I_t-(\sigma_2+\mu+\psi_3)I_t\},\\
I_{em}(t)-I_{em}(0) &= {}_0^{CF}I_t^{\varepsilon}\{\omega\lambda_1 I_m+\chi\lambda_2 I_e-(\tau_2+\mu+\psi_1+\psi_2)I_{em}\},\\
I_{et}(t)-I_{et}(0) &= {}_0^{CF}I_t^{\varepsilon}\{\vartheta\lambda_1 I_t+\rho\lambda_3 I_e-(\psi_1+\psi_3+\mu+\tau_3)I_{et}\},\\
I_{mt}(t)-I_{mt}(0) &= {}_0^{CF}I_t^{\varepsilon}\{\phi\lambda_2 I_t+\kappa\lambda_3 I_m-(\mu+\psi_2+\psi_3+\tau_1)I_{mt}\},\\
R_e(t)-R_e(0) &= {}_0^{CF}I_t^{\varepsilon}\{\sigma_3 I_e+e\tau_2 I_{em}+\tau_3 g(1-e)I_{et}-(\mu+\alpha)R_e\},\\
R_m(0)-R_m(0) &= {}_0^{CF}I_t^{\varepsilon}\{\sigma_1 I_m+\tau_1 e I_{mt}+\tau_2 g(1-e)I_{em}-(\gamma+\mu)R_m\},\\
R_t(t)-R_t(0) &= {}_0^{CF}I_t^{\varepsilon}\{\sigma_2 I_t+\tau_3 e I_{et}+\tau_1 g(1-e)I_{mt}-(\mu+\beta)R_t\},\\
R_{et}(t)-R_{et}(0) &= {}_0^{CF}I_t^{\varepsilon}\{\tau_3(1-e)(1-g)I_{et}-(\mu+\delta)R_{et}\},\\
R_{em}(t)-R_{em}(0) &= {}_0^{CF}I_t^{\varepsilon}\{\tau_2(1-g)(1-e)I_{em}-(\zeta+\mu)R_{em}\},\\
R_{mt}(t)-R_{mt}(t) &= {}_0^{CF}I_t^{\varepsilon}\{\tau_1(1-g)(1-e)I_{mt}-(\eta+\mu)R_{mt}\}.
\end{aligned} \tag{16}$$

For simplification, we consider the equations of the model (16)

$$
\begin{aligned}
H_1(t,S) &= \Pi + \alpha R_e + \beta R_t + \gamma R_m + \delta R_{et} + \zeta R_{em} + \eta R_{mt} - (\lambda_1+\lambda_2+\lambda_3+\mu)S,\\
H_2(t,I_m) &= \lambda_2 S - \omega\lambda_1 I_m - \kappa\lambda_3 I_m - (\sigma_1+\mu+\psi_2)I_m,\\
H_3(t,I_e) &= \lambda_1 S - \chi\lambda_2 I_e - \rho\lambda_3 I_e - (\sigma_3+\mu+\psi_1)I_e,\\
H_4(t,I_t) &= \lambda_3 S - \phi\lambda_2 I_t - \vartheta\lambda_1 I_t - (\sigma_2+\mu+\psi_3)I_t,\\
H_5(t,I_{em}) &= \omega\lambda_1 I_m + \chi\lambda_2 I_e - (\tau_2+\mu+\psi_1+\psi_2)I_{em},\\
H_6(t,I_{et}) &= \vartheta\lambda_1 I_t + \rho\lambda_3 I_e - (\psi_1+\psi_3+\mu+\tau_3)I_{et},\\
H_7(t,I_{mt}) &= \phi\lambda_2 I_t + \kappa\lambda_3 I_m - (\mu+\psi_2+\psi_3+\tau_1)I_{mt},\\
H_8(t,R_e) &= \sigma_3 I_e + e\tau_2 I_{em} + \tau_3 g(1-e)I_{et} - (\mu+\alpha)R_e,\\
H_9(t,R_m) &= \sigma_1 I_m + \tau_1 e I_{mt} + \tau_2 g(1-e)I_{em} - (\gamma+\mu)R_m,\\
H_{10}(t,R_t) &= \sigma_2 I_t + \tau_3 e I_{et} + \tau_1 g(1-e)I_{mt} - (\mu+\beta)R_t,\\
H_{11}(t,R_{et}) &= \tau_3(1-e)(1-g)I_{et} - (\mu+\delta)R_{et},\\
H_{12}(t,R_{em}) &= \tau_2(1-g)(1-e)I_{em} - (\zeta+\mu)R_{em},\\
H_{13}(t,R_{mt}) &= \tau_1(1-g)(1-e)I_{mt} - (\eta+\mu)R_{mt}.
\end{aligned}
\tag{17}
$$

14 Sep 2023 22:33:10 PDT
230505-Zaman Version 3 - Submitted to Rocky Mountain J. Math.

By utilizing the definition of fractional integral [20], we have

$$
\begin{aligned}
S(t) - S(0) &= \tfrac{2(1-\imath)}{(2-\imath)M(\imath)} H_1(t, S) + \tfrac{2\imath}{(2-\imath)M(\imath)} \int_0^t H_1(\zeta, S) d\zeta, \\
I_m(t) - I_m(0) &= \tfrac{2(1-\imath)}{(2-\imath)M(\imath)} H_2(t, I_m) + \tfrac{2\imath}{(2-\imath)M(\imath)} \int_0^t H_2(\zeta, I_m) d\zeta, \\
I_e(t) - I_e(0) &= \tfrac{2(1-\imath)}{(2-\imath)M(\imath)} H_3(t, I_e) + \tfrac{2\imath}{(2-\imath)M(\imath)} \int_0^t H_3(\zeta, I_e) d\zeta, \\
I_t(t) - I_t(0) &= \tfrac{2(1-\imath)}{(2-\imath)M(\imath)} H_4(t, I_t) + \tfrac{2\imath}{(2-\imath)M(\imath)} \int_0^t H_4(\zeta, I_t) d\zeta, \\
I_{em}(t) - I_{em}(0) &= \tfrac{2(1-\imath)}{(2-\imath)M(\imath)} H_5(t, I_{em}) + \tfrac{2\imath}{(2-\imath)M(\imath)} \int_0^t H_5(\zeta, I_{em}) d\zeta, \\
I_{et}(t) - I_{et}(0) &= \tfrac{2(1-\imath)}{(2-\imath)M(\imath)} H_6(t, I_{et}) + \tfrac{2\imath}{(2-\imath)M(\imath)} \int_0^t H_6(\zeta, I_{et}) d\zeta, \\
I_{mt}(t) - I_{mt}(0) &= \tfrac{2(1-\imath)}{(2-\imath)M(\imath)} H_7(t, I_{mt}) + \tfrac{2\imath}{(2-\imath)M(\imath)} \int_0^t H_7(\zeta, I_{mt}) d\zeta, \qquad (18) \\
R_m(t) - R_m(0) &= \tfrac{2(1-\imath)}{(2-\imath)M(\imath)} H_8(t, R_m) + \tfrac{2\imath}{(2-\imath)M(\imath)} \int_0^t H_8(\zeta, R_m) d\zeta, \\
R_e(t) - R_e(0) &= \tfrac{2(1-\imath)}{(2-\imath)M(\imath)} H_9(t, R_e) + \tfrac{2\imath}{(2-\imath)M(\imath)} \int_0^t H_9(\zeta, R_e) d\zeta, \\
R_t(t) - R_t(0) &= \tfrac{2(1-\imath)}{(2-\imath)M(\imath)} H_{10}(t, R_t) + \tfrac{2\imath}{(2-\imath)M(\imath)} \int_0^t H_{10}(\zeta, R_t) d\zeta, \\
R_{em}(t) - R_{em}(0) &= \tfrac{2(1-\imath)}{(2-\imath)M(\imath)} H_{11}(t, R_{em}) + \tfrac{2\imath}{(2-\imath)M(\imath)} \int_0^t H_{12}(\zeta, R_{em}) d\zeta, \\
R_{et}(t) - R_{et}(0) &= \tfrac{2(1-\imath)}{(2-\imath)M(\imath)} H_{12}(t, R_{et}) + \tfrac{2\imath}{(2-\imath)M(\imath)} \int_0^t H_{12}(\zeta, R_{et}) d\zeta, \\
R_{mt}(t) - R_{mt}(0) &= \tfrac{2(1-\imath)}{(2-\imath)M(\imath)} H_{13}(t, R_{mt}) + \tfrac{2\imath}{(2-\imath)M(\imath)} \int_0^t H_{13}(\zeta, R_{mt}) d\zeta.
\end{aligned}
$$

Theorem 4.1. *The kernels H_1, H_2, H_3, H_4, H_5 and H_6 hold the Lipschitz condition and contraction, if the inequalities $0 \leq \ell_i \leq 1$ holds for $i = 1(1)13$, where $\ell_1 = \{(c\Theta_1 + c\Theta_2 + a\Theta_3 + a\Theta_4 + b\Theta_5 + b\Theta_6 + \mu)\}$.*

Proof. Suppose S and S_1 are two functions, thus we have

$$
\begin{aligned}
\parallel H_1(t, S) - H_1(t, S_1) \parallel &= \parallel \{\Pi + \alpha R_e + \beta R_t + \gamma R_m + \delta R_{et} + \zeta R_{em} + \eta R_{mt} - (\lambda_1 + \lambda_2 + \lambda_3 + \mu)S\} - \{\Pi + \alpha R_e + \beta R_t \\
&+ \gamma R_m + \delta R_{et} + \zeta R_{em} + \eta R_{mt} - (\lambda_1 + \lambda_2 + \lambda_3 + \mu)S_1\} \parallel, \\
&= \parallel -\{(\lambda_1 + \lambda_2 + \lambda_3 + \mu)\}(S - S_1) \parallel, \\
&\leq \{(\lambda_1 + \lambda_2 + \lambda_3 + \mu)\} \parallel S - S_1 \parallel, \\
&\leq \{(c \parallel I_e \parallel + c \parallel I_{em} \parallel + a \parallel I_m \parallel + a \parallel I_{mt} \parallel + b \parallel I_t \parallel + b \parallel I_{et} \parallel + \mu)\} \parallel S - S_1 \parallel, \\
&\leq \{(\lambda_1 + \lambda_2 + \lambda_3 + \mu)\} \parallel S - S_1 \parallel, \\
&\leq \{(c\Theta_1 + c\Theta_2 + a\Theta_3 + a\Theta_4 + b\Theta_5 + b\Theta_6 + \mu)\} \parallel S - S_1 \parallel,
\end{aligned}
\qquad (19)
$$

Now by taking

$$\ell_1 = \{(c\Theta_1 + c\Theta_2 + a\Theta_3 + a\Theta_4 + b\Theta_5 + b\Theta_6 + \mu)\}$$

where $\parallel I_e \parallel \leq \Theta_1$, $\parallel I_{em} \parallel \leq \Theta_2$ $\parallel I_m \parallel \leq \Theta_3$, $\parallel I_{mt} \parallel \leq \Theta_4$ $\parallel I_t \parallel \leq \Theta_5$ and $\parallel I_{et} \parallel \leq \Theta_6$ all are bounded functions, so we have

$$\parallel H_1(t, S) - H_1(t, S_1) \parallel \leq \ell_1 \parallel S - S_1 \parallel . \qquad (20)$$

Thus, we can confirm that the Lipschitz condition is obtained for H_1 and for the rest of the cases, it can be easily verified, given that $0 \leq (c\Theta_1 + c\Theta_2 + a\Theta_3 + a\Theta_4 + b\Theta_5 + b\Theta_6 + \mu) \leq 1$ which provides a contraction.

$$\begin{aligned}
\| H_2(t, I_m) - H_2(t, I_{m1}) \| &\leq \ell_2 \| I_m - I_{m1} \| \\
\| H_3(t, I_e) - H_3(t, I_{e1}) \| &\leq \ell_3 \| I_e - I_{e1} \| \\
\| H_4(t, I_t) - H_4(t, I_{t1}) \| &\leq \ell_4 \| I_t - I_{t1} \| \\
\| H_5(t, I_{em}) - H_5(t, I_{em1}) \| &\leq \ell_5 \| I_{em} - I_{em1} \| \\
\| H_6(t, I_{et}) - H_6(t, I_{et1}) \| &\leq \ell_6 \| I_{et} - I_{et1} \| \\
\| H_7(t, I_{mt}) - H_7(t, I_{mt1}) \| &\leq \ell_7 \| I_{mt} - I_{mt1} \| \\
\| H_8(t, R_e) - H_8(t, R_{e1}) \| &\leq \ell_8 \| R_e - R_{e1} \| \\
\| H_9(t, R_m) - H_9(t, R_{m1}) \| &\leq \ell_9 \| R_m - R_{m1} \| \\
\| H_{10}(t, R_t) - H_{10}(t, R_{t1}) \| &\leq \ell_{10} \| R_t - R_{t1} \| \\
\| H_{11}(t, R_{em}) - H_{11}(t, R_{em1}) \| &\leq \ell_{11} \| R_{em} - R_{em1} \| \\
\| H_{12}(t, R_{et}) - H_{12}(t, R_{et1}) \| &\leq \ell_{12} \| R_{et} - R_{et1} \| \\
\| H_{13}(t, R_{mt}) - H_{13}(t, R_{mt1}) \| &\leq \ell_{13} \| R_{mt} - R_{mt1} \| .
\end{aligned} \tag{21}$$

□

The afore-mentioned equation (18), we get

$$\begin{aligned}
S(t) &= S(0) + \tfrac{2(1-\imath)}{(2-\imath)M(\imath)} H_1(t, S) + \tfrac{2\imath}{(2-\imath)M(\imath)} \int_0^t H_1(\zeta, S) d\zeta, \\
I_m(t) &= I_m(0) + \tfrac{2(1-\imath)}{(2-\imath)M(\imath)} H_2(t, I_m) + \tfrac{2\imath}{(2-\imath)M(\imath)} \int_0^t H_2(\zeta, I_m) d\zeta, \\
I_e(t) &= I_e(0) + \tfrac{2(1-\imath)}{(2-\imath)M(\imath)} H_3(t, I_e) + \tfrac{2\imath}{(2-\imath)M(\imath)} \int_0^t H_3(\zeta, I_e) d\zeta, \\
I_t(t) &= I_t(0) + \tfrac{2(1-\imath)}{(2-\imath)M(\imath)} H_4(t, I_t) + \tfrac{2\imath}{(2-\imath)M(\imath)} \int_0^t H_4(\zeta, I_t) d\zeta, \\
I_{em}(t) &= I_{em}(0) + \tfrac{2(1-\imath)}{(2-\imath)M(\imath)} H_5(t, I_{em}) + \tfrac{2\imath}{(2-\imath)M(\imath)} \int_0^t H_5(\zeta, I_{em}) d\zeta, \\
I_{et}(t) &= I_{et}(0) + \tfrac{2(1-\imath)}{(2-\imath)M(\imath)} H_6(t, I_{et}) + \tfrac{2\imath}{(2-\imath)M(\imath)} \int_0^t H_6(\zeta, I_{et}) d\zeta, \\
I_{mt}(t) &= I_{mt}(0) + \tfrac{2(1-\imath)}{(2-\imath)M(\imath)} H_7(t, I_{mt}) + \tfrac{2\imath}{(2-\imath)M(\imath)} \int_0^t H_7(\zeta, I_{mt}) d\zeta, \qquad (22) \\
R_m(t) &= R_m(0) + \tfrac{2(1-\imath)}{(2-\imath)M(\imath)} H_8(t, R_m) + \tfrac{2\imath}{(2-\imath)M(\imath)} \int_0^t H_8(\zeta, R_m) d\zeta, \\
R_e(t) &= R_e(0) + \tfrac{2(1-\imath)}{(2-\imath)M(\imath)} H_9(t, R_e) + \tfrac{2\imath}{(2-\imath)M(\imath)} \int_0^t H_9(\zeta, R_e) d\zeta, \\
R_t(t) &= R_t(0) + \tfrac{2(1-\imath)}{(2-\imath)M(\imath)} H_{10}(t, R_t) + \tfrac{2\imath}{(2-\imath)M(\imath)} \int_0^t H_{10}(\zeta, R_t) d\zeta, \\
R_{em}(t) &= R_{em}(0) + \tfrac{2(1-\imath)}{(2-\imath)M(\imath)} H_{11}(t, R_{em}) + \tfrac{2\imath}{(2-\imath)M(\imath)} \int_0^t H_{12}(\zeta, R_{em}) d\zeta, \\
R_{et}(t) &= R_{et}(0) + \tfrac{2(1-\imath)}{(2-\imath)M(\imath)} H_{12}(t, R_{et}) + \tfrac{2\imath}{(2-\imath)M(\imath)} \int_0^t H_{12}(\zeta, R_{et}) d\zeta, \\
R_{mt}(t) &= R_{mt}(0) + \tfrac{2(1-\imath)}{(2-\imath)M(\imath)} H_{13}(t, R_{mt}) + \tfrac{2\imath}{(2-\imath)M(\imath)} \int_0^t H_{13}(\zeta, R_{mt}) d\zeta.
\end{aligned}$$

We get the following recursive formula

$$\begin{aligned}
S_n(t) &= \frac{\Pi}{\mu} + \frac{2(1-\imath)}{(2-\imath)M(\imath)} H_1(t, S_{(n-1)}) + \frac{2\imath}{(2-\imath)M(\imath)} \int_0^t H_1(\zeta, S_{(n-1)}) d\zeta, \\
I_{m_n}(t) &= \frac{2(1-\imath)}{(2-\imath)M(\imath)} H_2(t, I_{m_{(n-1)}}) + \frac{2\imath}{(2-\imath)M(\imath)} \int_0^t H_2(\zeta, I_{m_{(n-1)}}) d\zeta, \\
I_{e_n}(t) &= \frac{2(1-\imath)}{(2-\imath)M(\imath)} H_3(t, I_{e_{(n-1)}}) + \frac{2\imath}{(2-\imath)M(\imath)} \int_0^t H_3(\zeta, I_{e_{(n-1)}}) d\zeta, \\
I_{t_n}(t) &= \frac{2(1-\imath)}{(2-\imath)M(\imath)} H_4(t, I_{t_{(n-1)}}) + \frac{2\imath}{(2-\imath)M(\imath)} \int_0^t H_4(\zeta, I_{t_{(n-1)}}) d\zeta, \\
I_{em_n}(t) &= \frac{2(1-\imath)}{(2-\imath)M(\imath)} H_5(t, I_{em_{(n-1)}}) + \frac{2\imath}{(2-\imath)M(\imath)} \int_0^t H_5(\zeta, I_{em_{(n-1)}}) d\zeta, \\
I_{et_n}(t) &= \frac{2(1-\imath)}{(2-\imath)M(\imath)} H_6(t, I_{et_{(n-1)}}) + \frac{2\imath}{(2-\imath)M(\imath)} \int_0^t H_6(\zeta, I_{et_{(n-1)}}) d\zeta, \\
I_{mt_n}(t) &= \frac{2(1-\imath)}{(2-\imath)M(\imath)} H_7(t, I_{mt_{(n-1)}}) + \frac{2\imath}{(2-\imath)M(\imath)} \int_0^t H_7(\zeta, I_{mt_{(n-1)}}) d\zeta, \qquad (23) \\
R_{m_n}(t) &= \frac{2(1-\imath)}{(2-\imath)M(\imath)} H_8(t, R_{m_{(n-1)}}) + \frac{2\imath}{(2-\imath)M(\imath)} \int_0^t H_8(\zeta, R_{m_{(n-1)}}) d\zeta, \\
R_{e_n}(t) &= \frac{2(1-\imath)}{(2-\imath)M(\imath)} H_9(t, R_{e_{(n-1)}}) + \frac{2\imath}{(2-\imath)M(\imath)} \int_0^t H_9(\zeta, R_{e_{(n-1)}}) d\zeta, \\
R_{t_n}(t) &= \frac{2(1-\imath)}{(2-\imath)M(\imath)} H_{10}(t, R_{t_{(n-1)}}) + \frac{2\imath}{(2-\imath)M(\imath)} \int_0^t H_{10}(\zeta, R_{t_{(n-1)}}) d\zeta, \\
R_{em_n}(t) &= \frac{2(1-\imath)}{(2-\imath)M(\imath)} H_{11}(t, R_{em_{(n-1)}}) + \frac{2\imath}{(2-\imath)M(\imath)} \int_0^t H_{12}(\zeta, R_{em_{(n-1)}}) d\zeta, \\
R_{et_n}(t) &= \frac{2(1-\imath)}{(2-\imath)M(\imath)} H_{12}(t, R_{et_{(n-1)}}) + \frac{2\imath}{(2-\imath)M(\imath)} \int_0^t H_{12}(\zeta, R_{et_{(n-1)}}) d\zeta, \\
R_{mt_n}(t) &= \frac{2(1-\imath)}{(2-\imath)M(\imath)} H_{13}(t, R_{mt_{(n-1)}}) + \frac{2\imath}{(2-\imath)M(\imath)} \int_0^t H_{13}(\zeta, R_{mt_{(n-1)}}) d\zeta.
\end{aligned}$$

14 Sep 2023 22:33:10 PDT
230505-Zaman Version 3 - Submitted to Rocky Mountain J. Math.

The difference between successive terms of the system (18) in recursive form is given below

$$\begin{aligned}
\Psi_{1n}(t)=S_n(t)-S_{n-1}(t) &= \tfrac{2(1-\imath)}{(2-\imath)M(\imath)}(H_1(t,S_{n-1})-H_1(t,S_{n-2}))+\tfrac{2\imath}{(2-\imath)M(\imath)}\int_0^t(H_1(\zeta,S_{n-1})-H_1(\zeta,S_{n-2})d\zeta,\\
\Psi_{2n}(t)=I_{m_n}(t)-I_{m_{(n-1)}}(t) &= \tfrac{2(1-\imath)}{(2-\imath)M(\imath)}(H_2(t,I_{m_{(n-1)}})-H_2(t,I_{m_{(n-2)}}))+\tfrac{2\imath}{(2-\imath)M(\imath)}\int_0^t(H_2(\zeta,I_{m_{(n-1)}})\\
&- H_2(\zeta,I_{m_{(n-2)}}))d\zeta,\\
\Psi_{3n}(t)=I_{e_n}(t)-I_{e_{(n-1)}}(t) &= \tfrac{2(1-\imath)}{(2-\imath)M(\imath)}(H_3(t,I_{e_{(n-1)}})\\
&- H_3(t,I_{e_{(n-2)}}))+\tfrac{2\imath}{(2-\imath)M(\imath)}\int_0^t(H_3(\zeta,I_{e_{(n-1)}})-H_3(\zeta,I_{e_{(n-2)}})d\zeta,\\
\Psi_{4n}(t)=I_{t_n}(t)-I_{t_{(n-1)}}(t) &= \tfrac{2(1-\imath)}{(2-\imath)M(\imath)}(H_4(t,I_{t_{(n-1)}})-H_4(t,I_{t_{(n-2)}}))+\tfrac{2\imath}{(2-\imath)M(\imath)}\int_0^t(H_4(\zeta,I_{t_{(n-1)}})\\
&- H_4(\zeta,I_{t_{(n-2)}}))d\zeta,\\
\Psi_{5n}(t)=I_{em_n}(t)-I_{em_{(n-1)}}(t) &= \tfrac{2(1-\imath)}{(2-\imath)M(\imath)}(H_5(t,I_{em_{(n-1)}})-H_5(t,I_{em_{(n-2)}}))+\tfrac{2\imath}{(2-\imath)M(\imath)}\int_0^t(H_5(\zeta,I_{em_{(n-1)}})\\
&- H_5(\zeta,I_{em_{(n-2)}}))d\zeta,\\
\Psi_{6n}(t)=I_{et_n}(t)-I_{et_{(n-1)}}(t) &= \tfrac{2(1-\imath)}{(2-\imath)M(\imath)}(H_6(t,I_{et_{(n-1)}})-H_6(t,I_{et_{(n-2)}}))+\tfrac{2\imath}{(2-\imath)M(\imath)}\int_0^t(H_6(\zeta,I_{et_{(n-1)}})\\
&- H_6(\zeta,I_{et_{(n-2)}}))d\zeta,\\
\Psi_{7n}(t)=I_{mt_n}(t)-I_{mt_{(n-1)}}(t) &= \tfrac{2(1-\imath)}{(2-\imath)M(\imath)}(H_7(t,I_{mt_{(n-1)}})-H_7(t,I_{mt_{(n-2)}}))+\tfrac{2\imath}{(2-\imath)M(\imath)}\int_0^t(H_7(\zeta,I_{mt_{(n-1)}})\\
&- H_7(\zeta,I_{mt_{(n-2)}}))d\zeta,\\
\Psi_{8n}(t)=R_{m_n}(t)-R_{m_{(n-1)}} &= \tfrac{2(1-\imath)}{(2-\imath)M(\imath)}(H_8(t,R_{m_{(n-1)}})-H_8(t,R_{m_{(n-2)}}))+\tfrac{2\imath}{(2-\imath)M(\imath)}\int_0^t(H_8(\zeta,R_{m_{(n-1)}})\\
&- H_8(\zeta,R_{m_{(n-2)}}))d\zeta,\\
\Psi_{9n}(t)=R_{e_n}(t)-R_{e_{(n-1)}} &= \tfrac{2(1-\imath)}{(2-\imath)M(\imath)}(H_9(t,R_{e_{(n-1)}})-H_9(t,R_{e_{(n-2)}}))+\tfrac{2\imath}{(2-\imath)M(\imath)}\int_0^t(H_9(\zeta,R_{e_{(n-1)}})\\
&- H_9(\zeta,R_{e_{(n-2)}}))d\zeta,\\
\Psi_{10n}(t)=R_{t_n}(t)-R_{t_{(n-1)}} &= \tfrac{2(1-\imath)}{(2-\imath)M(\imath)}(H_{10}(t,R_{t_{(n-1)}})-H_{10}(t,R_{t_{(n-2)}}))+\tfrac{2\imath}{(2-\imath)M(\imath)}\int_0^t(H_{10}(\zeta,R_{t_{(n-1)}})\\
&- H_{10}(\zeta,R_{t_{(n-2)}}))d\zeta,\\
\Psi_{11n}(t)=R_{em_n}(t)-R_{em_{(n-1)}} &= \tfrac{2(1-\imath)}{(2-\imath)M(\imath)}(H_{11}(t,R_{em_{(n-1)}})-H_{11}(t,R_{em_{(n-2)}}))+\tfrac{2\imath}{(2-\imath)M(\imath)}\int_0^t(H_{11}(\zeta,R_{em_{(n-1)}})\\
&- H_{11}(\zeta,R_{em_{(n-2)}}))d\zeta,\\
\Psi_{12n}(t)=R_{et_n}(t)-R_{et_{(n-1)}} &= \tfrac{2(1-\imath)}{(2-\imath)M(\imath)}(H_{12}(t,R_{et_{(n-1)}})-H_{12}(t,R_{et_{(n-2)}}))+\tfrac{2\imath}{(2-\imath)M(\imath)}\int_0^t(H_{12}(\zeta,R_{et_{(n-1)}})\\
&- H_{12}(\zeta,R_{et_{(n-2)}}))d\zeta,\\
\Psi_{13n}(t)=R_{mt_n}(t)-R_{mt_{(n-1)}} &= \tfrac{2(1-\imath)}{(2-\imath)M(\imath)}(H_{13}(t,R_{mt_{(n-1)}})-H_{13}(t,R_{mt_{(n-2)}}))+\tfrac{2\imath}{(2-\imath)M(\imath)}\int_0^t(H_{13}(\zeta,R_{mt_{(n-1)}})\\
&- H_{13}(\zeta,R_{mt_{(n-2)}}))d\zeta.
\end{aligned} \tag{24}$$

with the following initial conditions
$S_0=S(0)$, $I_{m(0)}=I_m(0)$, $I_{e(0)}=I_e(0)$, $I_{t(0)}=I_t(0)$, $I_{em(0)}=I_{em}(0)$, $I_{et(0)}=I_{et}(0)$, $I_{mt(0)}=I_{mt}(0)$, $R_{m(0)}=R_m(0)$, $R_{t(0)}=R_t(0)$, $R_{e(0)}=R_e(0)$, $R_{em(0)}=R_{em}(0)$, $R_{mt(0)}=R_{mt}(0)$, $R_{et(0)}=R_{et}(0)$,
By taking the norm of the second equation in the system (24),

$$\begin{aligned}
\parallel \Psi_{2n}(t)\parallel &= \parallel I_{m_{(n)}}(t)-I_{m_{(n-1)}}(t)\parallel=\parallel \tfrac{2(1-\imath)}{(2-\imath)M(\imath)}(H_2(t,I_{m_{(n-1)}})-H_2(t,I_{m_{(n-2)}}))+\tfrac{2\imath}{(2-\imath)M(\imath)}\int_0^t(H_2(\zeta,I_{m_{(n-1)}})\\
&- H_2(\zeta,I_{m_{(n-2)}}))d\zeta\parallel .
\end{aligned} \tag{25}$$

By utilizing the triangular inequality on equation (25), we obtain

$$\begin{aligned}\| I_{m_{(n)}}(t) - I_{m_{(n-1)}}(t) \| &\leq \tfrac{2(1-\imath)}{(2-\imath)M(\imath)} \| (H_2(t, I_{m_{(n-1)}}) - H_2(t, I_{m_{(n-2)}})) \| + \tfrac{2\imath}{(2-\imath)M(\imath)} \int_0^t \| (H_2(\zeta, I_{m_{(n-1)}}) \\ &- H_2(\zeta, I_{m_{(n-2)}}))d\zeta \| .\end{aligned} \tag{26}$$

By applying the Lipschitz condition to equation (26), we obtain

$$\| I_{m_{(n)}}(t) - I_{m_{(n-1)}}(t) \| \leq \frac{2(1-\imath)}{(2-\imath)M(\imath)} \ell_2 \| (I_{m_{(n-1)}} - I_{m_{(n-2)}}) \| + \frac{2\imath}{(2-\imath)M(\imath)} \int_0^t \ell_2 \| (I_{m_{(n-1)}} - I_{m_{(n-2)}}) \| d\zeta. \tag{27}$$

Thus, we have

$$\| \Psi_{2n} \| \leq \frac{2(1-\imath)}{(2-\imath)M(\imath)} \ell_2 \Psi_{2(n-1)} + \frac{2\imath}{(2-\imath)M(\imath)} \int_0^t \ell_2 \Psi_{2(n-1)} d\zeta. \tag{28}$$

Similarly, for other equations in (24), we have

$$\begin{aligned}
\| \Psi_{1n} \| &\leq \tfrac{2(1-\imath)}{(2-\imath)M(\imath)} \ell_1 \Psi_{1(n-1)} + \tfrac{2\imath}{(2-\imath)M(\imath)} \int_0^t \ell_1 \Psi_{1(n-1)} \| d\zeta, \\
\| \Psi_{3n} \| &\leq \tfrac{2(1-\imath)}{(2-\imath)M(\imath)} \ell_3 \Psi_{3(n-1)} + \tfrac{2\imath}{(2-\imath)M(\imath)} \int_0^t \ell_3 \Psi_{3(n-1)} \| d\zeta, \\
\| \Psi_{4n} \| &\leq \tfrac{2(1-\imath)}{(2-\imath)M(\imath)} \ell_4 \Psi_{4(n-1)} + \tfrac{2\imath}{(2-\imath)M(\imath)} \int_0^t \ell_4 \Psi_{4(n-1)} \| d\zeta, \\
\| \Psi_{5n} \| &\leq \tfrac{2(1-\imath)}{(2-\imath)M(\imath)} \ell_5 \Psi_{5(n-1)} + \tfrac{2\imath}{(2-\imath)M(\imath)} \int_0^t \ell_5 \Psi_{5(n-1)} \| d\zeta, \\
\| \Psi_{6n} \| &\leq \tfrac{2(1-\imath)}{(2-\imath)M(\imath)} \ell_6 \Psi_{6(n-1)} + \tfrac{2\imath}{(2-\imath)M(\imath)} \int_0^t \ell_6 \Psi_{6(n-1)} \| d\zeta, \\
\| \Psi_{7n} \| &\leq \tfrac{2(1-\imath)}{(2-\imath)M(\imath)} \ell_7 \Psi_{7(n-1)} + \tfrac{2\imath}{(2-\imath)M(\imath)} \int_0^t \ell_7 \Psi_{7(n-1)} \| d\zeta, \\
\| \Psi_{8n} \| &\leq \tfrac{2(1-\imath)}{(2-\imath)M(\imath)} \ell_8 \Psi_{8(n-1)} + \tfrac{2\imath}{(2-\imath)M(\imath)} \int_0^t \ell_8 \Psi_{8(n-1)} \| d\zeta, \\
\| \Psi_{9n} \| &\leq \tfrac{2(1-\imath)}{(2-\imath)M(\imath)} \ell_9 \Psi_{9(n-1)} + \tfrac{2\imath}{(2-\imath)M(\imath)} \int_0^t \ell_9 \Psi_{9(n-1)} \| d\zeta, \\
\| \Psi_{10n} \| &\leq \tfrac{2(1-\imath)}{(2-\imath)M(\imath)} \ell_{10} \Psi_{1(n-1)} + \tfrac{2\imath}{(2-\imath)M(\imath)} \int_0^t \ell_{10} \Psi_{10(n-1)} \| d\zeta, \\
\| \Psi_{11n} \| &\leq \tfrac{2(1-\imath)}{(2-\imath)} \ell_{11} \Psi_{11(n-1)} + \tfrac{2\imath}{(2-\imath)M(\imath)} \int_0^t \ell_{11} \Psi_{11(n-1)} \| d\zeta, \\
\| \Psi_{12n} \| &\leq \tfrac{2(1-\imath)}{(2-\imath)M(\imath)} \ell_{12} \Psi_{12(n-1)} + \tfrac{2\imath}{(2-\imath)M(\imath)} \int_0^t \ell_{12} \Psi_{12(n-1)} \| d\zeta, \\
\| \Psi_{13n} \| &\leq \tfrac{2(1-\imath)}{(2-\imath)M(\imath)} \ell_{13} \Psi_{13(n-1)} + \tfrac{2\imath}{(2-\imath)M(\imath)} \int_0^t \ell_{13} \Psi_{13(n-1)} \| d\zeta.
\end{aligned} \tag{29}$$

From above, we can write that

$$
\begin{aligned}
S_n(t) &= \Sigma_{i=n}^{n} \Psi_{1n}(t), \\
I_{m(n)}(t) &= \Sigma_{i=n}^{n} \Psi_{2n}(t), \\
I_{e(n)}(t) &= \Sigma_{i=n}^{n} \Psi_{3n}(t), \\
I_{t(n)}(t) &= \Sigma_{i=n}^{n} \Psi_{4n}(t), \\
I_{em(n)}(t) &= \Sigma_{i=n}^{n} \Psi_{5n}(t), \\
I_{et(n)}(t) &= \Sigma_{i=n}^{n} \Psi_{6n}(t), \\
I_{mt(n)}(t) &= \Sigma_{i=n}^{n} \Psi_{7n}(t), \\
R_{m(n)}(t) &= \Sigma_{i=n}^{n} \Psi_{8n}(t), \\
R_{e(n)}(t) &= \Sigma_{i=n}^{n} \Psi_{9n}(t), \\
R_{t(n)}(t) &= \Sigma_{i=n}^{n} \Psi_{10n}(t), \\
R_{em(n)}(t) &= \Sigma_{i=n}^{n} \Psi_{11n}(t), \\
R_{et(n)}(t) &= \Sigma_{i=n}^{n} \Psi_{12n}(t), \\
R_{mt(n)}(t) &= \Sigma_{i=n}^{n} \Psi_{13n}(t).
\end{aligned}
\tag{30}
$$

Theorem 4.2. *A coupled solution for the system of the proposed model exists if there exists a t_0 such that $i = 1, 2, 3, 4, ..., 13$.*

Proof. By verifying that the kernel satisfies the conditions given in (20) and (21), we can use the recursive technique to obtain the successive results for the equation (30).

$$
\begin{aligned}
\| \Psi_{1n} \| &\leq \| S_n(0) \| [(\tfrac{2(1-\imath)}{(2-\imath)M(\imath)}\ell_1) + \tfrac{2\imath}{(2-\imath)M(\imath)}\ell_1 t]^n, \\
\| \Psi_{2n} \| &\leq \| I_{m_n}(0) \| [\tfrac{(2(1-\imath)}{(2-\imath)M(\imath)}\ell_2) + \tfrac{(2\imath}{(2-\imath)M(\imath)}\ell_2 t]^n, \\
\| \Psi_{3n} \| &\leq \| I_{e_n}(0) \| [\tfrac{(2(1-\imath)}{(2-\imath)M(\imath)}\ell_3 + \tfrac{(2\imath}{(2-\imath)M(\imath)}\ell_3 t]^n, \\
\| \Psi_{4n} \| &\leq \| I_{t_n}(0) \| [(\tfrac{2(1-\imath)}{(2-\imath)}\ell_4) + \tfrac{2\imath}{(2-\imath)M(\imath)}\ell_4 t]^n, \\
\| \Psi_{5n} \| &\leq \| I_{em_n}(0) \| [(\tfrac{2(1-\imath)}{(2-\imath)M(\imath)}\ell_5) + \tfrac{2\imath}{(2-\imath)M(\imath)}\ell_5 t]^n, \\
\| \Psi_{6n} \| &\leq \| I_{et_n}(0) \| [(\tfrac{2(1-\imath)}{(2-\imath)M(\imath)}\ell_6) + \tfrac{2\imath}{(2-\imath)M(\imath)}\ell_6 t]^n, \\
\| \Psi_{7n} \| &\leq \| I_{mt_n}(0) \| [(\tfrac{2(1-\imath)}{(2-\imath)M(\imath)}\ell_7) + \tfrac{2\imath}{(2-\imath)}M(\imath)\ell_7 t]^n, \\
\| \Psi_{8n} \| &\leq \| R_{m_n}(0) \| [(\tfrac{2(1-\imath)}{(2-\imath)M(\imath)}\ell_8) + \tfrac{2\imath}{(2-\imath)M(\imath)}\ell_8 t]^n, \\
\| \Psi_{9n} \| &\leq \| R_{e_n}(0) \| [(\tfrac{2(1-\imath)}{(2-\imath)M(\imath)}\ell_9) + \tfrac{2\imath}{(2-\imath)M(\imath)}\ell_9 t]^n, \\
\| \Psi_{10n} \| &\leq \| R_{t_n}(0) \| [(\tfrac{2(1-\imath)}{(2-\imath)M(\imath)}\ell_{10}) + \tfrac{2\imath}{(2-\imath)M(\imath)}\ell_{10} t]^n, \\
\| \Psi_{11n} \| &\leq \| R_{em_n}(0) \| [(\tfrac{2(1-\imath)}{(2-\imath)M(\imath)}\ell_{11}) + \tfrac{2\imath}{(2-\imath)M(\imath)}\ell_{11} t]^n, \\
\| \Psi_{12n} \| &\leq \| R_{et_n}(0) \| [(\tfrac{2(1-\imath)}{(2-\imath)M(\imath)}\ell_{12}) + \tfrac{2\imath}{(2-\imath)M(\imath)}\ell_{12} t]^n, \\
\| \Psi_{13n} \| &\leq \| R_{mt_n}(0) \| [(\tfrac{2(1-\imath)}{(2-\imath)M(\imath)}\ell_{13}) + \tfrac{2\imath}{(2-\imath)M(\imath)}\ell_{13} t]^n.
\end{aligned}
\tag{31}
$$

Therefore, we can conclude that the system solution exists and also continuous. Further, to confirm that the above functions represent a solution of the model (6), we consider

$$\begin{aligned}
S(t)-S(0) &= S_n(t)-B_{1n}(t),\\
I_m(t)-I_m(0) &= I_{m(n)}(t)-B_{2n}(t),\\
I_e(t)-I_e(0) &= I_{e(n)}(t)-B_{3n}(t),\\
I_t(t)-I_t(0) &= I_{t(n)}(t)-B_{4n}(t),\\
I_{em}(t)-I_{em}(0) &= I_{em(n)}(t)-B_{5n}(t),\\
I_{et}(t)-I_{et}(0) &= I_{et(n)}(t)-B_{6n}(t),\\
I_{mt}(t)-I_{mt}(0) &= I_{mt(n)}(t)-B_{7n}(t),\\
R_m(t)-R_m(0) &= R_{m(n)}(t)-B_{8n}(t),\\
R_e(t)-R_e(0) &= R_{e(n)}(t)-B_{9n}(t),\\
R_t(t)-R_t(0) &= R_{t(n)}(t)-B_{10n}(t),\\
R_{em}(t)-R_{em}(0) &= R_{em(n)}(t)-B_{11n}(t),\\
R_{et}(t)-R_{et}(0) &= R_{et(n)}(t)-B_{12n}(t),\\
R_{mt}(t)-R_{mt}(0) &= R_{mt(n)}(t)-B_{13n}(t).
\end{aligned} \tag{32}$$

Taking norm of B_{2n}, we get

$$\begin{aligned}
\parallel B_{2n}\parallel &= \parallel \tfrac{2(1-\imath)}{(2-\imath)M(\imath)}(H_2(t,I_{m_{(n-1)}})-H_2(t,I_{m_{(n-2)}}))+\tfrac{2\imath}{2-\imath M(\imath)}\textstyle\int_0^t(H_2(\zeta,I_{m_{(n-1)}})-H_2(\zeta,I_{m_{(n-2)}}))d\zeta\parallel,\\
&\leq \tfrac{2(1-\imath)}{(2-\imath)M(\imath)}\parallel (H_2(t,I_{m_{(n-1)}})-H_2(t,I_{m_{(n-2)}}))\parallel+\tfrac{2\imath}{2-\imath M(\imath)}\textstyle\int_0^t\parallel (H_2(\zeta,I_{m_{(n-1)}})-H_2(\zeta,I_{m_{(n-2)}}))\parallel d\zeta,\\
&\leq \tfrac{2(1-\imath)}{(2-\imath)M(\imath)}\ell_2\parallel I_{m_{(n-1)}}-I_{m_{(n-2)}}\parallel+\tfrac{2\imath}{2-\imath M(\imath)}\ell_2 t\parallel I_{m(n-1)}-I_{m(n-2)}\parallel.
\end{aligned} \tag{33}$$

By applying the same procedure at t_1, we obtained

$$\parallel B_{2n}\parallel\leq\{\frac{2(1-\imath)}{2-\imath}+\frac{2\imath}{2-\imath M(\imath)}\}\ell_2^{n+1}b. \tag{34}$$

Applying limit to equation (34) as $n \to \infty$, we get $\| B_{2n}(t) \| \to 0$. In similar way we can proceed to show that

$$\begin{aligned}
\| B_{1n}(t) \| &\to 0,\\
\| B_{3n}(t) \| &\to 0,\\
\| B_{4n}(t) \| &\to 0,\\
\| B_{5n}(t) \| &\to 0,\\
\| B_{6n}(t) \| &\to 0,\\
\| B_{7n}(t) \| &\to 0,\\
\| B_{8n}(t) \| &\to 0,\\
\| B_{9n}(t) \| &\to 0,\\
\| B_{10n}(t) \| &\to 0,\\
\| B_{11n}(t) \| &\to 0,\\
\| B_{12n}(t) \| &\to 0,\\
\| B_{13n}(t) \| &\to 0.
\end{aligned} \tag{35}$$

Further, to establish the uniqueness of the solution, we assume the existence of another solution to the proposed model, denoted by $S_1(t)$, $I_{m1}(t)$, $I_{e1}(t)$, $I_{t1}(t)$, $I_{em1}(t)$, $I_{et1}(t)$, $I_{mt1}(t)$, $R_{m1}(t)$, $R_{e1}(t)$, $R_{t1}(t)$,$R_{em1}(t)$, $R_{et1}(t)$, and $R_{mt1}(t)$, and using the following proceeding, we have

$$I_m(t) - I_{m1}(t) = \frac{2(1-\imath)}{(2-\imath)M(\imath)}(H_2(t, I_m(t) - H_2(t, I_{m1}(t)) + \frac{2\imath}{2-\imath M(\imath)} \times \int_0^t (H_2(\zeta, I_m)(t) - H_2(\zeta, I_{m1})(t))d\zeta, \tag{36}$$

By applying the Lipschitz condition and taking the norm of equation (36), we obtain

$$\| I_m(t) - I_{m1}(t) \| = \frac{2(1-\imath)}{(2-\imath)M(\imath)}\ell_2 \| I_m(t) - I_{m1}(t) \| + \frac{2\imath}{(2-\imath)M(\imath)}\ell_2 t \| I_m(t) - I_{m1}(t) \| . \tag{37}$$

After some simplification, we get

$$\| I_m(t) - I_{m1}(t) \| (1 - \frac{2(1-\imath)}{(2-\imath)M(\imath)}\ell_2 + \frac{2\imath}{(2-\imath)M(\imath)}\ell_2 t) \leq 0. \tag{38}$$

□

Theorem 4.3. *The model* (6) *solution will be unique if* $\| I_m(t) - I_{m1}(t) \| (1 - \frac{2(1-\imath)}{(2-\imath)M(\imath)}\ell_2 + \frac{2\imath}{2-\imath M(\imath)}\ell_2 t) > 0$.

Proof. Let the condition (38) holds, then $\| I_m(t) - I_{m1}(t) \| (1 - \frac{2(1-\imath)}{(2-\imath)M(\imath)}\ell_2 + \frac{2\imath}{2-\imath M(\imath)}\ell_2 t) \leq 0$ which implies that $\| I_m(t) - I_{m1}(t) \| = 0$. Thus, we get $I_m(t) = I_{m1}(t)$. Similarly, we can prove for the remaining i.e

$$S(t) - S_1(t) = \frac{2(1-\imath)}{(2-\imath)M(\imath)}(H_1(t, S(t) - H_1(t, S_1(t)) + \frac{2\imath}{2-\imath M(\imath)} \times \int_0^t (H_1(\zeta, S)(t) - H_1(\zeta, S_1)(t))d\zeta, \tag{39}$$

By applying the Lipschitz condition and taking the norm of equation (39), we obtain

$$\| S(t) - S_1(t) \| = \frac{2(1-\imath)}{(2-\imath)M(\imath)}\ell_1 \| S(t) - S_1(t) \| + \frac{2\imath}{(2-\imath)M(\imath)}\ell_1 t \| S(t) - S_1(t) \| . \tag{40}$$

Can be written in simple form

$$\| S(t) - S_1(t) \| (1 - \frac{2(1-\imath)}{(2-\imath)M(\imath)}\ell_1 + \frac{2\imath}{(2-\imath)M(\imath)}\ell_1 t) \leq 0. \tag{41}$$

Which implies that $\| S(t) - S_1(t) \| = 0$, thus we get $S(t) = S_1(t)$

$$I_e(t) - I_{e1}(t) = \frac{2(1-\imath)}{(2-\imath)M(\imath)}(H_3(t, I_{e1}(t) - H_3(t, I_{e1}(t)) + \frac{2\imath}{2-\imath M(\imath)} \times \int_0^t (H_3(\zeta, I_e)(t) - H_3(\zeta, I_{e3})(t))d\zeta, \tag{42}$$

By applying the Lipschitz condition and taking the norm of equation (42), we obtain

$$\| I_e(t) - I_{e1}(t) \| = \frac{2(1-\imath)}{(2-\imath)M(\imath)}\ell_3 \| I_e(t) - I_{e1}(t) \| + \frac{2\imath}{(2-\imath)M(\imath)}\ell_3 t \| I_e(t) - I_{e1}(t) \| . \tag{43}$$

Which simplifies to

$$\| I_e(t) - I_{e1}(t) \| (1 - \frac{2(1-\imath)}{(2-\imath)M(\imath)}\ell_3 + \frac{2\imath}{(2-\imath)M(\imath)}\ell_3 t) \leq 0. \tag{44}$$

Which implies that $\| I_e(t) - I_{e1}(t) \|$, thus we get $I_e(t) = I_{e1}(t)$

$$I_t(t) - I_{t1}(t) = \frac{2(1-\imath)}{(2-\imath)M(\imath)}(H_4(t, I_{t1}(t) - H_4(t, I_{t1}(t)) + \frac{2\imath}{2-\imath M(\imath)} \times \int_0^t (H_4(\zeta, I_t)(t) - H_4(\zeta, I_{t3})(t))d\zeta, \tag{45}$$

By applying the Lipschitz condition and taking the norm of equation (45), we obtain

$$\| I_t(t) - I_{t1}(t) \| = \frac{2(1-\imath)}{(2-\imath)M(\imath)}\ell_4 \| I_t(t) - I_{t1}(t) \| + \frac{2\imath}{(2-\imath)M(\imath)}\ell_4 t \| I_t(t) - I_{t1}(t) \| . \tag{46}$$

Which simplifies to

$$\| I_t(t) - I_{t1}(t) \| (1 - \frac{2(1-\imath)}{(2-\imath)M(\imath)}\ell_4 + \frac{2\imath}{(2-\imath)M(\imath)}\ell_4 t) \leq 0. \tag{47}$$

Which implies that $\| I_t(t) - I_{t1}(t) \|$, thus we get $I_t(t) = I_{t1}(t)$

$$I_{em}(t) - I_{em1}(t) = \frac{2(1-\imath)}{(2-\imath)M(\imath)}(H_5(t, I_{em1}(t) - H_5(t, I_{em1}(t)) + \frac{2\imath}{2-\imath M(\imath)} \times \int_0^t (H_5(\zeta, I_{em})(t) - H_5(\zeta, I_{em3})(t))d\zeta, \tag{48}$$

By applying the Lipschitz condition and taking the norm of equation (48), we obtain

$$\| I_{em}(t) - I_{em1}(t) \| = \frac{2(1-\imath)}{(2-\imath)M(\imath)}\ell_5 \| I_{em}(t) - I_{em1}(t) \| + \frac{2\imath}{(2-\imath)M(\imath)}\ell_5 t \| I_{em}(t) - I_{em1}(t) \| . \tag{49}$$

Which simplifies to

$$\| I_{em}(t) - I_{em1}(t) \| (1 - \frac{2(1-\imath)}{(2-\imath)M(\imath)}\ell_5 + \frac{2\imath}{(2-\imath)M(\imath)}\ell_5 t) \leq 0. \tag{50}$$

Which implies that $\| I_{em}(t) - I_{em1}(t) \|$, thus we get $I_{em}(t) = I_{em1}(t)$

$$I_{et}(t) - I_{et1}(t) = \frac{2(1-\imath)}{(2-\imath)M(\imath)}(H_6(t, I_{et1}(t) - H_6(t, I_{et1}(t)) + \frac{2\imath}{2-\imath M(\imath)} \times \int_0^t (H_6(\zeta, I_{et})(t) - H_6(\zeta, I_{et1})(t))d\zeta, \tag{51}$$

By applying the Lipschitz condition and taking the norm of equation (51), we obtain

$$\| I_{et}(t) - I_{et1}(t) \| = \frac{2(1-\imath)}{(2-\imath)M(\imath)}\ell_6 \| I_{et}(t) - I_{et1}(t) \| + \frac{2\imath}{(2-\imath)M(\imath)}\ell_6 t \| I_{et}(t) - I_{et1}(t) \| . \tag{52}$$

After some simplification, we obtain

$$\| I_{et}(t) - I_{et1}(t) \| (1 - \frac{2(1-\imath)}{(2-\imath)M(\imath)}\ell_6 + \frac{2\imath}{(2-\imath)M(\imath)}\ell_6 t) \leq 0. \tag{53}$$

Which implies that $\| I_{et}(t) - I_{et1}(t) \|$, thus we get $I_{et}(t) = I_{et1}(t)$

$$I_{mt}(t)-I_{mt1}(t) = \frac{2(1-\imath)}{(2-\imath)M(\imath)}(H_7(t, I_{mt1}(t)-H_7(t, I_{mt1}(t))+\frac{2\imath}{2-\imath M(\imath)}\times\int_0^t (H_7(\zeta, I_{mt})(t)-H_7(\zeta, I_{mt1})(t))d\zeta, \tag{54}$$

By applying the Lipschitz condition and taking the norm of equation (54), we obtain

$$\| I_{mt}(t) - I_{mt1}(t) \|= \frac{2(1-\imath)}{(2-\imath)M(\imath)}\ell_7 \| I_{mt}(t) - I_{mt1}(t) \| +\frac{2\imath}{(2-\imath)M(\imath)}\ell_7 t \| I_{mt}(t) - I_{mt1}(t) \| . \tag{55}$$

After some simplification, we have

$$\| I_{mt}(t) - I_{mt1}(t) \| (1 - \frac{2(1-\imath)}{(2-\imath)M(\imath)}\ell_7 + \frac{2\imath}{(2-\imath)M(\imath)}\ell_7 t) \le 0. \tag{56}$$

Which implies that $\| I_{mt}(t) - I_{mt1}(t) \|$, thus we get $I_{mt}(t) = I_{mt1}(t)$

$$R_m(t)-R_{m1}(t) = \frac{2(1-\imath)}{(2-\imath)M(\imath)}(H_8(t, R_{m1}(t)-H_8(t, R_{m1}(t))+\frac{2\imath}{2-\imath M(\imath)}\times\int_0^t (H_8(\zeta, R_m)(t)-H_8(\zeta, R_{m1})(t))d\zeta, \tag{57}$$

By applying the Lipschitz condition and taking the norm of equation (58), we obtain

$$\| R_m(t) - R_{m1}(t) \|= \frac{2(1-\imath)}{(2-\imath)M(\imath)}\ell_8 \| R_m(t) - R_{m1}(t) \| +\frac{2\imath}{(2-\imath)M(\imath)}\ell_8 t \| R_m(t) - R_{m1}(t) \| . \tag{58}$$

After some simplification, we get

$$\| R_m(t) - R_{m1}(t) \| (1 - \frac{2(1-\imath)}{(2-\imath)M(\imath)}\ell_8 + \frac{2\imath}{(2-\imath)M(\imath)}\ell_8 t) \le 0. \tag{59}$$

Which implies that $\| R_m(t) - R_{m1}(t) \|$, thus we get $R_m(t) = R_{m1}(t)$

$$R_e(t) - R_{e1}(t) = \frac{2(1-\imath)}{(2-\imath)M(\imath)}(H_9(t, R_{e1}(t) - H_9(t, R_{e1}(t)) + \frac{2\imath}{2-\imath M(\imath)} \times \int_0^t (H_9(\zeta, R_e)(t) - H_9(\zeta, R_{e1})(t))d\zeta, \tag{60}$$

By applying the Lipschitz condition and taking the norm of equation (60), we obtain

$$\| R_e(t) - R_{e1}(t) \|= \frac{2(1-\imath)}{(2-\imath)M(\imath)}\ell_9 \| R_e(t) - R_{e1}(t) \| +\frac{2\imath}{(2-\imath)M(\imath)}\ell_9 t \| R_e(t) - R_{e1}(t) \| . \tag{61}$$

Which can further simplifies to

$$\| R_e(t) - R_{e1}(t) \| (1 - \frac{2(1-\imath)}{(2-\imath)M(\imath)}\ell_9 + \frac{2\imath}{(2-\imath)M(\imath)}\ell_9 t) \le 0. \tag{62}$$

Which implies that $\| R_e(t) - R_{e1}(t) \|$, thus we get $R_e(t) = R_{e1}(t)$

$$R_t(t)-R_{t1}(t) = \frac{2(1-\imath)}{(2-\imath)M(\imath)}(H_{10}(t, R_{t1}(t)-H_{10}(t, R_{t1}(t))+\frac{2\imath}{2-\imath M(\imath)}\times\int_0^t (H_{10}(\zeta, R_t)(t)-H_{10}(\zeta, R_{t1})(t))d\zeta, \tag{63}$$

By applying the Lipschitz condition and taking the norm of equation (63), we obtain

$$\| R_t(t) - R_{t1}(t) \|= \frac{2(1-\imath)}{(2-\imath)M(\imath)}\ell_{10} \| R_t(t) - R_{t1}(t) \| +\frac{2\imath}{(2-\imath)M(\imath)}\ell_{10} t \| R_t(t) - R_{t1}(t) \| . \tag{64}$$

Which simplifies to

$$\| R_t(t) - R_{t1}(t) \| (1 - \frac{2(1-\imath)}{(2-\imath)M(\imath)}\ell_{10} + \frac{2\imath}{(2-\imath)M(\imath)}\ell_{10} t) \le 0. \tag{65}$$

Which implies that $\| R_t(t) - R_{t1}(t) \|$, thus we get $R_t(t) = R_{t1}(t)$

$$R_{em}(t)-R_{em1}(t) = \frac{2(1-\imath)}{(2-\imath)M(\imath)}(H_{11}(t, R_{em1}(t)-H_{11}(t, R_{em1}(t))+\frac{2\imath}{2-\imath M(\imath)}\times\int_0^t (H_{11}(\zeta, R_{em})(t)-H_{11}(\zeta, R_{em1})(t))d\zeta, \tag{66}$$

By applying the Lipschitz condition and taking the norm of equation (**??**), we obtain

$$\| R_{em}(t) - R_{em1}(t) \| = \frac{2(1-\imath)}{(2-\imath)M(\imath)}\ell_{11} \| R_{em}(t) - R_{em1}(t) \| + \frac{2\imath}{(2-\imath)M(\imath)}\ell_{11}t \| R_{em}(t) - R_{em1}(t) \| . \quad (67)$$

After some simplification, we obtain

$$\| R_{em}(t) - R_{em1}(t) \| (1 - \frac{2(1-\imath)}{(2-\imath)M(\imath)}\ell_{11} + \frac{2\imath}{(2-\imath)M(\imath)}\ell_{11}t) \leq 0. \quad (68)$$

Which implies that $\| R_{em}(t) - R_{em1}(t) \|$, thus we get $R_{em}(t) = R_{em1}(t)$

$$R_{et}(t) - R_{et1}(t) = \frac{2(1-\imath)}{(2-\imath)M(\imath)}(H_{12}(t, R_{et1}(t) - H_{12}(t, R_{et1}(t)) + \frac{2\imath}{2-\imath M(\imath)} \times \int_0^t (H_{12}(\zeta, R_{et})(t) - H_{12}(\zeta, R_{et1})(t))d\zeta, \quad (69)$$

By applying the Lipschitz condition and taking the norm of equation (69), we obtain

$$\| R_{et}(t) - R_{et1}(t) \| = \frac{2(1-\imath)}{(2-\imath)M(\imath)}\ell_{12} \| R_{et}(t) - R_{et1}(t) \| + \frac{2\imath}{(2-\imath)M(\imath)}\ell_{12}t \| R_{et}(t) - R_{et1}(t) \| . \quad (70)$$

After some simplification, we get

$$\| R_{et}(t) - R_{et1}(t) \| (1 - \frac{2(1-\imath)}{(2-\imath)M(\imath)}\ell_{12} + \frac{2\imath}{(2-\imath)M(\imath)}\ell_{12}t) \leq 0. \quad (71)$$

Which implies that $\| R_{et}(t) - R_{et1}(t) \|$, thus we get $R_{et}(t) = R_{et1}(t)$

$$R_{mt}(t) - R_{mt1}(t) = \frac{2(1-\imath)}{(2-\imath)M(\imath)}(H_{13}(t, R_{mt1}(t) - H_{13}(t, R_{mt1}(t)) + \frac{2\imath}{2-\imath M(\imath)} \times \int_0^t (H_{13}(\zeta, R_{mt})(t) - H_{13}(\zeta, R_{mt1})(t))d\zeta, \quad (72)$$

By applying the Lipschitz condition and taking the norm of equation (72), we obtain

$$\| R_{mt}(t) - R_{mt1}(t) \| = \frac{2(1-\imath)}{(2-\imath)M(\imath)}\ell_{13} \| R_{mt}(t) - R_{mt1}(t) \| + \frac{2\imath}{(2-\imath)M(\imath)}\ell_{13}t \| R_{mt}(t) - R_{et1}(t) \| . \quad (73)$$

Which simplifies to

$$\| R_{mt}(t) - R_{mt1}(t) \| (1 - \frac{2(1-\imath)}{(2-\imath)M(\imath)}\ell_{13} + \frac{2\imath}{(2-\imath)M(\imath)}\ell_{13}t) \leq 0. \quad (74)$$

Which implies that $\| R_{mt}(t) - R_{mt1}(t) \|$, thus we get $R_{mt}(t) = R_{mt1}(t)$ □

5 Numerical Scheme and Simulations

In this section, we present numerical simulations of the proposed co-infections fractional order model, by taking into account the possible treatment, explore the impact of the fractional order $\imath$ and biologically significant parameters on the disease prevalence. To solve the proposed model (6) numerically, we have employed the technique of fractional Adams-Bashforth for the Caputo-Fabrizio (CF) fractional order derivative[26]. Specifically, we utilized the second equation of the system (6) and the system (17) along with the fundamental theorem of integration to derive the necessary numerical scheme.

$$I_m(t) - I_m(0) = \frac{1-\imath}{M(\imath)}H_2(t, I_m) + \frac{\imath}{M(\imath)}\int_0^\infty H_2(\imath, I_m)d\imath. \quad (75)$$

At $t = t_{n+1}$, we have

$$I_m(t_{n+1}) - I_m(0) = \frac{1-\imath}{M(\imath)}H_2(t_n, I_{m(n)}) + \frac{\imath}{M(\imath)}\int_0^{t_{n+1}} H_2(\imath, I_m)d\imath. \quad (76)$$

14 Sep 2023 22:33:10 PDT
230505-Zaman Version 3 - Submitted to Rocky Mountain J. Math.

and similarly

$$I_m(t_n) - I_m(0) = \frac{1-\imath}{M(\imath)} H_2(t_{n-1}, I_{m(n-1)}) + \frac{\imath}{M(\imath)} \int_0^{t_n} H_2(\imath, I_m) d\imath. \tag{77}$$

From the equation (76) and (77), we have

$$I_{m(n+1)} - I_{m(n)} = \frac{1-\imath}{M(\imath)} \{H_2(t_n, I_{m(n)}) - H_2(t_{n-1}, I_{m(n-1)})\} + \frac{\imath}{M(\imath)} \int_{t_n}^{t_{n+1}} H_2(\imath, I_m) d\imath. \tag{78}$$

To approximate $H_2(t, I_m)$, we can use Lagrange interpolation with a step size of $h = t_{i+1} - t_i$ and calculating the integral part in equation (78) over the interval $[t_n, t_{n+1}]$, to obtain

$$\begin{aligned} \textstyle\int_{t_n}^{t_{n+1}} H_2(t_n, I_{m(n)}) dt &= \textstyle\int_{t_n}^{t_{n+1}} [\frac{H_2(t_n, I_{m(n)})}{h}(t - t_{n-1}) - \frac{H_2(t_n, I_m)}{h}(t - t_n)] dt \\ &= \tfrac{3h}{2} H_2(t_n, I_{m(n)}) - \tfrac{h}{2} H_2(t_{n-1}, I_{m(n-1)}. \end{aligned} \tag{79}$$

By putting this approximated value in equation (78), we get

$$I_{m(n+1)} = I_{m(n)} + \{\frac{1-\imath}{M(\imath)} + \frac{3h\imath}{2M(\imath)}\}\{H_2(t_n, I_{m(n)}) - \{\frac{\imath}{M(\imath)} + \frac{h\imath}{2M(\imath)}\} H_2(t_{n-1}, I_{m(n-1)}). \tag{80}$$

The remaining equations can be written similarly as

$$\begin{aligned}
S_{(n+1)} &= S_{(n)} + \{\tfrac{1-\imath}{M(\imath)} + \tfrac{3h\imath}{2M(\imath)}\}\{H_1(t_n, S_{(n)}) - \{\tfrac{\imath}{M(\imath)} + \tfrac{h\imath}{2M(\imath)}\} H_1(t_{n-1}, S_{(n-1)}), \\
I_{e(n+1)} &= I_{e(n)} + \{\tfrac{1-\imath}{M(\imath)} + \tfrac{3h\imath}{2M(\imath)}\}\{H_3(t_n, I_{e(n)}) - \{\tfrac{\imath}{M(\imath)} + \tfrac{h\imath}{2M(\imath)}\} H_3(t_{n-1}, I_{e(n-1)}), \\
I_t t(n+1) &= I_{t(n)} + \{\tfrac{1-\imath}{M(\imath)} + \tfrac{3h\imath}{2M(\imath)}\}\{H_4(t_n, I_{t(n)}) - \{\tfrac{\imath}{M(\imath)} + \tfrac{h\imath}{2M(\imath)}\} H_4(t_{n-1}, I_{t(n-1)}), \\
I_{em(n+1)} &= I_{em(n)} + \{\tfrac{1-\imath}{M(\imath)} + \tfrac{3h\imath}{2M(\imath)}\}\{H_5(t_n, I_{em(n)}) - \{\tfrac{\imath}{M(\imath)} + \tfrac{h\imath}{2M(\imath)}\} H_5(t_{n-1}, I_{em(n-1)}), \\
I_{et(n+1)} &= I_{et(n)} + \{\tfrac{1-\imath}{M(\imath)} + \tfrac{3h\imath}{2M(\imath)}\}\{H_6(t_n, I_{et(n)}) - \{\tfrac{\imath}{M(\imath)} + \tfrac{h\imath}{2M(\imath)}\} H_6(t_{n-1}, I_{et(n-1)}), \\
I_{mt(n+1)} &= I_{mt(n)} + \{\tfrac{1-\imath}{M(\imath)} + \tfrac{3h\imath}{2M(\imath)}\}\{H_7(t_n, I_{mt(n)}) - \{\tfrac{\imath}{M(\imath)} + \tfrac{h\imath}{2M(\imath)}\} H_7(t_{n-1}, I_{mt(n-1)}), \\
R_{m(n+1)} &= R_{m(n)} + \{\tfrac{1-\imath}{M(\imath)} + \tfrac{3h\imath}{2M(\imath)}\}\{H_8(t_n, R_{m(n)}) - \{\tfrac{\imath}{M(\imath)} + \tfrac{h\imath}{2M(\imath)}\} H_8(t_{n-1}, R_{m(n-1)}), \\
R_{e(n+1)} &= R_{e(n)} + \{\tfrac{1-\imath}{M(\imath)} + \tfrac{3h\imath}{2M(\imath)}\}\{H_9(t_n, R_{e(n)}) - \{\tfrac{\imath}{M(\imath)} + \tfrac{h\imath}{2M(\imath)}\} H_9(t_{n-1}, R_{e(n-1)}), \\
R_{t(n+1)} &= R_{t(n)} + \{\tfrac{1-\imath}{M(\imath)} + \tfrac{3h\imath}{2M(\imath)}\}\{H_{10}(t_n, R_{t(n)}) - \{\tfrac{\imath}{M(\imath)} + \tfrac{h\imath}{2M(\imath)}\} H_{10}(t_{n-1}, R_{t(n-1)}), \\
R_{em(n+1)} &= R_{em(n)} + \{\tfrac{1-\imath}{M(\imath)} + \tfrac{3h\imath}{2M(\imath)}\}\{H_{11}(t_n, R_{em(n)}) - \{\tfrac{\imath}{M(\imath)} + \tfrac{h\imath}{2M(\imath)}\} H_{11}(t_{n-1}, R_{em(n-1)}), \\
R_{et(n+1)} &= R_{et(n)} + \{\tfrac{1-\imath}{M(\imath)} + \tfrac{3h\imath}{2M(\imath)}\}\{H_{12}(t_n, R_{et(n)}) - \{\tfrac{\imath}{M(\imath)} + \tfrac{h\imath}{2M(\imath)}\} H_{12}(t_{n-1}, R_{et(n-1)}), \\
R_{mt(n+1)} &= R_{mt(n)} + \{\tfrac{1-\imath}{M(\imath)} + \tfrac{3h\imath}{2M(\imath)}\}\{H_{13}(t_n, R_{mt(n)}) - \{\tfrac{\imath}{M(\imath)} + \tfrac{h\imath}{2M(\imath)}\} H_{13}(t_{n-1}, R_{mt(n-1)}).
\end{aligned} \tag{81}$$

The graphical results are obtained for various values of $\imath \in (0, 1]$ while the parameters used in simulations are $\Pi = 1$, $\alpha = 3 \times 10^{-3}$, $\beta = 3.96 \times 10^{-2}$, $\gamma = 2 \times 10^{-2}$, $\delta = 3.1 \times 10^{-3}$, $\zeta = 4 \times 10^{-4}$, $\eta = 0.05$, $\mu = 5.1 \times 10^{-3}$, $N = 50$, $\omega = 0.005$, $\kappa = 4 \times 10^{-4}$, $\sigma_1 = 0.0396$, $\sigma_2 = 0.02$, $\sigma_3 = 0.017$, $\psi_1 = 0.15$, $\psi_2 = 1.6 \times 10^{-3}$, $\psi_3 = 0.51$, $\tau_1 = 3.96 \times 10^{-2}$, $\tau_2 = 0.02$, $\tau_3 = 0.017$, $e = 0.00035$, $g = 1.2 \times 10^{-4}$, $a = 1 \times 10^{-8}$, $b = 1 \times 10^{-9}$, and $c = 1 \times 10^{-10}$. The time level is taken up to 30 days. Figure1, shows the dynamics of suspectable individuals for five different values of fractional order $\imath$. It is observed that the population in class S(t) increases for the decreasing the values of $\imath$. The effects of $\imath$ on the dynamics of remaining model classes i.e $(I_m, I_e, I_t, I_{em}, I_{et}, I_{mt})$ is depicted in Figures$(2-7)$. It is clear from the Figures(2-7) that the population in all infected classes decreases significantly for the decreasing the values of $\imath$. Also from figures(8,....,13) the

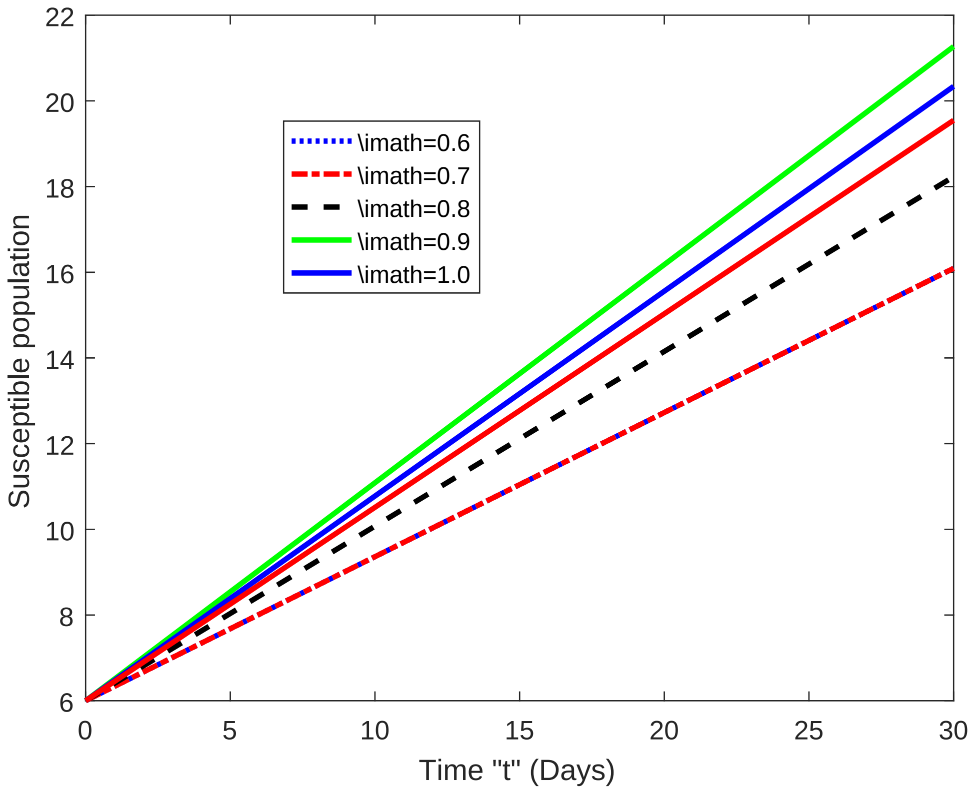

Figure 1: Simulation of susceptible human population

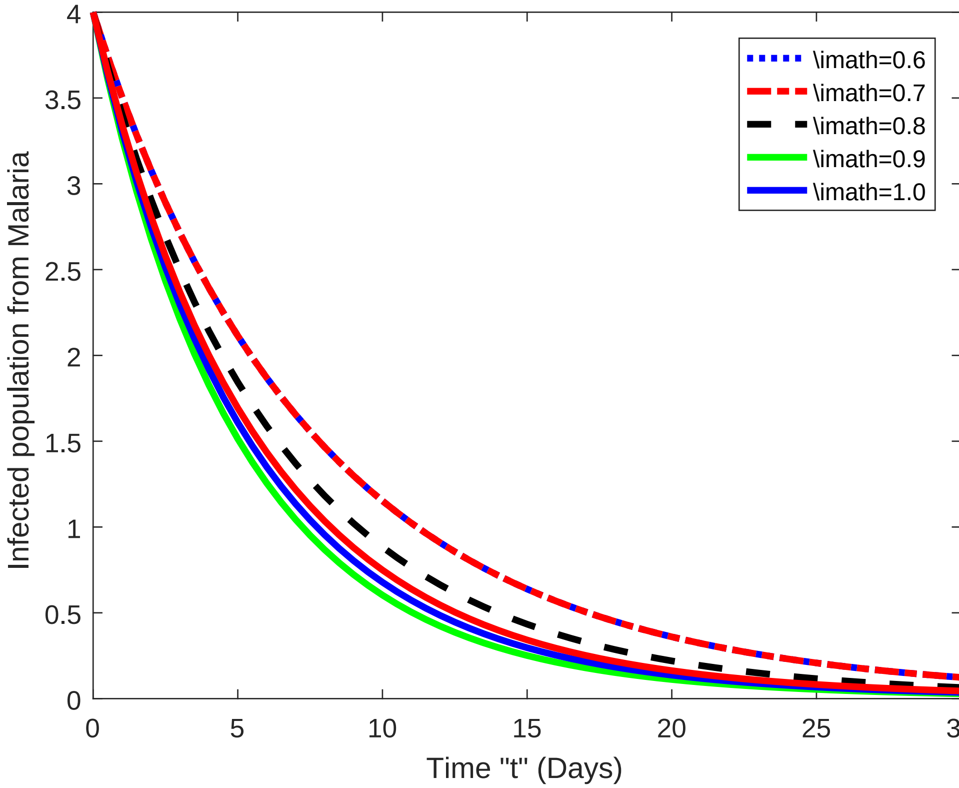

Figure 2: Simulation of infected human population from Malaria

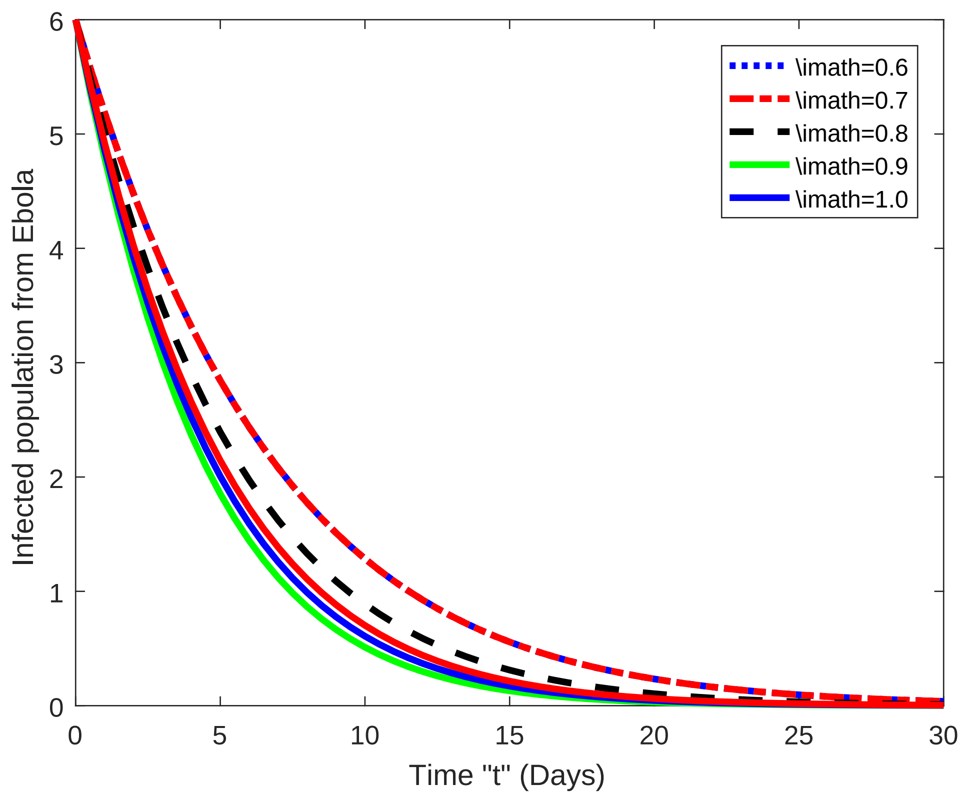

Figure 3: Simulation of infected human population from Ebola

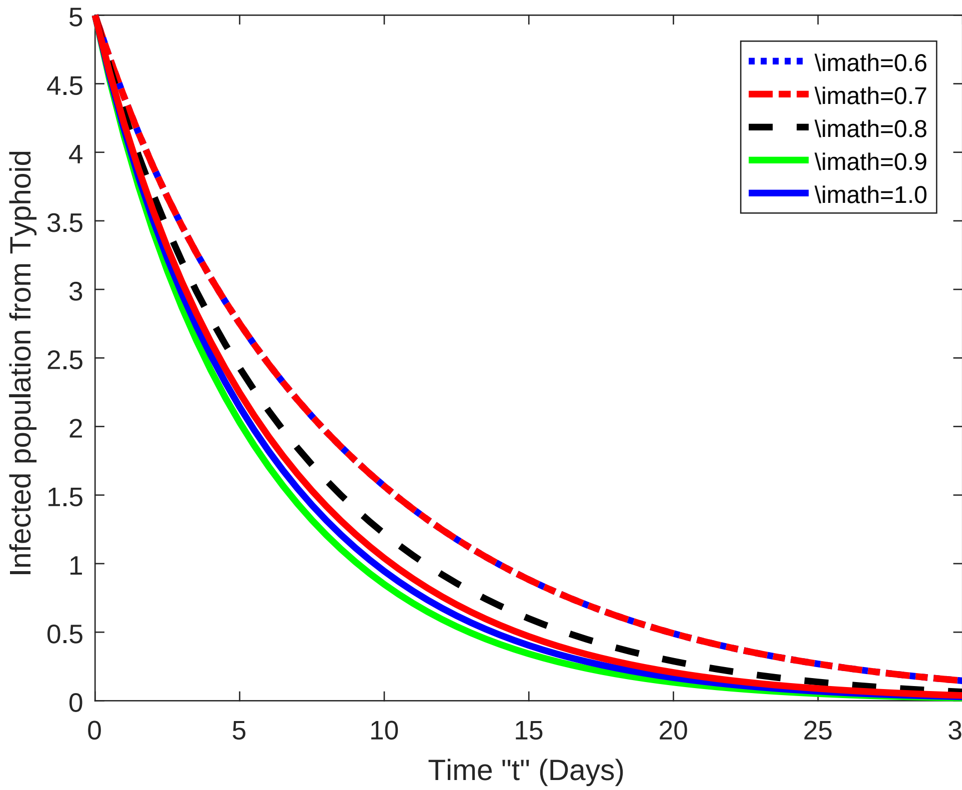

Figure 4: Simulation of infected human population from Typhoid

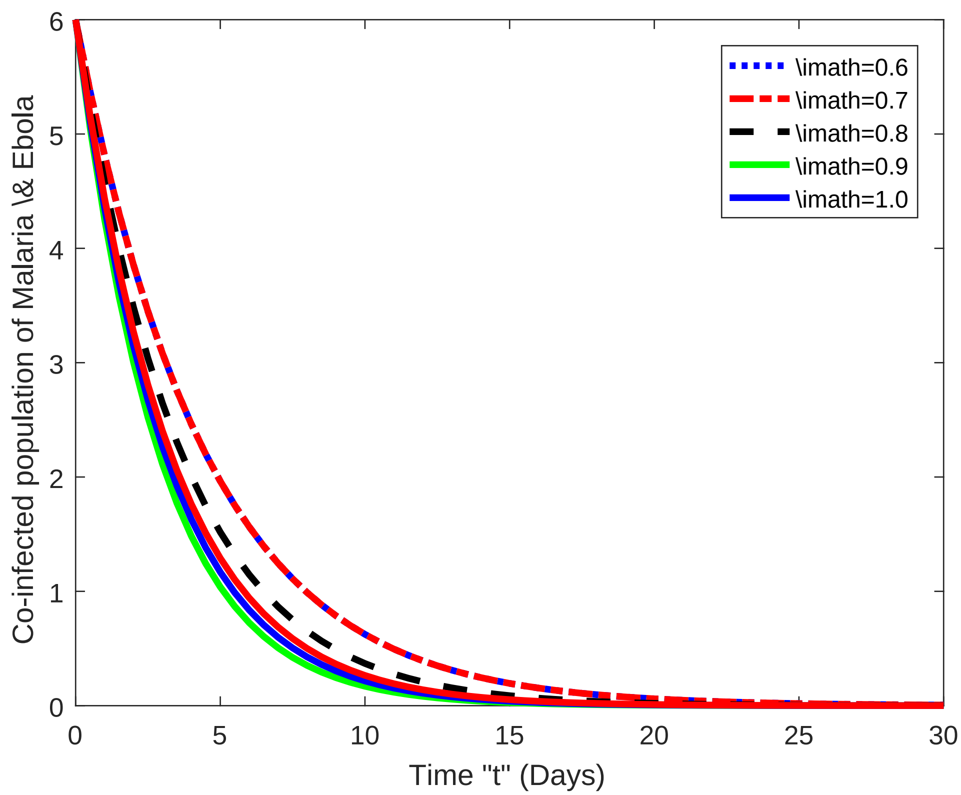

Figure 5: Simulation of co-infected human population from Malaria & Ebola

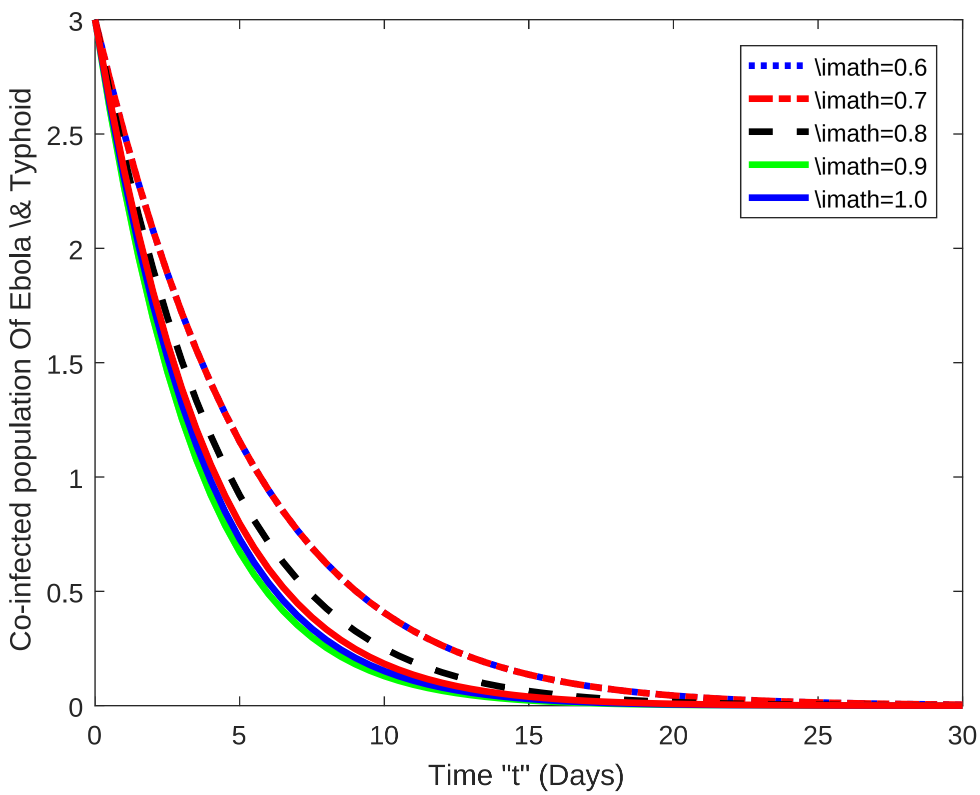

Figure 6: Simulation of co-infected human population from Ebola & Typhoid

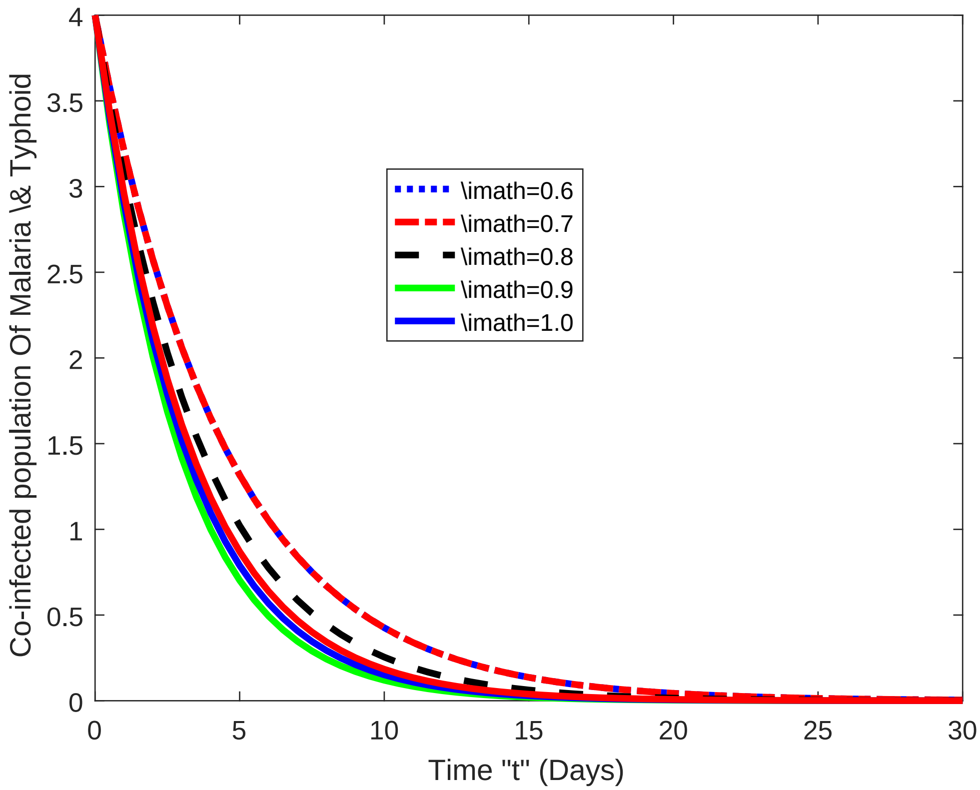

Figure 7: Simulation of co-infected human population from Malaria & Typhoid

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230505-Zaman Version 3 - Submitted to Rocky Mountain J. Math.

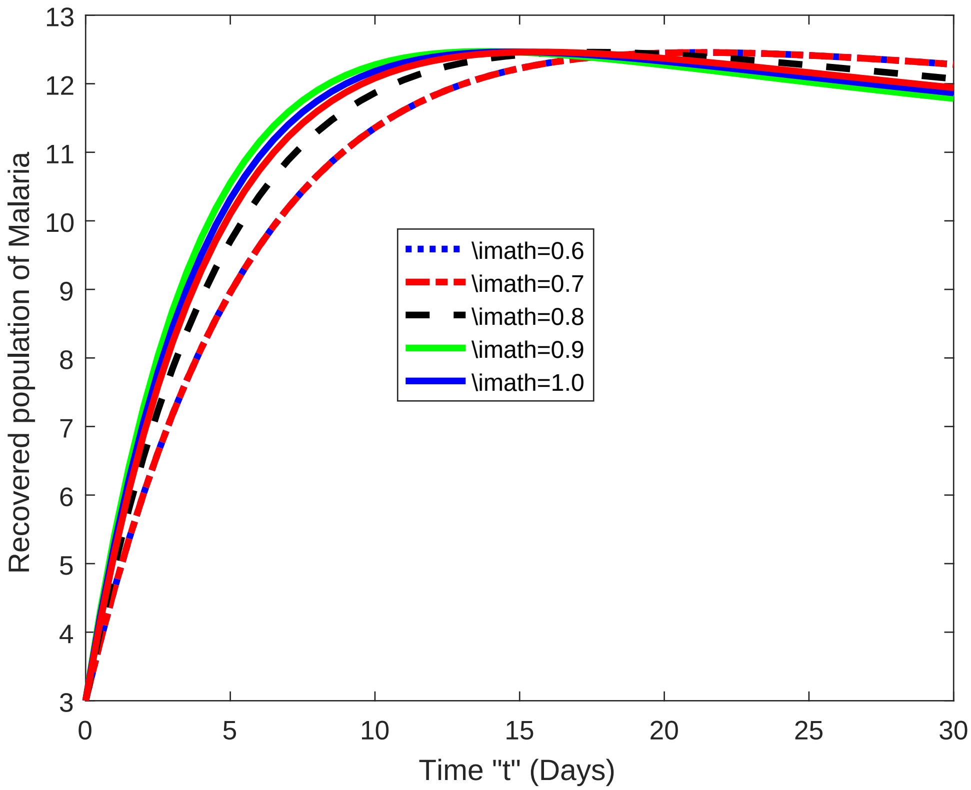

Figure 8: Simulation of recovered human population from Malaria

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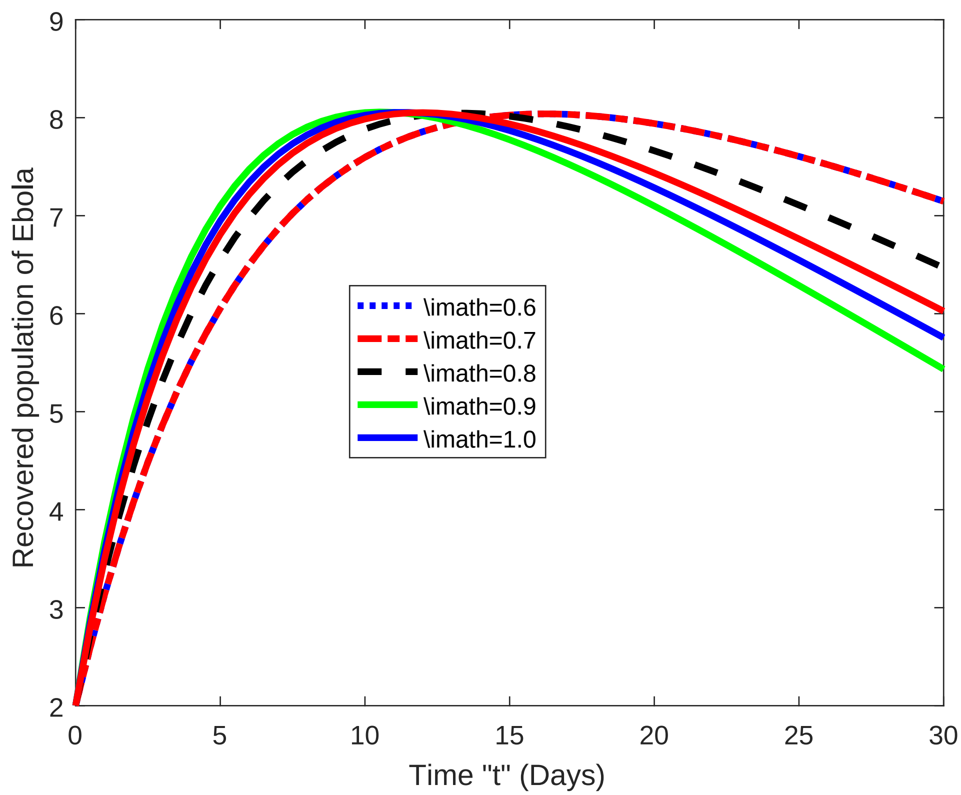

Figure 9: Simulation of recovered human population from Ebola

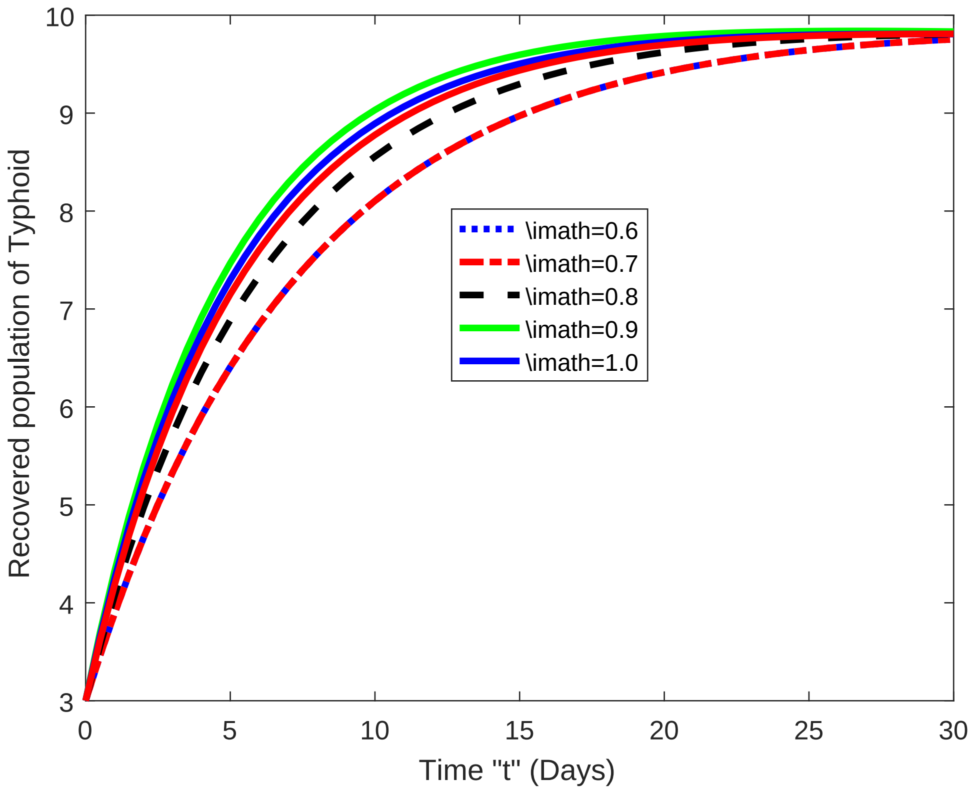

Figure 10: Simulation of recovered human population from Typhoid

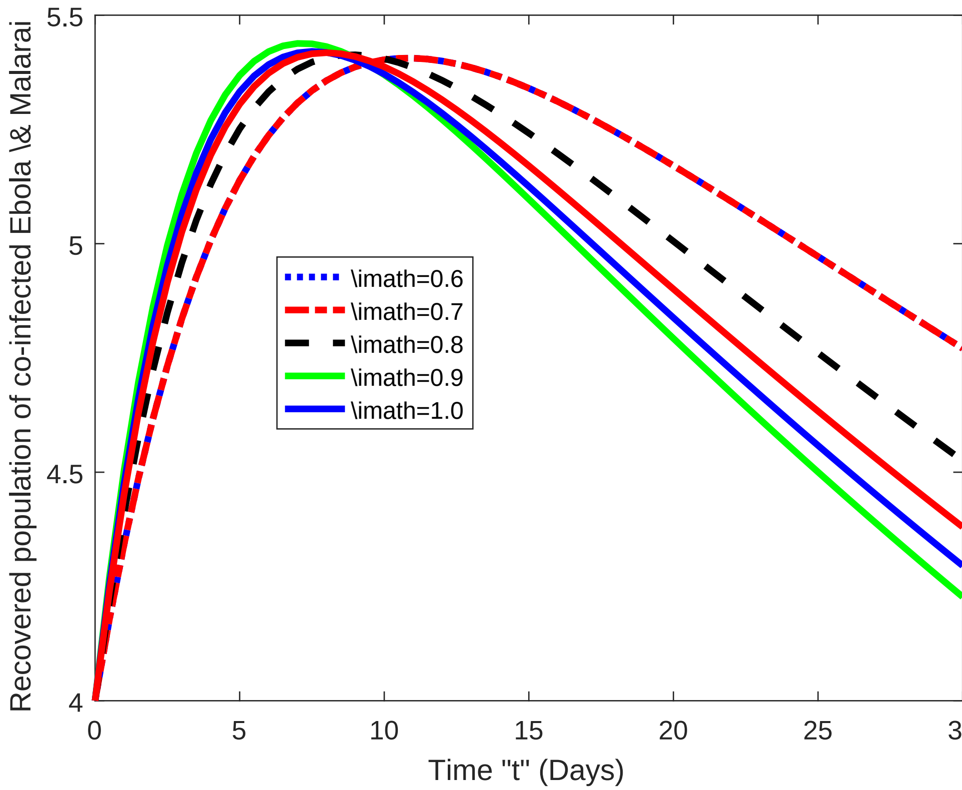

Figure 11: Simulation of recovered human population from Ebola & Malaria co-infection

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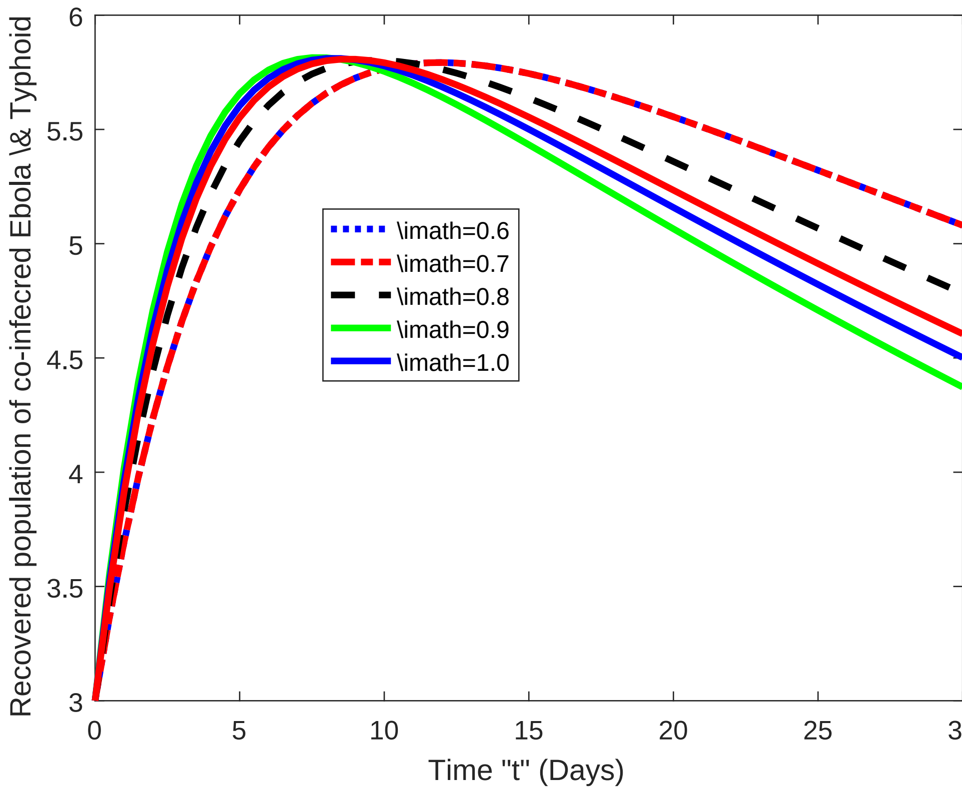

Figure 12: Simulation of recovered human population from Ebola & Typhoid co-infection

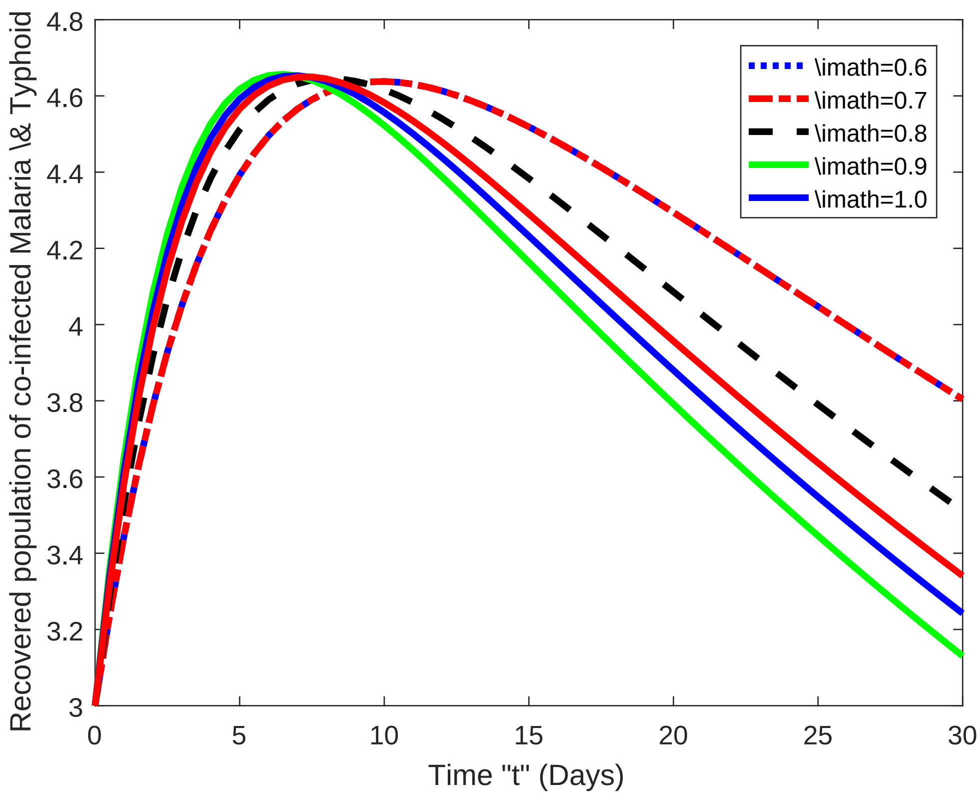

Figure 13: Simulation of recovered human population from Malaria & Typhoid co-infection

Parameter	value	Source
Π	1	Assumed
γ	2×10^{-2}	Assumed
β	3.96	Assumed
α	3×10^{-3}	Assumed
η	0.05	[30]
ζ	4×10^{-4}	Assumed
δ	3.1×10^{-2}	[33]
μ	5.1×10^{-3}	[30]
τ_1	3.96×10^{-2}	Assumed
τ_2	0.02	[31]
τ_3	0.017	Assumed
ψ_1	0.15	Assumed
ψ_2	1.6×10^{-3}	Assumed
ψ_3	0.51	Assumed
σ_1	0.0396	Assumed
σ_2	0.02	[34]
σ_3	0.017	[32]

population in all five recovered classes i.e $(R_e, R_m, R_t, R_{em}, R_{mt})$ increased for decreasing the values of $\imath$. It means that for decreasing the values of $\imath$, the disease endemic state moves to disease free state. Also this behavior become more biologically feasible for decreasing values of $\imath$ of CF operator used in the proposed model.

6 Conclusion

In this work, we developed multi-infections model by using the Caputo Fab-Raizo fractional order derivatives. First, we investigated the existence and uniqueness of the proposed fractional order model by using Lipschitz and the convolution functions theory. The positivity and boundedness of the solutions for the multi-infection type model are also established. For the numerical solution we applied an Adams-Bashforth method to calculate solution of the proposed fractional order model. Finally, to show the influence of fractional order and model parameters, a detailed numerical simulation for different values of fractional order is presented. It is clear from our numerical results that the population in all infected classes decreased significantly by decreasing different values of the given parameters, while the recovered classes increased. It means that for decreasing the values of the disease endemic state moves to disease free state, which showed the importance and convincing behavior of the fractional order and ensures that by including the memory effects in the model seems very appropriate for such an investigation. This research study helps to generate meaningful predictions regarding co-infections rather than prevention and management of multiple infections. Since it allows the proper information regarding infection transmission to be clarified, the fractional model offers a more acceptable solution than the integer scenario, as shown by the graphical representations.

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