Semi-analytical and Numerical Solution for Generalized Nonlinear Functional Integral Equations

Sukanta Halder[a,1], Deepmala[a,2]

[a] *Mathematics Discipline, PDPM-IIITDM Jabalpur, M.P. India*

[1] *Email: haldersukanta000@gmail.com,* [2] *Email: dmrai23@gmail.com,*

Abstract

In this article, we study two semi-analytical methods: the successive approximation method, i.e., the Picard method (PM), the Adomian decomposition method (ADM), and one numerical technique (NT) method for finding the approximate solution of Generalized Nonlinear Functional Integral Equations (GNFIE). The existence and uniqueness results are proved by applying Banach's contraction theorem. In some cases, it is difficult to find the integral when we use the ADM to find the approximate solution of certain nonlinear integral equations. To overcome this problem, some numerical techniques are applied based on GNFIE. Our existence results contain many functional integral equations as a special case. Finally, we discuss some examples and compare the methods along with the error analysis.

Keywords. Picard method, Nonlinear integral equation, Adomian decomposition method, Fixed point theorem, Convergence analysis, Error analysis.

MSC 2020. 45D05, 45G10, 65G99.

1 Introduction

Generalized Nonlinear Functional Integral Equations (GNFIE) have numerous applications in various fields, including the theory of neutron transport, the theory of radiative transfer, the kinetic theory of gases, and traffic theory. Numerous papers, monographs, and ap-

[2]Corresponding author.

plications frequently involve the GNFIE. For more details, see ([9]-[11], [12]-[14], [21]-[23], [25],[26],[28]-[31], and references therein).

In 1987, R. Rach [39] and N. Bellomo and D. Sarafyan [19] compared ADM with the Picard method (PM) on several examples. In 1999, Golberg [33] investigated whether the Adomian method for linear differential equations is equivalent to the traditional method of successive approximations (PM). But this is not true in general for all nonlinear differential equations. In 2010, El-Sayed et al. [31] studied the existence and uniqueness of solutions for the following quadratic integral equation (QIE):

$$z(\mu) = s(\mu) + h(\mu, z(\mu)) \int_0^\mu f(\upsilon, z(\upsilon)) d\upsilon,$$

where $h(\mu, z(\mu))$ and $f(\upsilon, z(\upsilon))$ are bounded functions and compared their results with ADM and Picard method.

In 2013, E.A.A. Ziada [43] proved the existence and uniqueness of a solution for the following QIE:

$$z(\mu) = s(\mu) + \left(\int_0^\mu \mathcal{K}_1(\mu, \upsilon) f(\upsilon, z(\upsilon)) d\upsilon \right) \left(\int_0^\mu \mathcal{K}_2(\mu, \upsilon) g(\upsilon, z(\upsilon)) d\upsilon \right),$$

and applied ADM and the repeated trapizoidal rule to solve the above equation with comparisons.

In this article, we prove the existence and uniqueness of the following GNFIE:

$$z(\mu) = s(\mu) + h(\mu, z(\mu)) \left(\int_0^\mu \mathcal{K}_1(\mu, \upsilon) f(\upsilon, z(\upsilon)) d\upsilon \right) \left(\int_0^\mu \mathcal{K}_2(\mu, \upsilon) g(\upsilon, z(\upsilon)) d\upsilon \right), \quad (1)$$

and apply ADM, the Picard method, and one numerical technique to solve the above equation. To solve the numerical examples, we use MATLAB 2021a software. Apart from this, a detailed comparison of these methods, including the error analysis, is discussed.

2 Main results

In this section, we investigate the existence and uniqueness of the solution to the following GNFIE:

$$z(\mu) = s(\mu) + h(\mu, z(\mu)) \left(\int_0^\mu \mathcal{K}_1(\mu, \upsilon) f(\upsilon, z(\upsilon)) d\upsilon \right) \left(\int_0^\mu \mathcal{K}_2(\mu, \upsilon) g(\upsilon, z(\upsilon)) d\upsilon \right). \quad (2)$$

Using the subsequent presumption:

(i) $s : \mathcal{I} \to \mathcal{R}_+$ is a continuous function on $\mathcal{I}$, where $\mathcal{I}= [0, 1]$ and $\mathcal{R}_+$=[0,+∞];

(ii) h, f and $g : \mathcal{I} \times \Omega \subset \mathcal{R}_+ \to \mathcal{R}_+$ are continuous and $\mathcal{P}=$ sup $\{|h(\mu, 0)| : \mu \in [0, 1]\}$, $\mathcal{Q}=$ sup $\{|f(\mu, 0)| : \mu \in [0, 1]\}$, $\mathcal{S}=$ sup $\{|g(\mu, 0)| : \mu \in [0, 1]\}$ and $|z(\mu)| \leqq \mathcal{H}$;

(iii) $\mathcal{K}_1, \mathcal{K}_2 : \mathcal{I} \times \mathcal{I} \to \mathcal{R}$ are continuous such that $K_i= \max_{\mu,\upsilon\in\mathcal{I}} |\mathcal{K}_i(\mu, \upsilon)|$, i=1,2;

(iv) h, f, and g satisfy the Lipschitz condition with Lipschitz constants $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ such that

$$\begin{aligned}
|h(\mu, z) - h(\mu, y)| &\leqq \mathcal{C}_1|z - y|, \\
|f(\mu, z) - f(\mu, y)| &\leqq \mathcal{C}_2|z - y|, \\
|g(\mu, z) - g(\mu, y)| &\leqq \mathcal{C}_3|z - y|.
\end{aligned}$$

Assuming $\mathcal{C} = \mathcal{C}(\mathcal{I})$ to be the space of all real-valued functions that are continuous on $\mathcal{I}$.

Consider the operator M as,

$$(Mz)(\mu) = s(\mu)+h(\mu, z(\mu)) \left(\int_0^\mu \mathcal{K}_1(\mu, \upsilon)\ f(\upsilon, z(\upsilon))d\upsilon\right) \left(\int_0^\mu \mathcal{K}_2(\mu, \upsilon)\ g(\upsilon, z(\upsilon))d\upsilon\right)\ \forall\ z \in \mathcal{C}.$$

Theorem 2.1. *Let $s(\mu) \in \mathcal{C}(\mathcal{I})$ and the presumptions (i)-(iv) be satisfied. If $K = K_1K_2[\mathcal{C}_1 (\mathcal{C}_2\mathcal{H} + \mathcal{Q}) (\mathcal{C}_3\mathcal{H} + \mathcal{S}) +\mathcal{C}_2 (\mathcal{C}_1\mathcal{H} + \mathcal{P}) (\mathcal{C}_3\mathcal{H} + \mathcal{S})+\mathcal{C}_3 (\mathcal{C}_1\mathcal{H} + \mathcal{P}) (\mathcal{C}_2\mathcal{H} + \mathcal{Q})]$ and $K < 1$, then the GNFIE (2) has a unique positive solution $z \in \mathcal{C}$.*

Proof. Clearly, the operator M maps $\mathcal{C}$ into $\mathcal{C}$.

Define $\mathcal{D}$ as a subset of $\mathcal{C}$,

$$\mathcal{D} = \{z \in \mathcal{C} : |z - s(\mu)| \leq \mathcal{M}\} \textit{ where } \mathcal{M} = K_1K_2 (\mathcal{C}_1\mathcal{H} + \mathcal{P}) (\mathcal{C}_2\mathcal{H} + \mathcal{Q}) (\mathcal{C}_3\mathcal{H} + \mathcal{S}) > 0.$$

So, the operator M maps $\mathcal{D}$ into $\mathcal{D}$. For $z \in \mathcal{D}$, we have

$$\begin{aligned}
|(Mz)(\mu) - s(\mu)| \leqq\ & |h(\mu, z(\mu))| \left(\int_0^\mu |\mathcal{K}_1(\mu, \upsilon)|\ |f(\upsilon, z(\upsilon))|d\upsilon\right) \left(\int_0^\mu |\mathcal{K}_2(\mu, \upsilon)|\ |g(\upsilon, z(\upsilon))|d\upsilon\right) \\
\leqq\ & K_1K_2 (|h(\mu, z(\mu)) - h(\mu, 0)| + |h(\mu, 0)|) \left(\int_0^\mu (|f(\upsilon, z(\upsilon)) - f(\upsilon, 0)| + |f(\upsilon, 0)|)\, d\upsilon\right) \\
& \left(\int_0^\mu (|g(\upsilon, z(\upsilon)) - g(\upsilon, 0)| + |g(\upsilon, 0)|)\, d\upsilon\right) \\
\leqq\ & K_1K_2 (\mathcal{C}_1|z(\mu)| + \mathcal{P}) (\mathcal{C}_2|z(\upsilon)| + \mathcal{Q}) (\mathcal{C}_3|z(\upsilon)| + \mathcal{S}) \left(\int_0^\mu d\upsilon\right) \left(\int_0^\mu d\upsilon\right)
\end{aligned}$$

$$
\begin{aligned}
&= K_1K_2\left(\mathcal{C}_1|z(\mu)|+\mathcal{P}\right)\left(\mathcal{C}_2|z(\upsilon)|+\mathcal{Q}\right)\left(\mathcal{C}_3|z(\upsilon)|+\mathcal{S}\right)\mu^2\\
&\leqq K_1K_2\left(\mathcal{C}_1\mathcal{H}+\mathcal{P}\right)\left(\mathcal{C}_2\mathcal{H}+\mathcal{Q}\right)\left(\mathcal{C}_3\mathcal{H}+\mathcal{S}\right)\mu^2\\
&\leqq K_1K_2\left(\mathcal{C}_1\mathcal{H}+\mathcal{P}\right)\left(\mathcal{C}_2\mathcal{H}+\mathcal{Q}\right)\left(\mathcal{C}_3\mathcal{H}+\mathcal{S}\right)
\end{aligned}
$$

Also, one can easily verify that $\mathcal{D}$ is a closed subset of $\mathcal{C}$. Now we prove that M is a contraction. For $z,\ y\in\mathcal{D}$, we get

$$
\begin{aligned}
(Mz)(\mu)-(My)(\mu) &= h(\mu,z(\mu))\left(\int_0^\mu \mathcal{K}_1(\mu,\upsilon)f(\upsilon,z(\upsilon))d\upsilon\right)\left(\int_0^\mu \mathcal{K}_2(\mu,\upsilon)g(\upsilon,z(\upsilon))d\upsilon\right)\\
&\quad -h(\mu,y(\mu))\left(\int_0^\mu \mathcal{K}_1(\mu,\upsilon)f(\upsilon,y(\upsilon))d\upsilon\right)\left(\int_0^\mu \mathcal{K}_2(\mu,\upsilon)g(\upsilon,y(\upsilon))d\upsilon\right)\\
&= h(\mu,z(\mu))\left(\int_0^\mu \mathcal{K}_1(\mu,\upsilon)f(\upsilon,z(\upsilon))d\upsilon\right)\left(\int_0^\mu \mathcal{K}_2(\mu,\upsilon)g(\upsilon,z(\upsilon))d\upsilon\right)\\
&\quad -h(\mu,y(\mu))\left(\int_0^\mu \mathcal{K}_1(\mu,\upsilon)f(\upsilon,y(\upsilon))d\upsilon\right)\left(\int_0^\mu \mathcal{K}_2(\mu,\upsilon)g(\upsilon,z(\upsilon))d\upsilon\right)\\
&\quad +h(\mu,y(\mu))\left(\int_0^\mu \mathcal{K}_1(\mu,\upsilon)f(\upsilon,y(\upsilon))d\upsilon\right)\left(\int_0^\mu \mathcal{K}_2(\mu,\upsilon)g(\upsilon,z(\upsilon))d\upsilon\right)\\
&\quad -h(\mu,y(\mu))\left(\int_0^\mu \mathcal{K}_1(\mu,\upsilon)f(\upsilon,y(\upsilon))d\upsilon\right)\left(\int_0^\mu \mathcal{K}_2(\mu,\upsilon)g(\upsilon,y(\upsilon))d\upsilon\right)\\
&= [h(\mu,z(\mu))-h(\mu,y(\mu))]\times\left(\int_0^\mu \mathcal{K}_1(\mu,\upsilon)f(\upsilon,z(\upsilon))d\upsilon\right)\left(\int_0^\mu \mathcal{K}_2(\mu,\upsilon)g(\upsilon,z(\upsilon))d\upsilon\right)\\
&\quad +h(\mu,y(\mu))\left(\int_0^\mu \mathcal{K}_1(\mu,\upsilon)\ [f(\upsilon,z(\upsilon))-f(\upsilon,y(\upsilon))]d\upsilon\right)\left(\int_0^\mu \mathcal{K}_2(\mu,\upsilon)\ g(\upsilon,z(\upsilon))d\upsilon\right)\\
&\quad +h(\mu,y(\mu))\left(\int_0^\mu \mathcal{K}_1(\mu,\upsilon)\ f(\upsilon,y(\upsilon))d\upsilon\right)\left(\int_0^\mu \mathcal{K}_2(\mu,\upsilon)\ [g(\upsilon,z(\upsilon))-g(\upsilon,y(\upsilon))]d\upsilon\right)\\
|(Mz)(\mu)-(My)(\mu)| &\leqq |h(\mu,z(\mu))-h(\mu,y(\mu))|\left(\int_0^\mu |\mathcal{K}_1(\mu,\upsilon)|\ |f(\upsilon,z(\upsilon))|d\upsilon\right)\left(\int_0^\mu |\mathcal{K}_2(\mu,\upsilon)|\ |g(\upsilon,z(\upsilon))|d\upsilon\right)\\
&\quad +|h(\mu,y(\mu))|\left(\int_0^\mu |\mathcal{K}_1(\mu,\upsilon)|\ |f(\upsilon,z(\upsilon))-f(\upsilon,y(\upsilon))|d\upsilon\right)\left(\int_0^\mu |\mathcal{K}_2(\mu,\upsilon)|\ |g(\upsilon,z(\upsilon))|d\upsilon\right)\\
&\quad +|h(\mu,y(\mu))|\left(\int_0^\mu |\mathcal{K}_1(\mu,\upsilon)|\ |f(\upsilon,y(\upsilon))|d\upsilon\right)\left(\int_0^\mu |\mathcal{K}_2(\mu,\upsilon)|\ |g(\upsilon,z(\upsilon))-g(\upsilon,y(\upsilon))|d\upsilon\right)\\
\therefore\quad \|Mz-My\| &= \max_{\mu\in\mathcal{I}}|(Mz)(\mu)-(My)(\mu)|\\
&\leqq K_1K_2\mathcal{C}_1\left(\mathcal{C}_2\mathcal{H}+\mathcal{Q}\right)\left(\mathcal{C}_3\mathcal{H}+\mathcal{S}\right)\|z-y\|\mu^2+K_1K_2\mathcal{C}_2\left(\mathcal{C}_1\mathcal{H}+\mathcal{P}\right)\left(\mathcal{C}_3\mathcal{H}+\mathcal{S}\right)\|z-y\|\mu^2\\
&\quad K_1K_2\mathcal{C}_3\left(\mathcal{C}_1\mathcal{H}+\mathcal{P}\right)\left(\mathcal{C}_2\mathcal{H}+\mathcal{Q}\right)\|z-y\|\mu^2\\
&\leqq K_1K_2[\mathcal{C}_1\left(\mathcal{C}_2\mathcal{H}+\mathcal{Q}\right)\left(\mathcal{C}_3\mathcal{H}+\mathcal{S}\right)+\mathcal{C}_2\left(\mathcal{C}_1\mathcal{H}+\mathcal{P}\right)\left(\mathcal{C}_3\mathcal{H}+\mathcal{S}\right)\\
&\quad +\mathcal{C}_3\left(\mathcal{C}_1\mathcal{H}+\mathcal{P}\right)\left(\mathcal{C}_2\mathcal{H}+\mathcal{Q}\right)]\ \|z-y\|\\
&= K\times\|z-y\|,
\end{aligned}
$$

where $K=K_1K_2\left[\mathcal{C}_1\left(\mathcal{C}_2\mathcal{H}+\mathcal{Q}\right)\left(\mathcal{C}_3\mathcal{H}+\mathcal{S}\right)+\mathcal{C}_2\left(\mathcal{C}_1\mathcal{H}+\mathcal{P}\right)\left(\mathcal{C}_3\mathcal{H}+\mathcal{S}\right)+\mathcal{C}_3\left(\mathcal{C}_1\mathcal{H}+\mathcal{P}\right)\left(\mathcal{C}_2\mathcal{H}+\mathcal{Q}\right)\right]$. If $K<1$, the operator M is a contraction. By Banach contraction theorem, M has a fixed point which is unique in $\mathcal{D}$. Hence, GNFIE (2) has a unique solution. □

3 Applications

Our proposed GNFIE contains several nonlinear integral equations as a special case.

• If $h(\mu, z(\mu)) = 1$, Eq. (1) becomes the subsequent nonlinear QIE, which was studied by E.A.A. Ziada in [43].

$$z(\mu) = s(\mu) + \left(\int_0^{\mu} \mathcal{K}_1(\mu, \upsilon) f(\upsilon, z(\upsilon)) d\upsilon\right) \left(\int_0^{\mu} \mathcal{K}_2(\mu, \upsilon) g(\upsilon, z(\upsilon)) d\upsilon\right).$$

• If we put $\mathcal{K}_1(\mu, \upsilon) = \mathcal{K}_2(\mu, \upsilon) = 1$ in Eq. (1), we get the following nonlinear GNFIE:

$$z(\mu) = s(\mu) + h(\mu, z(\mu)) \left(\int_0^{\mu} f(\upsilon, z(\upsilon)) d\upsilon\right) \left(\int_0^{\mu} g(\upsilon, z(\upsilon)) d\upsilon\right).$$

• If we put $\mathcal{K}_1(\mu, \upsilon) = \mathcal{K}_2(\mu, \upsilon) = \mathcal{K}(\mu, \upsilon)$ and $f(\upsilon, z(\upsilon))= g(\upsilon, z(\upsilon))$ in Eq. (1), we get the following nonlinear GNFIE:

$$z(\mu) = s(\mu) + h(\mu, z(\mu)) \left(\int_0^{\mu} \mathcal{K}(\mu, \upsilon) f(\upsilon, z(\upsilon)) d\upsilon\right)^2.$$

• If we put $h(\mu, z(\mu)) = 1; \mathcal{K}_1(\mu, \upsilon) = \mathcal{K}_2(\mu, \upsilon) = 1$ in Eq. (1), we get the following nonlinear QIE:

$$z(\mu) = s(\mu) + \left(\int_0^{\mu} f(\upsilon, z(\upsilon)) d\upsilon\right) \left(\int_0^{\mu} g(\upsilon, z(\upsilon)) d\upsilon\right).$$

• If $g(\upsilon, z(\upsilon)) = 1$ and $\mathcal{K}_2(\mu, \upsilon) = 1$, Eq. (1) transforms to the subsequent nonlinear GNFIE studied by E.A.A. Ziada [42].

$$z(\mu) = s(\mu) + h_1(\mu, z(\mu)) \left(\int_0^{\mu} \mathcal{K}_1(\mu, \upsilon) f(\upsilon, z(\upsilon)) d\upsilon\right).$$

• If $g(\upsilon, z(\upsilon)) = 1$ and $\mathcal{K}_1(\mu, \upsilon) = \mathcal{K}_2(\mu, \upsilon) = 1$, Eq. (1) transforms to the following nonlinear QIE studied by A.M.A El-Sayed et al. [31].

$$z(\mu) = s(\mu) + h_1(\mu, z(\mu)) \left(\int_0^{\mu} f(\upsilon, z(\upsilon)) d\upsilon\right). \tag{3}$$

Also, if $h_1(\mu, z(\mu)) = 1$ in Eq. (3), we get the following nonlinear Volterra integral equation [24]

$$z(\mu) = s(\mu) + \left(\int_0^{\mu} f(\upsilon, z(\upsilon)) d\upsilon\right).$$

• If $g(\upsilon, z(\upsilon)) = 1$, $s(\mu) = 0$ and $\mathcal{K}_1(\mu, \upsilon) = \mathcal{K}_2(\mu, \upsilon) = 1$, Eq. (1) transforms to the subsequent nonlinear QIE studied by Maleknejad et al. [38].

$$z(\mu) = h_1(\mu, z(\mu)) \left(\int_0^{\mu} f(\upsilon, z(\upsilon)) d\upsilon\right).$$

4 Picard method (PM)

Applying PM to GNFIE (2), the solution occurs by the sequence

$$
\begin{aligned}
z_0(\mu) &= s(\mu), \\
z_n(\mu) &= s(\mu) + h(\mu, z_{n-1}(\mu)) \left(\int_0^{\mu} \mathcal{K}_1(\mu, \upsilon)\ f(\upsilon, z_{n-1}(\upsilon)) d\upsilon \right) \times \\
&\left(\int_0^{\mu} \mathcal{K}_2(\mu, \upsilon)\ g(\upsilon, z_{n-1}(\upsilon)) d\upsilon \right); \quad n \geqq 1 \qquad (4)
\end{aligned}
$$

where $z_n(\mu)$ are continuous functions and we can write z_n as a sum of successive differences:

$$z_n = z_0 + \sum_{i=1}^{n} (z_i - z_{i-1}).$$

This implies that the convergence of the sequence $\{z_n\}$ is equivalent to the convergence of the infinite series $\sum_{i=1}^{\infty}(z_i - z_{i-1})$ and the solution will be the limit of the sequence $\{z_n\}$, i.e.,

$$z(\mu) = \lim_{n \to \infty} z_n(\mu).$$

If the series $\sum(z_i - z_{i-1})$ convergent, then the sequence $\{z_n(\mu)\}$ is uniformly converges to $z(\mu)$. To show uniform convergence of $\{z_n(\mu)\}$, we consider the series

$$\sum_{n=1}^{\infty} [z_n(\mu) - z_{n-1}(\mu)].$$

For n=1 in Eq. (4), we get

$$
\begin{aligned}
z_1(\mu) - z_0(\mu) &= h(\mu, z_0(\mu)) \left(\int_0^{\mu} \mathcal{K}_1(\mu, \upsilon)\ f(\upsilon, z_0(\upsilon)) d\upsilon \right) \left(\int_0^{\mu} \mathcal{K}_2(\mu, \upsilon)\ g(\upsilon, z_0(\upsilon)) d\upsilon \right) \\
\textit{and} \quad |z_1(\mu) - z_0(\mu)| &\leqq K_1 K_2 (\mathcal{C}_1 \mathcal{H} + \mathcal{P}) (\mathcal{C}_2 \mathcal{H} + \mathcal{Q}) (\mathcal{C}_3 \mathcal{H} + \mathcal{S}) \mu^2 = \mathcal{M}\ \mu^2. \qquad (5)
\end{aligned}
$$

Now, to find an estimation for $(z_n - z_{n-1}), \quad n \geqq 2$

$$
\begin{aligned}
z_n - z_{n-1} &= h(\mu, z_{n-1}(\mu)) \left(\int_0^{\mu} \mathcal{K}_1(\mu, \upsilon)\ f(\upsilon, z_{n-1}(\upsilon)) d\upsilon \right) \left(\int_0^{\mu} \mathcal{K}_2(\mu, \upsilon)\ g(\upsilon, z_{n-1}(\upsilon)) d\upsilon \right) \\
&\quad - h(\mu, z_{n-2}(\mu)) \left(\int_0^{\mu} \mathcal{K}_1(\mu, \upsilon)\ f(\upsilon, z_{n-2}(\upsilon)) d\upsilon \right) \left(\int_0^{\mu} \mathcal{K}_2(\mu, \upsilon)\ g(\upsilon, z_{n-2}(\upsilon)) d\upsilon \right) \\
&= h(\mu, z_{n-1}(\mu)) \left(\int_0^{\mu} \mathcal{K}_1(\mu, \upsilon)\ f(\upsilon, z_{n-1}(\upsilon)) d\upsilon \right) \left(\int_0^{\mu} \mathcal{K}_2(\mu, \upsilon)\ g(\upsilon, z_{n-1}(\upsilon)) d\upsilon \right) \\
&\quad - h(\mu, z_{n-2}(\mu)) \left(\int_0^{\mu} \mathcal{K}_1(\mu, \upsilon)\ f(\upsilon, z_{n-2}(\upsilon)) d\upsilon \right) \left(\int_0^{\mu} \mathcal{K}_2(\mu, \upsilon)\ g(\upsilon, z_{n-1}(\upsilon)) d\upsilon \right) \\
&\quad + h(\mu, z_{n-2}(\mu)) \left(\int_0^{\mu} \mathcal{K}_1(\mu, \upsilon)\ f(\upsilon, z_{n-2}(\upsilon)) d\upsilon \right) \left(\int_0^{\mu} \mathcal{K}_2(\mu, \upsilon)\ g(\upsilon, z_{n-1}(\upsilon)) d\upsilon \right) \\
&\quad - h(\mu, z_{n-2}(\mu)) \left(\int_0^{\mu} \mathcal{K}_1(\mu, \upsilon)\ f(\upsilon, z_{n-2}(\upsilon)) d\upsilon \right) \left(\int_0^{\mu} \mathcal{K}_2(\mu, \upsilon)\ g(\upsilon, z_{n-2}(\upsilon)) d\upsilon \right)
\end{aligned}
$$

29 Feb 2024 23:24:37 PST
230309-Deepmala Version 4 - Submitted to J. Integr. Eq. Appl.

$$
\begin{aligned}
z_n - z_{n-1} &= [h(\mu, z_{n-1}(\mu)) - h(\mu, z_{n-2}(\mu))] \times \left(\int_0^\mu \mathcal{K}_1(\mu,\upsilon)\ f(\upsilon, z_{n-1}(\upsilon))d\upsilon\right) \\
&\times \left(\int_0^\mu \mathcal{K}_2(\mu,\upsilon)\ g(\upsilon, z_{n-1}(\upsilon))d\upsilon\right) \\
&+h(\mu, z_{n-2}(\mu)) \left(\int_0^\mu \mathcal{K}_1(\mu,\upsilon)\ [f(\upsilon, z_{n-1}(\upsilon)) - f(\upsilon, z_{n-2}(\upsilon))]d\upsilon\right) \\
&\times \left(\int_0^\mu \mathcal{K}_2(\mu,\upsilon)\ g(\upsilon, z_{n-1}(\upsilon))d\upsilon\right) \\
&+h(\mu, z_{n-2}(\mu)) \left(\int_0^\mu \mathcal{K}_1(\mu,\upsilon)\ f(\upsilon, z_{n-2}(\upsilon))d\upsilon\right) \\
&\times \left(\int_0^\mu \mathcal{K}_2(\mu,\upsilon)\ [g(\upsilon, z_{n-1}(\upsilon)) - g(\upsilon, z_{n-2}(\upsilon))]d\upsilon\right) \\
\therefore\quad |z_n - z_{n-1}| &\leqq K_1K_2\mathcal{C}_1\left(\mathcal{C}_2\mathcal{H}+\mathcal{Q}\right)\left(\mathcal{C}_3\mathcal{H}+\mathcal{S}\right)\mu^2\ \|z_{n-1}-z_{n-2}\| \\
&+K_1K_2\mathcal{C}_2\left(\mathcal{C}_1\mathcal{H}+\mathcal{P}\right)\left(\mathcal{C}_3\mathcal{H}+\mathcal{S}\right)\mu\int_0^\mu |z_{n-1}(\upsilon)-z_{n-2}(\upsilon)|d\upsilon \\
&+K_1K_2\mathcal{C}_3\left(\mathcal{C}_1\mathcal{H}+\mathcal{P}\right)\left(\mathcal{C}_2\mathcal{H}+\mathcal{Q}\right)\mu\int_0^\mu |z_{n-1}(\upsilon)-z_{n-2}(\upsilon)|d\upsilon. \qquad (6)
\end{aligned}
$$

Putting n=2 in Eq. (6) and using Eq. (5), we get

$$
\begin{aligned}
|z_2 - z_1| &\leqq K_1K_2\mathcal{C}_1\left(\mathcal{C}_2\mathcal{H}+\mathcal{Q}\right)\left(\mathcal{C}_3\mathcal{H}+\mathcal{S}\right)\mu^2\ \|z_1-z_0\| \\
&\qquad + K_1K_2\mathcal{C}_2\left(\mathcal{C}_1\mathcal{H}+\mathcal{P}\right)\left(\mathcal{C}_3\mathcal{H}+\mathcal{S}\right)\mu\int_0^\mu |z_1(\upsilon)-z_0(\upsilon)|\ d\upsilon \\
&+K_1K_2\mathcal{C}_3\left(\mathcal{C}_1\mathcal{H}+\mathcal{P}\right)\left(\mathcal{C}_2\mathcal{H}+\mathcal{Q}\right)\mu\int_0^\mu |z_1(\upsilon)-z_0(\upsilon)|\ d\upsilon. \\
&\leqq \mathcal{M}K_1K_2\mathcal{C}_1\left(\mathcal{C}_2\mathcal{H}+\mathcal{Q}\right)\left(\mathcal{C}_3\mathcal{H}+\mathcal{S}\right)\mu^4 + \mathcal{M}K_1K_2\mathcal{C}_2\left(\mathcal{C}_1\mathcal{H}+\mathcal{P}\right)\left(\mathcal{C}_3\mathcal{H}+\mathcal{S}\right)\frac{1}{3}\mu^4 \\
&\mathcal{M}K_1K_2\mathcal{C}_3\left(\mathcal{C}_1\mathcal{H}+\mathcal{P}\right)\left(\mathcal{C}_2\mathcal{H}+\mathcal{Q}\right)\frac{1}{3}\mu^4 \\
&= \mathcal{M}K_1K_2\left[\mathcal{C}_1\left(\mathcal{C}_2\mathcal{H}+\mathcal{Q}\right)\left(\mathcal{C}_3\mathcal{H}+\mathcal{S}\right)+\frac{1}{3}\mathcal{C}_2\left(\mathcal{C}_3\mathcal{H}+\mathcal{S}\left(\mathcal{C}_1\mathcal{H}+\mathcal{P}\right)\right)+\frac{1}{3}\mathcal{C}_3\left(\mathcal{C}_1\mathcal{H}+\mathcal{P}\right)\left(\mathcal{C}_2\mathcal{H}+\mathcal{Q}\right)\right]\mu^4.
\end{aligned}
$$

Putting n=3 in Eq. (6) and using Eq. (5), we get

$$
\begin{aligned}
|z_3 - z_2| &\leqq K_1K_2\mathcal{C}_1\left(\mathcal{C}_2\mathcal{H}+\mathcal{Q}\right)\left(\mathcal{C}_3\mathcal{H}+\mathcal{S}\right)\mu^2\ \|z_2-z_1\| \\
&\qquad + K_1K_2\mathcal{C}_2\left(\mathcal{C}_1\mathcal{H}+\mathcal{P}\right)\left(\mathcal{C}_3\mathcal{H}+\mathcal{S}\right)\mu\int_0^\mu |z_2(\upsilon)-z_1(\upsilon)|\ d\upsilon \\
&+K_1K_2\mathcal{C}_3\left(\mathcal{C}_1\mathcal{H}+\mathcal{P}\right)\left(\mathcal{C}_2\mathcal{H}+\mathcal{Q}\right)\mu\int_0^\mu |z_2(\upsilon)-z_1(\upsilon)|\ d\upsilon. \\
&\leqq \mathcal{M}(K_1K_2)^2\left[\mathcal{C}_1\left(\mathcal{C}_2\mathcal{H}+\mathcal{Q}\right)\left(\mathcal{C}_3\mathcal{H}+\mathcal{S}\right)+\frac{1}{3}\mathcal{C}_2\left(\mathcal{C}_3\mathcal{H}+\mathcal{S}\left(\mathcal{C}_1\mathcal{H}+\mathcal{P}\right)\right)+\frac{1}{3}\mathcal{C}_3\left(\mathcal{C}_1\mathcal{H}+\mathcal{P}\right)\left(\mathcal{C}_2\mathcal{H}+\mathcal{Q}\right)\right] \\
&\times\left[\mathcal{C}_1\left(\mathcal{C}_2\mathcal{H}+\mathcal{Q}\right)\left(\mathcal{C}_3\mathcal{H}+\mathcal{S}\right)+\frac{1}{5}\mathcal{C}_2\left(\mathcal{C}_3\mathcal{H}+\mathcal{S}\left(\mathcal{C}_1\mathcal{H}+\mathcal{P}\right)\right)+\frac{1}{5}\mathcal{C}_3\left(\mathcal{C}_1\mathcal{H}+\mathcal{P}\right)\left(\mathcal{C}_2\mathcal{H}+\mathcal{Q}\right)\right]\mu^6.
\end{aligned}
$$

29 Feb 2024 23:24:37 PST
230309-Deepmala Version 4 - Submitted to J. Integr. Eq. Appl.

By continuing the same procedure, we get

$$
\begin{aligned}
|z_n - z_{n-1}| &\leqq \mathcal{M}(K_1K_2)^{n-1}\left[\mathcal{C}_1\left(\mathcal{C}_2\mathcal{H}+\mathcal{Q}\right)\left(\mathcal{C}_3\mathcal{H}+\mathcal{S}\right)+\frac{1}{3}\mathcal{C}_2\left(\mathcal{C}_3\mathcal{H}+\mathcal{S}\right)\left(\mathcal{C}_1\mathcal{H}+\mathcal{P}\right)+\frac{1}{3}\mathcal{C}_3\left(\mathcal{C}_1\mathcal{H}+\mathcal{P}\right)\left(\mathcal{C}_2\mathcal{H}+\mathcal{Q}\right)\right] \\
&\quad \times\left[\mathcal{C}_1\left(\mathcal{C}_2\mathcal{H}+\mathcal{Q}\right)\left(\mathcal{C}_3\mathcal{H}+\mathcal{S}\right)+\frac{1}{5}\mathcal{C}_2\left(\mathcal{C}_3\mathcal{H}+\mathcal{S}\right)\left(\mathcal{C}_1\mathcal{H}+\mathcal{P}\right)+\frac{1}{5}\mathcal{C}_3\left(\mathcal{C}_1\mathcal{H}+\mathcal{P}\right)\left(\mathcal{C}_2\mathcal{H}+\mathcal{Q}\right)\right]\times \ldots \times \\
&\quad \left[\mathcal{C}_1\left(\mathcal{C}_2\mathcal{H}+\mathcal{Q}\right)\left(\mathcal{C}_3\mathcal{H}+\mathcal{S}\right)+\frac{1}{2n-1}\mathcal{C}_2\left(\mathcal{C}_3\mathcal{H}+\mathcal{S}\right)\left(\mathcal{C}_1\mathcal{H}+\mathcal{P}\right)+\frac{1}{2n-1}\mathcal{C}_3\left(\mathcal{C}_1\mathcal{H}+\mathcal{P}\right)\left(\mathcal{C}_2\mathcal{H}+\mathcal{Q}\right)\right]\mu^{2n} \\
&\leqq \mathcal{M}(K_1K_2)^{n-1}\left[\mathcal{C}_1\left(\mathcal{C}_2\mathcal{H}+\mathcal{Q}\right)\left(\mathcal{C}_3\mathcal{H}+\mathcal{S}\right)+\mathcal{C}_2\left(\mathcal{C}_3\mathcal{H}+\mathcal{S}\right)\left(\mathcal{C}_1\mathcal{H}+\mathcal{P}\right)+\mathcal{C}_3\left(\mathcal{C}_1\mathcal{H}+\mathcal{P}\right)\left(\mathcal{C}_2\mathcal{H}+\mathcal{Q}\right)\right]\times ..\times \\
&\quad \left[\mathcal{C}_1\left(\mathcal{C}_2\mathcal{H}+\mathcal{Q}\right)\left(\mathcal{C}_3\mathcal{H}+\mathcal{S}\right)+\mathcal{C}_2\left(\mathcal{C}_3\mathcal{H}+\mathcal{S}\right)\left(\mathcal{C}_1\mathcal{H}+\mathcal{P}\right)+\mathcal{C}_3\left(\mathcal{C}_1\mathcal{H}+\mathcal{P}\right)\left(\mathcal{C}_2\mathcal{H}+\mathcal{Q}\right)\right] \\
&= \mathcal{M}\left[K_1K_2\left\{\mathcal{C}_1\left(\mathcal{C}_2\mathcal{H}+\mathcal{Q}\right)\left(\mathcal{C}_3\mathcal{H}+\mathcal{S}\right)+\mathcal{C}_2\left(\mathcal{C}_3\mathcal{H}+\mathcal{S}\right)\left(\mathcal{C}_1\mathcal{H}+\mathcal{P}\right)+\mathcal{C}_3\left(\mathcal{C}_1\mathcal{H}+\mathcal{P}\right)\left(\mathcal{C}_2\mathcal{H}+\mathcal{Q}\right)\right\}\right]^{n-1}.
\end{aligned}
$$

Since, $K = \left[K_1K_2\left\{\mathcal{C}_1\left(\mathcal{C}_2\mathcal{H}+\mathcal{Q}\right)\left(\mathcal{C}_3\mathcal{H}+\mathcal{S}\right)+\mathcal{C}_2\left(\mathcal{C}_3\mathcal{H}+\mathcal{S}\right)\left(\mathcal{C}_1\mathcal{H}+\mathcal{P}\right)+\mathcal{C}_3\left(\mathcal{C}_1\mathcal{H}+\mathcal{P}\right)\left(\mathcal{C}_2\mathcal{H}+\mathcal{Q}\right)\right\}\right]$ and $K < 1$ also $\mathcal{M} > 0$, so the convergence of the series

$$\sum_{i=1}^{\infty}[z_n(\mu) - z_{n-1}(\mu)]$$

is uniform. Hence, $z_n(\mu)$ is uniformly convergent.

Since $h(\mu, z)$, $f(\mu, z)$ and $g(\mu, z)$ are continuous at z, so

$$
\begin{aligned}
z(\mu) &= s(\mu) + \lim_{n\to\infty} h(\mu, z_n(\mu))\left(\int_0^{\mu}\mathcal{K}_1(\mu, \upsilon)\ f(\upsilon, z_n(\upsilon))d\upsilon\right)\left(\int_0^{\mu}\mathcal{K}_2(\mu, \upsilon)\ g(\upsilon, z_n(\upsilon))d\upsilon\right) \\
&= s(\mu) + h(\mu, z(\mu))\left(\int_0^{\mu}\mathcal{K}_1(\mu, \upsilon)\ f(\upsilon, z(\upsilon))d\upsilon\right)\left(\int_0^{\mu}\mathcal{K}_2(\mu, \upsilon)\ g(\upsilon, z(\upsilon))d\upsilon\right).
\end{aligned}
$$

Therefore, the solution exists.

Now to show the uniqueness of the solution, assume $x_1(\mu)$ is another solution of Eq. (2) which is continuous. Thus,

$$x_1(\mu) = s(\mu) + h(\mu, x_1(\mu))\left(\int_0^{\mu}\mathcal{K}_1(\mu, \upsilon)\ f(\upsilon, x_1(\upsilon))d\upsilon\right)\left(\int_0^{\mu}\mathcal{K}_2(\mu, \upsilon)\ g(\upsilon, x_1(\upsilon))d\upsilon\right), \forall\ \mu \in [0, 1] \quad (7)$$

and

$$
\begin{aligned}
x_1(\mu) - z_n(\mu) &= h(\mu, x_1(\mu))\left(\int_0^{\mu}\mathcal{K}_1(\mu, \upsilon)\ f(\upsilon, x_1(\upsilon))d\upsilon\right)\left(\int_0^{\mu}\mathcal{K}_1(\mu, \upsilon)\ g(\upsilon, x_1(\upsilon))d\upsilon\right) \\
&\quad - h(\mu, z_{n-1}(\mu))\left(\int_0^{\mu}\mathcal{K}_1(\mu, \upsilon)\ f(\upsilon, z_{n-1}(\upsilon))d\upsilon\right)\left(\int_0^{\mu}\mathcal{K}_2(\mu, \upsilon)\ g(\upsilon, z_{n-1}(\upsilon))d\upsilon\right)
\end{aligned}
$$

$$
\begin{aligned}
&= h(\mu, x_1(\mu))\left(\int_0^\mu \mathcal{K}_1(\mu,\upsilon)\ f(\upsilon, x_1(\upsilon))d\upsilon\right)\left(\int_0^\mu \mathcal{K}_2(\mu,\upsilon)\ g(\upsilon, x_1(\upsilon))d\upsilon\right) \\
&\quad - h(\mu, z_{n-1}(\mu))\left(\int_0^\mu \mathcal{K}_1(\mu,\upsilon)\ f(\upsilon, z_{n-1}(\upsilon))d\upsilon\right)\left(\int_0^\mu \mathcal{K}_2(\mu,\upsilon) g(\upsilon, x_1(\upsilon))d\upsilon\right) \\
&\quad + h(\mu, z_{n-1}(\mu))\left(\int_0^\mu \mathcal{K}_1(\mu,\upsilon)\ f(\upsilon, z_{n-1}(\upsilon))d\upsilon\right)\left(\int_0^\mu \mathcal{K}_2(\mu,\upsilon)\ g(\upsilon, x_1(\upsilon))d\upsilon\right) \\
&\quad - h(\mu, z_{n-1}(\mu))\left(\int_0^\mu \mathcal{K}_1(\mu,\upsilon)\ f(\upsilon, z_{n-1}(\upsilon))d\upsilon\right)\left(\int_0^\mu \mathcal{K}_2(\mu,\upsilon)\ g(\upsilon, z_{n-1}(\upsilon))d\upsilon\right).
\end{aligned}
$$

$$
\begin{aligned}
|x_1(\mu) - z_n(\mu)| \leqq\ & K_1K_2\mathcal{C}_1\left(\mathcal{C}_2\mathcal{H}+\mathcal{Q}\right)\left(\mathcal{C}_3\mathcal{H}+\mathcal{S}\right)\mu^2\ \|z_1 - z_{n-1}\| \\
& + K_1K_2\mathcal{C}_2\left(\mathcal{C}_1\mathcal{H}+\mathcal{P}\right)\left(\mathcal{C}_3\mathcal{H}+\mathcal{S}\right)\mu\int_0^\mu |z_1(\upsilon) - z_{n-1}(\upsilon)|d\upsilon \\
& + K_1K_2\mathcal{C}_3\left(\mathcal{C}_1\mathcal{H}+\mathcal{P}\right)\left(\mathcal{C}_2\mathcal{H}+\mathcal{Q}\right)\mu\int_0^\mu |z_1(\upsilon) - z_{n-1}(\upsilon)|d\upsilon. \qquad (8)
\end{aligned}
$$

From Eq. (7), we obtain

$$
|x_1(\mu) - s(\mu)| \leqq K_1K_2\left(\mathcal{C}_1\mathcal{H}+\mathcal{P}\right)\left(\mathcal{C}_2\mathcal{H}+\mathcal{Q}\right)\left(\mathcal{C}_3\mathcal{H}+\mathcal{S}\right)\mu^2 = \mathcal{M}\ \mu^2 \leqq \mathcal{M}. \qquad (9)
$$

From Eq. (8) and (9), we get

$$
|x_1(\mu) - z_n(\mu)| \leqq \mathcal{M}\ \left[K_1K_2\left\{\mathcal{C}_1\left(\mathcal{C}_2\mathcal{H}+\mathcal{Q}\right)\left(\mathcal{C}_3\mathcal{H}+\mathcal{S}\right)+\mathcal{C}_2\left(\mathcal{C}_3\mathcal{H}+\mathcal{S}\right)\left(\mathcal{C}_1\mathcal{H}+\mathcal{P}\right)+\mathcal{C}_3\left(\mathcal{C}_1\mathcal{H}+\mathcal{P}\right)\left(\mathcal{C}_2\mathcal{H}+\mathcal{Q}\right)\right\}\right]^{n-1}.
$$

Hence,

$$
\lim_{n\to\infty} z_n(\mu) = x_1(\mu) = z(\mu). \qquad \square
$$

5 Adomian's Decomposition Method (ADM)

The ADM was first introduced and developed by George Adomian [1] in the 1970s and 1990s. It is a semi-analytical method for solving a wide class of linear and nonlinear ODEs, PDEs, integral equations, functional equations, integro-differential equations, and differential delay equations. For more details, refer to ([1]-[6] and references therein). ADM is a type of algorithm based on the decomposition technique to build approximate solutions and numerical simulations for real-world problems in applied sciences and engineering without restrictive assumptions such as those required by linearization, perturbation, temporary assumptions, guessing the initial terms or a set of basis functions, and so forth. For details, one may refer ([1]-[6],[22],[23], and references therein). It provides the solution as an infinite series that converges faster towards accurate solutions. In 1989, Cherruault [22] provided the first proof of convergence for the ADM using fixed point theorems for abstract functional equations. After that, Abbaoui and Cherruault ([5],[7]), Himoun et al. ([34],[35]), Hosseini and Nasabzadeh [36] have further explored the convergence of the ADM. Additionally,

29 Feb 2024 23:24:37 PST
230309-Deepmala Version 4 - Submitted to J. Integr. Eq. Appl.

Babolian and Biazar [16] introduced the order of convergence, while Boumenir and Gordon [20] discussed the rate of convergence, and El-Kalla [32] presented another perspective on error analysis. In 2011, Abdelrazae and Pelinovsky [8] provided a rigorous proof of convergence for the ADM, utilizing the Cauchy-Kovalevskaya theorem for initial value problems. In references ([16]-[18],[40]), application and convergence of ADM to several forms of integral equations were examined.

In this section, our main objective is to construct an algorithm using ADM (motivated by the work of Fu et al. [41]) to find the solution of our proposed GNFIE:

$$z(\mu) = s(\mu) + h(\mu, z(\mu)) \left(\int_0^{\mu} \mathcal{K}_1(\mu, \upsilon) f(\upsilon, z(\upsilon)) d\upsilon \right) \left(\int_0^{\mu} \mathcal{K}_2(\mu, \upsilon) g(\upsilon, z(\upsilon)) d\upsilon \right), \tag{10}$$

where the functions $s,\ h,\ f,\ g$ and kernel $\mathcal{K}_1, \mathcal{K}_2$ satisfies the condition $[(i) - (iv)]$ defined in section 2.

The ADM representing the solution $z(\mu)$ into a series form

$$z(\mu) = \sum_{i=0}^{\infty} z_i(\mu), \tag{11}$$

and the nonlinear functions $h(\mu, z(\mu)),\ f(\upsilon, z(\upsilon))$ and $g(\upsilon, z(\upsilon))$ decomposed as follows:

$$h(\mu, z(\mu)) = \sum_{i=0}^{\infty} U_i(\mu),\ \ f(\upsilon, z(\upsilon)) = \sum_{i=0}^{\infty} V_i(\upsilon),\ \ and\ g(\upsilon, z(\upsilon)) = \sum_{i=0}^{\infty} W_i(\upsilon). \tag{12}$$

Here $U_i(\mu)$, depending on $z_0(\mu), z_1(\mu), ..., z_i(\mu)$ is the Adomian's polynomial of $h(\mu, z(\mu))$ defined as follows

$$U_i(\mu) \;=\; \frac{1}{i!} \frac{d^i}{d\lambda^i} \left[h\left(\mu, \sum_{n=0}^{\infty} \lambda^n z_n(\mu) \right) \right]_{\lambda=0}, \quad i \geqq 0. \tag{13}$$

Similarly we can define $V_i(\upsilon)$ and $W_i(\upsilon)$. Thus the nonlinear term in Eq. (10) is decomposed into

$$\mathcal{N}(z(\mu)) = h(\mu, z(\mu)) \left(\int_0^{\mu} \mathcal{K}_1(\mu, \upsilon) f(\upsilon, z(\upsilon)) d\upsilon \right) \left(\int_0^{\mu} \mathcal{K}_2(\mu, \upsilon) g(\upsilon, z(\upsilon)) d\upsilon \right) = \sum_{i=0}^{\infty} B_i(\mu), \tag{14}$$

where the Adomian polynomials are

$$B_i(\mu) = \sum_{j=0}^{i} \sum_{n=0}^{j} U_n(\mu) \left(\int_0^{\mu} \mathcal{K}_1(\mu, \upsilon) V_{j-n}(\upsilon) d\upsilon \right) \left(\int_0^{\mu} \mathcal{K}_2(\mu, \upsilon) W_{i-j}(\upsilon) d\upsilon \right). \tag{15}$$

Therefore, the Adomian recurrence scheme for the solution of Eq. (10) is

$$z_0(\mu) \;=\; s(\mu), \tag{16}$$

$$z_{i+1}(\mu) \;=\; B_i(\mu), \quad i \geqq 0. \tag{17}$$

6 Numerical Technique (NT)

Sometimes the evaluation of integrals in ADM is very difficult or impossible to calculate. For that reason, finding the series solution terms of the ADM is challenging for the researchers. So this numerical technique will be helpful to overcome this problem. In 2005, E. Babolian [15] applied this method to the second kind of linear Volterra integral equations. In this section, we apply this numerical technique to solve the GNFIE (2). Applying ADM to Eq. (2), we get

$$\begin{aligned} z_0(\mu) &= s(\mu), \\ z_{n+1}(\mu) &= h(\mu, z_n(\mu)) \left(\int_0^\mu \mathcal{K}_1(\mu, \upsilon)\ f(\upsilon, z_n(\upsilon)) d\upsilon \right) \left(\int_0^\mu \mathcal{K}_2(\mu, \upsilon)\ g(\upsilon, z_n(\upsilon)) d\upsilon \right). \end{aligned} \tag{18}$$

Now to approximate the integral terms in (18), we will use the numerical technique discussed in [27]. We take a regular mesh in μ and setting $\mu = \mu_i = ih_*$, where h_* is the step size $= \frac{1}{n}$. Hence, the integral in (18) can be approximated as,

$$\left(\int_0^{\mu_i} \mathcal{K}_1(\mu_i, \upsilon)\ f(\upsilon, z_n(\upsilon)) d\upsilon \right) \left(\int_0^{\mu_i} \mathcal{K}_2(\mu_i, \upsilon)\ g(\upsilon, z_n(\upsilon)) d\upsilon \right) \cong \left(h_* \sum_{j=0}^{i} \alpha_{ij}\ \mathcal{K}_1(\mu_i, \upsilon_j)\ f(\upsilon_j, z_n(\upsilon_j)) \right) \times \left(h_* \sum_{j=0}^{i} \alpha_{ij}\ \mathcal{K}_2(\mu_i, \upsilon_j)\ g(\upsilon_j, z_n(\upsilon_j)) \right),$$

where $\mu_i = \upsilon_i$, $i = 0, 1, 2, \ldots.n$.

So we have

$$\begin{aligned} & z_0(\mu_i) = s(\mu_i), \\ & z_{n+1}(\mu_i) \cong h(\mu_i, z_n(\mu_i)) \left(h_* \sum_{j=0}^{i} \alpha_{ij}\ \mathcal{K}_1(\mu_i, \upsilon_j)\ f(\upsilon_j, z_n(\upsilon_j)) \right) \left(h_* \sum_{j=0}^{i} \alpha_{ij}\ \mathcal{K}_2(\mu_i, \upsilon_j)\ g(\upsilon_j, z_n(\upsilon_j)) \right), \end{aligned} \tag{19}$$

i= 0,1,2 ... and n=0,1,2... .

We can choose suitable weights α_{ij} for each i and $j = 0, 1, ..i$, which represent the weights for $(i+1)$ points quadrature rules of Newton-Cotes type for the interval $[0, ih_*]$.

Now we apply the above three methods to solve the following examples.

7 Numerical Examples

Example 7.1. *Consider following GNFIE*

$$z(\mu) = \left(\mu^2 - \frac{\mu^{17}}{1350} \right) + z(\mu) \left(\int_0^\mu \upsilon z^2(\upsilon) d\upsilon \right) \left(\int_0^\mu \frac{\upsilon^2}{25} z^3(\upsilon) d\upsilon \right), \tag{20}$$

and $z(\mu) = \mu^2$ *is the exact solution of this equation.*

Applying successive approximation method (PM) to the Eq. (20)*, we get*

$$\begin{aligned} z_0(\mu) &= \left(\mu^2 - \frac{\mu^{17}}{1350}\right), \\ z_n(\mu) &= \left(\mu^2 - \frac{\mu^{17}}{1350}\right) + z_{n-1}(\mu)\left(\int_0^\mu \upsilon z_{n-1}^2(\upsilon)d\upsilon\right)\left(\int_0^\mu \frac{\upsilon^2}{25} z_{n-1}^3(\upsilon)d\upsilon\right), \quad n = 1, 2, ..., \end{aligned}$$

and the solution is

$$z(\mu) = \lim_{n\to\infty} z_n(\mu).$$

Applying Adomian decomposition method (ADM) to the Eq. (20)*, we get*

$$\begin{aligned} z_0(\mu) &= \left(\mu^2 - \frac{\mu^{17}}{1350}\right), \\ z_{i+1}(\mu) &= B_i(\mu), \quad i \geqq 0, \end{aligned}$$

where

$$B_i(\mu) = \sum_{j=0}^{i}\sum_{n=0}^{j} z_n(\mu)\left(\int_0^\mu \upsilon A_{j-n}(\upsilon)d\upsilon\right)\left(\int_0^\mu \frac{\upsilon^2}{25} D_{i-j}(\upsilon)d\upsilon\right)$$

is the Adomian polynomial of the nonlinear term

$$\mathcal{N}(z(\mu)) = z(\mu)\left(\int_0^\mu \upsilon z^2(\upsilon)d\upsilon\right)\left(\int_0^\mu \frac{\upsilon^2}{25} z^3(\upsilon)d\upsilon\right)$$

in Eq. (20) *and*

$$A_k(\upsilon) = \sum_{i=0}^{k} z_i(\upsilon)z_{k-i}(\upsilon), \quad D_k(\upsilon) = \sum_{j=0}^{k}\sum_{i=0}^{j} z_i(\upsilon)z_{j-i}(\upsilon)z_{k-j}(\upsilon),$$

are the Adomian polynomial of z^2 *and* z^3 *respectively.*

The approximate solution upto $(p+1)^{th}$ *terms by ADM is*

$$z(\mu) = \sum_{i=0}^{p} z_i(\mu).$$

If we apply the discussed Numerical Technique (NT) to Eq. (20)*, we get*

$$\begin{aligned} z_0(\mu_i) &= \left(\mu_i^2 - \frac{\mu_i^{17}}{1350}\right), \\ z_{n+1}(\mu_i) &\cong z_n(\mu_i)\left(h_* \sum_{j=0}^{i} \alpha_{ij}\ \upsilon_j\ z_n^2(\upsilon_j)\right)\left(h_* \sum_{j=0}^{i} \alpha_{ij}\ \frac{\upsilon_j^2}{25} z_n^3(\upsilon_j)\right), \end{aligned}$$

where $\mu_i = \upsilon_i$ *for* $i = 0, 1, 2, ...n$.

Table (1) *compares the absolute error of the ADM, PM, and NT methods. Figure* (1) *shows the exact solution and the approximate solution of these three methods, along with an absolute error graph.*

μ	$\lvert z_{exact} - z_{Picard}\rvert$	$\lvert z_{exact} - z_{ADM}\rvert$	$\lvert z_{exact} - z_{NT}\rvert$
0.1	2.8536e-74	9.0405e-74	9.2592e-20
0.2	1.3159e-55	4.1692e-55	2.5379e-15
0.3	1.0887e-44	3.4492e-44	1.0109e-12
0.4	6.0689e-37	1.9227e-36	7.2382e-11
0.5	6.1877e-31	1.9604e-30	2.0105e-09
0.6	5.0207e-26	1.5906e-25	3.0570e-08
0.7	7.1034e-22	2.2504e-21	3.0612e-07
0.8	2.7987e-18	8.8660e-18	2.2548e-06
0.9	4.1530e-15	1.3151e-14	1.3084e-05
1.0	2.8516e-12	9.0126e-12	6.1778e-05

Table 1: **Comparison of absolute error for ADM, PM and NT methods**

Example 7.2. *Consider following GNFIE*

$$z(\mu) = \left(\mu^3 - \frac{\mu^{25}}{1760} - \frac{\mu^{26}}{2200}\right) + \frac{\mu^3}{10} z^2(\mu) \left(\int_0^{\mu} (\upsilon + 1)\ z(\upsilon) d\upsilon\right) \left(\frac{1}{4}\int_0^{\mu} (\upsilon\mu)\ z^3(\upsilon) d\upsilon\right), \tag{21}$$

and $z(\mu) = \mu^3$ *is the exact solution of this equation. Applying Picard method (PM) to the Eq.* (21), *we get*

$$\begin{aligned} z_0(\mu) &= \left(\mu^3 - \frac{\mu^{25}}{1760} - \frac{\mu^{26}}{2200}\right), \\ z_n(\mu) &= \left(\mu^3 - \frac{\mu^{25}}{1760} - \frac{\mu^{26}}{2200}\right) + \frac{\mu^3}{10} z_{n-1}^2(\mu) \left(\int_0^{\mu} (\upsilon + 1)\ z_{n-1}(\upsilon) d\upsilon\right) \left(\frac{1}{4}\int_0^{\mu} (\upsilon\mu)\ z_{n-1}^3(\upsilon) d\upsilon\right), \quad n = 1, 2, ..., \end{aligned}$$

and the solution is

$$z(\mu) = \lim_{n\to\infty} z_n(\mu).$$

Applying ADM to the Eq. (21), *we get*

$$\begin{aligned} z_0(\mu) &= \left(\mu^3 - \frac{\mu^{25}}{1760} - \frac{\mu^{26}}{2200}\right), \\ z_{i+1}(\mu) &= \frac{\mu^3}{10} B_i(\mu), \quad i \geqq 0, \end{aligned}$$

where

$$B_i(\mu) = \sum_{j=0}^{i}\sum_{n=0}^{j} A_n(\mu) \left(\int_0^{\mu} (\upsilon + 1) z_{j-n}(\upsilon) d\upsilon\right) \left(\int_0^{\mu} \frac{\upsilon\mu}{4} D_{i-j}(\upsilon) d\upsilon\right)$$

29 Feb 2024 23:24:37 PST
230309-Deepmala Version 4 - Submitted to J. Integr. Eq. Appl.

is the Adomian polynomial of the nonlinear term

$$\mathcal{N}(z(\mu)) = z^2(\mu)\left(\int_0^{\mu}(\upsilon+1)\ z(\upsilon)d\upsilon\right)\left(\frac{1}{4}\int_0^{\mu}(\upsilon\mu)\ z^3(\upsilon)d\upsilon\right)$$

in Eq. (21) *and*

$$A_k(\upsilon) = \sum_{i=0}^{k} z_i(\upsilon)z_{k-i}(\upsilon), \quad D_k(\upsilon) = \sum_{j=0}^{k}\sum_{i=0}^{j} z_i(\upsilon)z_{j-i}(\upsilon)z_{k-j}(\upsilon),$$

are the Adomian polynomial of z^2 *and* z^3 *respectively.*

The approximate solution upto $(p+1)^{th}$ *terms by ADM is*

$$z(\mu) = \sum_{i=0}^{p} z_i(\mu).$$

Applying NT method to the Eq. (21)*, we get*

$$\begin{aligned} z_0(\mu_i) &= \left(\mu_i^3 - \frac{\mu_i^{25}}{1760} - \frac{\mu_i^{26}}{2200}\right), \\ z_{n+1}(\mu_i) &\cong \frac{\mu_i^3}{10} z_n^2(\mu_i)\left(h_*\sum_{j=0}^{i}\alpha_{ij}\ (\upsilon_j+1)\ z_n(\upsilon_j)\right)\left(\mu_i\ (h_*/4)\sum_{j=0}^{i}\alpha_{ij}\ \upsilon_j z_n^3(\upsilon_j)\right), \end{aligned}$$

where $\mu_i = \upsilon_i$ *for* $i = 0, 1, 2, ...n.$

Table (2) *compares the absolute error of the ADM, PM, and NT methods. Figure* (2) *shows the exact solution and the approximate solution with these three methods, along with an absolute error graph.*

Example 7.3. *Consider following GNFIE*

$$z(\mu) = \left(\mu - \frac{\mu^6}{20}(1+\mu-e^{\mu})\right) + \frac{z^2(\mu)}{5}\left(\int_0^{\mu}(\upsilon-\mu)\ e^{z(\upsilon)}d\upsilon\right)\left(\int_0^{\mu}\upsilon^2\ z(\upsilon)d\upsilon\right), \tag{22}$$

and $z(\mu) = \mu$ *is the exact solution of this equation.*

Applying PM to Eq. (22)*, we get*

$$\begin{aligned} z_0(\mu) &= \left(\mu - \frac{\mu^6}{20}(1+\mu-e^{\mu})\right), \\ z_n(\mu) &= \left(\mu - \frac{\mu^6}{20}(1+\mu-e^{\mu})\right) + \frac{z_{n-1}^2(\mu)}{5}\left(\int_0^{\mu}(\upsilon-\mu)\ e^{z_{n-1}(\upsilon)}d\upsilon\right)\left(\int_0^{\mu}\upsilon^2\ z_{n-1}(\upsilon)d\upsilon\right), \quad n = 1, 2, ..., \end{aligned}$$

and the solution is

$$z(\mu) = \lim_{n\to\infty} z_n(\mu).$$

Applying ADM to Eq. (22)*, we get*

$$\begin{aligned} z_0(\mu) &= \left(\mu - \frac{\mu^6}{20}(1+\mu-e^{\mu})\right), \\ z_i(\mu) &= \frac{1}{5}B_i(\mu), \quad i \geqq 0, \end{aligned}$$

μ	$\lvert z_{exact} - z_{Picard}\rvert$	$\lvert z_{exact} - z_{ADM}\rvert$	$\lvert z_{exact} - z_{NT}\rvert$
0.1	2.9814e-103	6.8526e-76	4.4205e-28
0.2	9.8204e-76	4.8603e-55	1.0978e-21
0.3	1.3557e-59	8.1636e-43	1.1138e-16
0.4	4.0743e-48	4.0115e-34	1.9898e-13
0.5	3.3960e-39	2.2749e-27	6.1306e-11
0.6	6.8062e-32	7.6553e-22	6.4826e-09
0.7	1.0385e-25	3.6620e-17	3.3135e-07
0.8	2.4015e-20	4.1948e-13	9.9875e-06
0.9	1.3121e-15	1.6063e-09	0.00020154
1.0	2.2928e-11	2.5114e-06	0.00297882

Table 2: **Comparison of absolute error for ADM, PM and NT methods**

where

$$B_i(\mu) = \sum_{j=0}^{i}\sum_{n=0}^{j} A_n(\mu)\left(\int_0^{\mu}(\upsilon-\mu)B_{j-n}(\upsilon)d\upsilon\right)\left(\int_0^{\mu}\upsilon^2 z_{i-j}(\upsilon)d\upsilon\right)$$

is the Adomian polynomial of the nonlinear term

$$\mathcal{N}(z(\mu)) = z^2(\mu)\left(\int_0^{\mu}(\upsilon-\mu)\ e^{z(\upsilon)}d\upsilon\right)\left(\int_0^{\mu}\upsilon^2\ z(\upsilon)d\upsilon\right),$$

in Eq. (21) *and*

$$A_k(\upsilon) = \sum_{i=0}^{k} z_i(\upsilon)z_{k-i}(\upsilon),$$

is the Adomian polynomial of z^2*. Also the Adomian polynomials for* $f(\upsilon, z(\upsilon)) = e^{z(\upsilon)}$ *are*

$$\begin{aligned} B_0(\upsilon) &= e^{z_0(\upsilon)}, \\ B_1(\upsilon) &= z_1(\upsilon)e^{z_0(\upsilon)}, \\ B_2(\upsilon) &= \frac{1}{2}z_1^2(\upsilon)e^{z_0(\upsilon)} + z_2(\upsilon)e^{z_0(\upsilon)}, \\ &\ldots\ldots \end{aligned}$$

The approximate solution upto $(p+1)^{th}$ *terms by ADM is*

$$z(\mu) = \sum_{i=0}^{p} z_i(\mu).$$

Applying NT to Eq. (22), we get

$$\begin{aligned} z_0(\mu_i) &= \left(\mu_i - \frac{\mu_i^6}{20}(1+\mu_i - e^{\mu_i})\right), \\ z_{n+1}(\mu_i) &\cong \frac{z_n^2(\mu_i)}{5}\left(h_* \sum_{j=0}^{i} \alpha_{ij}\ (\upsilon_j - \mu_i)\ e^{z_n(\upsilon_j)}\right)\left(h_* \sum_{j=0}^{i} \alpha_{ij}\ \upsilon_j^2 z_n(\upsilon_j)\right), \end{aligned}$$

where $\mu_i = \upsilon_i$ for $i = 0, 1, 2, \ldots n$.

Table (3) compares the absolute error of the ADM, PM, and NT methods. Figure (3) shows the exact solution and the approximate solution with these three methods, along with an absolute error graph.

μ	$\lvert z_{exact} - z_{Picard}\rvert$	$\lvert z_{exact} - z_{ADM}\rvert$	$\lvert z_{exact} - z_{NT}\rvert$
0.1	9.0827e-27	1.2073e-26	2.4145e-10
0.2	4.2235e-20	5.6167e-20	1.5718e-08
0.3	3.5097e-16	4.6696e-16	1.8003e-07
0.4	2.1894e-13	2.9145e-13	1.0161e-06
0.5	3.3082e-11	4.4072e-11	3.9392e-06
0.6	2.0408e-09	2.7221e-09	1.2674e-05
0.7	6.7922e-08	9.0825e-08	4.1146e-05
0.8	1.4416e-06	1.9378e-06	1.6049e-04
0.9	2.1756e-05	2.9564e-05	7.3709e-04
1.0	2.5178e-04	3.4983e-04	0.0035

Table 3: **Comparison of absolute error for ADM, PM and NT methods**

8 Conclusion

In this article, we discussed the solvability of a generalized nonlinear functional integral equation. Some examples are provided and solved by two semi-analytical methods and one numerical technique (NT) method. A detailed comparison of these three methods is also provided, along with three different examples. We have shown that all three methods give a solution that is almost close to the exact solution, but based on the absolute error graph, we conclude that the Picard method (PM) provides a more accurate solution compared to the ADM and NT methods, respectively. In the future, one may compare our results with some other methods to find more accurate solutions.

Acknowledgment

The research work of the first author is supported by the University Grant Commission (UGC), Government of India under the JRF fellowship reference No. 201610139246 (CSIR-UGC NET NOV 2020).

Competing Interests

The authors declare that there is not any conflict of interests regarding the publication of this manuscript.

References

[1] Adomian G, *Stochastic system, Academic press, New York (1983).*

[2] Adomian G, *Nonlinear stochastic operator equations, Academic press, San Diego (1986).*

[3] Adomian G, *Nonlinear stochastic systems: theory and applications to physics, Kluwer, Dordrecht (1989).*

[4] Adomian G, Rach R, Mayer R, *Modified decomposition, J. Math. Comput. 23 (1992) 17-23.*

[5] Abbaoui K, Cherruault Y, *Convergence of Adomians method applied to differential equations, Comput. Math. Appl. 28 (1994) 103-109.*

[6] Adomian G,*Solving frontier problems of physics: the decomposition method, Kluwer, Dordrecht (1995).*

[7] Abbaoui K, Cherruault Y, *Convergence of Adomians method applied to nonlinear equations, Math Comput. Model. 20 (1994) 69-73.*

[8] Abdelrazec A, Pelinovsky D, *Convergence of the Adomian decomposition method for initial-value problems, Numer. Methods Partial Differ. Equ. 27(4) (2011) 749-766.*

[9] Amar Deep, Deepmala, Mohsen Rabbani, *A numerical method for solvability of some non-linear functional integral equations, Appl. Math. Comput. 402 (2021) 125637.*

[10] Amar Deep, Deepmala, Ezzati R, *Application of Petryshyns fixed point theorem to solvability for functional integral equations, Appl. Math. Comput. 395 (2021) 125878.*

[11] Amar Deep, Deepmala, Roshan JR, *Solvability for generalized nonlinear functional integral equations in Banach spaces with applications, J. Integral Equations Applications 33 (1) Spring (2021) 19 - 30.*

[12] Banás J, Lecko M, El-Sayed WG, *Existence theorems of some quadratic integral equation, J. Math. Anal. Appl 227 (1998) 276-279.*

[13] Banás J, Martinon A, *Monotonic solutions of a quadratic integral equation of Volterra type, Comput. Math. Appl. 47 (2004) 271-279.*

[14] Banás J, Rocha Martin J, *On the solution of a quadratic integral equation of Hammerstein type, Math. Comput. Modelling. 43 (2006) 97-104.*

[15] Babolian E, Davari A, *Numerical implementation of Adomian decomposition method for linear Volterra integral equations of the second kind, J. Appl. Math. Comput. 165 (2005) 223-227.*

[16] Babolian E, Biazar J, *On the order of convergence of Adomian method, Appl. Math. and Comput., 130 (2002), 383-387.*

[17] Babolian E, Biazar J, Vahidi AR, *The decomposition method applied to systems of Fredholm integral equations of the second kind, Appl. Math. and Comput., 148 (2004), 443-452.*

[18] Biazar J, Babolian E, *Solution of a system of nonlinear Voltera integral equations of the second kind, Far. East J. Math. Sci., 2, No. 6 (2000), 935-945.*

[19] Bellomo N, Sarafyan D, *On Adomians decomposition method and some comparisons with Picards iterative scheme, J. Math. Anal. Appl. 123 (1987) 389-400.*

[20] Boumenir A, Gordon M, *The rate of convergence for the decomposition method, Numer. Funct. Anal. Optim. 25 (2004) 15-25.*

[21] Chandrasekher S, *Radiative Transfer, Dover New York 1960.*

[22] Cherruault Y, *Convergence of Adomian method, Kybernetes 18 (1989) 31-38.*

[23] Cherruault Y, Adomian G, Abbaoui K and Rach R, *Further remarks on convergence of decomposition method, Int. J. of Bio-Medical Computing. 38 (1995) 89-93.*

[24] Corduneanu C, *Integral equations and applications. New York: Cambridge University Press (1990).*

[25] Deepmala, Pathak HK, *A study on some problems on existence of solutions for some nonlinear functional-integral equations, Acta Math. Sci. 33 (5) (2013) 13051313.*

[26] Deepmala, Pathak HK, *Study on existence of solutions for some nonlinear functional-integral equations with applications, Math. Commu. 18 (2013) 97107.*

[27] Delves LM, Mohamad JL, *Computational Methods for Integral Equations, Cambridge University Press (1985).*

[28] El-Sayed AMA, Saleh MM, Ziada EAA, *Numerical and analytic solution for nonlinear quadratic integral equations, Math. Sci. Res. J. 12(8) (2008) 183-191.*

[29] El-Sayed AMA, Hashem HHG,*Carathodory type theorem for a nonlinear quadratic integral equation, Math. Sci. Res. J. 12(4) (2008) 71-95.*

[30] El-Sayed AMA and Hashem HHG, *Solvability of nonlinear Hammerstein quadratic integral equations, J. Nonlinear Sci. Appl. 2(3) (2009) 152-160.*

[31] El-Sayed AMA, Hashem HHG, Ziada EAA, *Picard and Adomian Methods for quadratic integral equation, Computational and Applied Mathematics 29 (3) (2010) 2576-2580.*

[32] El-kalla IL, *Convergence of Adomians method applied to a class of Volterra type integro-differential equations, Int. J. Differ. Equ. Appl. 10 (2) (2005) 225234.*

[33] Golberg MA, *A note on the decomposition method for operator equation, Appl. Math. Comput. 106 (1999) 215-220.*

[34] Himoun N, Abbaoui K, Cherrualt Y, *New results of convergence of Adomian's method, Kybernetes 28 (1999) 423-429.*

[35] Himoun N, Abbaoui K, Cherrualt Y, *Short new results on Adomian method, Kybernetes 32 (2003) 523-539.*

[36] Hosseini M, Nasabzadeh H, *On convergence of Adomian decomposition method, Appl. Math. Comput. 182 (2006) 536-543.*

[37] Hu S, Khavani M, Zhuang W, *Integral equations arising in the kinetic theory of gases, Appl. Anal. 34 (1989) 261-266.*

[38] Maleknejad K, Nouri K, Mollapourasl R, *Investigation on the existence of solutions for some nonlinear functional-integral equations, Nonlinear Analysis 71(12) (2009) 15751578.*

[39] Rach R, *On the Adomian (decomposition) method and comparisons with Picards method, J. Math. Anal. Appl. 128(2) (1987) 480-483.*

[40] Sadeghi Goghary H, Javadi Sh, Babolian E, *Restarted Adomian method for system of nonlinear Volterra integral equations, Appl. Math. and Comput., 161 (2005), 745-751.*

[41] Fu SZ, Wang Z, Duan JS, *Solution of quadratic integral equations by the Adomian decomposition method, CMES-Comput. Model. Eng. Sci. 92(4) (2013) 369-385.*

[42] Ziada EAA, *Numerical solution for nonlinear quadratic integral equation, Journal of Fractional Calculus and Applications 4(3S) (2013) 1-11.*

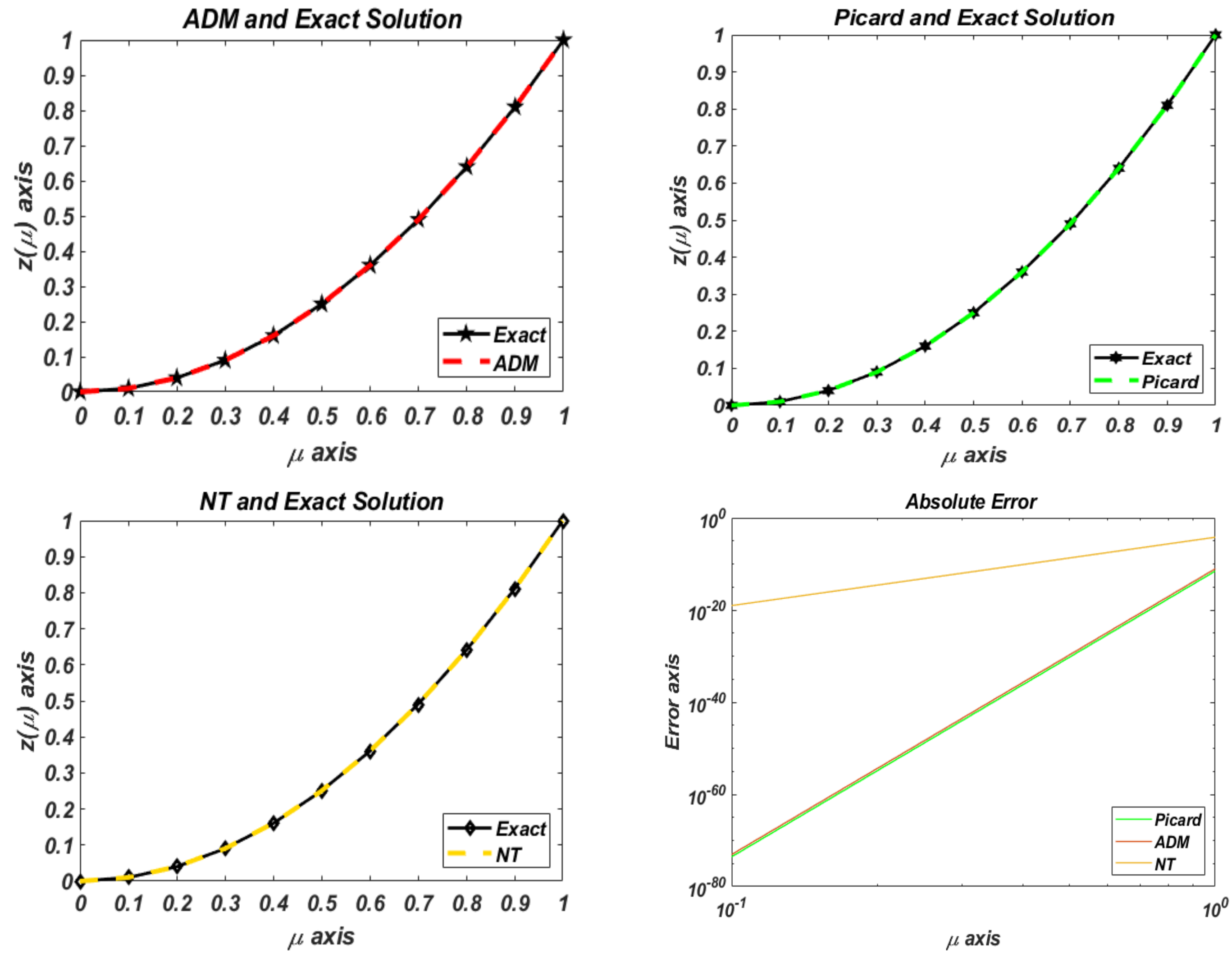

Figure 1: **Exact Solution and Approximate Solution with Absolute Error graph.**

[43] Ziada EAA, *Adomian solution of a nonlinear quadratic integral equation, Journal of Egyptian mathematical society 21 (2013) 52-56.*

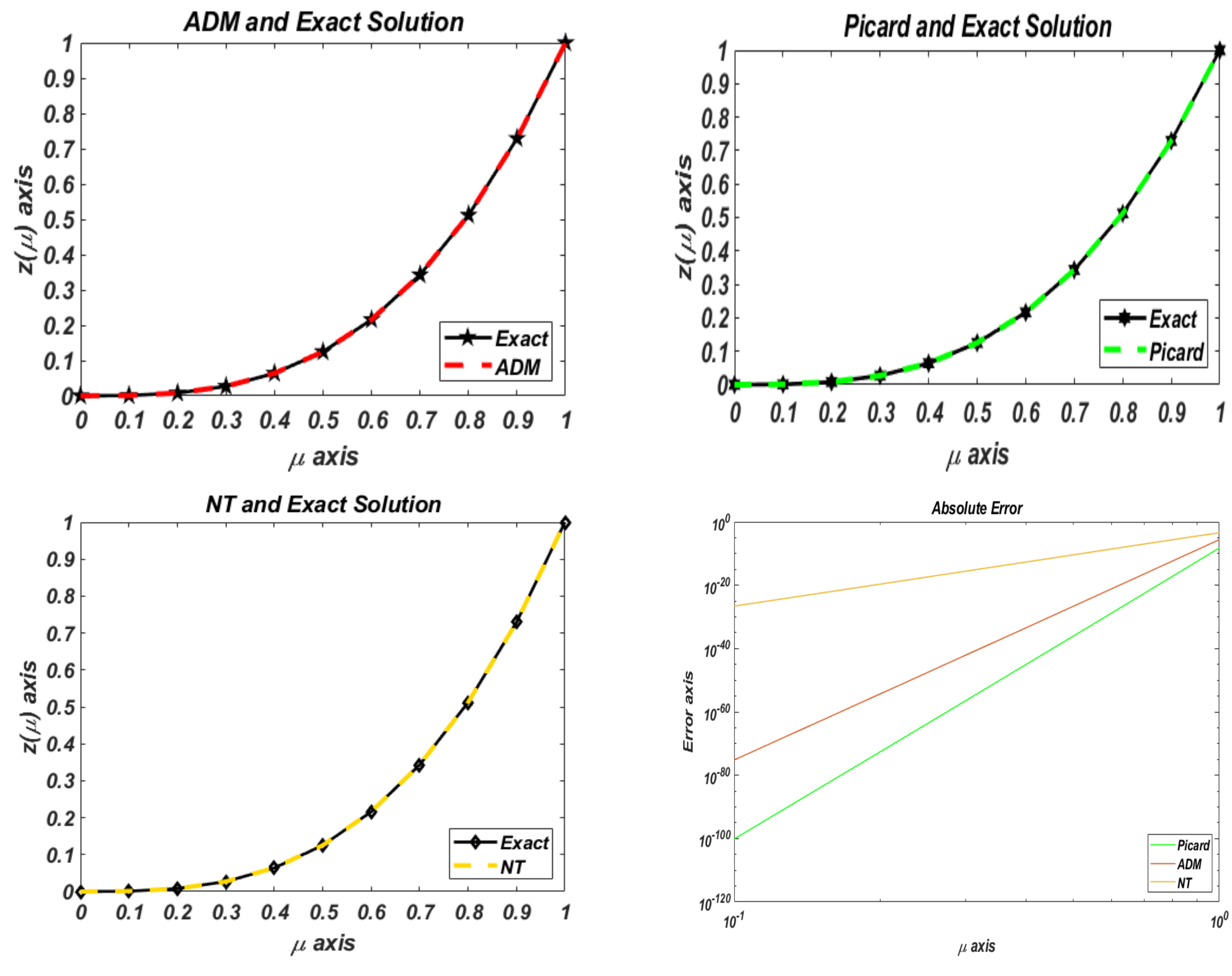

Figure 2: **Exact Solution and Approximate Solution with Absolute Error graph.**

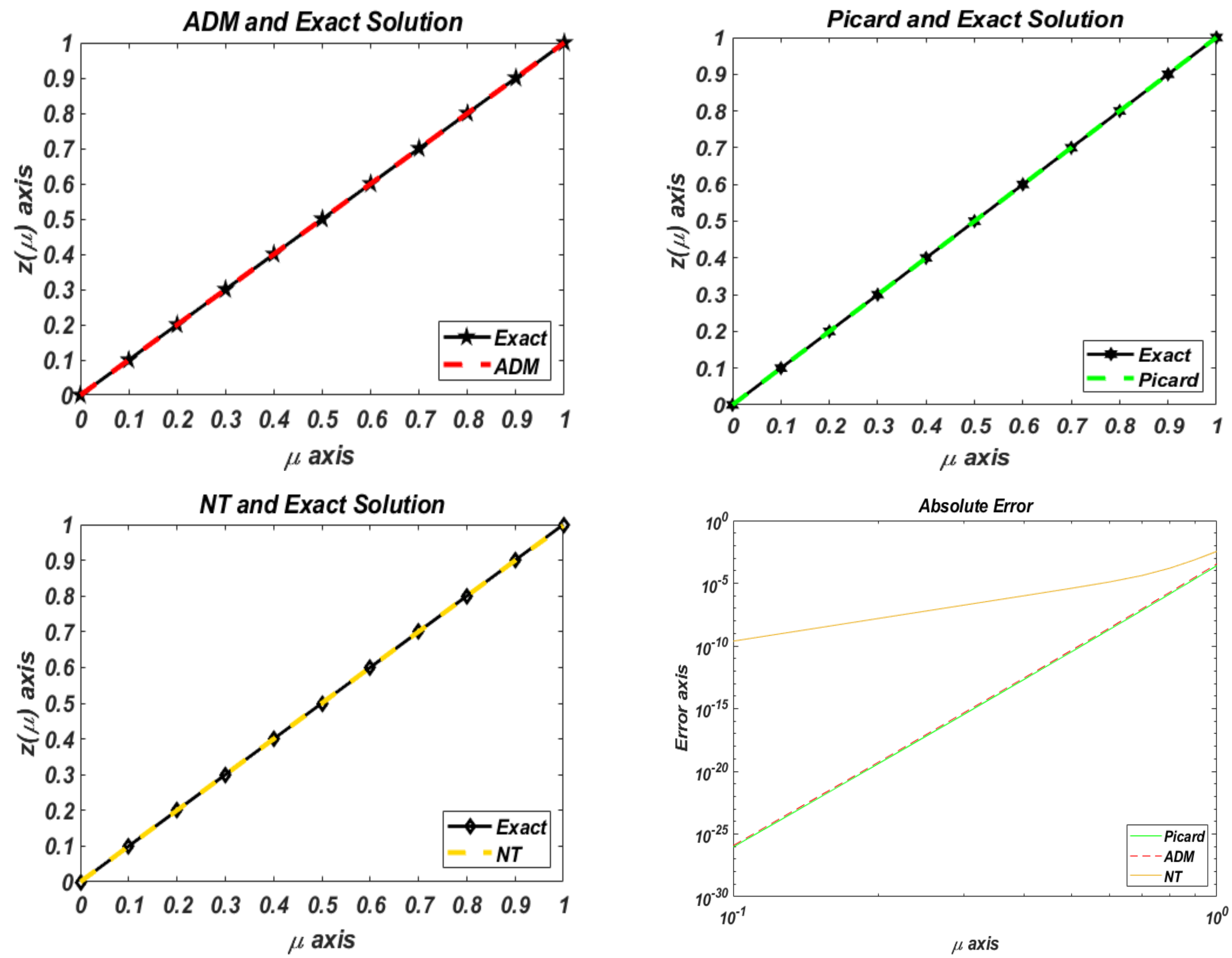

Figure 3: **Exact Solution and Approximate Solution with Absolute Error graph.**